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NON-REGULAR TANGENTIAL BEHAVIOUR OF A MONOTONE MEASURE

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THE FINAL VERSION IS DIFFERENT IN SOME DETAILS, INCL. EQUATION NUMBERING.

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Abstract

A Radon measure μ on $\mathbb{R}^n$ is said to be k-monotone if $r \mapsto \frac{\mu(B(x,r))}{r^k}$ is a non-decreasing function on $(0,\infty)$ for every $x \in \mathbb{R}^n$. (If $\mu = \mathcal{H}^k \llcorner A$ is the k-dimensional Hausdorff measure restricted to a k-dimensional minimal surface A then this important property is expressed by the monotonicity formula.) We give an example of a 1-monotone measure in $\mathbb{R}^2$ with non-unique and non-conical tangent measures at a point. Furthermore we show this measure can be of the form $\mathcal{H}^k \llcorner A$.

1. *Introduction*

Monotonicity formula and *tangent planes* are important tools for the regularity results in geometric measure theory (see e.g. [**9**]) which are, however, limited to low dimension or codimension generalized surfaces (varifolds etc.). Clarifying the relations between the two notions could help to solve the general case which remains open. For an attempt to proof some regularity of monotone measures see [**5**]. In this paper we give an example of a monotone measure with bad tangential behaviour at a point.

Definition. Let μ be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that μ is *k-monotone* if the function $r \mapsto \frac{\mu(B(x,r))}{r^k}$ is non-decreasing on $(0,\infty)$ for every $x \in \mathbb{R}^n$.

An important example of a k-monotone measure is $\mathcal{H}^k \llcorner A$ where A is a k-dimensional minimal surface in $\mathbb{R}^n$.

The tangent measures of μ at $a \in \mathbb{R}^n$ are defined by blowing up μ by sequences of expansive homotheties around a, normalizing suitably and taking weak limits. The map $T_{a,r}$ that blows up $B(a,r)$ to $B(0,1)$ is given by

$$T_{a,r}(x) = (x-a)/r.$$

If μ is a k-monotone measure then μ has a finite k-dimensional density $\theta^k_a\mu = \lim_{r\searrow 0} \frac{\mu(B(a,r))}{r^k\omega_k}$, (where $\omega_k = \mathcal{L}^k B_{\mathbb{R}^k}(0,1)$), for every point a. Therefore it is natural to normalize the blow-up by multiplier $\frac{1}{r^k}$. If $\theta^k_a\mu = 1$ then the following definition coincides with the one given in [**7**, 14.1] where arbitrary normalization multipliers are allowed. Tangent measures were introduced in [**8**].

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Definition. Let μ be a Radon measure on $\mathbb{R}^n$, $a \in \operatorname{spt}\mu$ and $k \le n$. We say that ν is a *k-tangent measure of μ at a* and write $\nu \in \operatorname{Tan}_a^k \mu$ if ν is a non-zero Radon measure on $\mathbb{R}^n$ and if there exists a sequence (r_i) of positive numbers such that $r_i \to 0$ and

$$\frac{1}{r^k} T_{a,r_i\,\#}(\mu) \to \nu \qquad \text{weakly as } i \to \infty,$$

i.e., for every continuous function ϕ on $\mathbb{R}^n$ with compact support

$$\lim_{i\to\infty} \frac{1}{r^k} \int \phi \,\mathrm{d}T_{a,r_i\,\#}(\mu) = \lim_{i\to\infty} \frac{1}{r^k} \int \phi((x-a)/r_i)\,\mathrm{d}\mu x = \int \phi \,\mathrm{d}\nu.$$

Definition. A Radon measure ν on $\mathbb{R}^n$ is said to be *k-conical* if $\lambda^{-k}\nu(\lambda A) = \nu(A)$ for every Borel $A \subset \mathbb{R}^n$ and every $\lambda > 0$. (Equivalently: $\lambda^{-k} T_{0,\lambda\,\#}(\nu) = \nu$ for every $\lambda > 0$.) If $k = 1$ we drop the prefix and write simply *monotone*, *tangent*, *conical*, respectively.

If μ is a Borel measure in $\mathbb{R}^n$ and h a Borel measurable function then $h\mu$ (written also as $\frac{1}{\|\cdot\|}\mu$ in the case $h(x) = \frac{1}{\|x\|}$) denotes the measure which has density h with respect to μ (i.e., $(h\mu)(M) = \int_M h \,\mathrm{d}\mu$). If A is a Borel set, then $\mu \llcorner A$ is the restriction of μ to A ($(\mu \llcorner A)(M) = \mu(M \cap A)$). $\mathcal{H}^1$ stands for one-dimensional Hausdorff measure, $\mathcal{L}^2$ is the Lebesgue measure in $\mathbb{R}^2$.

The example is based on the idea that non-monotonicity of a measure can be sometimes compensated by adding a measure which is abundant in monotonicity. For example, if $h|_{\mathbb{R}\times\{0\}}$ is a $\frac{\pi}{2}$-Lipschitz function and $h|_{\mathbb{R}\times(\mathbb{R}\setminus\{0\})} = 0$ then $\mathcal{L}^2 + h\mathcal{H}^1$ is always a monotone measure in $\mathbb{R}^2$, although $h\mathcal{H}^1$ itself is monotone if and only if $h|_{\mathbb{R}\times\{0\}}$ is convex. The strength of monotonicity of $\mathcal{L}^2$ (its ability of compensation) is expressed by $\frac{\mathrm{d}}{\mathrm{d}r}(\mathcal{L}^2(B(x,r))/r) = \pi > 0$.

For a more interesting example, let γ be a simple closed $\mathcal{C}^2$ curve in $\mathbb{R}^2$. Then there is a number $K > 0$ such that $K\mathcal{L}^2 + \mathcal{H}^1 \llcorner \langle\gamma\rangle$ is a monotone measure. Using a nice recent result of Robert Černý [**1**] (see Section 4), $K\mathcal{L}^2$ can be replaced by the measure $\mathcal{H}^1 \llcorner (\varepsilon\mathbb{Z} \times \mathbb{R})$, $\varepsilon > 0$, supported by vertical lines provided γ has non-zero curvature everywhere.

However, the idea of compensation is not itself powerful enough and we need to do a bit more to get a monotone measure with bad tangential behavior. Naturally, we will use $\frac{1}{\|\cdot\|}\mathcal{L}^2$ — a Radon measure which is able to compensate much more than $\mathcal{L}^2$ near to the origin. Further ingredient is symmetry which together with smoothness of our objects helps us to evade terms of order of $\|x\|$ (see the way we use Taylor's formula in Lemma 3), which can not be compensated by $\frac{1}{\|\cdot\|}\mathcal{L}^2$.

The main result of this paper is

Theorem 1. *There is a monotone measure on $\mathbb{R}^2$ which has non-conical (and hence non-unique) tangents at 0. There is also a monotone measure μ on $\mathbb{R}^2$, such that all tangent measures to μ are conical but the tangent measure to μ at 0 is non-unique.*

Theorem 1 follows from Proposition 1 as explained in Examples 1 and 2 in Section 3. For an improvement of the first part of Theorem with μ in the form $\mathcal{H}^1 \llcorner A$ see Section 5. This μ is $\mathcal{H}^1$ restricted to the union of a two-branch logarithmic spiral and finitely many lines through the origin and therefore it negatively answers question in the beginning of [**5**, Remark 5.5]. Indeed, μ is concentrated (in the sense of [**5**]), monotone and satisfies $\mu(B(0,r)) = 2cr$ for $r > 0$, ($c \in \mathbb{R}$), but μ is not conical. Another question in [**5**, the paragraph above Proposition 5.7] which asks about uniqueness of tangent measures (referring to the uniqueness of limits in Prop. 5.4) is solved by the same μ.

Section 4 contains some remarks on more recent development.

2. *Strong monotonicity of* $\frac{1}{\|\cdot\|}\mathcal{L}^2$

The following two lemmata are technical. However, the second lemma quantifies the monotonicity of $\nu = \frac{1}{\|\cdot\|}\mathcal{L}^2$ and it will be used in the proof of Theorem 1.

For $\alpha \in (-\pi/2, \pi/2)$, let $L_\alpha = \{(t\cos\alpha, t\sin\alpha) : t \in \mathbb{R}\}$, denote the lines through the origin. Since for every $x \in \mathbb{R}^2$, $r > 0$ and $A = B(x,r)$,

$$\nu A = \int_A \frac{1}{\|x\|}\,\mathrm{d}\mathcal{L}^2(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{H}^1(L_\alpha \cap A)\,\mathrm{d}\alpha$$

we are interested in the monotonicity of the lines. If $x = (x_1, 0)$ is fixed with $x_1 \geq 0$ (we loose no generality with this assuption) it is represented by the quantity

$$h(\alpha, r) = \frac{\mathcal{H}^1(L_\alpha \cap B(x,r))}{r} = \begin{cases} \frac{2\sqrt{r^2 - d(\alpha)^2}}{r} & |\alpha| \in (0, \alpha_1(r)) \\ 0 & |\alpha| \in [\alpha_1(r), \pi/2). \end{cases}$$

where $d(\alpha) = \|x\|\sin|\alpha|$ and $\alpha_1(r) = \arcsin\frac{r}{\|x\|}$ if $r < \|x\|$ and $\alpha_1(r) = \pi/2$ otherwise.

Lemma 1 (An integrable majorant). *If $r, s > 0$ and $r \neq s$ then*

$$\left|\frac{h(\alpha,s) - h(\alpha,r)}{s - r}\right| \leq \phi_r(\alpha) := \begin{cases} \frac{2}{r}\sqrt{\frac{r + d(\alpha)}{r - d(\alpha)}} & |\alpha| \in (0, \alpha_1(r)), \\ \frac{2}{\sqrt{r}\sqrt{d(\alpha) - r}} & |\alpha| \in (\alpha_1(r), \pi/2). \end{cases}$$

If $\alpha_1(r) \neq \pi/2$, i.e., $r < \|x\|$ then ϕ_r is obviously an integrable function.

Proof.

Since the derivative $\frac{\mathrm{d}}{\mathrm{d}\xi}h(\alpha,\xi) = \frac{d(\alpha)^2}{\xi^2\sqrt{\xi^2 - d(\alpha)^2}}$ is positive and nonincreasing for $\xi \in (d(\alpha), \infty)$,

$$\left|\frac{h(\alpha,s) - h(\alpha,r)}{s - r}\right| \leq \left|\frac{h(\alpha, d(\alpha)) - h(\alpha,r)}{d(\alpha) - r}\right| = \frac{h(\alpha,r)}{r - d(\alpha)} = \phi_r(\alpha)$$

whenever $r, s \in [d(\alpha), \infty)$, $r \neq s$, $r \neq d(\alpha)$. The estimate is also true for $0 < s < d(\alpha) < r$ because $h(\alpha, \cdot)$ is constant zero on $(0, d(\alpha)]$.

In the remaining case $0 < r < d(\alpha) \leq s$ it is convenient to substitute $d(\alpha) = r + a$ and $s = r + a + b$ with $a > 0$, $b \geq 0$. We want to show

$$\frac{2\sqrt{s^2 - d(\alpha)^2}}{s(s-r)} \leq \frac{2}{\sqrt{r}\sqrt{d(\alpha) - r}}$$

which is equivalent to

$$(b^2 + 2ab + 2br)ra \leq ((a+b)^2 + r(a+b))^2$$

and this is the the sum of obvious inequalities

$$0 \le (a+b)^4, \qquad (b^2+2ab)ra \le 2(a+b)^3 r, \qquad 2abr^2 \le (a+b)^2 r^2. \qquad \square$$

LEMMA 2 (Strong monotonicity). *For* $\nu = \frac{1}{\|\cdot\|}\mathcal{L}^2$, $x \in \mathbb{R}^2$, $r > 0$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(x,r))}{r} \ge \frac{\pi}{2}\frac{\|x\|^2}{r^3} \qquad \text{if } \|x\| \le r, \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(x,r))}{r} \ge \frac{\pi}{2C^2}\frac{1}{r} \qquad \text{if } C \ge \pi \text{ and } \frac{\|x\|}{C} \le r \le C\|x\|, \tag{2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(x,r))}{r} \ge \frac{1}{2\|x\|} \qquad \text{if } r \le \|x\|. \tag{2.3}$$

Proof. We will prove the first and last assertions, (2.2) is their trivial corollary. In the case $x = (0,0)$ we have $\nu(B(0,r)) = 2\pi r$ and $\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(0,r))}{r} = 0$. Fix $x \in \mathbb{R}^2 \setminus \{(0,0)\}$. Without any loss of generality $x = (x_1, 0)$ with $x_1 > 0$.

Assume first that $r \in (0, \|x\|)$. By Lemma 1 and dominated convergence theorem,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(x,r))}{r} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}}{\mathrm{d}r} h(\alpha,r)\,\mathrm{d}\alpha \\ &= \int_{-\alpha_1}^{\alpha_1} \frac{2d(\alpha)^2}{r^2\sqrt{r^2-d(\alpha)^2}}\,\mathrm{d}\alpha \ge \int_{-\alpha_1}^{\alpha_1} \frac{2d(\alpha)^2}{r^3}\,\mathrm{d}\alpha,\end{aligned} \tag{2.4}$$

where $\alpha_1 = \alpha_1(r)$. For $|\alpha| \in (\alpha_1/2, \alpha_1)$ we have $d(\alpha) \ge \frac{1}{2}\|x\| \sin\alpha_1 = \frac{1}{2}r$ therefore $2d(\alpha)^2 r^{-3} \ge (2r)^{-1} \ge (2\alpha_1\|x\|)^{-1}$ and thus (2.3) follows from (2.4).

Assuming $r > \|x\|$, we have (2.4) with $\alpha_1 = \frac{\pi}{2}$. For $|\alpha| \in (\frac{\pi}{4}, \frac{\pi}{2})$, $\frac{\sqrt{2}}{2}\|x\| \le d(\alpha)$ and we obtain (2.1).

The remaining case is $r = \|x\| > 0$ when $\frac{\mathrm{d}}{\mathrm{d}r}\frac{\nu(B(x,r))}{r} = \infty$. This is not important for this paper but we proof it for the sake of completness. Since $f(r) := \frac{\nu B(x,r)}{r}$ is continuous on $(0,\infty)$ we have by (2.4) and Fatou's lemma

$$f'_+(\|x\|) = \lim_{s \searrow \|x\|} f'(s) = \lim_{s \searrow \|x\|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2d(\alpha)^2}{s^2\sqrt{s^2-d(\alpha)^2}}\,\mathrm{d}\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\sin^2|\alpha|}{\|x\|\cos|\alpha|}\,\mathrm{d}\alpha = \infty.$$

The integrand is decreasing in s and estimating it by its value at $s = \|x\|$ we obtain

$$f'_-(\|x\|) = \lim_{s \nearrow \|x\|} f'(s) \ge \lim_{s \nearrow \|x\|} \int_{-\alpha_1(s)}^{\alpha_1(s)} \frac{2\sin^2|\alpha|}{\|x\|\cos|\alpha|}\,\mathrm{d}\alpha = \infty$$

because $\alpha_1(s) \to \pi/2$. $\square$

3. *Proof of Theorem 1*

The measure sought after will be of the form $\mu + \frac{K}{\|\cdot\|}\mathcal{L}^2$. First we specify the measure $\mu = \mu((b_n),(a_n))$, together with the parameters q, (b_n), (a_n) and the functions h_1, h_2 which we will use throughout the section. We proof Proposition 1 which assures the measure is monotone for a suitable constant $K = K(q)$. Then Examples 1 and 2 explain how the tangent measures depend on parameters (b_n) and (a_n).

Construction. Now let us fix ones for all an arbitrary function $\phi_0 \in \mathcal{C}^2(\mathbb{R})$ with $0 \le \phi_0 \le 1$, $\phi_0(x) = 0$ for $x \le 0$ and $\phi_0(x) = 1$ for $x \ge 1$. Let $q \in (0,1)$ and let $(b_n)_{n\in\mathbb{Z}}$, $(a_n)_{n\in\mathbb{Z}}$ be sequences of positive numbers such that $b_n \le qa_n$ and $a_{n+1} \le qb_n$ for every $n \in \mathbb{Z}$. Let $h_1, h_2 \in \mathcal{C}^2(\mathbb{R}\setminus\{0\})$ be defined by

$$h_1(x) = \begin{cases} \phi_0\left(\frac{x-b_n}{a_n-b_n}\right) & x \in [b_n, a_n],\ \ n \text{ even} \\ \phi_0\left(1-\frac{x-b_n}{a_n-b_n}\right) & x \in [b_n, a_n],\ \ n \text{ odd} \\ 0 & x \in [a_{n+1}, b_n],\ \ n \text{ even} \\ 1 & x \in [a_{n+1}, b_n],\ \ n \text{ odd} \\ h_1(-x) & x < 0 \end{cases}$$
$$h_2(x) = 1 - h_1(x) \qquad x \in \mathbb{R}\setminus\{0\}.$$

Let $L_1 = \max_{[0,1]} |\phi_0'|/(1-q)$ and $L_2 = \max_{[0,1]} |\phi_0''|/(1-q)^2$. Recall $a_n - b_n \ge (1-q)a_n$ to see that, for $\xi \in [b_n, a_n]$ $(n \in \mathbb{Z})$, actually for all $\xi \ne 0$,

$$|h_1'(\xi)| \le L_1/|\xi| \quad \text{and} \quad |h_1''(\xi)| \le L_2/\xi^2. \tag{3.1}$$

In particular, if $a > 0$ then h_1 is $\frac{L_1}{a}$-Lipschitz on $[a, \infty)$.

Let

$$h(x_1, x_2) = \begin{cases} h_1(x_1) & \text{if } x_1 \ne 0,\ x_2 = 0, \\ h_2(x_2) & \text{if } x_1 = 0,\ x_2 \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu = \mu((b_n), (a_n)) := h\mathcal{H}^1$ be the one-dimensional Hausdorff measure on x-axis with density h_1 plus the one-dimensional Hausdorff measure on y-axis with density h_2, i.e.,

$$\mu(A) = \int\limits_{\{x_1:(x_1,0)\in A\}} h_1(x_1)\,\mathrm{d}x_1 + \int\limits_{\{x_2:(0,x_2)\in A\}} h_2(x_2)\,\mathrm{d}x_2 \tag{3.2}$$

for every Borel set $A \subset \mathbb{R}^2$.

Proposition 1. *There exists a constant $K = K(q) > 0$ such that $\mu + \frac{K}{\|\cdot\|}\mathcal{L}^2$ is a monotone measure.*

The proof comes on page 7 after we prepare estimates for μ corresponding to each of the three cases of Lemma 2.

Lemma 3. *Let $c \in (0,1)$. Then there exists a constant $K_1 = K_1(q,c)$ such that if $x \in \mathbb{R}^2$, $r > 0$ and $\|x\| < cr$ then*

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} \ge -K_1\frac{\|x\|^2}{r^3}.$$

Proof. If a_n/b_n does not depend on n and c is small enough we can use the similarity to restrict r, and hence also the pairs (x,r), to a compact set. The estimated quantity (denoted by Φ below) is a $\mathcal{C}^2$ and even function which is zero at $x = 0$, therefore it is at most of the order of $\|x\|^2$. If the distances to the origin of the intersections of $\partial B(x,r)$ with the axes are in $[b_{n+2k}, a_{n+2k}]$ (or close to it), k even, and $\alpha := a_n/a_{n+2k}$ we have $\Psi(x,r)/r = \Psi(\alpha x, \alpha r)/\alpha r$. Thus $\Phi(x,r) = \alpha\Phi(\alpha x, \alpha r)$

is at most of the order $\alpha\|\alpha x\|^2 \sim \|x\|^2/r^3$. We proceed by explaining technical details of the proof in a way not limited to this particular case.

For $\|x\| < r$, $x = (x_1, x_2)$, let $q_1^{\pm}(x,r) = x_1 \pm \sqrt{r^2 - x_2^2}$, $q_2^{\pm}(x,r) = x_2 \pm \sqrt{r^2 - x_1^2}$ be the coordinates of the intersection points of $\partial B(x,r)$ and the axes.

By the triangle inequality in $\mathbb{R}^2$ we get, for $\|x\| < cr$, $(1-c)r \le r - \|x\| \le |q_i^{\pm}(x,r)| \le r + \|x\| \le (1+c)r$ for $i = 1,2$ and for an arbitrary choice of sign $\pm$, i.e., $|q_i^{\pm}(x,r)| \in I := [(1-c)r, (1+c)r]$.

Let

$$\Psi(x,r) := \mu(B(x,r)) = \int_{q_1^-(x,r)}^{q_1^+(x,r)} h_1 + \int_{q_2^-(x,r)}^{q_2^+(x,r)} h_2,$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Psi(x,r) =& \big(h_1(q_1^-(x,r)) + h_1(q_1^+(x,r))\big)\frac{r}{\sqrt{r^2 - x_2^2}} \\ &+ \big(h_2(q_2^-(x,r)) + h_2(q_2^+(x,r))\big)\frac{r}{\sqrt{r^2 - x_1^2}}.\end{aligned}$$

We want to estimate from below the function

$$\Phi(x,r) := \frac{\mathrm{d}}{\mathrm{d}r}\frac{\Psi(x,r)}{r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Psi(x,r) - \frac{1}{r^2}\Psi(x,r) \tag{3.3}$$

Clearly, $\Phi(\cdot, r)$ is $\mathcal{C}^2$-smooth on $\{x : \|x\| < cr\}$ and its second differential $\mathrm{D}_x^2\Phi$ can be expressed using the values of h_i and the first and second derivatives of h_i and $q_i^{\pm}$. Using (3.1) for $\xi \in I$ it is easy to calculate there is a constant $K_1 = K_1(L_1, L_2, c) = K_1(q,c)$ such that $\|\mathrm{D}_x^2\Phi(x,r)\| \le K_1/r^3$ whenever $\|x\| < cr$. Note that

$$\begin{aligned}\Psi((-x_1,x_2),r) &= \Psi((x_1,x_2),r) = \Psi((x_1,-x_2),r)\\ \Phi((-x_1,x_2),r) &= \Phi((x_1,x_2),r) = \Phi((x_1,-x_2),r)\end{aligned}$$

hence $\mathrm{D}_x\Phi(0,r) = 0$. Furthermore $\Psi(0,r) = \int_{-r}^{r}(h_1 + h_2) = 2r$ and therefore $\Phi(0,r) = 0$. Using Taylor's formula w.r. to x we see that $|\Phi(x,r)| \le K_1\|x\|^2/r^3$ whenever $\|x\| < cr$. □

LEMMA 4. *For every $x \in \mathbb{R}^2$ and every $r > 0$, $|x_1| \ne r \ne |x_2|$*

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} \ge -\frac{4}{r}.$$

Proof. Clearly $\mu(B(x,r)) \le 4r$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\mu(B(x,r)) - \frac{1}{r^2}\mu(B(x,r)) \ge -\frac{1}{r^2}\mu(B(x,r)) \ge -\frac{4}{r}. \quad \square$$

LEMMA 5. *Let $c \in (0,1)$. Then there exists a constant $K_2 = K_2(q,c)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} \ge -K_2\frac{1}{\|x\|}$$

whenever $x = (x_1,x_2) \in \mathbb{R}^2$, $0 < r < c\|x\|$ and $|x_1| \ne r \ne |x_2|$.

Proof. For simplicity, assume that $B(x,r)$ does not intersect the y-axis, i.e., the second term in (3.2) is zero. Assume $B(x,r)$ intersects the x-axis (otherwise $\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} = 0$). Let $q_1^{\pm} = q_1^{\pm}(r) = x_1 \pm \sqrt{r^2 - x_2^2}$ like in the proof of Lemma 3;

now obviously $\operatorname{sgn} q_1^- = \operatorname{sgn} q_1^+$. Without loss of generality $q_1^- > 0$ and $q_1^+ > 0$. We have

$$\mu(B(x,r)) = \int_{q_1^-(r)}^{q_1^+(r)} h_1$$

and (cf. (3.3))

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} = \frac{1}{r}\frac{r}{\sqrt{r^2-x_2^2}}\left(h_1(q_1^-)+h_1(q_1^+)\right) - \frac{1}{r^2}\int_{q_1^-}^{q_1^+} h_1.$$

By $h_1 \geq 0$, $r^2/\sqrt{r^2-x_2^2} \geq r \geq \sqrt{r^2-x_2^2} = (q_1^+ - q_1^-)/2$ and by the Lipschitz continuity of h_1 (see p. 5) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mu(B(x,r))}{r} &\geq \frac{1}{r^2}\int_{q_1^-}^{q_1^+}\frac{h_1(q_1^-)-h_1(t)+h_1(q_1^+)-h_1(t)}{2}\,\mathrm{d}t\\ &\geq -\frac{1}{r^2}\frac{L}{q_1^-}(q_1^+-q_1^-)^2 \geq -\frac{1}{r^2}\frac{L}{\|x\|(1-c)}4r^2 = -\frac{4L_1}{1-c}\frac{1}{\|x\|}. \qquad \square\end{aligned}$$

Proof of Proposition 1. Let $c \in (0, 1/\pi)$ be arbitrary, $C = \frac{1}{c}$. Let $K = \max(\frac{2}{\pi}K_1, \frac{8}{\pi}C^2, 2K_2)$ and let $\mu_K = \mu + \frac{K}{\|\cdot\|}\mathcal{L}^2$. Then, by Lemma 3–5 together with Lemma 2, for all $x = (x_1, x_2) \in \mathbb{R}^2$, the continuous function $r \mapsto \mu_K(B(x,r))/r$ has a positive derivative on $(0,\infty) \setminus \{|x_1|, |x_2|, \|x\|\}$. Hence μ_K is a monotone measure. $\square$

In what follows, $K = K(q)$ denotes the constant from Proposition 1. Note that $\frac{1}{\|\cdot\|}\mathcal{L}^2$ is 1-conical and therefore it plays only a trivial role in $\operatorname{Tan}^1_{(0,0)}$.

Example 1. Let $q \in (0,1)$ and $a_n = q^{2n}$, $b_n = qa_n$ for $n \in \mathbb{Z}$. Then $\mu_K := \mu((b_n),(a_n)) + \frac{K}{\|\cdot\|}\mathcal{L}^2$ is a monotone measure in $\mathbb{R}^2$, that has non-conical (and non-unique) tangent measures at $(0,0)$. Indeed,

$$\operatorname{Tan}^1_{(0,0)}(\mu_K) = \{\lambda^{-1}T_{(0,0),\lambda\,\#}(\mu_K) : \lambda > 0\}.$$

Proof. Obviously $q^{-2k}T_{(0,0),q^{2k}\,\#}(\mu_K) = \mu_K$ hence taking $r_k = q^{2k}$ we see the inclusion $\supset$ must be true. The oposite inclusion is an easy excercise on weak convergence of measures. $\square$

Example 2. (a) Let $q \in (0,1)$ and $a_n = q^{2n^3}$, $b_n = qa_n$ for $n \in \mathbb{Z}$. Then $\mu_K := \mu((b_n),(a_n)) + \frac{K}{\|\cdot\|}\mathcal{L}^2$ is a monotone measure in $\mathbb{R}^2$. Conical measures

$$\nu_1 = \mathcal{H}^1 \llcorner (\mathbb{R}\times\{0\}) + \frac{K}{\|\cdot\|}\mathcal{L}^2 \quad\text{and}\quad \nu_2 = \mathcal{H}^1 \llcorner (\{0\}\times\mathbb{R}) + \frac{K}{\|\cdot\|}\mathcal{L}^2$$

are tangent measures of μ_K at $(0,0)$. Moreover, every $\nu \in \operatorname{Tan}^1_{(0,0)}(\mu_K)$ agrees with ν_1 or ν_2 in a neighbourhood of $(0,0)$.

(b) Let $q \in (0,1)$ and $a_n = q^{(2n)^3}$, $b_n = q^{(2n+1)^3}$ for $n \in \mathbb{Z}$. Then $\mu_K := \mu((b_n),(a_n)) + \frac{K}{\|\cdot\|}\mathcal{L}^2$ is a monotone measure in $\mathbb{R}^2$. Moreover

$$\operatorname{Tan}^1_{(0,0)}(\mu_K) = \{\alpha\nu_1 + (1-\alpha)\nu_2 : \alpha \in [0,1]\}$$

where ν_1, ν_2 are conical measures defined in (a). Thus μ_K is a monotone measure with conical and non-unique tangent measures at $(0,0)$.

Proof sketch. (a) Let $r_n = q^{2n^3+6n}$. For every $n > 2$, $r_n^{-1} T_{(0,0),r_n\,\#}(\mu_K)$ agrees on $B(0, q^{-6n+1}) \setminus B(0, q^{6n^2+2})$ with ν_1 if n is odd and with ν_2 if n is even. Thus $\nu_1, \nu_2 \in \mathrm{Tan}^1_{(0,0)}(\mu_K)$.

On the other hand, assume $\mu_i = r_i^{-1} T_{(0,0),r_i\,\#}(\mu_K)$ converge weakly to $\nu \in \mathrm{Tan}^1_{(0,0)}(\mu)$ and $r_i \searrow 0$. If there is a sequence $(n_i) \subset \mathbb{Z}$ such that r_i/a_{n_i} stays bounded and bounded away from zero, we may assume (passing to a subsequence if necessary) that $\varrho := \lim r_i/a_{n_i}$ exists; then $\lim n_i = \infty$. If $0 < c < 1 < C$ are arbitrarily fixed, we see that μ_i agrees either with ν_1 or ν_2 on $B(0, C\varrho q) \setminus B(0, c\varrho q^{2n_i^3})$ if i is so large that $r_i/a_{n_i} \in (c\varrho, C\varrho)$. In this case ν agrees with one of the ν_k on $B(0, C\varrho q)$ and with the other one on $\mathbb{R}^2 \setminus B(0, C\varrho)$ with a transition layer inbetween.

If there is no such (n_i) then, for every $R > 1$, μ_i agrees with ν_1 or ν_2 on $B(0, R) \setminus B(0, 1/R)$ for all i large enough and hence $\nu \in \{\nu_1, \nu_2\}$.

(b) Assume $r_i \searrow 0$ and $\mu_i = r_i^{-1} T_{(0,0),r_i\,\#}(\mu_K)$ converge weakly to ν. Fix $R > 1$. The density of the x_1-axis part of μ_i is $h_1^{(i)}(t) = h_1(r_i t)$. Let $\alpha_i = h_1^{(i)}(1)$. From the construction (cf. p. 5) we see that $h_1'(\xi)\xi \to 0$, for $\xi \to 0$. Hence $h_1^{(i)}$ is uniformly close to α_i on the interval $[1/R, R]$ for large i. By the weak convergence $\alpha := \lim \alpha_i$ exists. Obviously, $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ on $B(0, R) \setminus B(0, 1/R)$, hence on $\mathbb{R}^2$.

On the other hand, any $\alpha \in [0, 1]$ can be obtained by a suitable choice of (r_i). □

4. *Some remarks*

Higher dimension and codimension. It is easy to transform our example to higher dimensions: If μ is k-monotone measure on $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N$ then it is k-monotone in the bigger space $\mathbb{R}^N$, $N > n$, since, for $x = (x_0, x_1) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^{N-n}$ and $r > \|x_1\|$,

$$r \mapsto \frac{\mu(B(x, r))}{r^k} = \frac{\mu B(x_0, r_0)}{r_0^k}\bigg|_{r_0 = \sqrt{r^2 - \|x_1\|^2}} \left(1 - \frac{\|x_1\|^2}{r^2}\right)^{\frac{k}{2}}$$

is a product of two nonnegative nondecreasing functions. Furthermore, if μ is a k-monotone measure on $\mathbb{R}^n$ then $\mu \times \mathcal{L}^l$ is $(k+l)$-monotone in $\mathbb{R}^{n+l}$ by the above observation and a simple Fubini type argument.

Size of singular set. It remains open, for a given k-monotone measure μ on $\mathbb{R}^n$, how large can be the set of points $a \in \mathbb{R}^n$ such that $\mathrm{Tan}^k_a\,\mu$ contains two different conical measures or a non-conical measure. By our result it follows that if $1 \le k \le n-1$, the $(k-1)$-dimensional Hausdorff measure of this set can be positive.

Measure density restriction. We would like to know if monotonicity is sufficient for Allard's regularity theorem. (For a modest regularity result for monotone measures see [**5**, 8.5].) Hence an important question is whether there is, for every small $\varepsilon > 0$, a monotone measure μ with non-unique tangent behaviour and with the density $\theta^k_a\mu \in [1, 1+\varepsilon)$ for every $a \in \operatorname{spt}\mu$. An example for certain relatively "small" ε is the subject of papers in preparation by the author and Robert Černý ([**3**, **4**], cf. also [**1**]). In particular we are able to achive ε arbitrarily close to 2 in $\mathbb{R}^2$ and arbitrarily close to 1 in $\mathbb{R}^3$. Meanwhile the next section provides an example for a large ε.

5. *A concentrated measure*

In order to replace continuous measure $\frac{1}{\|\cdot\|}\mathcal{L}^2$ by $\mathcal{H}^1$ restricted to lines we have to combine the methods of Section 3 with a local version of monotonicity.

Local monotonicity of the logarithmic spiral was first observed by Bernd Kirchheim [**6**]. It also follows from a recent general result of Černý [**2**] which implies: *If* $\gamma\colon [a,b] \to \mathbb{R}^2$ *is an injective* C^2 *curve with nonzero curvature everywhere, then* $\mu = \mathcal{H}^1 \llcorner \langle\gamma\rangle$ *is a "locally monotone" measure, i.e., for every* $x_0 \in \mathbb{R}^2 \setminus \{\gamma(a), \gamma(b)\}$ *there is* $r_0 > 0$ *such that for every* $x \in B(x_0, r_0)$*, the function* $r \mapsto \mu(B(x,r))/r$ *is non-decreasing on* $(0, r_0)$*.* (In case γ is possibly non-injective, the result is also true if μ accounts for multiplicity, i.e., $\mu(A) := \int_{\gamma^{-1}(A)} |\gamma'(t)|\,\mathrm{d}t$ for Borel A.)

THEOREM 2. *There is a closed set* $A \subset \mathbb{R}^2$ *such that* $\mathcal{H}^1 \llcorner A$ *is a monotone measure and its tangent measures at* 0 *are non-conical and non-unique.*

Proof sketch. Let $\gamma(t) = (e^t \cos t, e^t \sin t)$, and $A_0 = \{\gamma(t) : t \in \mathbb{R}\}$. Find $r_0 > 0$ corresponding to the local monotonicity for $\gamma|_{[-1,1]}$ near $\gamma(0)$. Thus the fucntion $r \mapsto (\mathcal{H}^1 \llcorner A)(B(x,r))/r$ is non-decreasing on $r \in (0, r_0)$ if $x \in B(\gamma(0), r_0)$. Obviously the same fucntion is non-decreasing for $r \in (0, r_1)$ and every x satisfying $\|x\| = \|\gamma(0)\|$ provided $r_1 \in (0, r_0)$ is choosen so small that $B(x, r_1)$ does not intersect $\langle\gamma\rangle$ if $\|x\| = \|\gamma(0)\|$ and $\|x - \gamma(0)\| \geq r_0$. By the "selfsimilarity" of the spiral we have the monotonicity up to the radius $r = r_1\|x\|$ for general $x \in \mathbb{R}^2$. This is a very good replacement for Lemma 5. For $\mu = \mathcal{H}^1 \llcorner (A_0 \cup -A_0)$, it is also possible to modify the proofs to obtain statements similar to those in Lemma 3 (there is $c \in (0,1)$ and K_1 with specified properties) and Lemma 4.

Choose $k \in \mathbb{N}$ with $k > 10$ and $1/k < r_1/100\pi$ and let A_1 be the union of k lines through the origin, more precisely $A_1 = \bigcup_{i=1}^{k} \{(t \cos \pi i/k, t \sin \pi i/k) : t \in \mathbb{R}\}$. For $\nu = (10/k)\mathcal{H}^1 \llcorner A_1$, Lemma 2 remains valid for all x and r with $r \geq r_1\|x\|$ because in this case there is at least about 100 lines intersecting $B(x,r)$.

If k is chosen big enough compared to K_1 we have that $\mu_k := \mu + (k/10)\nu = \mathcal{H}^1 \llcorner A$, $A = A_0 \cup -A_0 \cup A_1$, is a monotone measure. Furthermore, it is very easy to see that the tangent measures are precisely μ_k and its blow-ups: $\mathrm{Tan}^1_{(0,0)}(\mu_k) = \{\frac{1}{\lambda} T_{(0,0),\lambda\,\#}(\mu_k) : \lambda > 0\}$. □

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