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EXCESSIVE EXTRAPOLATIONS OF EINSTEIN'S EQUATIONS

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Cordially dedicated to Assoc. Prof. Jan Chleboun on his 60th birthday

Abstract: The standard cosmological model is surprisingly quite thoroughly investigated even though it possesses many paradoxes. We present several arguments indicating why excessive extrapolations of Einstein's equations to cosmological distances are questionable. First, we show how to express explicitly the first of Einstein's 10 partial differential equations to demonstrate their extremely large complexity. Therefore, it would be very difficult to find their solution for two or more bodies to model, e.g., the evolution of the Solar system. Further, we present some unexpected failures of the Schwarzschild and Friedmann solution of these equations. Then we explain why application of Einstein's equations to the whole universe represents incorrect extrapolations that lead to dark matter, dark energy, and several unrealistic situations. Finally, we give 10 further arguments showing why celebrated Einstein's equations do not describe reality well.

Keywords: Einstein's equations, Schwarzschild solution, Friedmann equation, modeling error, incorrect extrapolations, dark matter, dark energy

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1. Introduction

Einstein's field equations of general relativity consist of 10 equations (cf. [13])

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad \mu,\nu = 0,1,2,3, \tag{1}$$

for 10 components of the unknown symmetric metric tensor $g_{\mu\nu}$ (sometimes called *gravitational potentials*) of one timelike coordinate $x^0 = ct$ and three Cartesian or curvilinear space coordinates x^1, x^2, x^3, i.e. $g_{\mu\nu} = g_{\mu\nu}(x^0, x^1, x^2, x^3)$ (for simplicity the dependence of all functions from (1) on these coordinates will be nowhere indicated), where $c = 299\,792\,458$ m/s is the speed of light in vacuum, $G = 6.674 \cdot 10^{-11}$ $\mathrm{m^3 kg^{-1} s^{-2}}$ is the gravitational constant,

$$R_{\mu\nu} = \sum_{\varkappa=0}^{3} R^{\varkappa}{}_{\mu\varkappa\nu} \tag{2}$$

is the symmetric *Ricci tensor*,

$$R = \sum_{\mu,\nu=0}^{3} g^{\mu\nu} R_{\mu\nu} \tag{3}$$

is the *Ricci scalar* (i.e. the scalar curvature), $T_{\mu\nu}$ is the symmetric *tensor of density of energy and momentum*,

$$R^{\varkappa}{}_{\mu\sigma\nu} = \frac{\partial \Gamma^{\varkappa}{}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\varkappa}{}_{\mu\sigma}}{\partial x^{\nu}} + \sum_{\lambda=0}^{3} \Gamma^{\lambda}{}_{\mu\nu} \Gamma^{\varkappa}{}_{\lambda\sigma} - \sum_{\lambda=0}^{3} \Gamma^{\lambda}{}_{\mu\sigma} \Gamma^{\varkappa}{}_{\lambda\nu}, \quad \varkappa,\ \mu,\ \sigma,\ \nu = 0,1,2,3, \tag{4}$$

is the *Riemann curvature tensor* that has 20 independent components from the total number of $4^4 = 256$ components due to several symmetries

$$R^{\varkappa}{}_{\mu\sigma\nu} + R^{\varkappa}{}_{\sigma\nu\mu} + R^{\varkappa}{}_{\nu\mu\sigma} = 0, \quad R_{\lambda\mu\sigma\nu} = -R_{\mu\lambda\sigma\nu} = -R_{\lambda\mu\nu\sigma}, \quad R_{\lambda\mu\sigma\nu} = \sum_{\varkappa} g_{\lambda\varkappa} R^{\varkappa}{}_{\mu\sigma\nu},$$

where the first equality is called the *first Bianchi identity* and

$$\Gamma^{\mu}{}_{\varkappa\sigma} = \frac{1}{2} \sum_{\nu=0}^{3} g^{\mu\nu} \Big(\frac{\partial g_{\varkappa\nu}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\varkappa}} - \frac{\partial g_{\varkappa\sigma}}{\partial x^{\nu}} \Big) = \frac{1}{2} \sum_{\nu=0}^{3} g^{\mu\nu} (g_{\varkappa\nu,\sigma} + g_{\sigma\nu,\varkappa} - g_{\varkappa\sigma,\nu}) \tag{5}$$

are the *Christoffel symbols of the second kind* (also called the *connection coefficients*). All derivatives are supposed to be classical (for simplicity we shall write $g_{\varkappa\nu,\sigma} := \partial g_{\varkappa\nu}/\partial x^{\sigma}$ to reduce notation). From this and the relation $g_{\varkappa\sigma} = g_{\sigma\varkappa}$ one can derive the symmetry

$$\Gamma^{\mu}{}_{\varkappa\sigma} = \Gamma^{\mu}{}_{\sigma\varkappa} \quad \text{for } \mu = 0,1,2,3.$$

Thus altogether we have $40 = 4 \times (1+2+3+4)$ independent components. Finally, the contravariant symmetric 4×4 metric tensor $g^{\mu\nu}$ is inverse to the covariant metric tensor $g_{\mu\nu}$, i.e.

$$g^{\mu\nu} = \frac{g^{*}_{\mu\nu}}{\det(g_{\mu\nu})}, \qquad \det(g_{\mu\nu}) := \sum_{\pi \in S_4} (-1)^{\operatorname{sgn}\pi} g_{0\nu_0} g_{1\nu_1} g_{2\nu_2} g_{3\nu_3}, \tag{6}$$

where the entries $g^{*}_{\mu\nu}$ form the 4×4 matrix of 3×3 algebraic adjoints of $g_{\mu\nu}$, S_4 is the symmetric group of 24 permutations π of indices $(\nu_0, \nu_1, \nu_2, \nu_3)$, $\operatorname{sgn}\pi = 0$ for an even permutation and $\operatorname{sgn}\pi = 1$ for an odd permutation.

Theorem 1. *If $g_{\mu\nu}$ is a solution of* (1), *then* $(-g_{\mu\nu})$ *also solves* (1).

Proof. From (5), we find that the Christoffel symbols remain the same if we replace $g_{\mu\nu}$ by $(-g_{\mu\nu})$, namely,

$$\Gamma^{\mu}{}_{\varkappa\sigma} = \frac{1}{2}(-g^{\mu\nu})\Big(-\frac{\partial g_{\varkappa\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\sigma\nu}}{\partial x^{\varkappa}} + \frac{\partial g_{\varkappa\sigma}}{\partial x^{\nu}}\Big).$$

Using (2) and (4), we find that the Ricci tensor $R_{\mu\nu}$ in (1) does not change as well. Concerning the second term on the left-hand side of (1), we observe from (3) that $\big(-\frac{1}{2}Rg_{\mu\nu}\big)$ also remains unchanged if we replace $g_{\mu\nu}$ by $(-g_{\mu\nu})$. $\square$

2. On explicit form of the first Einstein equation

In this section we want to point out the extreme complexity of Einstein's equations. In (1), the dependence of the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$ on the metric tensor $g_{\mu\nu}$ is not indicated. Therefore, Einstein's equations (1) seem to be quite simple. To avoid this deceptive opinion, we will now derive the explicit form of the first Einstein equation.

We will consider only the simplest case when $T_{\mu\nu} = 0$ (and without the cosmological constant, cf. (17)). Multiplying (1) by $g^{\mu\nu}$ and summing over all μ and ν, we obtain by (3) that

$$0 = \sum_{\mu,\nu=0}^{3} g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R\sum_{\mu,\nu=0}^{3} g^{\mu\nu}g_{\mu\nu} = R - \frac{1}{2}R\sum_{\mu=0}^{3}\delta^{\mu}{}_{\mu} = R - \frac{1}{2}4R,$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta. Thus, $R = 0$ and Einstein's vacuum equations can be rewritten as[1]

$$R_{\mu\nu} = 0.$$

Applying (2) and (4), we can express the first Einstein equation as follows

$$0 = R_{00} = \sum_{\varkappa=0}^{3} R^{\varkappa}{}_{0\varkappa 0} = \sum_{\varkappa=0}^{3}\left(\Gamma^{\varkappa}{}_{00,\varkappa} - \Gamma^{\varkappa}{}_{0\varkappa,0} + \sum_{\lambda=0}^{3}(\Gamma^{\lambda}{}_{00}\Gamma^{\varkappa}{}_{\lambda\varkappa} - \Gamma^{\lambda}{}_{0\varkappa}\Gamma^{\varkappa}{}_{0\lambda})\right)$$

$$\begin{aligned}
=&\underline{\Gamma^{0}{}_{00,0}} + \Gamma^{1}{}_{00,1} + \Gamma^{2}{}_{00,2} + \Gamma^{3}{}_{00,3} - \underline{\Gamma^{0}{}_{00,0}} - \Gamma^{1}{}_{01,0} - \Gamma^{2}{}_{02,0} - \Gamma^{3}{}_{03,0}\\
&+ \underline{\Gamma^{0}{}_{00}(\Gamma^{0}{}_{00}} + \Gamma^{1}{}_{01} + \Gamma^{2}{}_{02} + \Gamma^{3}{}_{03}) + \underline{\underline{\Gamma^{1}{}_{00}(\Gamma^{0}{}_{10}}} + \Gamma^{1}{}_{11} + \Gamma^{2}{}_{12} + \Gamma^{3}{}_{13})\\
&+ \underline{\underline{\Gamma^{2}{}_{00}(\Gamma^{0}{}_{20}}} + \Gamma^{1}{}_{21} + \Gamma^{2}{}_{22} + \Gamma^{3}{}_{23}) + \underline{\Gamma^{3}{}_{00}(\Gamma^{0}{}_{30}} + \Gamma^{1}{}_{31} + \Gamma^{2}{}_{32} + \Gamma^{3}{}_{33})\\
&- \underline{\Gamma^{0}{}_{00}\Gamma^{0}{}_{00}} - \Gamma^{0}{}_{01}\Gamma^{1}{}_{00} - \Gamma^{0}{}_{02}\Gamma^{2}{}_{00} - \Gamma^{0}{}_{03}\Gamma^{3}{}_{00} - \underline{\underline{\Gamma^{1}{}_{00}\Gamma^{0}{}_{01}}} - \Gamma^{1}{}_{01}\Gamma^{1}{}_{01}\\
&- \Gamma^{1}{}_{02}\Gamma^{2}{}_{01} - \Gamma^{1}{}_{03}\Gamma^{3}{}_{01}\\
&- \underline{\underline{\Gamma^{2}{}_{00}\Gamma^{0}{}_{02}}} - \Gamma^{2}{}_{01}\Gamma^{1}{}_{02} - \Gamma^{2}{}_{02}\Gamma^{2}{}_{02} - \Gamma^{2}{}_{03}\Gamma^{3}{}_{02} - \underline{\Gamma^{3}{}_{00}\Gamma^{0}{}_{03}} - \Gamma^{3}{}_{01}\Gamma^{1}{}_{03}\\
&- \Gamma^{3}{}_{02}\Gamma^{2}{}_{03} - \Gamma^{3}{}_{03}\Gamma^{3}{}_{03},
\end{aligned}$$

[1] Concerning nonuniqueness expressed by Theorem 1, we observe from (5) that we can add any constant to any component of $g_{\mu\nu}$ and Einstein's equations $R_{\mu\nu} = 0$ will still be valid.

where the underlined terms cancel. Hence, the first Einstein equation can be rewritten by means of the Christoffel symbols in the following way:

$$
\begin{aligned}
0 =&\Gamma^1{}_{00,1} + \Gamma^2{}_{00,2} + \Gamma^3{}_{00,3} - \Gamma^1{}_{01,0} - \Gamma^2{}_{02,0} - \Gamma^3{}_{03,0} \\
&+ \Gamma^0{}_{00}(\Gamma^1{}_{01} + \Gamma^2{}_{02} + \Gamma^3{}_{03}) + \Gamma^1{}_{00}(-\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) \\
&+ \Gamma^2{}_{00}(-\Gamma^0{}_{20} + \Gamma^1{}_{21} + \Gamma^2{}_{22} + \Gamma^3{}_{23}) + \Gamma^3{}_{00}(-\Gamma^0{}_{30} + \Gamma^1{}_{31} + \Gamma^2{}_{32} + \Gamma^3{}_{33}) \\
&- 2\Gamma^1{}_{02}\Gamma^2{}_{01} - 2\Gamma^1{}_{03}\Gamma^3{}_{01} - 2\Gamma^2{}_{03}\Gamma^3{}_{02} - (\Gamma^1{}_{01})^2 - (\Gamma^2{}_{02})^2 - (\Gamma^3{}_{03})^2. \qquad (7)
\end{aligned}
$$

Using (5), we obtain

$$
\begin{aligned}
2\Gamma^\mu{}_{\varkappa\sigma} =&g^{\mu 0}(g_{\varkappa 0,\sigma} + g_{\sigma 0,\varkappa} - g_{\varkappa\sigma,0}) + g^{\mu 1}(g_{\varkappa 1,\sigma} + g_{\sigma 1,\varkappa} - g_{\varkappa\sigma,1}) \\
&+ g^{\mu 2}(g_{\varkappa 2,\sigma} + g_{\sigma 2,\varkappa} - g_{\varkappa\sigma,2}) + g^{\mu 3}(g_{\varkappa 3,\sigma} + g_{\sigma 3,\varkappa} - g_{\varkappa\sigma,3})
\end{aligned}
$$

and thus by (7) we can express the first Einstein equation $R_{00} = 0$ by means of the metric coefficients and their first and second order derivatives as follows:

$$
\begin{aligned}
0 =&4R_{00} = 2[g^{10}_{,1}g_{00,0} + g^{11}_{,1}(2g_{01,0} - g_{00,1}) + g^{12}_{,1}(2g_{02,0} - g_{00,2}) + g^{13}_{,1}(2g_{03,0} - g_{00,3}) \qquad (8) \\
&+ g^{10}g_{00,01} + g^{11}(2g_{01,01} - g_{00,11}) + g^{12}(2g_{02,01} - g_{00,21}) + g^{13}(2g_{03,01} - g_{00,31}) \qquad (9) \\
&+ g^{20}_{,2}g_{00,0} + g^{21}_{,2}(2g_{01,0} - g_{00,1}) + g^{22}_{,2}(2g_{02,0} - g_{00,2}) + g^{23}_{,2}(2g_{03,0} - g_{00,3}) \\
&+ g^{20}g_{00,02} + g^{21}(2g_{01,02} - g_{00,12}) + g^{22}(2g_{02,02} - g_{00,22}) + g^{23}(2g_{03,02} - g_{00,32}) \\
&+ g^{30}_{,3}g_{00,0} + g^{31}_{,3}(2g_{01,0} - g_{00,1}) + g^{32}_{,3}(2g_{02,0} - g_{00,2}) + g^{33}_{,3}(2g_{03,0} - g_{00,3}) \\
&+ g^{30}g_{00,03} + g^{31}(2g_{01,03} - g_{00,13}) + g^{32}(2g_{02,03} - g_{00,23}) + g^{33}(2g_{03,03} - g_{00,33}) \\
&- g^{10}_{,0}g_{00,1} - g^{11}_{,0}g_{11,0} - g^{12}_{,0}(g_{02,1} + g_{12,0} - g_{01,2}) - g^{13}_{,0}(g_{03,1} + g_{13,0} - g_{01,3}) \\
&- g^{10}g_{00,10} - g^{11}g_{11,00} - g^{12}(g_{02,10} + g_{12,00} - g_{01,20}) - g^{13}(g_{03,10} + g_{13,00} - g_{01,30}) \\
&- g^{20}_{,0}g_{00,2} - g^{21}_{,0}(g_{01,2} + g_{21,0} - g_{02,1}) - g^{22}_{,0}g_{22,0} - g^{23}_{,0}(g_{03,2} + g_{23,0} - g_{02,3}) \\
&- g^{20}g_{00,20} - g^{21}(g_{01,20} + g_{21,00} - g_{02,10}) - g^{22}g_{22,00} - g^{23}(g_{03,20} + g_{23,00} - g_{02,30}) \\
&- g^{30}_{,0}g_{00,3} - g^{31}_{,0}(g_{01,3} + g_{31,0} - g_{03,1}) - g^{32}_{,0}(g_{02,3} + g_{32,0} - g_{03,2}) - g^{33}_{,0}g_{33,0} \\
&- g^{30}g_{00,30} - g^{31}(g_{01,30} + g_{31,00} - g_{03,10}) - g^{32}(g_{02,30} + g_{32,00} - g_{03,20}) - g^{33}g_{33,00}]
\end{aligned}
$$

$$
\begin{aligned}
&+ (g^{00} g_{00,0} - g^{01} g_{00,1} - g^{02} g_{00,2} - g^{03} g_{00,3}) \\
&\times [g^{10}(2g_{10,1} - g_{11,0}) + g^{11} g_{11,1} + g^{12}(2g_{12,1} - g_{11,2}) + g^{13}(2g_{13,1} - g_{11,3}) \\
&+ g^{20}(g_{10,2} + g_{20,1} - g_{12,0}) + g^{21} g_{11,2} + g^{22} g_{22,1} + g^{23}(g_{13,2} + g_{23,1} - g_{12,3}) \\
&+ g^{30}(g_{10,3} + g_{30,1} - g_{13,0}) + g^{31} g_{11,3} + g^{32}(g_{12,3} + g_{32,1} - g_{13,2}) + g^{33} g_{33,1}]
\end{aligned}
$$

$$
\begin{aligned}
&+ (g^{10} g_{00,0} + g^{11} g_{11,1} - g^{12} g_{11,2} - g^{13} g_{11,3}) \\
&\times [-g^{00} g_{00,1} - g^{01} g_{11,0} - g^{02}(g_{12,0} + g_{02,1} - g_{10,2}) - g^{03}(g_{13,0} + g_{03,1} - g_{10,3}) \\
&+ g^{10}(2g_{10,1} - g_{11,0}) + g^{11} g_{11,1} + g^{12}(2g_{12,1} - g_{11,2}) + g^{13}(2g_{13,1} - g_{11,3}) \\
&+ g^{20}(g_{10,2} + g_{20,1} - g_{12,0}) + g^{21} g_{11,2} + g^{22} g_{22,1} + g^{23}(g_{13,2} + g_{23,1} - g_{12,3}) \\
&+ g^{30}(g_{10,3} + g_{30,1} - g_{13,0}) + g^{31} g_{11,3} + g^{32}(g_{12,3} + g_{32,1} - g_{13,2}) + g^{33} g_{33,1}]
\end{aligned}
$$

$$
\begin{aligned}
&+ [g^{20} g_{00,0} + g^{21}(2g_{01,0} - g_{00,1}) + g^{22}(2g_{02,0} - g_{00,2}) + g^{23}(2g_{03,0} - g_{00,3})] \\
&\times [-g^{00} g_{00,2} - g^{01}(g_{21,0} + g_{01,2} - g_{20,1}) - g^{02} g_{22,0} - g^{03}(g_{23,0} + g_{03,2} - g_{20,3}) \\
&+ g^{10}(g_{20,1} + g_{10,2} - g_{21,0}) + g^{11} g_{11,2} + g^{12} g_{22,1} + g^{13}(g_{23,1} + g_{13,2} - g_{21,3}) \\
&+ g^{20}(2g_{20,2} - g_{22,0}) + g^{21}(2g_{21,2} - g_{22,1}) + g^{22} g_{22,2} + g^{23}(2g_{23,2} - g_{22,3}) \\
&+ g^{30}(g_{20,3} + g_{30,2} - g_{23,0}) + g^{31}(g_{21,3} + g_{31,2} - g_{23,1}) + g^{32} g_{22,3} + g^{33} g_{33,2}]
\end{aligned}
$$

$$
\begin{aligned}
&+ [g^{30} g_{00,0} + g^{31}(2g_{01,0} - g_{00,1}) + g^{32}(2g_{02,0} - g_{00,2}) + g^{33}(2g_{03,0} - g_{00,3})] \\
&\times [-g^{00} g_{00,3} - g^{01} g_{01,3} - g^{02}(g_{32,0} + g_{02,3} - g_{30,2}) - g^{03} g_{33,0} \\
&+ g^{10}(g_{30,1} + g_{10,3} - g_{31,0}) + g^{11} g_{11,3} + g^{12}(g_{32,1} + g_{12,3} - g_{31,2}) + g^{13} g_{33,1} \\
&+ g^{20}(g_{30,2} + g_{20,3} - g_{32,0}) + g^{21}(g_{31,2} + g_{21,3} - g_{32,1}) + g^{22} g_{22,3} + g^{23} g_{33,2} \\
&+ g^{30}(2g_{30,3} - g_{33,0}) + g^{31}(2g_{31,3} - g_{33,1}) + g^{32}(2g_{32,3} - g_{33,2}) + g^{33} g_{33,3}]
\end{aligned}
$$

$$
\begin{aligned}
&- 2[g^{10} g_{00,2} + g^{11}(g_{01,2} + g_{21,0} - g_{02,1}) + g^{12} g_{22,0} + g^{13}(g_{03,2} + g_{23,0} - g_{02,3})] \\
&\times [g^{20} g_{00,1} + g^{21} g_{11,0} + g^{22}(g_{02,1} + g_{12,0} - g_{01,2}) + g^{23}(g_{03,1} + g_{13,0} - g_{01,3})]
\end{aligned}
$$

$$
\begin{aligned}
&- 2[g^{10} g_{00,3} + g^{11}(g_{01,3} + g_{31,0} - g_{03,1}) + g^{12}(g_{02,3} + g_{32,0} - g_{03,2}) + g^{13} g_{33,0}] \\
&\times [g^{30} g_{00,1} + g^{31} g_{11,0} + g^{32}(g_{02,1} + g_{12,0} - g_{01,2}) + g^{33}(g_{03,1} + g_{13,0} - g_{01,3})]
\end{aligned}
$$

$$
\begin{aligned}
&- 2[g^{20} g_{00,3} + g^{21}(g_{01,3} + g_{31,0} - g_{03,1}) + g^{22}(g_{02,3} + g_{32,0} - g_{03,2}) + g^{23} g_{33,0}] \\
&\times [g^{30} g_{00,2} + g^{31}(g_{01,2} + g_{21,0} - g_{02,1}) + g^{32} g_{22,0} + g^{33}(g_{03,2} + g_{23,0} - g_{02,3})]
\end{aligned}
$$

$$
\begin{aligned}
&- [g^{10} g_{00,1} + g^{11} g_{11,0} + g^{12}(g_{02,1} + g_{12,0} - g_{01,2}) + g^{13}(g_{03,1} + g_{13,0} - g_{01,3})]^2 \\
&- [g^{20} g_{00,2} + g^{21}(g_{01,2} + g_{21,0} - g_{02,1}) + g^{22} g_{22,0} + g^{23}(g_{03,2} + g_{23,0} - g_{02,3})]^2
\end{aligned}
$$

$$- [g^{30}g_{00,3} + g^{31}(g_{01,3} + g_{31,0} - g_{03,1}) + g^{32}(g_{02,3} + g_{32,0} - g_{03,2}) + g^{33}g_{33,0}]^2. \quad (10)$$

Now we should substitute (6) to all entries with double upper indices to (10). For instance, the entry g^{11} in line (9) could be rewritten by means of the Sarrus rule for 3×3 symmetric matrices g^*_{11} by

$$\begin{aligned} g^{11} =& \frac{g^*_{11}}{\det(g_{\mu\nu})} \\ =& \frac{g_{00}g_{22}g_{33} + 2g_{02}g_{03}g_{23} - g_{00}(g_{23})^2 - g_{22}(g_{03})^2 - g_{33}(g_{02})^2}{\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}}, \end{aligned} \quad (11)$$

where the sum in the denominator contains $4! = 24$ terms. Note that the optimal expression for the minimum number of arithmetic operations to calculate the inverse of a 4×4 matrix is not known, yet. The other nine entries g^{00}, g^{01}, g^{02}, g^{03}, g^{12}, g^{13}, g^{22}, g^{23}, and g^{33} can be expressed similarly.

However, we have to evaluate also the first derivatives of $g^{\mu\nu}$. Consider for instance the entry $g^{11}_{,1}$ in line (8). Then by (11) we get

$$\begin{aligned} g^{11}_{,1} =& \frac{\partial}{\partial x_1}\Big(\frac{g^*_{11}}{\det(g_{\mu\nu})}\Big) \\ =& \Big(\frac{1}{\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}} \\ & \qquad \times (g_{00}g_{22}g_{33} + 2g_{02}g_{03}g_{23} - g_{00}(g_{23})^2 - g_{22}(g_{03})^2 - g_{33}(g_{02})^2)\Big)_{,1} \\ =& \Big[\big(g_{00,1}g_{22}g_{33} + 2g_{02,1}g_{03}g_{23} - g_{00,1}(g_{23})^2 - g_{22,1}(g_{03})^2 - g_{33,1}(g_{02})^2 + g_{00}g_{22,1}g_{33} \\ & + 2g_{02}g_{03,1}g_{23} + g_{00}g_{22}g_{33,1} + 2g_{02}g_{03}g_{23,1} - 2g_{00}g_{23,1} - 2g_{22}g_{03,1} - 2g_{33}g_{02,1}\big) \\ & \times \Big(\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}\Big) \\ & - \big(g_{00}g_{22}g_{33} + 2g_{02}g_{03}g_{23} - g_{00}(g_{23})^2 - g_{22}(g_{03})^2 - g_{33}(g_{02})^2\big) \\ & \times \sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi}(g_{0\nu_0,1}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3} + g_{0\nu_0}g_{1\nu_1,1}g_{2\nu_2}g_{3\nu_3} + g_{0\nu_0}g_{1\nu_1}g_{2\nu_2,1}g_{3\nu_3} \\ & + g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3,1})\Big]\Big(\sum_{\pi\in S_4}(-1)^{\operatorname{sgn}\pi} g_{0\nu_0}g_{1\nu_1}g_{2\nu_2}g_{3\nu_3}\Big)^{-2}. \end{aligned} \quad (12)$$

Substituting all $g^{\mu\nu}$ and also its first derivatives into (10), we get the explicit form of the first Einstein equation $R_{00} = 0$ of the second order for 10 unknowns $g_{00}, g_{01}, g_{02}, \dots, g_{33}$. It is evident that such an equation is extremely complicated. Relation (8) takes only 4 lines, relation (10) takes 40 lines and after substitution of determinants into (10) the equation $R_{00} = 0$ takes several pages. The other nine equations $R_{\mu\nu} = 0$ can be expressed similarly. Therefore, the explicit form of all 10 Einstein equations will occupy a huge amount of pages. **This fact prevents**

us to verify whether Einstein's equations describe, for instance, the Solar system better than Newtonian mechanics by N-body simulations.

For comparison note that the Laplace equation $\Delta u = 0$ has only three terms $\partial^2 u/\partial x_i^2$ on its left-hand side, $i = 1, 2, 3$, and the famous Navier-Stokes equations 24 terms.

3. Non-differentiability of the Schwarzschild composite solution

In 1915, Karl Schwarzschild wrote to Albert Einstein that he has found a solution [47] for the case $T_{\mu\nu} = 0$ (for the English translation of Schwarzschild's original letter by R. A. Rydin see [12]). It can be written as the following diagonal tensor

$$g_{\mu\mu} = \mathrm{diag}\Big(-\frac{r-S}{r}, \frac{r}{r-S}, r^2 \sin^2\theta, r^2\Big), \tag{13}$$

$g_{\mu\nu} = 0$ for $\mu \neq \nu$, where $r > S$, the constant S is given by (14) below, (r, φ, θ) are the standard spherical coordinates, $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$, i.e.,

$$\begin{aligned} x^1 =& r \sin\theta \cos\varphi, \\ x^2 =& r \sin\theta \sin\varphi, \\ x^3 =& r \cos\theta. \end{aligned}$$

Schwarzschild assumed that the gravitational field has the following properties: it is static (meaning that it does not change over time), it is spherically symmetric, the spacetime is empty, and the spacetime is asymptotically flat. For a fixed nonrotating ball in vacuum with mass $M > 0$ and with a spherically symmetric mass distribution we set

$$S = \frac{2MG}{c^2} \tag{14}$$

which is called the *Schwarzschild gravitational radius* and (13) is called the *exterior Schwarzschild metric.*

In 1916 Karl Schwarzschild (see [48]) found the first nonvacuum solution[2] of Einstein's equations (1). He assumed that the ball with coordinate radius $r_0 > 0$ is formed by an ideal incompressible non-rotating fluid with constant density to avoid a possible internal mechanical stress that may have a non-negligible influence on the resulting gravitational field. He also assumed zero pressure at the surface. Then the corresponding metric is (see e.g. [15], [16, p. 529], [49, p. 213], [53])

$$g_{\mu\mu} = \mathrm{diag}\bigg(-\frac{1}{4}\bigg(3\sqrt{1-\frac{S}{r_0}} - \sqrt{1-\frac{Sr^2}{r_0^3}}\,\bigg)^2, \frac{r_0^3}{r_0^3 - Sr^2}, r^2\sin^2\theta, r^2\bigg), \tag{15}$$

[2]The Schwarzschild solution is static. On the other hand, the well-known Kerr metric [39, p. 878] is stationary, which means that there exists a coordinate system where we can express the metric tensor independent of the time coordinate. Every static solution is stationary, but not vice versa.

where $r \in [0, r_0]$, S is given by (14), and we assume that $r_0 > S$. The corresponding metric tensor is called the *interior Schwarzschild solution.* It is again a static solution and the corresponding right-hand side $T_{\mu\nu}$ is also a diagonal tensor for which $T_{00} = 0$, since it does not change over time.

Using (13) and (15), we can easily verify that the exterior and interior metric have the same values for $r = r_0$, i.e., each component $g_{\mu\mu} = g_{\mu\mu}(r)$ is a continuous function on $[0, \infty)$ for $\mu = 0, 1, 2, 3$. However, the first derivatives of g_{11} of the exterior and interior Schwarzschild solution do not match (see Figure 1), since they have a jump on the common boundary $r = r_0$. Note that the 2nd order Einstein equations contain classical derivatives of $g_{\mu\nu}$ which are supposed to be continuous in definition (5). Therefore, the corresponding space manifold described by the Riemann curvature tensor is not differentiable, since the tangent hyperplane for $r = r_0$ cannot be uniquely defined. All Riemannian manifolds must be locally flat which is not true in this particular case (see also [26]).

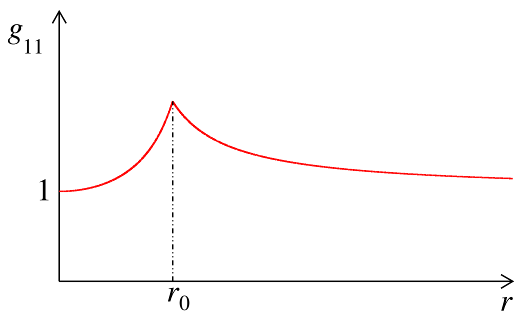

Figure 1: The behavior of the non-differentiable component $g_{11} = g_{11}(r)$ of the metric tensor from (13) and (15). The first derivative $(\partial g_{11}/\partial r)(r_0)$ is not defined. The piecewise rational function g_{11} cannot be smoothed near r_0, since then Einstein's equations (1) would not be valid in a close neighborhood of r_0.

From (13) we observe that the one-sided limit of the derivative of the component $g_{11}(r) = r/(r - S)$ of the exterior solution is negative

$$\lim_{r \to r_0^+} \frac{\partial g_{11}}{\partial r}(r) < 0,$$

whereas the component $g_{11}(r)$ of the interior solution (15) is an increasing function on $[0, r_0]$ (cf. Figure 1). It is increasing even for a variable spherically symmetric density $\rho = \rho(r)$, see [8, 39]. Consequently, the Schwarzschild solution cannot be used inside the ball with radius $r_1 > r_0$ to model our Sun or any other star with radius r_0 together with its spherically symmetric vacuum neighborhood (see Figure 2 and (5)). This is a serious drawback, since the composite metric tensor (13)+(15) is not differentiable for $r = r_0$. Consequently, (13) and (15) are only local solutions and together they do not form a global solution in the ball with radius r_1.

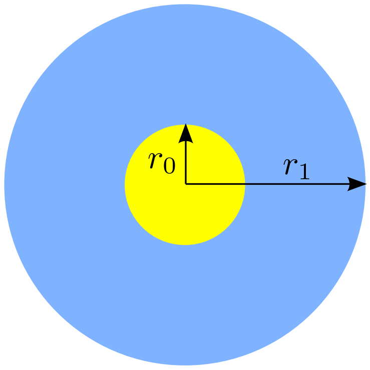

Figure 2: Spherical shell $\{(x,y,z) \in \mathbb{E}^3 \mid r_0^2 \le x^2+y^2+z^2 \le r_1^2\}$ is the region between two concentric spheres.

Similarly, the function $u(x) = |x|$ is a local classical solution of the second order ordinary differential equation $u'' = 0$ on the intervals $[-1,0]$ and $[0,1]$, but it is not a global solution over the interval $[-1,1]$. It is not even a weak solution there.

Example 1. For comparison, we also note that the first order classical derivatives of the Newton potential u for the situation sketched in Figure 2 are continuous. It is described by the Poisson equation

$$\Delta u = 4\pi G\rho,$$

where ρ is the mass density. Let the right-hand $f = 4\pi G\rho$ side be spherically symmetric and such that $f(r) = 1$ for $r \in [0,1]$ and $f(r) = 0$ otherwise. The Laplace operator in spherical coordinates reads (see [44, Sect. 7.2])

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \operatorname{cotan}\theta \frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\Big).$$

The sum in parenthesis on the right-hand side is zero for the spherically symmetric case. By the well-known method of variations of constants, we find the following solution of the above Poisson equation

$$u(r,\varphi,\theta) = \frac{1}{6}r^2 - \frac{1}{2} \quad \text{for } r \in [0,1],$$

$$u(r,\varphi,\theta) = -\frac{1}{3r} \quad \text{otherwise.}$$

Hence, both u and $\partial u/\partial r$ are continuous at $r_0 = 1$. □

Finally, let us emphasize that the covariant divergence of the right-hand side of (1) has to be zero, see (30) and [39, p. 146]. Therefore, the covariant divergence of the left-hand side of (1) is zero, too. However, this requires the existence of the third order derivatives of the metric tensor $g_{\mu\nu}$ at r_0.

4. Another unexpected property of the Schwarzschild solution

For positive numbers $r_0 < r_1$ consider a spherical shell with interior radius r_0 and exterior radius r_1 (see Figure 2). Its volume in the Euclidean space $\mathbb{E}^3$ is clearly given by

$$V = \frac{4}{3}\pi(r_1^3 - r_0^3). \tag{16}$$

Now we will derive a formula for the proper volume $\tilde{V}$ of the spherical shell with coordinate radii $r_0 < r_1$ in a curved space around the mass ball with coordinate radius r_0. Here the tilde indicates a curved space. By (13) we find that the exterior spatial volume element is equal to

$$\mathrm{d}\widetilde{V} = \sqrt{\frac{r}{r-S}}\,\mathrm{d}r \cdot (r \sin\theta\,\mathrm{d}\varphi) \cdot (r\,\mathrm{d}\theta).$$

Therefore, the *proper (relativistic) volume* is defined as

$$\widetilde{V} = \int_{r_0}^{r_1} r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r \cdot \int_0^{\pi} \Big(\int_0^{2\pi} \sin\theta\,\mathrm{d}\varphi \Big) \mathrm{d}\theta = 4\pi \int_{r_0}^{r_1} r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r. \tag{17}$$

Theorem 2. *If $M > 0$ and $r_0 > S$ are any fixed numbers satisfying* (14), *then*

$$\widetilde{V} - V \to \infty \quad \text{as } r_1 \to \infty.$$

Proof. By differentiation, we can easily check that

$$\int r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r = \Big(\frac{r^2}{3} + \frac{5Sr}{12} + \frac{5S^2}{8} \Big) \sqrt{r(r-S)} + \frac{5S^3}{16} \ln(2\sqrt{r(r-S)} + 2r - S).$$

From this, (17), and (16) we get

$$\begin{aligned}
\widetilde{V} - V =& 4\pi \int_{r_0}^{r_1} r^2 \sqrt{\frac{r}{r-S}}\,\mathrm{d}r - \frac{4}{3}\pi(r_1^3 - r_0^3) \\
=& \frac{4\pi}{3} \Big[\Big(r_1^2 + \frac{5Sr_1}{4} + \frac{15S^2}{8} \Big) \sqrt{r_1(r_1 - S)} + \frac{15S^3}{16} \ln(2\sqrt{r_1(r_1-S)} + 2r_1 - S) \\
& - \Big(r_0^2 + \frac{5Sr_0}{4} + \frac{15S^2}{8} \Big) \sqrt{r_0(r_0 - S)} - \frac{15S^3}{16} \ln(2\sqrt{r_0(r_0-S)} + 2r_0 - S) \\
& - r_1^3 + r_0^3 \Big].
\end{aligned} \tag{18}$$

Since

$$r_1 > r_0 > S$$

and since the logarithmic function is increasing, the difference of the two terms containing ln in (18) is positive.[3] Thus from the inequality

$$\sqrt{r_1(r_1 - S)} > r_1 - S$$

we obtain the following lower bound

$$\widetilde{V} - V > \Big(r_1^2 + \frac{5Sr_1}{4} + \frac{15S^2}{8}\Big)(r_1 - S) - r_1^3 + C = \frac{Sr_1^2}{4} + \frac{5S^2 r_1}{8} + \overline{C},$$

where C contains all remaining terms not depending on r_1 and where $\overline{C} = C - 15S^3/8$. Letting $r_1 \to \infty$, we obtain the statement of the theorem. □

We observe that the difference of volumes $\tilde{V} - V$ increases over all limits for $r_1 \to \infty$, which is a quite surprising property. Namely, Theorem 2 can be applied for instance to a billiard ball or a small steel ball bearing (see Example 2 below) or an imperceptible pinhead, since the mass $M > 0$ can be arbitrarily small. Consequently, a natural question arises: How large can r_1 be so that the relativistic relation (17) approximates reality well.

Example 2. Setting $M = 0.033$ kg, $r_0 = 0.01$ m, and $r_1 = 5 \cdot 10^{20}$ m, which is the radius of our Galaxy, we find that $S = 5 \cdot 10^{-29}$ m and by (18) the difference

$$\widetilde{V} - V \approx 10\,000 \text{ km}^3.$$

This is about 10^{19} times more than the volume of the ball itself. From this we see that the use of Einstein's equations to galactic distances is questionable. Further drawbacks of the Schwarzschild metric are surveyed in [20]. □

5. Division by zero in the Friedmann normalized equation

Einstein's equations (1) were derived for a local description of the universe. They were "tested" in a neighborhood of the Sun [12, 24, 39]. However, in 1917 Einstein applied his equations to the whole universe [14]. At that time he did not know what is its real size. Now we know that the size of the observable universe is at least 10^{15} astronomical units.

We will show that these excessive extrapolations by many orders of magnitude may lead to a division by zero in Einstein's equations. To avoid a gravitational collapse of the whole universe Einstein introduced a new form of his equations (see [14])

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{19}$$

with non-zero cosmological constant Λ.

[3] The argument in parenthesis after ln is dimensionless, because $\ln a - \ln b = \ln(a/b)$.

At present there are thousands of papers on the cosmological constant Λ. However, for the time being, we do not know any of its significant digit (nor even its sign). The standard cosmological model (see [42]) assumes that

$$\Lambda \approx 10^{-52}\ \mathrm{m}^{-2}. \tag{20}$$

Einstein's equations were not developed for a dynamical evolution of the universe. This was done later in 1922 by Alexander Friedmann [18, 19] who derived from the first Einstein equation the following ordinary differential equation for the *expansion function* $a = a(t)$

$$\dot{a}^2 = \frac{8\pi G\rho a^2}{3} + \frac{\Lambda c^2 a^2}{3} - kc^2, \tag{21}$$

where the dot denotes the time derivative, $k \in \{-1, 0, 1\}$ is the curvature index, and $\rho = \rho(t) > 0$ is the mean mass density. At present it is assumed that (21) should be considered only for $t > \tau$, where

$$\tau \approx 380\,000\ \mathrm{yr}$$

is the time of decoupling of the cosmic microwave background radiation (CMB). Note that Friedmann derived (21) exactly from (19) without any approximations, i.e., (21) is a direct mathematical consequence of Einstein's equations for a homogeneous and isotropic universe which is described by a maximally symmetric manifold for $k \in \{-1, 0, 1\}$. In particular, it is a consequence only of the first Einstein equation (8)–(12) enriched by the cosmological constant (see [30] for a detailed proof),

$$\boxed{\text{Einstein's equations} \ + \ \text{maximum symmetry} \ \Longrightarrow \ \text{Friedmann equation.}} \tag{22}$$

We shall suppose that

$$\dot{a}(\tau) > 0, \tag{23}$$

since the universe was expanding at time τ. Furthermore, assume that $\dot{a}(t) \neq 0$ for all $t > \tau$ and divide equation (21) by $\dot{a}^2$. Then the Friedmann equation reads

$$\Omega_{\mathrm{M}}(t) + \Omega_{\Lambda}(t) + \Omega_k(t) = 1 \quad \text{for all } t > \tau, \tag{24}$$

where

$$\Omega_{\mathrm{M}}(t) = \frac{8\pi G\rho(t)}{3H^2(t)} > 0, \quad \Omega_{\Lambda}(t) = \frac{\Lambda c^2}{3H^2(t)}, \quad \text{and} \quad \Omega_k(t) = -\frac{kc^2}{a^2(t)H^2(t)}, \tag{25}$$

are normalized cosmological parameters called (see [41, p. 58], [43, p. 37]) the *density of dark and baryonic matter, density of dark (or vaccum) energy,* and the *curvature parameter*, respectively, and

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{26}$$

is the *Hubble-Lemaître parameter*. Note that (24) is really a differential equation, since the derivative $\dot{a}$ is hidden in the Hubble-Lemaître parameter.

In the literature on cosmology, the division of (21) by the square $\dot{a}^2 \geq 0$ is usually done without any preliminary warning that we may possibly divide by zero which may lead to various paradoxes. For instance, we see by (25) and (26) that

$$\dot{a}(t) \to 0 \quad \Longrightarrow \quad \Omega_{\mathrm{M}}(t) \to \infty \quad \text{and} \quad \Omega_{\Lambda}(t) \to \pm\infty \quad \text{for} \quad \Lambda \neq 0, \tag{27}$$

corresponding e.g. to an oscillating (cyclic) universe, or a loitering universe or a bouncing universe by de Sitter with $\rho \equiv 0$ and $\Lambda > 0$, see Figure 3,

$$a(t) = \frac{1}{\alpha}\cosh(\alpha ct) \quad \text{for } \alpha = \sqrt{\frac{\Lambda}{3}}.$$

In the last case the Friedmann equation (21) is satisfied for $k = 1$, namely,

$$(\dot{a}(t))^2 = c^2\sinh^2(\alpha ct) = c^2\cosh^2(\alpha ct) - c^2 = c^2\alpha^2 a^2(t) - c^2 = \frac{\Lambda c^2}{3}a^2(t) - kc^2$$

and we have $\dot{a}(0) = 0$. Note that de Sitter solution does not describe reality well due to the unrealistic assumption $\rho \equiv 0$.

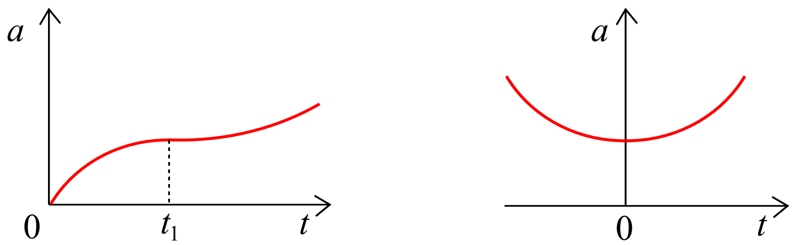

Figure 3: Graph of the expansion function $a = a(t)$ for 1) a loitering universe for which there exists $t_1 > 0$ such that $\dot{a}(t_1) = 0$, amd for 2) a bouncing universe.

The Einstein static solution (see Figure 4)

$$a(t) \equiv \Lambda^{-1/2}$$

with $\Lambda > 0$ satisfies $\dot{a} \equiv 0$ which leads to division by zero in (25), too. The Friedmann equation (21) is satisfied again for $k = 1$ and $\rho = \Lambda c^2/(4\pi G)$, namely,

$$0 \equiv \dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + \frac{\Lambda c^2}{3}a^2 - c^2 = \frac{8\pi G}{3}\frac{\Lambda c^2}{4\pi G}\frac{1}{\Lambda} + \frac{\Lambda c^2}{3}\frac{1}{\Lambda} - c^2 = 0.$$

However, the Einstein static solution also does not describe reality well, since the universe is expanding and the condition (23) does not hold. Moreover, this solution is unstable [6, 9, 14, 35], i.e., small fluctuations can make it either expand or contract (cf. Figure 3 for $a(t_1) = \Lambda^{-1/2}$).

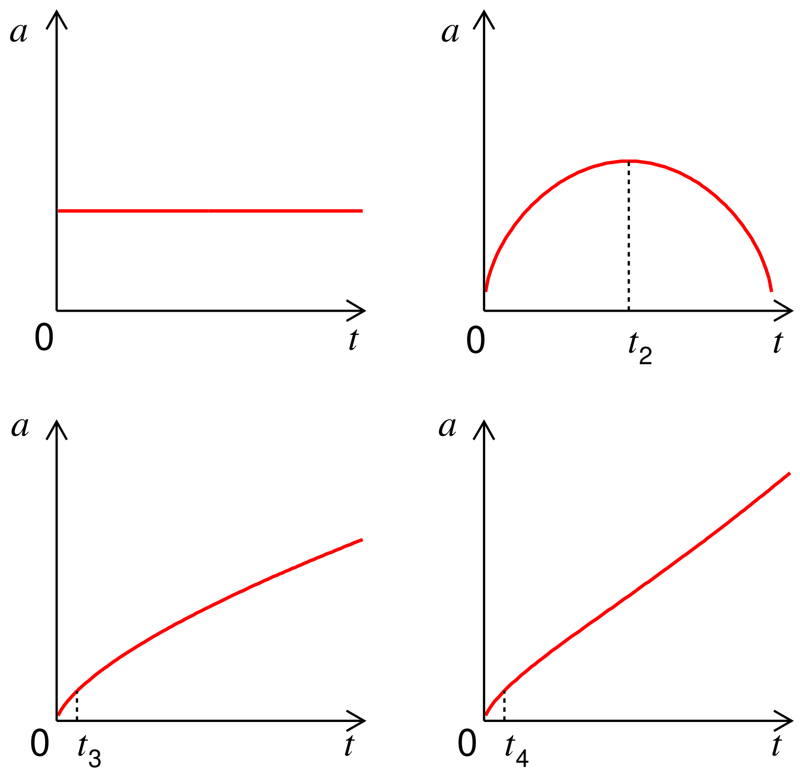

Figure 4: The expansion function for 1) the Einstein static universe, 2) the cyclic universe (the expansion stops at some time $t_2 > 0$ and then starts to shrink), 3) the universe with zero cosmological constant, and for 4) the currently proposed expansion of the universe with a positive cosmological constant. Here $t_4 = \tau$ denotes the time instant of the origin of the cosmic microwave background radiation.

Is there really an infinite density of dark matter and dark energy, when $a = a(t)$ reaches its extremal values? The true density of baryonic matter is surely finite. For an oscillating universe (see the upper right of Figure 4), when the density of dark energy reaches an infinite value (27), the universe starts to shrink. This is a quite paradoxical result, see [30, p. 169].

The behavior of the curvature parameter is also strange. Applying (26), we see that $\Omega_k(t) = -kc^2/\dot{a}^2(t)$, i.e., $\Omega_k(t) \approx 0$ when the universe has originated (cf. Figure 4).

In [26] we show that division by zero in (25) may appear for Λ negative, vanishing, and also positive. If $\Lambda < 0$, we always divide by zero in (25).

In the model with $\Lambda = 0$ we again divide by zero in (25) if $k = 1$ and $\rho = \rho(t)$ is larger than the so-called critical density $\rho_{\text{crit}}(t) = 3H^2(t)/(8\pi G)$. For $k = -1$ the expansion function is strictly convex and increasing for $t > t_3 > 0$ (see the bottom left part of Figure 4).

If $\Lambda > 0$ then the division by zero in (25) may appear for $k = 1$. Otherwise the expansion function satisfying (23) changes from strictly concave to strictly convex on the interval (9,14) Gyr (see the bottom right part of Figure 4).

6. Incorrect extrapolations lead to dark matter and dark energy

By the scientific results of the Planck satellite [43], our universe is composed of about 68 % of some mysterious dark energy (i.e., the present value $\Omega_\Lambda = 0.68$ in (25)), 27 % of some exotic dark matter, and less than 5 % of ordinary baryonic matter. In truth, it is more likely that the measured data just indicate that the extrapolation is wrong, since it requires one to introduce some hypothetical dark matter and dark energy.[4]

The above cosmological parameters were obtained by the three seemingly independent methods of Baryonic Acoustic Oscillations (BAO), Cosmic Microwave Background Radiation[5] (CMB), and Supernovae type Ia explosions (SNe). However, these methods are not independent, since they are all based on the same normalized Friedmann equation (24).

According to the standard cosmological model [43], our universe contains more dark matter than ordinary baryonic matter and

$$\text{the ratio of masses of dark matter to baryonic matter} \approx \mathbf{6:1}. \tag{28}$$

In [25] we present ten arguments showing that this proclaimed amount of dark matter is highly overestimated. The ratio (28) was again obtained from the standard ΛCDM cosmological model which is based on excessive extrapolations [32]. For instance, most of the observed galaxies have spiral structure. If these galaxies would contain six times more uniformly distributed nonbaryonic matter than baryonic matter, then they could not exhibit such a high symmetry of structured baryonic matter. Moreover, their disks would be more thicker due to dark matter halos.

In [28] we suggest that nonbaryonic dark matter need not be taken into account to explain the observed rapid rotation of spiral galaxies. The main reason is a special form of the gravitational potential of a flat disk which guarantees large orbital velocities of stars at the galaxy edge. In particular, we proved that a star orbiting a central mass point along a circular trajectory of radius R has a smaller speed than if it were to orbit a flat disk of radius R and the same mass with an arbitrary rotationally symmetric density distribution, see also [25, 33, 52].

At the end of the 20th century (when Perlmutter et al. published their famous paper [42]) it was thought that red dwarfs of the spectral class M form only 3 % of the total number of stars, see [5, p. 93]. Nevertheless, the Gaia satellite estimated that

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[4]There are hundreds of popularization books on dark matter and dark energy, since people like mysteries. This trend is difficult to stop, since almost nobody would buy a book stating that there is no dark matter and dark energy.

[5]For a trustworthy criticism of this method we refer to [50].

red dwarfs are in the vast majority — about 75 %. This large proportion essentially contributes to invisible baryonic matter.

The density of dark and baryonic matter is defined by means of the Friedmann equation by (25), i.e. via Einstein's equation, cf. (22). However, sometimes Newton laws are used to describe dynamic manifestations of galaxy clusters. Einstein's equations should thus not be substituted by Newtonian mechanics to "prove" the existence of dark matter.

For the luminosity distance of supernovae type Ia explosions, Perlmutter et al. [42, p. 566] used a formula which was derived from the Friedmann equation. This distance thus essentially depends on the fact whether Einstein's equations on cosmological distances sufficiently well approximate reality, since the Friedmann equation (21) is a mathematical consequence of (19) for a homogeneous and isotropic universe, see (22).

If this approximation is poor, the luminosity distances are not correct. Moreover, the method SNe treats type Ia supernovae as standard candles. However, they cannot be considered in this way due to a possible large extinction of light from the supernova [51]. This essentially depends on its location in the host galaxy, if it is at its edge or in the middle completely surrounded by galactic gas and dust. It also depends on the direction of the supernova rotational axis. In this way we may receive several orders of magnitude weaker light.

Further, let us introduce the dimensionless *deceleration parameter*

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}H^{-2} = -\dot{H}H^{-2} - 1,$$

where the second equality follows directly from (26). From this we see that the deceleration parameter $q_0 = q(t_0)$ at the present time t_0 appears at the quadratic term in the Taylor expansion (see e.g. [39, p. 781], [44, p. 652])

$$\begin{aligned} a(t) &= a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + \dots \\ &= a(t_0)\Big(1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 + \dots\Big), \end{aligned} \tag{29}$$

where $H_0 = H(t_0)$ is the present value of $H(t)$ called the *Hubble constant*. In the paper [45, p. 110], a negative value of the parameter

$$q_0 \approx -0.6$$

was found, i.e., a is strictly convex in a neighborhood of t_0 and the expansion of the universe accelerates (see the last graph in Figure 4). Using (29), we observe that the linear term is much larger than the quadratic term for t close to t_0, namely,

$$|H_0(t - t_0)| \gg \frac{1}{2}|q_0|H_0^2(t - t_0)^2 = \frac{1}{2}|q_0|\frac{\Lambda c^2}{3\Omega_\Lambda(t_0)}(t - t_0)^2,$$

where the last equality is due to (25). The single quadratic term is so small that the linear term in (29) essentially dominates not only in a close neighborhood of t_0, but also for the other $t < t_0$,

$$0.3|H_0(t - t_0)| > \frac{1}{2}|q_0|H_0^2(t - t_0)^2,$$

where $\frac{1}{2}|q_0| = 0.3$.

7. Further counter-arguments

We shall present 10 further counter-arguments showing that Einstein's equations do not describe reality well, especially on cosmological distances.

7.1. Problems with initial and boundary conditions

It is generally impossible to prescribe explicitly any appropriate initial and boundary conditions for non-spherically symmetric regions (e.g. on a cube) for $g_{\mu\nu}$ which satisfies (1) or (19). The reason is that spacetime tells matter how to move and matter tells spacetime how to curve [39]. So the initial space manifold is a priori not known for nontrivial cases. Thus we have serious problems to prove the uniqueness and also the existence of the solution of Einstein's equations and compare it with reality. Furthermore, suitable function spaces, where we look for the true solution, are usually not specified in the literature.

Let us also point out that Einstein's equations are fully deterministic whereas the universe (with its biological systems) does not operate solely gravitationally due to quantum phenomena. Their effects can be observed not only on microscopic scales. For instance, in our brain we can decide to change the trajectory of an asteroid in arbitrary direction by the famous kinetic impactor method. Hence, the evolution of the real world is very unstable with respect to initial conditions including our decision. Imperceptible quantum fluctuations may thus cause large changes of trajectories of celestial bodies and this process is definitely not described by Einstein's equations.

7.2. Nonuniqueness of the topology

The knowledge of the metric tensor $g_{\mu\nu}$ does not determine uniquely the topology of the corresponding space-time manifold. For instance, the Euclidean space $\mathbb{E}^3$ has obviously the same metric $g_{\mu\nu} = \delta_{\mu\nu}$, $\mu, \nu = 1, 2, 3$, as $\mathbb{S}^1 \times \mathbb{E}^2$, but different topology for a time-independent case with $T_{\mu\nu} = 0$ in (1). Here $\mathbb{S}^1$ stands for the unit circle. Hence, solving Einstein's equations does not mean that we obtain the shape of the associated manifold. Other examples can be found in [39, p. 725].

7.3. Law of conservation of energy

In general relativity the energy-momentum conservation is true only locally, which is expressed in the covariant divergence form as

$$T^{\mu\nu}{}_{;\nu} := \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\lambda\nu} T^{\lambda\nu} + \Gamma^\nu{}_{\lambda\nu} T^{\mu\lambda} = 0, \tag{30}$$

see e.g. [39, p. 146]. So the law of conservation of energy holds for Einstein's equations (19). However, in [29] we presented 10 independent observational arguments showing that the Solar system slowly expands and the expansion rate is comparable to H_0, cf. (29). For instance, the measured mean speed of the Moon from the Earth is 3.84 cm/yr while tidal forces can explain only one half of this value. The corresponding remainder is approximately equal to $0.67H_0$, see [10], [11], [29, p. 187].

Slight violation of the laws of conservation of energy and of momentum in static spacetime can easily explain a wide range of puzzles such as the faint young Sun paradox, the formation of Neptune and Uranus closer to the Sun, the existence of rivers on Mars, the paradox of tidal forces of the Moon, the paradox of the large orbital momentum of the Moon, Triton and Charon, migration of planets, the slow rotation of Mercury, the absence of its moons, rapid orbital expansion of Titan [34], etc.

In [29, Chapt. 16] we also show that galaxies themselves slightly expand at rate comparable with H_0. For instance, by [37, Sect. 8] the observed conservative expansion rate of the Milky Way is $0.6 - 1$ kpc/Gyr, which is approximately 600–1000 m/s and the Hubble constant recalculated on the diameter D of our Galaxy is $H_0 = 2\,\mathrm{km}/(\mathrm{s}D)$, see [29, p. 241].

The angular momentum of spiral galaxies is also not conserved, cf. [36, 40]. This is naturally expressed by the galactic angular momentum paradox: *How is it possible that spiral galaxies (originating from small random fluctuations in a hot homogeneous and isotropic universe) rotate so fast?*

7.4. Modeling error

The difference between physical reality and the solution of Einstein's equations is called the modeling error. In Figure 5 there is a general computational scheme of numerical solutions of problems of mathematical physics on a computer [29], where we always commit three basic errors: modeling error $e_0 = e_0(t)$, discretization error $e_1 = e_1(t)$, and rounding errors $e_2 = e_2(t)$. The size of e_0 is often not taken into account in the theory of general relativity even though it can be much larger than $e_1 + e_2$.

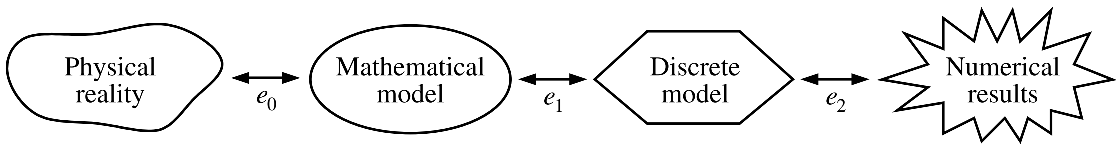

Figure 5: Modeling error $e_0(t)$ is the difference between physical reality and the solution of its mathematical model (mathematical-physical description). The discretization error $e_1(t)$ is the difference between solutions of the mathematical model and the discrete computer model. Finally, in $e_2(t)$ rounding errors (or iteration errors) are included.

Setting $A = \frac{1}{3}\Lambda c^2$ and $B = -kc^2$, the Friedmann equation (21) can be rewritten as the following simple autonomous ordinary differential equation with constant coefficients

$$\dot{a}^2 = Aa^2 + B + \frac{C}{a}, \tag{31}$$

where $C = \frac{8}{3}\pi G\rho a^3 > 0$ is constant by the law of conservation of mass for zero pressure, i.e. $\rho(t)a^3(t) = \rho(t_0)a^3(t_0)$ for all $t > \tau$, where t_0 is the present time (see [31, p. 100]). E.g. $C = \frac{2}{3}c^2\sqrt{\Lambda}$ for the Einstein universe and $C = 0$ for the de Sitter universe, where $k = 1$ and $\Lambda > 0$. However, the analytical solution of (31) is not known when $ABC \neq 0$, in general. Therefore, we cannot separate particular terms of the sum e_0+e_1 to establish the modeling error e_0. Hence, from equation (31) we should not make any categorical conclusions about the deep past and the future of the universe, about its age, origin, size, curvature, composition, expansion speed, etc., as is often done.

Moreover, we should not perform the backward integration of the Friedmann equation close to the Big Bang (cf. Figure 4), when quantum phenomena played an essential role, since they are not described by Einstein's equations.

7.5. Scale non-invariance

No equation of mathematical physics describes reality absolutely exactly on any scale. The reason is that the laws of physics are not unchanged under a change of scale, in general. For instance, Einstein's equations (19) are not scale-invariant, since they are highly nonlinear and contain fixed physical constants Λ, c, and G. They do not describe phenomena at an atomic level in a trustworthy manner. In this case, the modeling is very large (see Figure 6). If the curvature index $k = 1$, then the corresponding space manifold (hypersphere) is bounded, i.e., the scale invariance again cannot hold.

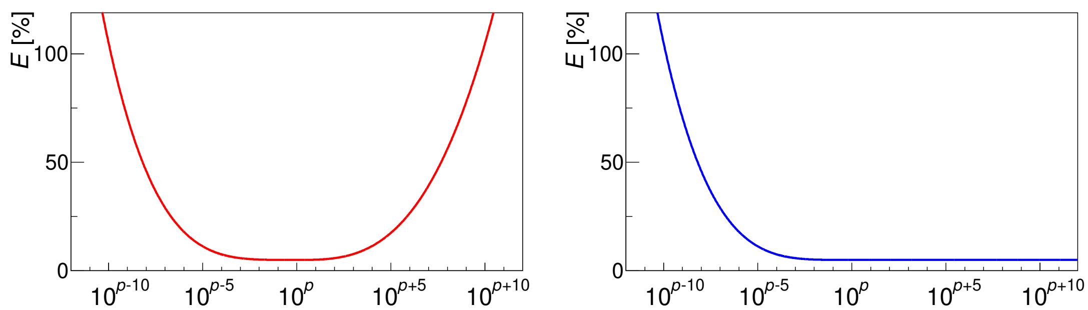

Figure 6: Left: A schematic illustration of a general behavior of the relative modeling error E for equations of mathematical physics. The horizontal axis has a logarithmic scale and p is the exponent for which the modeling error is the smallest. Right: Behavior of the modeling error for Einstein's equations promoted by the standard cosmological model.

The standard cosmological model is based on the unrealistic assumption that Einstein's equations are scale invariant, i.e., one can apply them to arbitrarily large objects — like the whole universe. This is, of course, an unjustified extrapolation, see Section 5.

7.6. Unconvincing general relativity tests

Classical tests of the theory of general relativity [38, 39], such as bending of light, Mercury's perihelion shift, gravitational redshift, and also Shapiro's fourth test of general relativity, are usually being verified by very simple algebraic formulae derived by various simplifications and approximations from the exterior Schwarzschild solution (13) without any guaranteed estimates of the modeling error. However, we cannot verify the validity of Einstein's equations (1) by means of the Schwarzschild solution.[6] This could be used only to disprove their validity (more precisely, to disprove their good approximation properties of reality).

For instance, Mercury's perihelion shift is thought to be one of the fundamental tests of the validity of the general theory of relativity. Recall Einstein's formula for this shift [12] during one period $T = 7.6005 \cdot 10^6$ s,

$$\varepsilon = 24\pi^3 \frac{a^2}{T^2c^2(1-e^2)} = 5.012 \cdot 10^{-7} \text{ rad}, \tag{32}$$

where $e = 0.2056$ is the eccentricity of its elliptic orbit and $a = 57.909 \cdot 10^9$ m the length of its semimajor axis. From this we find an incredibly small perihelion advance

$$\mathcal{E} = 43'' \text{ per century.}$$

Let us point out that formula (32) was published already in 1898 by Paul Gerber [21]. In [12], the planet Mercury is replaced by a massless point (which is again called Mercury) that does not curve the surrounding spacetime[7] and the influence of the other planets is not taken into account. According to [17, p. 147], the relativistic perihelion shift 43″ is in excellent agreement with observed values. However, the observed perihelion shift is $\mathcal{O} \approx 575''$ per century due to the gravitational tug of other planets. From this value general relativists subtract the value $\mathcal{C} \approx 532''$ calculated by Newtonian mechanics with infinite speed of gravity. This has to be done numerically, since the analytical solution of the n-body problem corresponding to the Solar system is not known. Hence, we necessarily meet all kinds of errors marked in Figure 4. In spite of that general relativists claim that

$$\mathcal{E} = \mathcal{O} - \mathcal{C}.$$

[6] Similarly, good approximation properties of reality modeled by the Laplace equation $\Delta u = 0$ cannot be verified by testing its linear solution $u(x_1, x_2, x_3) = x_1 + x_2 + x_3$, since there exist infinitely many other equations having the same solution.

[7] For comparison note that the Earth (18× heavier than Mercury) curves the surrounding spacetime by maintaining the Moon's motion at a speed of 1 km/s at a very large distance of 384 400 km. The generalized 3rd law of Kepler yields the difference 13 000 km in positions after one century if Mercury is replaced by a massless point.

However, the quantities $\mathcal{O}$ and $\mathcal{C}$ are not uniquely defined. Thus, the proposed relativistic shift $43''$ per century comes from the subtraction of two quite inexact numbers of almost equal magnitude (Observed minus Calculated). As such, this shift is highly uncertain and may not correspond to reality. We present a thorough numerical analysis of this ill-conditioned problem in [24].

The observed twist of line of apsides of binary pulsars does not imply that Einstein's equations describe reality well. The reason is that we do not know any of their solutions for two massive bodies, since they are extremely complicated, cf. (8)–(12). Nevertheless, we can apply formula (32) to the star S2 orbiting the super-massive black hole Sgr A*. For the corresponding values $a \approx 970$ au $\approx 145 \cdot 10^{12}$ m, $e \approx 0.885$, and the period $T \approx 16.052$ yr $\approx 507 \cdot 10^6$ s (see [1]) formula (32) yields a relatively large value $\varepsilon = 10.74$ arc minutes. However, we see that formula (32) contains squares of a, e, and T, so it is very sensitive to their precise determination. Moreover, the trajectory of S2 is seen only in the projection on the celestial sphere, so it is difficult to get precise values[8] of a and e, see [27]. Hence, we have to wait for several periods to verify, whether the proposed value $\varepsilon = 10.74'$ corresponds to reality.

7.7. Hubble-Lemaître constant

In 2016, the Planck Collaboration stated that

$$H_0 = 66.93 \pm 0.62 \text{ km}\,\text{s}^{-1}\text{Mpc}^{-1}. \tag{33}$$

This value is based on the standard cosmological model (i.e. Einstein's equations), while the Gaia and Hubble Space Telescope measurements of Cepheids and RR Lyrae yield the 10 % larger value $H_0 = 73.52 \pm 1.62$ km s^{-1}Mpc^{-1}, see [46]. We again get a disagreement between theory and observations. For a substantial discordance in estimating another cosmological parameter σ_8, see [22].

7.8. The age of the universe

The age of the universe was derived from the ΛCDM model up to four significant digits

$$t_0 = 13.79 \text{ Gyr} \tag{34}$$

using the backward integration of the Friedmann equation (21) and the present value of the Hubble-Lemaître constant (33). Nevertheless, from such a simple equation we should not make any categorical conclusions about the real age of the universe, since it was derived from Einstein's equations by excessive extrapolations to cosmological scales. For instance, by [7] the star HD 140283 is 14.46 ± 0.8 Gyr old, which contradicts (34). Moreover, this star is quite close to our Sun, so it is very probable that there are older stars in the whole universe.

[8]The angular size of the (projected) semimajor axis a was measured quite precisely, but the Earth-Sgr A* distance is known only approximately, i.e., the presented value of a in meters is very rough.

The existence of super-massive black holes $\approx 10^{10}$ $M_\odot$ at distances $z \approx 7$, when the universe was only 700 000 yr old, also indicates that its real age is probably higher than (34).

7.9. Time delay variables

Einstein's equations do not contain delays (in time variables) corresponding to the finite speed of gravity. This does not allow us to properly treat aberration effects. The actual angle of gravitational aberration has to be necessarily positive, since the zero aberration angle would contradict causality [29]. In fact, the causality principle should be prior to the law of conservation of energy.

7.10. Measurements of vacuum energy

The main argument against the proposed amount of vacuum (dark) energy is the 120-order-of-magnitude discrepancy between the measured and theoretically derived density of vacuum energy (see [2]). From this it is evident that the standard cosmological model, which is a direct mathematical consequence of Einstein's equations (22), does not approximate reality well.

For further cosmological paradoxes we refer e.g. to [3, 4].

8. Concluding remarks

The main problem of the standard cosmological model lies in the hidden assumption that Einstein's equations describe the evolution of the whole universe very well. Unfortunately, this unjustified assumption sits at the origin of all paradoxes of the current cosmology. It resembles the situation of the PhD thesis of J. N. He defined the so-called canal surfaces and proved many surprising lemmas and theorems about them. Later it was found that his set of canal surfaces is empty.

In 1922 astronomers had no idea about the real size of the universe, because galaxies (other than the Milky Way) were discovered by Edwin Hubble in 1925, see [23]. Their typical size is about 10^{10} astronomical units, and the size of the observable universe is at least five orders of magnitude larger. In spite of that, Alexander Friedmann [18] (and before him also Albert Einstein [14]) applied Einstein's equations to the whole universe, even though they are being tested on much smaller scales.

This excessive extrapolation has caused the current crisis of the standard cosmological model.

The standard cosmological model assumes that time flows completely uniformly from the Big Bang on (cf. Subsection 7.8). However, it is important to realize that in the observable universe we actually look in any direction into the vast spacetime singularity. The more distant the objects that are observed, the more it seems to us that time passes more slowly due to the cosmological redshift. For instance, if there were a huge clock placed at $z = 1$ from the Earth, then we would see that it runs twice as slow. For the largest currently observed distance corresponding to the CMB with redshift $z = 1089$, a similar clock would seem to tick $1090\times$ slower than on the Earth. Moreover, time flows more slowly close to dense massive objects which applies to the early universe when also quantum phenomena were present.

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