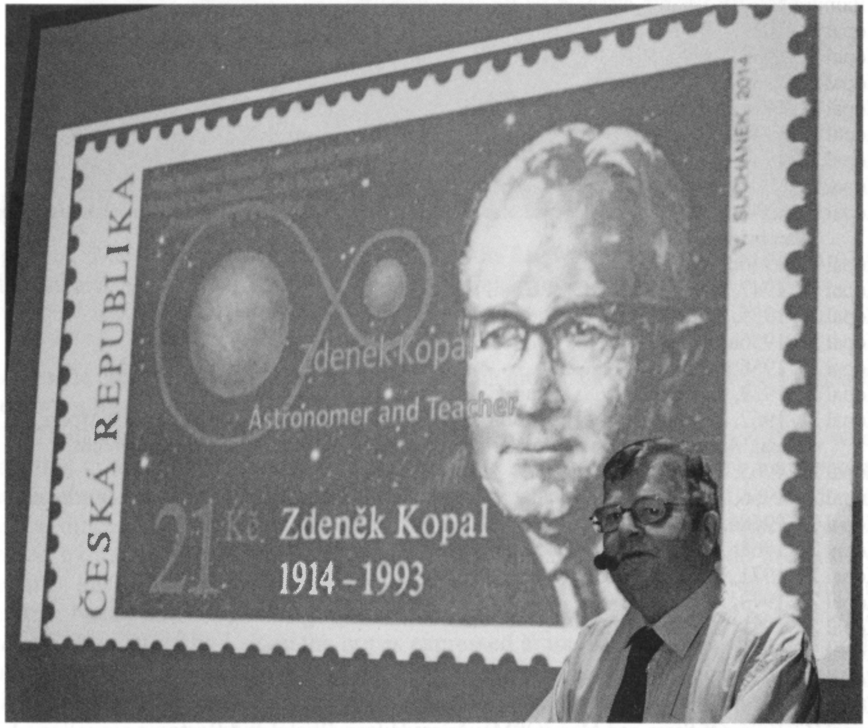

Ed Budding shows the commemorative postal stamp of Zdeněk Kopal.

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Zdeněk Kopal: Numerical Analyst

Michal Křížek

Institute of Mathematics, Academy of Sciences, Žitná 25, CZ-115 67 Prague 1, Czech Republic; `krizek@cesnet.cz`

Abstract. We give a brief overview of Zdeněk Kopal's life, his activities in the Czech Astronomical Society, his collaboration with Vladimír Vand, and his studies at Charles University, Cambridge, Harvard, and MIT. Then we survey Kopal's professional life. He published 26 monographs and 20 conference proceedings. We will concentrate on Kopal's extensive monograph *Numerical Analysis* (1955, 1961) that is widely accepted to be the first comprehensive textbook on numerical methods. It describes, for instance, methods for polynomial interpolation, numerical differentiation and integration, numerical solution of ordinary differential equations with initial or boundary conditions, and numerical solution of integral and integro-differential equations. Special emphasis will be laid on error analysis. Kopal himself applied numerical methods to celestial mechanics, in particular to the N-body problem. He also used Fourier analysis to investigate light curves of close binaries to discover their properties. This is, in fact, a problem from mathematical analysis.

1. Short Biography

Zdeněk Kopal was born on April 4, 1914, in Litomyšl, the central part of the Czech Republic. His father, Josef Kopal (1883–1966), was a member of the Royal Bohemian Society of Learning, the Bohemian Academy of the Sciences and the Arts, and Professor of the Charles University. Zdeněk's mother, Ludmila Kopalová (1884–1973), was a housewife as was quite usual at that time. Her father (Josef Lelek), who was a school teacher interested mainly in natural sciences (such as biology), very much influenced his grandson Zdeněk. In 1938, Zdeněk Kopal married his classmate Alena Müldnerová with whom he had three daughters: Georgiana, Zdenka, and Eva. There exists a very rich literature devoted to Kopal's life (see, e.g., Grygar 2004; Kopal 1991, 2014; Skřivánek et al. 1994; Šolcová & Křížek 2004, 2011). So we will very briefly recall only a few special issues and then concentrate on Kopal as a numerical analyst.

In 1923, Kopal's family moved to Prague. At the age of 14, Zdeněk constructed his first telescope and started to visit the Štefánik Observatory (see Figure 2). In 1946, he wrote in *Říše hvězd* (= Realm of the Stars), in the journal of the Czech Astronomical Society:

> ...*I will never forget that the Czech Astronomical Society and the Štefanik Observatory in Prague were my kindergarten.*

At the age of 15, Kopal became a member of the Czech Astronomical Society (CAS). Rostislav Rajchl, in Říše Hvězd, 10 (1929), p. 180, emphasizes that *Zd. Kopal made many maps of the celestial sphere with variable stars.* According to Říše Hvězd, 11 (1930), p. 130, the number of observations of variable stars provided by leading

Figure 1.: Zdeněk Kopal during his stay in Prague in 1964 (photo by Antonín Rükl).

observers was as follows: Kopal 2268, Kadavý 590, Rajchl 339, Izera 220, Černov 167, ..., which nicely illustrates Kopal's activity in this area. At the age of 16, he became the chair of the Variable Stars Section of the CAS. Together with Hubert Slouka, he wanted to establish a Mathematical Section of the CAS. Unfortunately, this good idea was never realized. At the age of 19, he translated the monograph *The Mysterious Universe* by James Jeans into the Czech language.

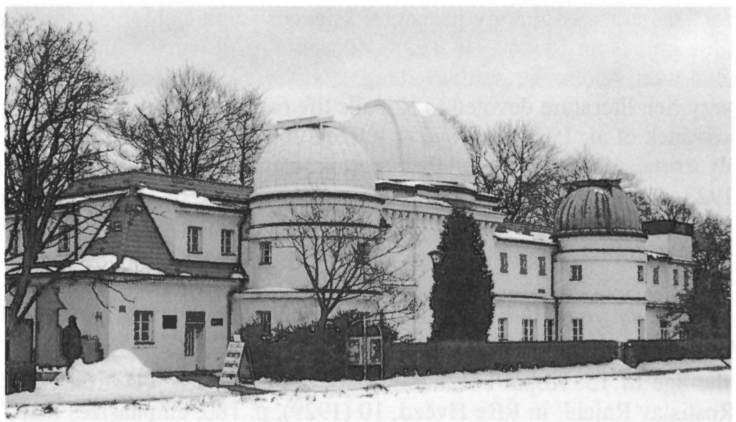

Figure 2.: Štefánik Observatory in Prague.

Zdeněk Kopal graduated from gymnasium in Smíchov (Drtinova street) in June 1933. The teacher dearest to his heart was Dr. Ladislav Klír, who taught him mathematics for four years. Kopal in (Kopal 1991, p. 52) recollects:

His constant message that mathematics was 'truth and beauty' I have always tried to imitate and pass on to others in the course of my own teaching career in different parts of the world. To be inspired and guided by such a teacher in the formative years of one's life was a gift which can never be repaid in the full ...

During the period 1933–1937, Kopal studied mathematics, physics, and astronomy at the Faculty of Science of Charles University in Prague. His teachers were, e.g., E. Čech, V. Dolejšek, V. Hlavatý, W.W. Heinrich, J. Heyrovský, V. Nechvíle, F. Nušl, F. Záviška. In 1935, Kopal attended the General Assembly of the IAU in Paris and became an IAU member. The next year he organized an expedition to Japan to observe a total Solar eclipse. In 1937 he obtained the prestigious Denis scholarship at Cambridge, UK; his advisor was Sir Arthur Eddington. In 1938, Kopal received his PhD from Charles University and then went to Harvard Observatory in the USA, where he had gotten another scholarship. From 1942, he worked at the Massachusetts Institute of Technology (MIT) and in 1948 became Professor of Numerical Analysis at MIT. In 1951, Kopal returned to Europe. He became the head of the Department of Astronomy in Manchester. In the late fifties, he started to collaborate with NASA on the Apollo mission. In order to find possible landing sites on the Moon, he organized the taking of about 60,000 photos of its surface using the 60 cm refractor at the Pic-du-Midi Observatory in France (Kopal 1991, p. 271).

Zdeněk Kopal returned for the first time (after the war) to Czechoslovakia in the Spring of 1957. He wanted to meet several Czech astronomers. After 1957 he came quite often to his homeland. In 1967 Kopal visited the Institute of Mathematics of the Czechoslovak Academy of Sciences, and met Prof. Ivo Babuška, a leading expert on numerical analysis in Prague. In the same year Kopal was elected as the Honorary member of the Czech Astronomical Society. In 1974 he became Honorary Doctor in Patras (Greece) and also in Kraków (Poland). Kopal died in Wilmslow (close to Manchester) in 1993. His tomb is situated in Vyšehrad in Prague (Šolcová & Křížek 2004).

2. Bibliography of Zdeněk Kopal

During his professional life, Kopal published about 400 research papers, 26 monographs, and 20 conference proceedings — their list is given in Šolcová & Křížek (2004). Eighty of his mathematics papers were reviewed by the Zentralblatt für Mathematik.

At the age of 16, Kopal wrote his first popularization paper in the Czech journal *Říše hvězd* (Kopal 1930). In 1931 he published his first scientific paper on variable stars (Kopal & Kadavý 1931), a modified version of which appeared in Astronomische Nachrichten.

At the age of 19, Kopal published together with Vladimír Vand the *Atlas of Variable Stars* (Kopal & Vand 1933), which had a remarkable success abroad, both among professionals as well as in amateur circles. Shortly after its publication, one could read in the journal of CAS, *Říše hvězd*, 16 (1938), p. 118:

We are asking the members of the CAS, who own the Atlas and no longer need it, to kindly sell their copy back to the CAS.

The first edition was sold out, and was acquired by the University Observatory in Berlin and the Yerkes Observatory, among other institutions. The Atlas contains 28 charts.

Kopal published eight monographs about close binary systems Kopal (1946, 1950, 1956, 1959, 1978, 1979b, 1989, 1972) and also eight monographs about the Moon (Kopal 1961, 1965, 1966, 1968, 1969, 1971, 1974a,b), three monographs about the Solar system (Kopal 1972, 1979a, 1984), two monographs about our Universe (Kopal 1970b, 1976), a monograph about artificial satellites (Kopal 1960), and one about space telescopes (Kopal 1970a).

From 1940 on, Kopal began publishing numerical analysis papers mostly about the investigation of light curves by means of the Fourier transform. Many of his papers also deal with the Roche lobes of a close binary system. The first edition of Kopal's monograph *Numerical Analysis* is widely considered to be the first systematic monograph about numerical methods (see Kopal 1955, 556 pp.). In the Preface, Kopal writes:

The present volume contains, in brief, an introduction to the numerical analysis of the functions of a single real variable, based on courses given by the writer to students of science and engineering at the Massachusetts Institute of Technology in Cambridge, U.S.A., between 1947 and 1951. The aim of the book has been to develop, in a systematic manner, the analytical basis of such numerical processes as are necessary for an algebraization and numerical solution of a wide range of problems of the infinitesimal calculus which are encountered daily in physics or engineering, but are not amenable to solution by formal mathematical methods.

The contents of this monograph are devoted to methods for polynomial interpolation, numerical differentiation, numerical solution of ordinary differential equations with initial or boundary conditions, numerical quadrature, and numerical solution of integral and integro-differential equations. Kopal only briefly mentions methods for solving large systems of linear algebraic equations, since this topic was treated in many other books. Each chapter contains a splendid account of the history of the subject with extensive bibliographical notes. The author precedes the main body of the text with an historical and philosophical introductory chapter.

The second edition of *Numerical Analysis* was published by John Wiley & Sons, New York, 1961, 600 pp. It contains a new chapter presenting systematic applications of the operational calculus to numerical analysis, particularly those based on the use of rational, rather than polynomial, approximations. The associated Padé tables have been added to Appendix. The second edition was translated into Chinese and published in Shanghai.

3. General Computational Scheme

The general computational scheme for solving problems of mathematical physics is sketched in Figure 3. Since no equation describes physical reality absolutely exactly, we always make an error e_0 (*modeling error*). Mathematical models are usually expressed as infinite dimensional problems. They are given by ordinary or partial differential equations, integro-differential or integral equations, systems of these equations, variational inequalities, systems of differential-algebraic equations, etc. Their solution can be searched for instance in the space of continuous functions whose dimension is infinite. For computer implementation we have to approximate them by finite dimensional problems, which yields the error e_1 (*discretization error*). This error may also

include the error of numerical integration, error of approximation of the boundary of the examined region, etc. Finally, the error e_2 arises during a computation of the discrete model. It contains, of course, round-off errors, but may include other errors (such as iteration errors, etc.).

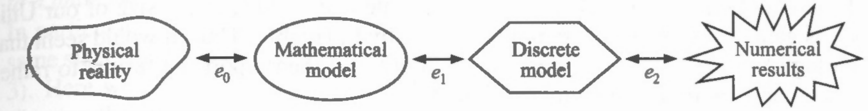

Figure 3.: General computational scheme.

Numerical analysis investigates only the errors e_1 and e_2, but not e_0. In a spite of that, let us take now a closer look also at the modeling error which may be essential, but is often ignored.

3.1. Modeling Error e_0

Without exception every equation in mathematical physics has limitations regarding the size of the objects investigated. For instance, the standard heat conduction equation (stationary or non-stationary) approximates very well the true temperature in solids of size about 1 meter, which can be verified by direct measurements (see Křížek & Neittaanmäki 1996). However, in applying the heat equation on the atomic level in a cube 10^{-10} m on a side, we get nonsensical numbers, and similarly in a cube with size 10^{10} m, which would in fact immediately collapse into a back hole (note that the diameter of our Sun is 1.4×10^9 m). The same is true for linear elasticity equations, semiconductor equations, supra-conductivity equations, Navier–Stokes equations for fluids, Maxwell's equations, Korteweg-de Vries equations, magneto-hydro-dynamic equations, and so on. Analogously, we should not apply Keplerian laws on scales of 10^{-10} m or, similarly, the Schrödinger equation to objects that have the size of a cat. Similar arguments apply also for time scales. For instance, the standard predator-prey equation or the N-body problem on the interval of 10^{10} years do not describe reality well. Therefore, in any calculation we must take into account the modeling error, which is small only under some restrictive space-time limits.

Despite this, when deriving the standard cosmological Friedmann equation, the Einstein equations are applied to the entire Universe. This is considered as a platitude and almost no one deals with the question of whether it is justified to perform such a fearless extrapolation without any observational support. It should be noted that general relativity was "checked" only for much smaller scales such as the Solar System (slowdown of electromagnetic waves and bending of light in the gravitational field of the Sun, measuring the curvature of spacetime near the rotating Earth by means of the Lense-Thirring precession effect, perihelion advance of Mercury's orbit, etc.). Note that galaxies have a diameter on the order of 10^{10} astronomical units and the Universe is at least five orders of magnitude larger.

Fritz Zwicky (Zwicky 1933) and Vera Rubin (Rubin 1997) made use of overly simplistic models and insufficient data involving a number of selection effects to predict dark matter (Křížek et al. 2014). For instance, at the end of the last century, astronomers believed that only 3 % of all stars are red dwarfs (Binney & Merrifield 1998,

p. 93). However, from *Hipparcos* data and other sources we now know that 70–80% of all stars are red dwarfs. Therefore, the proposed ratio 27 : 5 between the amount of dark matter and baryonic matter may be considerably overestimated. Moreover, the (normalized) Friedmann equation was derived under an incorrect extrapolation ignoring the modeling error (Křížek & Somer 2014, p. 442). Of course, in 1922 when A. Friedmann published his famous paper (Friedmann 1922), he had no idea of the size of our Universe, as galaxies were discovered only in 1924 by E. Hubble. Thus, it would seem that cosmologists solve incorrectly-derived equations and then report the results to rather high accuracy (up to three significant digits).

3.2. Discretization Error e_1

The difference between the true solution of a mathematical model and its numerical approximation is often estimated using various *a priori* or *a posteriori* error estimates. This step is very important, since real-life problems are often nonlinear, large scale, ill-conditioned, or non-stable. They may have multiple solutions, singularities, etc. If we do not perform a reliable error analysis, we do not know, in fact, how close the numerical solution is to the exact one.

To be more concrete, consider the Gaussian quadrature rule

$$\int_a^b g(x)dx \approx \sum_{i=1}^{n} w_i g(a_i), \tag{1}$$

where g is a continuous function, the nodes a_i are roots of Legendre polynomials, and $w_i \in \mathbb{R}$ are appropriate weights. The approximation in Eq.(1) is exact for all polynomials g of degree $2n-1$. For example, choosing $n = 2$,

$$[a, b] = [-1, 1],\ w_1 = w_2 = 1,\ a_i = \pm\sqrt{3}/3,$$

the Gaussian quadrature is exact for all cubic polynomials and thus its order of accuracy is four. In Kopal (1955), he gives a list of all coefficients up to $n = 16$. For large n, the discretization error is much smaller than rounding errors (depending on the computer).

Z. Kopal presents also a long series of numerical methods of various orders for solving the initial value problem for ordinary differential equations

$$\dot{y} = f(x, y),$$

$$y(x_0) = y_0,$$

where $\dot{y} = \dot{y}(t)$ stands for the time derivative of the searched solution $y = y(t)$, f is a given function, and y_0 is the initial condition. He establishes sufficient conditions under which these methods converge (see, e.g., Kopal 1955, p. 130), and examines the order of truncation (discretization) errors. For higher order methods the total numerical error is usually not due to truncation, but to round-off (see Kopal 1955, p. 194). Let us point out that, at present, there are many popular modifications of these methods, such as the six-stage fifth-order modification of the Runge-Kutta method by Cash & Karp (1990) for initial value problems with rapidly varying right-hand sides.

3.3. Round-Off Errors e_2

Zdeněk Kopal devoted considerable effort to studying the propagation and detection of rounding errors in tabular differences. In Kopal (1955, p. 69) he emphasizes:

... *all numerical data with which we shall have to deal can be given and used only to a finite number of significant figures.*

In particular, we should carefully examine any subtraction of numbers of almost the same size, which may lead to catastrophic loss of accuracy (see, e.g., Rice 1993, Chapt. 3). Here we recall two shocking examples from Muller et al. (2009) and Cuyt (2001) showing that we should never underestimate the impact of rounding errors.

Example 1. Let $a_1 = \mathrm{e} - 1 = 1.718281828\ldots$, where e is the Euler number, and consider the thoroughly innocent-looking sequence

$$a_n = n\,(a_{n-1} - 1) \text{ for } n = 2, 3, \ldots, 25. \tag{2}$$

By induction we easily find that

$$a_n = n!\Big(\frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\Big).$$

From this we see that the sequence (a_n) is decreasing and its limit is one, i.e., all a_n have a very reasonable size in the interval $(1, 2)$. However, performing only 24 subtractions and 24 multiplications in Eq.(2) with a different number of significant digits in MATLAB, we get the following numbers:

$\tilde{a}_{25} = -1.306946 \times 10^{13}$ in real-precision arithmetic (6 bytes),
$\hat{a}_{25} = 1.201807 \times 10^{9}$ in double-precision arithmetics (8 bytes),
$\overline{a}_{25} = -7.355732 \times 10^{6}$ in extended-precision arithmetic (10 bytes).

So let us extend the number of significant digits. In arithmetic with D decimal digits, the first 12 significant digits are:

D	$a_{25}(D)$
20	615990.413139
21	−4457.98859386
22	−4457.98859386
23	195.374419140
24	40.2623187072
25	−6.27131142281
26	1.48429359885

The values for $D = 21$ and $D = 22$ are the same, since the 21st decimal digit of the Euler number e is equal to zero. Only for $D \geq 30$ do numerical results start to resemble the exact value

$$a_{25} = 1.03993872967\ldots$$

For instance, if $D = 30$, we get $a_{25}(30) = 1.039897\ldots$

Let us take a closer look at the main reason for this strange behavior. Denote by ε_i the rounding error at the ith step. We then have

$$\tilde{a}_1 = \mathrm{e} - 1 + \varepsilon_1 = a_1 + \varepsilon_1,$$
$$\tilde{a}_2 = 2(\tilde{a}_1 - 1) + \varepsilon_2 = 2(a_1 + \varepsilon_1 - 1) + \varepsilon_2 = a_2 + 2!\varepsilon_1 + \varepsilon_2,$$
$$\dots$$
$$\tilde{a}_{25} = 25(\tilde{a}_{24} - 1) + \varepsilon_{25} = a_{25} + 25!\varepsilon_1 + 25!\varepsilon_2/2! + 25!\varepsilon_3/3! + \dots + \varepsilon_{25}.$$

Therefore, the total rounding error is $a_{25} - \tilde{a}_{25} = -25!(\varepsilon_1 + \varepsilon_2/2! + \varepsilon_3/3! + \dots + \varepsilon_{25}/25!)$ and its size depends particularly on the several initial rounding errors $\varepsilon_1, \varepsilon_2, \dots$ This sophisticated example illustrates why it is necessary to check subtraction of two numbers of almost the same size during numerical calculation, such as in Eq.(2), where the rounding error grows exponentially with n.

Example 2. Evaluate

$$u(x,y) = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + \frac{x}{2y} \tag{3}$$

at $x = 77617.0$ and $y = 33096.0$, i.e., this is a polynomial of the 8th degree plus the trivial rational function $x/(2y)$. We observe that no recurrence relation as in Eq.(2) is evaluated and we perform only three subtractions and a few other arithmetic operations. Contrary to Example 1, we get almost the same numbers:

$u(x,y) = 1.172603$ in single-precision arithmetics (4 bytes),
$u(x,y) = 1.1726039400531$ in double-precision arithmetic (8 bytes),
$u(x,y) = 1.172603940053178$ in extended-precision arithmetic (10 bytes)

on an outmoded IMB 370 computer (see Rump 1988). The MAPLE program with

$$D = 7, 8, 9, 10, 12, 18, 26, 27$$

decimal digits produces very similar results. However, we should not rejoice over the above results, since the exact value is

$$u(x,y) = -0.827396\dots \text{NEGATIVE!}$$

Numerical results by MAPLE and MATLAB only begin to be realistic starting at $D = 37, 38, \dots$ and $D \geq 30$, respectively. A detailed numerical analysis of this catastrophic behavior of rounding errors is presented by Cuyt (2001).

Thus, we should keep in mind that a very small amount of roundings (Eqs.(2) and (3)) may completely destroy the exact solution (Rump 2010). This fact should be taken into account in long-term numerical simulations of the N-body problem.

4. Total Error of the N-body Problem

4.1. Modeling Error e_0

For an integer $N \geq 2$ consider an isolated system of N point-mass bodies that mutually interact only gravitationally. Let $r_i = r_i(t)$, $i \in \{1, \dots, N\}$ be the so-called radius-vector in $\mathbb{R}^3$ describing the position of the ith body with mass m_i at time $t \geq 0$. Denoting the

gravitational constant by G, the classical N-body problem is described by the following nonlinear system of ordinary differential equations for the unknown trajectories r_i,

$$\ddot{r}_i = G \sum_{i \neq j}^{N} \frac{m_j(r_j - r_i)}{|r_j - r_i|^3}, \tag{4}$$

with some initial conditions at $t_0 = 0$ and over a time interval $[0, T]$ in which it is assumed the bodies do not collide. This system is autonomous, since the right-hand side of Eq.(4) does not explicitly depend on time.

Can we believe numerical simulations of the evolution of the Solar System based on Eq.(4) for billions of years in the past or future? The answer is NO for several reasons. Newtonian mechanics does not allow any delay given by the finite speed of gravitational interaction. Eq.(4) is a system of ordinary differential equation whose solution in the interval $[0, T]$ depends only on the value at point $t_0 = 0$, and not on the history. An extremely small modeling error $\varepsilon > 0$ during one year may yield an error of order $10^9\varepsilon$ after 1 Gyr that may be quite large (Křížek 2012). Also, various non-gravitational forces are not included in Eq.(4). These facts are usually not taken into account, e.g., in the well-known NICE model. The corresponding calculations over a time interval of several Gyr resemble a weather forecast for many months in advance.

The system in Eq.(4) is not Lyapunov stable, i.e., extremely small changes of initial conditions or other perturbations cause very large changes in the final state provided the time interval is long enough. The right-hand side of Eq.(4) does not satisfy the famous Caratheodory conditions. The nonlinear system Eq.(4) also has many unrealistic solutions; see, for example, Chenciner & Montgomery (2000) and Saari & Xia (1995).

4.2. Discretization Error e_1

For $N > 2$ Isaac Newton stated:

An exact solution exceeds, if I am not mistaken, the force of any human mind.

Henri Poincaré knew that the N-body problem for $N > 2$ has an analytical solution in a few special cases, but there is no explicit formula for its general solution. Therefore, this problem must be solved numerically by means of Runge-Kutta methods, multi-step methods, symplectic methods, etc.

Z. Kopal (1955) in his *Numerical Analysis* on p. 16 wrote:

... the Finnish mathematician Sundman, Acta Math. 36 (1913), 105, has succeeded at last in establishing the solution of the three-body problem (under certain restrictions) in terms of the series which converge absolutely and uniformly for all time. Yet — alas — their convergence turned out to be tantalizingly slow; so slow, in fact, that the sum of approximately one million terms would be needed to establish the positions of the planets for the time increment of one day to the degree of precision attainable by the well-known and simple asymptotic expansions.

If an integration step h gives almost the same numerical results as $h/2$ the integration of the system Eq.(4) is usually stopped. However, this heuristic criterion need not work properly. Here we present another way how to check whether our numerical results are good. It is based on backward integration of Eq.(4) as sketched in Figure 4. Let $r = (r_1, \dots, r_N)$ denote a vector with $3N$ entries and let $f = f(r)$ stand for the right-hand hand side of Eq.(4).

Theorem. *Let $r = r(t)$ be the unique solution of the system*

$$\ddot{r} = f(r) \tag{5}$$

in the interval $[0, T]$ *with given initial conditions*

$$r(0) = r_0 \quad \text{and} \quad \dot{r}(0) = v_0. \tag{6}$$

Then the function $s = s(t)$ *defined by*

$$s(t) = r(T - t) \tag{7}$$

solves the same system Eq.(5) with initial conditions $s(0) = r(T)$, $\dot{s}(0) = -\dot{r}(T)$, *and we have*

$$s(T) = r_0 \quad \text{and} \quad \dot{s}(T) = -v_0. \tag{8}$$

P r o o f. According to Eq.(5) and Eq.(7), we obtain

$$\ddot{s}(t) = (-\dot{r}(T-t))^{\cdot} = \ddot{r}(T-t) = f(r(T-t)) = f(s(t)).$$

We see that s satisfies the same system Eq.(5) as r. For the final conditions, by Eq.(6) and Eq.(7) we obtain the relations in Eq.(8),

$$s(T) = r(0) = r_0 \quad \text{a} \quad \dot{s}(T) = -\dot{r}(T-T) = -\dot{r}(0) = -v_0. \quad \square$$

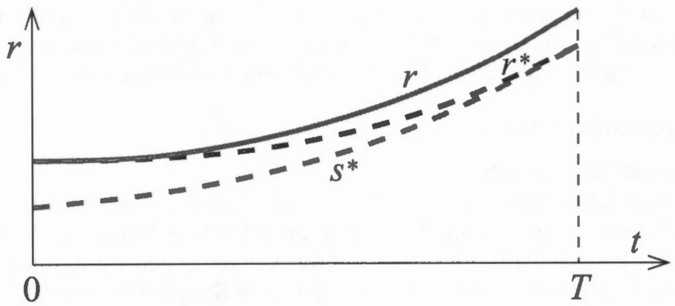

Figure 4.: Application of the theorem, as described in the text, to a numerical solution of the N-body problem. The symbol r stands for the true solution, r^* is the numerical solution, and s^* is the control backward solution.

The above theorem can be applied to long-term intervals as follows. Denote by r^* and s^* a numerical solution of the system Eq.(4) with initial conditions Eq.(7) and

$$s(0) = r^*(T) \quad \text{and} \quad \dot{s}(0) = -\dot{r}^*(T),$$

respectively. If $\delta > 0$ is a given tolerance and

$$|s^*(T) - r_0| + |\dot{s}^*(T) + v_0| \gg \delta,$$

then it cannot hold that $|r(T) - r^*(T)| + |\dot{r}(T) - \dot{r}^*(T)| < \delta$, where r is the exact solution, i.e., the numerical error of the original problem from Eq.(4) and Eq.(7) would be also large, as is schematically depicted in Figure 4.

Finally, let us point our that the previous theorem and computational strategy can be easily generalized to the evolution of partial differential equations.

4.3. Round-off Errors e_2

From Eq.(4) we observe that when two bodies are close to each other ($r_j \approx r_i$), which is an important case, e.g., in space aeronautics, then we subtract in the denominator two numbers of almost the same size. This may lead to a catastrophic loss of accuracy (see, e.g., Examples 1 and 2).

Moreover, numerical error usually grows exponentially with time. This fact has serious consequences. For instance, on April 13, 2029, the distance of the asteroid Apophis from the Earth will be about 30,000 km. However, for the next approach in 2036 the distance cannot be reliably calculated, since the system Eq.(4) is unstable.

Also note that examples showing a deterministic chaos are usually incorrectly understood, since rounding errors are not taken into account (Muller et al. 2009). In many of these examples chaos appears just due to rounding.

5. Final Remarks

During his life Professor Zdeněk Kopal received many prizes and other expressions of recognition, such as the golden medal of the Czechoslovak Academy of Sciences (1968) and the silver medal of the Charles University (1991). He was nominated to the Greek Academy in Athens (1976) and was appointed honorary citizen of the town of Delf (1978) and also of the town of Litomyšl (1991). Asteroid No. 2628 was named *Kopal* by the decision of the International Astronomical Society.

Z. Kopal admired three scientists: I. Newton, C. F. Gauss, and H. J. Poincaré (see Kopal 1991, p. 461). Also many famous artists, writers, and scientists influenced him, such as L. Baarová, Van Dyk, A. Einstein, A. Jirásek, G. Lemaître, H. N. Russell, H. Shapley, V. Vand, N. Wiener, F. Zwicky, and others.

M. V. Wilkers, D. J. Wheeler, and S. Gill wrote in the Preface of their monograph 'The preparation of programs for an electronic digital computer', Cambridge, 1951:

We would also like to express our gratitude to Dr. Z. Kopal for the help he has freely given in proofreading and in seeing the book through the press.

Also, Anthony Ralston in his famous book 'A First Course in Numerical Analysis', McGraw-Hill, 1965, thanks Zdeněk Kopal as follows:

I have a debt to many numerical analysts both path and present. But my debt is particularly great to three men, each the author of a standard text in numerical analysis. I have had the privilege of being a student of two of them — Zdenek Kopal and F. B. Hildebrand — a colleague of the third — R. W. Hamming. To anyone familiar with their books, my debt will be particularly clear.

Professor Zdeněk Kopal was one of several outstanding personalities whose contributions exceeded national borders in crucial — for the twentieth century — developments of applied mathematics and astronomy.

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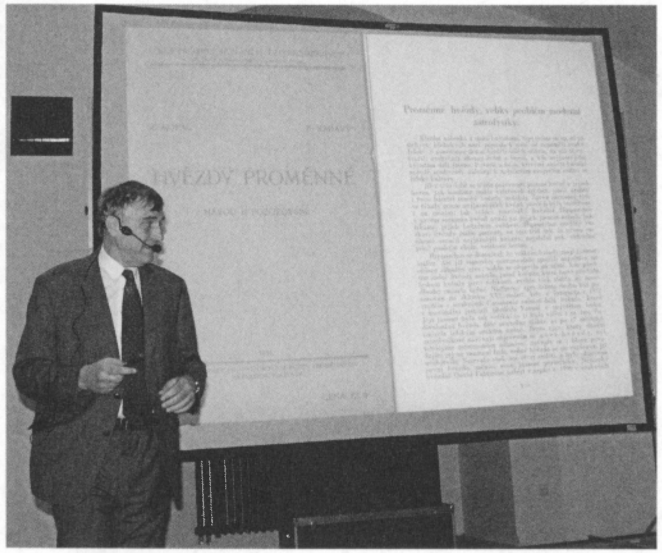

Michal Křížek shows one of the early publications of Zdeněk Kopal.

Monument of a binary star at the birthplace of Zdeněk Kopal.

Zdenka Smith (left) and Georgiana Rudge (right), daughters of Zdeněk Kopal.