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LETTER

Possible distribution of mass inside a black hole. Is there any upper limit on mass density?

Michal Křížek[1]

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Abstract The maximum mass of a neutron star is about three solar masses. In this case the radius of such neutron star is approximately equal to the Schwarzschild radius. Adding a small amount of matter to this star, a black hole arises. Thus its interior could contain a star with neutron or quark density just below the event horizon instead of the proposed point singularity. We also show that the Hawking miniature black hole evaporation is improbable, since it would yield unrealistic mean mass densities.

Keywords Black hole · Neutron star · Relativistic volume · Chandrasekhar limit · TOV limit

1 Introduction

Generally, it is believed that the center of any real black hole contains a point singularity with infinite mass density. This seems to be only a mathematical idealization, since no physical quantity can attain infinite values. In this article we suggest that the hidden center of any black hole could be occupied by a neutron-like star composed of neutrons and/or quark-gluon plasma.

The mass of the neutron is (see Patrignani et al. 2016)

$$m = 1.675 \cdot 10^{-27} \text{ kg} \tag{1}$$

and its interaction diameter is given by $d = \sqrt{4\sigma/\pi} \approx 1.65 \cdot 10^{-15}$ m, where σ is the effective cross section. Assuming spherical symmetry, the neutron density is approximately

$$\rho_{\rm n} \approx \frac{m}{\frac{1}{6}\pi d^3} \approx 0.712 \cdot 10^{18} \text{ kg/m}^3. \tag{2}$$

In our Galaxy, more than 2000 neutron stars are already known. By Burgio et al. (2002) most of their masses fall in a relatively narrow interval from $1.45M_\odot$ to $1.65M_\odot$, where $M_\odot = 2 \cdot 10^{30}$ kg is the mass of the Sun. Their mass is usually greater than the *Chandrasekhar limit* $1.4M_\odot$ for white dwarfs. However, rarely there exist neutron stars with masses close to $M_\odot$ (see Lattimer and Prakash 2004), since during the collapse of a white dwarf onto a neutron star, some amount of matter is thrown away.

The mass of each neutron star is smaller than the *Tolman–Oppenheimer–Volkoff limit* which is approximately (see e.g. Abbott et al. 2017; Lattimer and Prakash 2004; Linares et al. 2018)

$$M := 3M_\odot. \tag{3}$$

The neutron star mass density is likely not constant and increases towards the center. Thus, the mean density of a neutron star should be even higher than (2), i.e. denser than an atomic nucleus. It is assumed that the central part of any neutron star is occupied by a quark-gluon plasma which is a Fermi liquid satisfying the Pauli Exclusion Principle. Therefore, the central quark density could be several times higher than (2).

Assuming, for example, a double mean density $\rho = 2\rho_{\rm n}$, we find for

$$M = \frac{4}{3}\pi R^3 \rho$$

✉ M. Křížek
krizek@math.cas.cz

1 Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

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given by (3) that the maximum radius of such a neutron star is about

$$R = \sqrt[3]{\frac{3M}{4\pi\rho}} \approx 10 \text{ km}. \tag{4}$$

Recalling the definition of the Schwarzschild radius

$$S = \frac{2GM}{c^2}, \tag{5}$$

we find that S corresponding to $M = 3M_\odot$ has a very similar size as (4), namely

$$S = 9 \text{ km}.$$

In Sects. 2 and 3 we will derive finer estimates that take into account that the space inside the neutron star is curved and thus it has a larger relativistic volume than the Euclidean volume of a massless ball with the same circumference (cf. Fig. 1). Adding a small amount of mass to the TOV limit in (3), the body becomes a black hole, but its central part below the event horizon could still remain in the state of neutron matter and/or quark-gluon plasma, see Sect. 4. It need not collapse to a point singularity. In the Appendix, we present a purely geometrical method for estimating the mass of the black hole Sgr A*.

2 Interior Schwarzschild solution

If o is the circumference of a mass ball, then its apparent radius

$$R = \frac{o}{2\pi} \tag{6}$$

is called the *coordinate radius*. The first nonvacuum solution of Einstein's equations was found by Schwarzschild (1916). He supposed that the ball with coordinate radius $R > 0$ contains an ideal incompressible fluid. In this way, we may avoid a possible internal mechanical stress in the solid that could have a nonnegligible influence on the resulting gravitational field. Using Ellis (2012) (see also Stephani 2004, p. 213; Florides 1974, p. 529; https://en.wikipedia.org/wiki/Interior_Schwarzschild_metric) the corresponding time independent static metric (i.e. $\mathrm{d}t = 0$) is defined by

$$\mathrm{d}l^2 = \frac{R^3}{R^3 - Sr^2}\,\mathrm{d}r^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2 + r^2\,\mathrm{d}\theta^2, \tag{7}$$

where $r \in [0, R]$, $\varphi \in [0, 2\pi)$, and $\theta \in [0, \pi]$ are the standard spherical coordinates and S is given by (5).

The equality (7) is said to be the *interior Schwarzschild solution*, see Stephani (2004, p. 213). To avoid the division by zero in the first coefficient on the right-hand side of (7), we assume that

$$\frac{R^3}{R^3 - Sr^2} = \left(1 - \frac{Sr^2}{R^3}\right)^{-1} > 0 \quad \text{for all } r \in [0, R].$$

Hence (cf. (4) and (5)),

$$R > S. \tag{8}$$

Now we shall determine the relativistic radius of the ball. For $\mathrm{d}\varphi = 0$ and $\mathrm{d}\theta = 0$ the identity (7) clearly reduces to

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - \alpha^2 r^2}, \tag{9}$$

where

$$\alpha = \sqrt{\frac{S}{R^3}}. \tag{10}$$

For $r \in [0, \alpha^{-1})$ one can easily check that

$$F(r) = \frac{1}{\alpha}\arcsin(\alpha r) \tag{11}$$

is a primitive function of

$$f(r) = \frac{1}{\sqrt{1 - \alpha^2 r^2}}. \tag{12}$$

According to (8) and (10), we observe that

$$R < R\sqrt{\frac{R}{S}} = \alpha^{-1}.$$

Using (9), (11), and (12), we shall introduce the *relativistic (proper) radius* of the homogeneous mass ball given by the relation

$$\tilde{R} = \int_0^R \frac{\mathrm{d}r}{\sqrt{1 - \alpha^2 r^2}} = \frac{1}{\alpha}\arcsin(\alpha R). \tag{13}$$

Here the tilde indicates that we deal with a curved space.

For example, for an idealized homogeneous neutron star with radius $R = 10$ km as in (4), we obtain by (5), (10), and (13) that $\tilde{R} = 13.2$ km (see Fig. 1), which is in a good agreement with Steiner et al. (2013).

3 Relativistic volume of a homogeneous mass ball

In this section, we first derive a relation for the relativistic volume of a homogeneous mass ball. From (7) and (9), we see that the volume element is

$$\mathrm{d}\tilde{V} = \frac{\mathrm{d}r}{\sqrt{1 - \alpha^2 r^2}} \cdot (r \sin\theta\,\mathrm{d}\varphi) \cdot (r\,\mathrm{d}\theta). \tag{14}$$

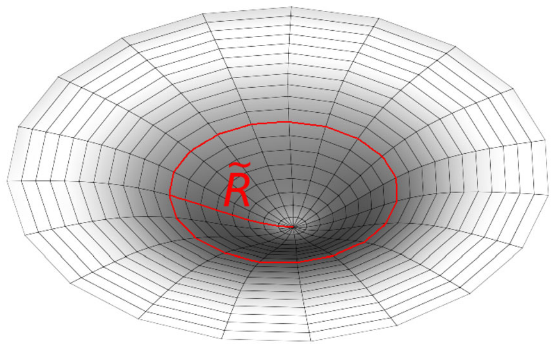

Fig. 1 Relativistic radius $\tilde{R}$ is greater than the coordinate radius R defined by (6)

One can easily verify by differentiation that

$$H(r) = \frac{1}{2\alpha^3} \arcsin \alpha r - \frac{r}{2\alpha^2}\sqrt{1-\alpha^2 r^2} \tag{15}$$

is a primitive function of

$$h(r) = \frac{r^2}{\sqrt{1-\alpha^2 r^2}}$$

on the interval $[0, \alpha^{-1})$, i.e. $\mathrm{d}H(r)/\mathrm{d}r = h(r)$, cf. Peebles (1993, p. 298).

According to (13), (15), and (14), the *relativistic (proper) volume* of the homogeneous ball is given by

$$\begin{aligned}\tilde{V} &= \int_0^R \frac{r^2\,\mathrm{d}r}{\sqrt{1-\alpha^2 r^2}} \cdot \int_0^\pi \left(\int_0^{2\pi} \sin\theta\,\mathrm{d}\varphi\right)\mathrm{d}\theta \\ &= H(R)\cdot 4\pi \\ &= \frac{2\pi}{\alpha^2}\left(\frac{\arcsin(\alpha R)}{\alpha} - R\sqrt{1-\alpha^2 R^2}\right) \\ &= \frac{2\pi}{\alpha^2}\left(\tilde{R} - R\sqrt{1-\alpha^2 R^2}\right).\end{aligned} \tag{16}$$

For a fixed $R > 0$ it can be derived by l'Hospital's rule and definition (10) that

$$\tilde{V} \to V := \frac{4}{3}\pi R^3 \quad \text{for } \alpha \to 0. \tag{17}$$

Consider now an idealized neutron star described by a homogeneous ball with mass $M = 3M_\odot = 6 \cdot 10^{30}$ kg and coordinate radius $R = 10$ km (cf. (3)–(4)). Then by (16) and (17) we get quite a large relativistic effect, namely, the ratio between the relativistic volume and the Euclidean volume equals

$$\frac{\tilde{V}}{V} = \frac{3}{2S}\left(\tilde{R} - R\sqrt{1-S/R}\right) = 1.666. \tag{18}$$

Consequently, the relativistic volume $\tilde{V}$ is 67% larger than the Euclidean volume V defined in (17). This means that the mean mass density $\tilde{\rho} = M/\tilde{V}$ is smaller than $\rho = M/V$ and $\tilde{\rho}/\rho = 0.6$.

Table 1 The Schwarzschild radius $R_\bullet$ and the mean mass density $\rho_\bullet$ of various objects

	$R_\bullet$ (in meters)	$\rho_\bullet$ (in kg/m^3)
1 kg	10^{-27}	10^{80}
Earth	0.009	10^{30}
$M_\odot$	3000	10^{20}
$3M_\odot$	9000	10^{18}
Sgr A*	10^{10}	10^{6}
M87*	$1.5 \cdot 10^{13}$	1

The matter inside a mass ball defends to gravitational compression in such a way that it increases its relativistic volume $\tilde{V}$ when R is fixed and M increases. In other words, the greater M is, the greater is $\tilde{V}$ for a fixed circumference and fixed V.

Since the density gradient of a real neutron star is not known, formula (18) represents only a lower estimate. The true ratio $\tilde{V}/V$ is probably larger. Moreover, the relativistic volume extension could be sufficient to prevent density increase above the neutron or quark densities. The definition of mass density under strong gravitational field conditions is not an evident quantity, because the volume is influenced by the curved geometry of space under gravity. For derivations of density expressions under such conditions we refer to Fischer (2017) (see also Fahr and Sokaliwska 2012).

4 The interior of a black hole

It is estimated that there are approximately 10^7 black holes in our Galaxy and 10^{18} in the whole observable universe. First, let us recall the well-known formula for their coordinate Schwarzschild radius

$$R_\bullet = \frac{2G}{c^2} M_\bullet \tag{19}$$

for a given mass $M_\bullet$. Then the corresponding mean mass density is

$$\rho_\bullet \approx \text{const.} \frac{M_\bullet}{R_\bullet^3} \approx \text{Const.} \frac{c^6}{G^3}\frac{M_\bullet}{M_\bullet^3} \approx 10^{80}\frac{1}{M_\bullet^2} \tag{20}$$

with some small (cf. (18)) dimensionless constants const. and Const. In Table 1 we present hypothetical values of black hole mean densities of various objects.

The values in Table 1 associated to the central black hole Sgr A* of the Milky Way are derived in Appendix (see (25) and (26)). The last line of Table 1 corresponds to the supermassive black hole in the center of the M87 galaxy (see

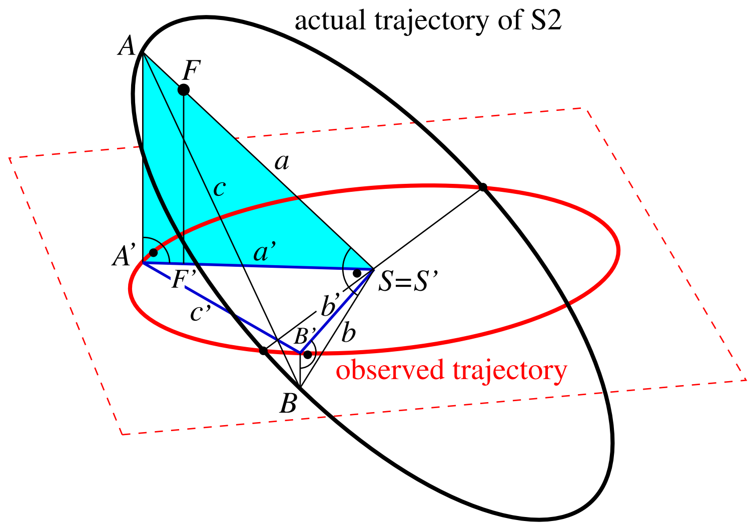

Fig. 2 Actual and observed trajectory of the star S2. The segments a, b, and $c = \sqrt{a^2 + b^2}$ are orthogonally projected on a', b', and c'

Akiyama et al. 2019). By (19) and (20) its mean mass density is $\rho_\bullet \approx 1$ kg/m^3, since this black hole is about 1500× more massive than Sgr A*. This mean density is thus like the density of air. Note that the mean mass density of our universe is only a few protons per m^3, i.e., approximately 10^{-27} kg/m^3 (cf. (1)). Note that the universe cannot be treated as a black hole, since it has a completely different topology (of maximally symmetric manifolds) than ordinary black holes.

The limit values $M_\odot$ and $3M_\odot$ appearing in Table 1 were investigated in previous sections. From Table 1 we further observe that the Earth being a black hole would have an incredibly large mean density, 12 orders higher than the neutron density (2). Moreover, 1 kg ball would have by (20) density 10^{80} kg/m^3.

Anxieties that in CERN miniature black holes could be produced are unjustified, since they would have unrealistic density. Another argument is that maximum energy of each proton in LHC is about 10^{12} eV, while from the universe we receive protons having over 10^{20} eV and nothing happens. Namely, in our neighborhood we do not observe any miniature black holes, even though the surface of the Earth and other planets is continually bombarded by particles of cosmic rays with extremely high energies.

The existence of Hawking's miniature black holes due to the Hawking radiation is thus also very improbable. By (20) their density would require to be much larger than (2), see Table 1.

So we can conclude that gravity cannot compress objects with mass smaller than $M_\odot$ to densities larger than the neutron density (2) or quark density. On the other hand, the interior of black holes with mass larger than $3M_\odot$ could contain below the event horizon a star ⊙ composed of dense neutron or quark matter instead of a point singularity.

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Appendix: Calculation of the mass of Sgr A*

According to Schödel et al. (2002), the star S2 orbits the supermassive black hole Sgr A* in the center of our Galaxy along elliptic trajectory. We shall assume that the Newtonian mechanics is a good approximation of reality in this case. By present measurements, S2 is about 26 500 ly from us and its orbital period is

$$T = 16.08 \text{ yr} \approx 507 \cdot 10^6 \text{ s}. \tag{21}$$

First we show that the eccentricity e of the actual elliptical trajectory can surprisingly be derived from the observed trajectory.

Denote by F' the point corresponding to the strong X-ray source Sgr A*, which is the orthogonal projection of the focus F of the actual orbit. Consider the line S'F' and denote by A' its intersection with the projected orbit. The semimajor axis a that contains the focus F is then projected to the line segment A'S'. Therefore, we have (see Fig. 2)

$$e = \frac{\varepsilon}{a} = \frac{|FS|}{|AS|} = \frac{|F'S'|}{|A'S'|}, \tag{22}$$

where the ratio on the right-hand side can be evaluated, $|\cdot|$ denotes the length, and $\varepsilon = |FS| = \sqrt{a^2 - b^2}$ is the linear eccentricity. For the observed trajectory illustrated in Fig. 3

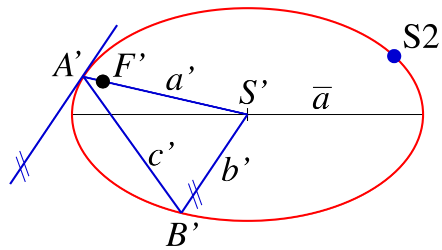

Fig. 3 Projection of S2-orbit on the celestial sphere. Its semimajor axis $\overline{a} = 4.55$ ld (light days) is less than a

we get by (22) that the eccentricity of the actual trajectory is $e = 0.885$.

Further, we construct the point B' on the observed orbit so that the line B'S' is parallel with the tangent at A' and that the angle A'S'B' is nonobtuse. Then the triangles ABS, AA'S, and BB'S are right and we have

$$a^2 + b^2 = c^2 = c'^2 + \left(\sqrt{a^2 - a'^2} + \sqrt{b^2 - b'^2}\right)^2,$$

where $a' = |A'S'|$, $b' = |B'S'|$, and $c' = |A'B'|$. From this we obtain

$$a'^2 + b'^2 - c'^2 = 2\sqrt{a^2 - a'^2}\sqrt{b^2 - b'^2}.$$

Squaring this equation, the substitution $b^2 = (1 - e^2)a^2$ leads to quartic equation for one unknown a,

$$\left(1 - e^2\right)\left(a^2\right)^2 - \left[\left(1 - e^2\right)a'^2 + b'^2\right]a^2 + a'^2b'^2 - \frac{1}{4}\left(a'^2 + b'^2 - c'^2\right)^2 = 0. \tag{23}$$

Since this equation does not contain any cubic and linear term, it is, in fact, a quadratic equation for a^2.

By angular measurements we know that $a' = 3.99$ ld, $b' = 2.49$ ld, and $c' = 4.01$ ld. Substituting these data into (23), we get

$$a = 5.61 \text{ ld} = 970 \text{ au} = 145 \cdot 10^{12} \text{ m}. \tag{24}$$

The second positive solution of (23) is not physical, since it is smaller than a'. From (21), (24), and Kepler's third law we obtain

$$M_\bullet = \frac{4\pi^2 a^3}{GT^2} \approx 7 \cdot 10^{36} \text{ kg} \approx 3.5 \cdot 10^6 \, M_\odot.$$

The resulting mass is, of course, very sensitive on precise measurements of a and T. The corresponding Schwarzschild radius is

$$R_\bullet = \frac{2GM_\bullet}{c^2} \approx 10^{10} \text{ m} \approx 0.07 \text{ au} \tag{25}$$

and the mean mass density

$$\rho_\bullet = \frac{M_\bullet}{V_{\text{relativistic}}} < \frac{M_\bullet}{\frac{4}{3}\pi R_\bullet^3} \approx 1.67 \cdot 10^6 \text{ kg/m}^3. \tag{26}$$

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