

Radical-theoretic approach to ring theory

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The Köthe nilradical

Throughout the rest of this talk A stands for a fixed ring, unless otherwise stated.

Nilpotents are bad¹: let us quotient them out from A :

Definition (Köthe, 1930)

The Köthe radical (nilradical) $N(A)$ is the largest nil-ideal in A . i.e.

$$N(A) = \sum \{I \triangleleft A : I \text{ is a nil ring}\}.$$

Some authors prefer the name *upper nilradical* for $N(A)$.

¹are they?

The Köthe nilradical

One can describe $N(A)$ (using Zornification argument) by:

$$N(A) = \bigcap \{I \triangleleft A : A/I \text{ is prime}^2 \text{ with no non-zero nil ideals}\}$$

Examples:

- $N(A[x_1, \dots, x_n])$ consists of all polynomials with nilpotent coefficients.
- Given a $*$ -ring R such that $a^*a = 0$ implies that $a = 0$. Then $N(R) = 0$. In particular, R can be any ring of operators on a Hilbert space closed under taking adjoints (self-adjoint).

Open problem (Köthe, 1930)

Does $N(A)$ contain all nil left-ideals in A ?

²A ring R is prime if every $a, b \in R$ and for each $r \in R$ $arb = 0$ implies that either $a = 0$ or $b = 0$.

The Baer radical

By virtue of the aforementioned description of $N(A)$ it is natural to define *the Baer radical*:

Definition (Baer, 1943)

$$\beta(A) = \bigcap \{I \triangleleft A : A/I \text{ is prime}\}.$$

Again, some authors prefer the name *lower radical* or *prime radical* for $\beta(A)$ and call rings with $\beta(A) = 0$ *semiprime*.

Theorem

- The ring A is (semi-)prime if and only if $M_n(A)$ is (semi-)prime, $n > 0$.
- $\beta(M_n(A)) = M_n(\beta(A))$.

The celebrated Jacobson radical

Define in A a new operation \circ by $a \circ b = a + b - ab$ and say that $I \triangleleft A$ is *quasi-regular* if $(I, \circ, 0)$ is a group.

Remark

An idempotent element e of a ring is quasi-regular if and only if $e = 0$. Indeed, if $e = e^2$ and x is its quasi-inverse, then $x + e - xe = 0$, thus $xe + e^2 - xe^2 = 0$. Therefore $e = e^2 = 0$.

The *Jacobson radical* $J(A)$ is defined as

Definition (Jacobson, 1945)

The (unique maximal) quasi-regular ideal

$$J(A) = \sum \{I \triangleleft A : I \text{ is quasi-regular}\}.$$

The celebrated Jacobson radical

Thus we have a simple but useful observation:

Corollary

A Jacobson radical ring ($A = J(A)$) does not contain non-zero idempotents.

We have the “intersection” characterisation of $J(A)$: Recall that a ring R is (*left-*) *primitive*, if R contains a maximal left ideal I such that $xR \subseteq I$ implies $x = 0$. In this language

Theorem

$$J(A) = \bigcap \{I \triangleleft A : A/I \text{ is primitive}\}$$

The Brown–McCoy radical

Say that a ring R is a G -ring if for each $a \in R$ we have

$$a \in \{ax - x + ya - y : x, y \in R\}.$$

Definition (Brown–McCoy, 1947)

The Brown–McCoy radical of A is

$$\begin{aligned} G(A) &= \sum \{I \triangleleft A : I \text{ a } G\text{-ring}\} \\ &= \bigcap \{I \triangleleft A : A/I \text{ is a unital simple ring}\} \end{aligned}$$

Are these notions really different?

- Observe that if A is nilpotent, then $N(A) = \beta(A) = J(A) = G(A)$.

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- Are there other defined radical-like ideals of a given ring? Incidentally, ...**yes!** But they need not agree with the already mentioned when derived at nilpotent rings.
- Before we proceed to the relevant counterexamples let us summarize further similarities between these notions.

Subdirect decompositions

Given a family $\{R_t : t \in T\}$ of rings. Recall that the *subdirect sum* $\sum_{t \in T}^{\text{sub}} R_t$ is the subring of the usual direct product $\prod_{t \in T} R_t$ such that each induced projection $p_t s$ is surjective, where

- p_t is the projection from $\prod_{t \in T} R_t$ onto R_t ,
- $s: \sum_{t \in T}^{\text{sub}} R_t \rightarrow \prod_{t \in T} R_t$ is the inclusion mapping.

One should have in mind the next fact:

Theorem

A ring R is a subdirect sum of the family of rings $\{R_t : t \in T\}$ if and only if for each $t \in T$ there exists ideal I_t such that $R/I_t = R_t$ and $\bigcap_{s \in S} I_s = 0$

Subdirect decompositions

We have

- $\beta(A) = 0 \Leftrightarrow A = \sum_{t \in T}^{\text{sub}} \{A/I : \text{each } A/I \text{ is a prime ring}\}$
- $N(A) = 0 \Leftrightarrow A = \sum_{t \in T}^{\text{sub}} \{A/I : \text{each } A/I \text{ is a prime ring with no non-zero nil ideals}\}$
- $J(A) = 0 \Leftrightarrow A = \sum_{t \in T}^{\text{sub}} \{A/I : \text{each } A/I \text{ is a primitive ring}\}$
- $G(A) = 0 \Leftrightarrow A = \sum_{t \in T}^{\text{sub}} \{A/I : \text{each } A/I \text{ is a simple unital ring}\}$

Wedderburn-Artin Structure Theorem

For a left-artinian ring A (i.e. A satisfies the descending chain condition on left-ideals) we have

$$\beta(A) = 0 \leftrightarrow N(A) = 0 \leftrightarrow J(A) = 0 \leftrightarrow G(A) = 0$$

and the subdirect decomposition on a A has finitely many summands

$$A = A_1 \boxplus A_2 \boxplus \dots \boxplus A_n$$

where each A_i is an artinian ring isomorphic to a matrix ring over a division ring.

Jacobson density theorem

A subring R of the ring $\text{Hom}(V)$ (where V is a vector space) is said to be **dense**³ in $\text{Hom}(V)$ if for each n -tuples (a_1, \dots, a_n) and (b_1, \dots, b_n) there exists a linear map $T \in R$ such that $Ta_i = b_i$.

Theorem (Jacobson density theorem)

A ring is primitive if and only if and only if it is isomorphic to a dense subring of $\text{Hom}(V)$, where V is a vector space over a division ring.

One may think about the von Neumann bicommutant theorem as a operator-theoretic counterpart to this theorem.

³it should not be confused with the topological notion of density: finite rank operators on a separable Hilbert space H are dense in this sense in $\text{Hom}(H)$ but are not topologically dense even in $\mathcal{B}(H)$.

Distinguishing the radicals

It is well known that if V is a countably infinite dimensional vector space over a field, then the family \mathcal{F}_V of finite rank operators on V is a simple non-unital subring of $\text{Hom}(V)$.

Using the Jacobson Density Theorem we see that:

$$0 = J(\mathcal{F}_V) \neq \mathcal{F}_V = G(\mathcal{F}_V)$$

Now consider the subring A of \mathbb{Q} given by

$$A = \left\{ \frac{2x}{2y+1} : x, y \in \mathbb{Z}, \gcd(2x, 2y+1) = 1 \right\}$$

Distinguishing the radicals

For any $a = \frac{2x}{2y+1} \in A$ the equation $a \circ z = a + z - az = 0$ has a solution

$$z = \frac{a}{a-1} = \frac{2x}{2(x-y-1)+1} \in A$$

whence A is quasi regular. Note that A has no non-zero nilpotent elements, thus

$$0 = N(A) \neq A = J(A).$$

To show that $\beta(A) \neq N(A)$ for some A it suffices to construct a nil ring which is prime as well. The construction was done by Baer and, independently, by Zelmanov and is beyond the scope of this talk. On the other hand, if A is noetherian, then $\beta(A) = N(A)$.

Distinguishing the radicals

Recall that A is noetherian if it satisfies the ascending chain condition. In this case we have

Theorem (Levitzki)

If A is noetherian, then $N(A)$ is nilpotent

Indeed, we note that the set of nil ideals of A has a maximal element, say N . Since the sum of nilpotent ideals is nilpotent as well, N contains every nilpotent ideal. Suppose that $N(A/N) \neq 0$. The ring A/N is noetherian. Since each noetherian ring with a non-zero nil one-side ideal has a non-zero nilpotent ideal (this is highly non-trivial!) A/N has a non-zero nilpotent ideal I/N , where I is a suitable ideal of A .

Distinguishing the radicals

In that case $I^k \subseteq N$ for some k , so I is nil, hence $I = N$, but this is absurd as $I/N = 0$.

We have *proved* that $N(A) \subseteq N$. The opposite inclusion is clear, thus we have

Corollary

If A is noetherian, then $N(A) = \beta(A)$.

We have also

Theorem

If A is commutative, then

$$N(A) = \beta(A) = \sqrt{A} := \{a \in A : a \text{ is nilpotent}\}.$$

Radical theory in analysis

In the unital case,

- $J(A) = \bigcap \{I \triangleleft A : I \text{ is maximal left-ideal}\}$
- $G(A) = \bigcap \{I \triangleleft A : I \text{ is a maximal ideal}\}$

For a Banach space E , we have $J(\mathcal{B}(E)) = 0$, however the Brown-McCoy radical of $\mathcal{B}(E)$ need not be trivial - take:

- $E = \ell^2$ (Calkin)
- $E = c_0$ or $E = \ell^p$, $p \in [1, \infty)$ (Gohberg, Markus and Feldman)
- $E = C[0, \omega^\omega]$ (folklore?)
- $E =$ the James space (Laustsen)
- $E = C[0, \omega_1]$ (K. and Laustsen)

For each E above, the space $\mathcal{B}(E)$ has the unique maximal ideal.

Radical theory in analysis

Yood observed that the Jacobson radical of $\mathcal{B}(E)/\overline{\mathcal{F}}(E)$ is non-zero for certain Banach spaces E . Kleinecke then defined the ideal of *inessential operators* by

$$\mathcal{E}(E) = \pi^{-1}J(\mathcal{B}(E)/\overline{\mathcal{F}}(E)).$$

On the other hand, one can prove that

$$\pi^{-1}G(\mathcal{B}(E)/\overline{\mathcal{F}}(E)) = G(\mathcal{B}(E))$$

for each infinite-dimensional Banach space E .

Radical theory in analysis

Using the Gelfand-Naimark theorem one can prove that Jacobson radical of each C^* -algebra is trivial. Nevertheless, there are commutative radical Banach algebras (i.e. equal to its Jacobson radical).

One can prove using the Spectral Radius Formula that a good example is $L^1[0, 1]$ with the multiplication given by

$$(fg)(x) = \int_0^x f(x-y)g(y)dy.$$

Of course, a pathological example can be any Banach space E with $x \cdot y = 0$ for each $x, y \in E$. The theory of radical Banach algebras is a huge branch of modern functional analysis.

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