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Unexpected Illusions in the Special Theory of Relativity

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Abstract

We show that the Doppler effect and aberration of light can produce more dominant and entirely opposite effects for relativistic speeds than those predicted by the Special Theory of Relativity, in particular, time dilatation and length contraction. For instance, an observer will always measure a higher frequency of an approaching clock than the same clock has at rest. We also show that under certain conditions, an approaching bar on a photo may seem to have a larger length for a relativistic speed than at rest.

Keywords: Lorentz transformation, theory of groups, inertial systems, time dilatation, length contraction

1 Introduction

According to Newton's first law of inertia, a body will remain at rest or in uniform motion in a straight line unless acted upon by an external force. This fundamental physical principle serves to introduce inertial systems in the Special Theory of Relativity (STR), see [6, p. 211]. Consider a fixed coordinate system S with orthogonal axes x, y, z containing a fixed system of hypothetical synchronized clocks that define the time coordinate $t \in (-\infty, \infty)$ of a uniformly flowing time. This can be, in fact, interpreted so that all clocks are synchronized by an infinite speed of signal. The coordinate system S is called *inertial* if it obeys Newton's first law of motion.

Let S' be another coordinate system with orthogonal axes x', y', z' which are for simplicity parallel with x, y, z and have the same scale at rest, see [15]. The time $t' \in (-\infty, \infty)$ in S' is introduced similarly using a fixed system of synchronized clocks in S' having also the same time scale at rest. Let the origin of S' move along the x axis at a constant speed $v \in (-c, c)$, where c is the speed of light in vacuum, see Fig. 1.

The Lorentz transformation (see [10]) is a fundamental tool of the STR. The parameter defined by

$$\gamma_v = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \geq 1 \tag{1}$$

is called the *Lorentz factor*. Note that its reciprocal value fulfills the equation for the unit circle $(v/c)^2 + (1/\gamma_v)^2 = 1$. Points of the spacetime $\mathbb{R}^4$ are called *events*. Unless otherwise stated, we will restrict ourselves to one pair of the above described inertial systems such that the encounter of the origins of S and S' determines the beginning of time counting in the first and in the second inertial system, respectively, i.e., $t = 0$ in S and $t' = 0$ in S'. In this special case the *Lorentz transformation* [10, p. 185] has the form $\mathcal{L}_v \colon \mathbb{R}^4 \to \mathbb{R}^4$,

$$\begin{aligned} x' &= \gamma_v (x - vt), && (2)\\ y' &= y, \\ z' &= z, \\ t' &= \gamma_v \left(t - \frac{v}{c^2} x \right), && (3) \end{aligned}$$

where $x, y, z, t \in (-\infty, \infty)$, and the last equality expresses how to transform a uniformly flowing proper time during transition from S to S'. Events which are simultaneous in S are given by the identity $t \equiv t_0$, where t_0 is a fixed constant. Let us emphasize that any two different events which are simultaneous in S are not causally connected. Thus, one can verify that they were really simultaneous only when their future light cones intersect.

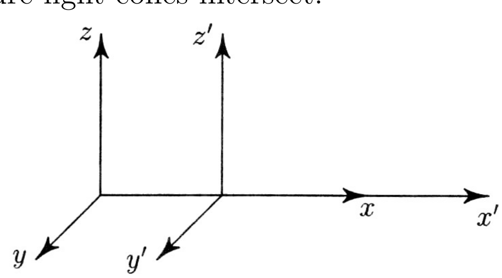

Fig. 1. The inertial system S' is moving at speed $v \in (-c, c)$ with respect to the system S.

By (3) we see that the time t' depends not only on t but also on the position x, i.e., t' is not constant and thus the corresponding events do not have to be simultaneous in S' for $v \neq 0$.

Notice that the right-hand sides of relations (2) and (3) are linear functions in variables x and t for any fixed v. Thus, for $\boldsymbol{x} = (ct, x, y, z)$ and $\boldsymbol{x'} = (ct', x', y', z')$ the Lorentz transformation can be rewritten into the matrix form

$$\boldsymbol{x'} = \boldsymbol{L}_v \boldsymbol{x},$$

where

$$\boldsymbol{L}_v = \begin{pmatrix} \gamma_v & -\frac{v}{c}\gamma_v & 0 & 0 \\ -\frac{v}{c}\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

is a block diagonal symmetric and positive definite matrix. Note that the unit of physical dimension of all entries of the vectors $\boldsymbol{x}$ and $\boldsymbol{x'}$ is one meter.

The inverse matrix $\boldsymbol{L}_v^{-1}$ has a similar form as $\boldsymbol{L}_v$, only the two minus signs in (4) being replaced by plus sign. Therefore, the Lorentz transformation $\mathcal{L}_v$ is a one-to-one mapping from $\mathbb{R}^4$ onto $\mathbb{R}^4$ for $v \in (-c, c)$.

Let us point out that in the limiting case $|v| = c$, the matrix (4) becomes singular, since its two first rows are linearly dependent. Consequently, the Lorentz transformation should not be applied to the surface of the light cone. Its inverse does not exist.

2 Time dilatation

The relation (3) is to be understood only as the time which we would record at the moment when the two clocks in S and S' are passing each other at one single x-coordinate (e.g. at the origin). By definition, all clocks in each inertial system at rest show the same time in the whole infinite three-dimensional space (e.g. at the beginning and at the end of a motionless bar). So, when we are exactly in the middle between any two fixed clocks, they will show us the same time.

Consider a fixed time interval

$$\Delta t' = t'_2 - t'_1,$$

where t'_i are space independent coordinates in S'. For an arbitrary fixed point x in S we determine the corresponding t_2 and t_1 from formulae

$$t'_2 = \gamma_v \left(t_2 - \frac{v}{c^2} x \right), \quad t'_1 = \gamma_v \left(t_1 - \frac{v}{c^2} x \right),$$

cf. (3), and we set $\Delta t = t_2 - t_1$. From this we get the well-known *time dilatation* (see e.g. [8, p. 430])

$$\Delta t' = \gamma_v \left(t_2 - \frac{v}{c^2} x - t_1 + \frac{v}{c^2} x \right) = \gamma_v \Delta t. \tag{5}$$

By (1) we see that $\Delta t' > \Delta t$ for any $v \neq 0$ independently of the sign of v. The relation (5) actually expresses that the time, measured by a clock in a moving system S', runs slower than the time measured by a clock that is at rest with respect to S. Hence, the clock at rest is fastest.

Remark 1. The time dilatation is verified by means of the transverse relativistic Doppler effect, see e.g. [3]. This effect was first measured by Ives already in 1938, see [9]. In classical mechanics, this transverse effect does not occur because it is given by time dilation (5) only. Note that the Hafele-Keating experiment [7] with two atomic clocks in airplanes and one on Earth is not too credible, since none of the corresponding three systems was inertial.

The non-relativistic longitudinal Doppler effect [4] is described by the relation

$$f_v = \frac{c}{c - v} f, \tag{6}$$

where f is the source frequency at rest, v is the speed of the source approaching an observer along the axis x, f_v is the frequency measured by the observer, and c is the speed of signal. For relativistic speeds this relationship needs to be corrected by time dilation [6]. All physical processes including clock speed in S' will run, by (3), slower when observed from S. Thus by (6), the new relation will be of the form

$$f_v = \frac{c}{c - v} f', \tag{7}$$

where c is the speed of light and

$$f' = \gamma_v^{-1} f \tag{8}$$

corresponds to the lower frequency calculated from (5). By (7), (8), and (1) we obtain a relativistic Doppler relation for the frequency detected in S (see [5]),

$$f_v = \frac{1}{1 - \dfrac{v}{c}} f' = \frac{\gamma_v^{-1}}{1 - \dfrac{v}{c}} f = \frac{\sqrt{1 - \left(\dfrac{v}{c}\right)^2}}{1 - \dfrac{v}{c}} f = \sqrt{\frac{c + v}{c - v}} f. \tag{9}$$

From this we immediately get the following theorem.

Theorem 1. *For any $v \in (0, c)$ we have that $f_v / f' > f_v / f > 1$. Moreover, $f_v / f \to \infty$ as $v \to c$.*

Consequently, the Doppler effect manifests more than the time dilation itself, whenever the clock approaches the observer. Hence, the higher the speed v, the greater the Doppler effect. Special relativity effects for large v are of higher order in $\frac{v}{c}$ than those arising from the Doppler effect.

It is therefore very important to distinguish consistently between reconstructions (calculations by means of the Lorentz transformation) and observations (e.g., measurements, detections, photos, videos, and recording in general). The notion "observer" in the STR is thus somewhat confusing. It should not be a person who only applies relations (2)–(3). The observer performs real observations and measurements including all effects together as it is usually understood, i.e., the observer measures incoming frequencies, velocities, etc.

Example 1. Suppose that a clock will be approaching the origin of S at relativistic speed $v = 0.8c$. Its proper time will pass slower than on clocks fixed in the system S, since by (1) and (5) we have

$$\gamma_v = \frac{1}{\sqrt{1 - 0.64}} = \frac{5}{3}$$

and

$$\Delta t' = \frac{5}{3}\Delta t.$$

However, substituting $v = 0.8c$ into (9), we find that

$$f_v = 3f \quad \text{and} \quad f_v = 5f', \tag{10}$$

i.e., the observer at the origin of S will detect a $3\times$ higher (blue-shifted) frequency than the same clock has at rest in the system S and even a $5\times$ higher frequency than the time dilatation predicts by (8). This may seem to be paradoxical, since **the observer sees an opposite effect than the STR claims** due to the Doppler phenomenon.

Remark 2. For a clock receding the origin by the speed $(-v)$, the observer will detect by (9) a $3\times$ lower (red-shifted) frequency than f. So there is a jump in these constant frequencies at the origin and only in this single point an observer could theoretically detect the proper frequency f'.

3 Length contraction

Lorentz's length contraction is an immediate consequence of the Lorentz transformation. On the horizontal axis x' consider a fixed bar which is at rest in the system S'. Denote its length by

$$\Delta x' = x_2' - x_1', \tag{11}$$

where x_i' are fixed time independent coordinates of its ends in S'. For an arbitrary fixed time instant t in S we determine the corresponding x_2 and x_1 from formulae

$$x_2' = \gamma_v(x_2 - vt), \quad x_1' = \gamma_v(x_1 - vt),$$

cf. (2), and we set $\Delta x = x_2 - x_1$. Substituting this into (11), we obtain (cf. (5))

$$\Delta x' = \gamma_v(x_2 - vt - x_1 + vt) = \gamma_v \Delta x. \tag{12}$$

Denoting $\ell_0 = \Delta x'$ and $\ell = \Delta x$, we get by (1) the well-known *length contraction*

$$\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}}. \tag{13}$$

Hence, the bar at rest has the greatest length.

In 1959, Roger Penrose published a paper [11] (inspired by [12]) describing why we should see a quickly flying, non-rotating ball in a photo again like a ball. In the same year, his thoughts were elaborated in more detail by James Terrell [13] using light aberration. In the article [14], Weisskopf describes an apparent deformation of a quickly flying cube on a photo. Here is a specific example showing the substantial effect of light aberration for relativistic speeds.

Example 2. Consider a bar with length $\ell_0 = 1$ m. Assume that it moves from the left to the right along the axis x by the constant speed $v = 0.8c$ and that its front end just reached the origin of the coordinate system S. By (13), the bar is shortened to

$$\ell = \ell_0 \sqrt{1 - 0.64} = 0.6 \text{ m},$$

and thus the length of the straight line segment AD in Fig. 2 is $|AD| = 0.4$ m. We will photograph this bar from the axis z by a fixed nonrotating camera C which is placed at the distance

$$d = 0.75 \text{ m} \tag{14}$$

from the origin. Using the similarity of right triangles from Fig. 2, we find that $|BD| = |AD| d/\ell_0 = 0.3$ m. Therefore,

$$|AB| = \sqrt{|AD|^2 + |BD|^2} = \sqrt{0.4^2 + 0.3^2} = 0.5$$

measured in meters. The segment on the hypotenuse from B to C has the same length in meters as d in (14),

$$\sqrt{1^2 + 0.75^2} - |AB| = 1.25 - 0.5 = d. \tag{15}$$

To avoid blurred photos, we assume that our idealized camera can take pictures within 1 picosecond. During this time period, the light will fly 0.3 mm only and a possible blurring will not play a significant role. For simplicity, we shall analyze only that photo, in which the front end of the bar just reached the coordinate origin of S. However, the rear end of the bar will be on the photo farther than ℓ, since the light from the front end flies along a shorter distance d than the light from the rear end (see Fig. 2).

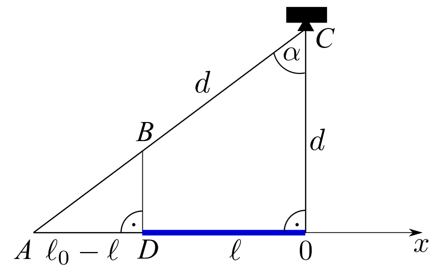

Fig. 2. The length of the legs of the larger (or smaller) right triangle is 1 and 0.75 (or 0.4 and 0.3) meters. The ratio between the lengths of sides of the both triangles is 5 : 4 : 3. Due to light aberration the flying bar from the left to the right at the speed $0.8c$ has the same length in the photo as the same bar at rest. A photon emitted to the camera from the rear end of the moving bar will always have the same x-coordinate as this rear end.

That is why there will be recorded photons on the photo from the rear end of the bar that were emitted earlier than those from the front end. During the time period, when the rear end of the bar moves from A to D, a photon pointing from A to the camera C will travel the distance $|AB|$, since $v/c = |AD|/|AB| = 0.8$. Hence, thanks to light aberration and (15), the moving bar will have on the photo the same length as the fixed one meter long bar.

Let us point out that a photon will travel the distance d from the origin to the camera C during the time period $\Delta t = d/c$. During this period, the bar will shift about $v\Delta t = 0.8d = 0.6$ m, i.e., it will be placed entirely to the right of point 0 (see Fig. 3).

Now we shall look for such a position F of the front end of the bar that a photon emitted from F arrives to the camera C at the same time as a photon emitted from the rear end of the bar when it is at the origin 0. In other words, we will determine the x-coordinate of F in Fig. 3 so that $|CG| = d$. Using the Pythagorean theorem, we find that

$$x^2 + d^2 = (d + |FG|)^2. \tag{16}$$

Moreover, we have

$$|FG| = \frac{c}{v}|EF| = \frac{c}{v}(\ell - x).$$

Substituting this into (16), we get the quadratic equation $3x^2 - 20x + 9 = 0$ whose admissible solution is $x_1 = \frac{1}{3}(10 - \sqrt{73}) = 0.485\ldots$ measured in meters. Hence, the bar on photo will be shorter than $\ell = 0.6$ m.

Example 3. Let again $\ell_0 = 1$ m and $v = 0.8c$. Hence, $\ell = 0.6$ m. This time, however, we place the camera C closer to the axis x, i.e. $d < 0.75$ m. We shall again analyze the image where the right end of the bar is at the origin. The left end of the bar will shift from the point $A = (-a, 0)$ to the point $(-\ell, 0)$ during the time period $\Delta t = (a - \ell)/v$. During this period, a photon will travel the distance $c\Delta t$ from the point A to the camera. From the Pythagorean theorem $a^2 + d^2 = ((a - \ell)c/v + d)^2$ we can easily derive the following inverse formula

$$d = \frac{a^2\left(\frac{v}{c} - \frac{c}{v}\right) + 2a\ell\frac{c}{v} - \ell^2\frac{c}{v}}{2(a-\ell)}.$$

For instance, when $a = 2$ m we obtain $d = \frac{15}{56} = 0.26\ldots$ m. So if we place the camera on the axis z at a distance of 26 cm from the origin, the one meter flying bar will appear extended in the photo as two meters long. Similarly for $v = 0.9$ and $d = 4$ cm we even get $a = 4$ m. The main reason for these surprising phenomena is that photons, which simultaneously passed through the lens, were not emitted simultaneously in S.

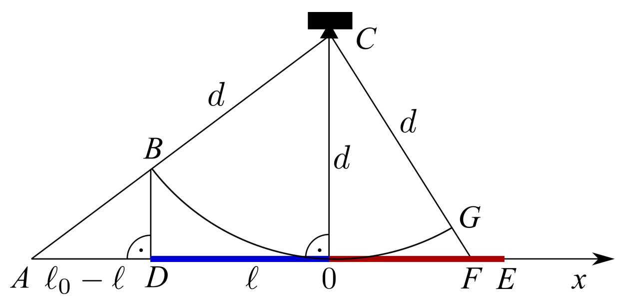

Fig. 3. Left: The front end reaches the origin. Right: The rear end reaches the origin.

Example 4. Now let us consider a 2m long moving bar ($\ell_0 = 2$ m) again with speed $v = 0.8c$, and let $d = \frac{15}{26} = 0.26\ldots$ m as in Example 3. When the bar is exactly symmetric with respect to the origin (cf. Fig. 3 for another value of d), the length of its left part will be 2m on the photo due to Example 3. Since its right part has positive length, **we will see that the bar is longer than 2m on the photo.**

Approaching objects are manifested by blue shift (i.e. shortening the wave length). However, due to aberration they may seem to be prolonged, which is paradoxical. On the other hand, receding objects that are manifested by red shift may seem to be shortened.

4 Concluding remarks

The Special Theory of Relativity has a number of unexpected claims that contradict our intuition. According to the STR, no experiment can be made to decide whether the body is at rest or moving. All inertial systems for describing physical phenomena are equivalent and there is no preferred inertial system. However, at present we know that the cosmic microwave background radiation (CMB) actually determines a certain kind of a fixed reference system in our neighborhood. Thus there arise speculations whether the principle of relativity in the real universe holds.

It is often said that the Lorentz transformation for low speeds, $|v| \ll c$, changes into the Galileo transformation

$$\begin{aligned} x' &= x - vt, \\ y' &= y, \\ z' &= z, \\ t' &= t. \end{aligned}$$

This is not true (see [2]), since for an arbitrarily small fixed $v > 0$ we can always find x such that the term vx/c^2 in (3) will dominate significantly over t. However, from (2)–(3) it follows that the Lorentz transformation changes into the Galileo transformation for a fixed v, if we treat c as a parameter and assume that $c \to \infty$. However, for an infinite speed of light there would be no Doppler effect nor aberration of light.

Remark 3. For $\vec{x} = (x, y, z)$ and a constant velocity vector $\vec{v} \in \mathbb{R}^3$ with length $|\vec{v}| \in (0, c)$, the *general Lorentz transformation* is of the form (see e.g. [8, p. 434])

$$\vec{x}\,' = \vec{x} + \left(\frac{\gamma - 1}{|\vec{v}|^2} \vec{v} \cdot \vec{x} - \gamma t \right) \vec{v}, \tag{17}$$

$$t' = \gamma \left(t - \frac{\vec{v} \cdot \vec{x}}{c^2} \right). \tag{18}$$

Here the Lorentz factor γ is defined similarly as in (1), only v^2 needs to be rewritten as $|\vec{v}|^2$. It is easy to find that for nonzero $\vec{v} = (v, 0, 0)$, where $v \in (-c, c)$, relations (17)–(18) change to (2)–(3). In general, the Einstein addition of velocities is neither commutative nor associative [1, 16].

We conclude by stating that the longitudinal Doppler effect and aberration of light may cause that we observe completely opposite phenomena than those predicted by the Special Theory of Relativity by meas of (2)–(3). Note that relations (2)–(3) represent only a transformation of spacetime coordinates of points from one inertial system into spacetime coordinates of the second inertial system. We saw that some other effects than time dilatation and length contraction can manifest stronger and that they cannot be shielded in any way. For a visualization of several further accompanying effects (like nonlinear distortion) we refer to www.spacetimetravel.org.

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