Domácí úkol č. 3

Zadáno: 22. a 24. 10.
Deadline: 6. 11. a 31. 10.

1. Najděte limitu
$$\lim_{x\to 0} \frac{\operatorname{cotg}(a+2x) - 2\operatorname{cotg}(a+x) + \operatorname{cotg} a}{x^2},$$
kde $a \in \mathbb{R}$, $\sin a \neq 0$.

   Hint: Pužijte chytře vzorec pro $\operatorname{cotg} A - \operatorname{cotg} B$ a následně v průběhu výpočtu i $\sin A - \sin B$

2. Najděte limitu
$$\lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x}.$$

Řešení

1.
$$\lim_{x\to 0} \frac{\operatorname{cotg}(a+2x) - 2\operatorname{cotg}(a+x) + \operatorname{cotg} a}{x^2} =$$
$$= \lim_{x\to 0} \frac{\operatorname{cotg}(a+2x) - \operatorname{cotg}(a+x) + \operatorname{cotg} a - \operatorname{cotg}(a+x)}{x^2} =$$
$$= \text{použijeme} \qquad \operatorname{cotg} A - \operatorname{cotg} B = \frac{\sin(B-A)}{\sin B \sin A}$$
$$= \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sin(-x)}{\sin(a+2x)\sin(a+x)} + \frac{\sin(x)}{\sin(a)\sin(a+x)}\right) =$$
$$= \lim_{x\to 0} \frac{1}{x^2} \frac{-\sin(x)\sin(a) + \sin(x)\sin(a+2x)}{\sin(a+2x)\sin(a+x)\sin(a)} =$$
$$= \lim_{x\to 0} \frac{\sin(x)}{x} \lim_{x\to 0} \frac{1}{\sin(a+2x)\sin(a+x)\sin(a)} \lim_{x\to 0} \frac{\sin(a+2x) - \sin(a)}{x} =$$
$$= \frac{1}{\sin^3(a)} \lim_{x\to 0} \frac{\sin(a+2x) - \sin(a)}{x} =$$
$$= \text{použijeme} \qquad \sin A - \sin B = 2\cos\frac{A+B}{2} \sin\frac{A-B}{2}$$
$$= \frac{1}{\sin^3(a)} \lim_{x\to 0} \frac{2\cos(a+x)\sin(x)}{x} = \frac{2\cos(a)}{\sin^3(a)}$$

2.

$$
\begin{aligned}
&\lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = \\
= \quad & \lim_{x\to\frac{\pi}{4}} \exp\left(\operatorname{tg} 2x \ln \operatorname{tg} x\right) = \\
= \quad & \exp \lim_{x\to\frac{\pi}{4}} \left(\operatorname{tg} 2x \ln \operatorname{tg} x\right) = \\
= \quad & \exp \lim_{x\to\frac{\pi}{4}} \left(\operatorname{tg} 2x \frac{\ln \operatorname{tg} x}{\operatorname{tg} x - 1} (\operatorname{tg} x - 1)\right) = \\
= \quad & \exp \lim_{x\to\frac{\pi}{4}} \frac{\ln \operatorname{tg} x}{\operatorname{tg} x - 1} \lim_{x\to\frac{\pi}{4}} \left(\operatorname{tg} 2x (\operatorname{tg} x - 1)\right) = \\
= \quad & \exp \lim_{x\to\frac{\pi}{4}} \left(\frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x} \frac{\sin x - \cos x}{\cos x}\right) = \\
= \quad & \exp \lim_{x\to\frac{\pi}{4}} \left(\frac{-2 \sin x}{\cos x + \sin x}\right) = \exp(-1) = \frac{1}{\mathrm{e}}
\end{aligned}
$$