Note

How to Imagine a Closed Universe

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Around 1900, the renowned astronomer Karl Schwarzschild came up with the idea that the geometry of the universe may not correspond to infinite three-dimensional Euclidean space $\mathbf{E}^3$, but that instead, it could be described as a closed three-dimensional sphere at a given time, i.e., the three-dimensional surface of a four-dimensional ball of radius $r > 0$ [2]:

$$\mathbf{S}^3 = \{(x, y, z, w) \in \mathbf{E}^4 | x^2 + y^2 + z^2 + w^2 = r^2\}.$$

See also [1]. For simplicity, we shall assume that $r = 1$. Since $\mathbf{S}^3$ is a curved manifold, it is difficult to visualize it. In [1], p. 112], we list ten ways to imagine it. One option is to use a stereographic projection, which preserves angles but not distances.

Please note that the term "closed universe" has a completely different meaning in astronomy from what it has in mathematics. For example, the unbounded interval $[0,\infty)$ is a closed set in $\mathbf{E}^1$, while a closed universe is bounded (closed within itself). On the other hand, the bounded interval $(0, 1)$ is open in $\mathbf{E}^1$, but the open universe is unbounded.

Now consider six great circles on the unit sphere $\mathbf{S}^3$ that intersect perpendicularly at eight points with coordinates $(\pm1, 0, 0, 0)$, $(0, \pm1, 0, 0)$, $(0, 0, \pm1, 0)$, and $(0, 0, 0, \pm1)$:

$$x^2 + y^2 = 1,\; x^2 + z^2 = 1,\; y^2 + z^2 = 1,$$
$$x^2 + w^2 = 1,\; y^2 + w^2 = 1,\; z^2 + w^2 = 1.$$

For instance, let us denote by $N = (0, 0, 0, 1)$ the north pole and by $S = (0, 0, 0, -1)$ the south pole of the sphere $\mathbf{S}^3$. Then the equator of $\mathbf{S}^3$ lies in the hyperplane $w = 0$, which is a two-dimensional sphere $\{(x,y,z,w) \in \mathbf{E}^4 | x^2 + y^2 + z^2 = 1\}$, within which lie the first three great circles that we are considering. The next three intersect perpendicularly at the north pole N and south pole S.

The stereographic projection of the above six circles into three-dimensional Euclidean space $\mathbf{E}^3$ can be constructed in the same way as in the case of two-dimensional space. We will project these circles from the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which lies on the sphere $\mathbf{S}^3$ and is symmetrically located with respect to these circles. Their centers do not belong to $\mathbf{S}^3$ and therefore are not displayed anywhere in the stereographic projection.

It is well known that Ptolemy's theorem on stereographic projection can be generalized to any dimension. Therefore, the six great circles of $\mathbf{S}^3$ in stereographic projection into three-dimensional space $\mathbf{E}^3$ are mapped to circles that intersect at right angles; see Figure 1. The resulting model can be printed on a standard 3D printer; see [3].[1]

Using this model, we can demonstrate that if we start at the north pole N and travel along any great circle in any direction, we will end up back at the north pole N via the south pole S. So we will cross the equator twice. This situation is similar to what we encounter on the Earth's surface, where the two main directions at the North Pole are perpendicular to each other. However, at the north pole N of the sphere $\mathbf{S}^3$, there are three directions that are mutually perpendicular: up–down, right–left, and forward–backward.

Figure 1. A rounded model of a closed universe in which distances are not preserved but angles remain the same.

[1] A program for accomplishing this is available at https://www.thingiverse.com/thing:1570212.

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