Conference *Cosmology on Small Scales 2020*
Michal Křížek and Yurii Dumin (Eds.)
Institute of Mathematics CAS, Prague

A CRITICAL REVIEW OF PARADOXES IN THE SPECIAL THEORY OF RELATIVITY

Michal Křížek

Institute of Mathematics, Czech Academy of Sciences
Žitná 25, CZ-115 67 Prague 1, Czech Republic
krizek@math.cas.cz

Abstract: We show that the Doppler effect and aberration of light can produce more dominant and entirely opposite effects for relativistic speeds than those predicted by the Special Theory of Relativity, in particular, the clock paradox, time dilatation, and length contraction. For instance, an observer will measure a higher frequency of an approaching clock than the same clock has at rest. We also prove that under certain conditions an approaching bar on a photo may seem to have a larger length for a relativistic speed than at rest.*

Keywords: Lorentz transformation, theory of groups, inertial systems, time dilatation, length contraction, twin paradox

PACS: 03.30.+p

1. Introduction

According to Newton's first law of inertia, a body will remain at rest or in uniform motion in a straight line unless acted upon by an external force. This fundamental physical principle serves to introduce the so-called inertial systems in the Special Theory of Relativity (STR), see [7, p. 211]. Consider a fixed coordinate system S with orthogonal axes x, y, z containing a fixed system of hypothetical synchronized clocks[1] that define the time coordinate $t \in (-\infty, \infty)$ of a uniformly flowing time. The coordinate system S is called *inertial* if it obeys Newton's first law of motion.

Let S' be another coordinate system with orthogonal axes x', y', z' which are for simplicity parallel with x, y, z and have the same scale at rest, see [25]. The time $t' \in (-\infty, \infty)$ in S' is introduced similarly using a fixed system of synchronized clocks in S' having also the same time scale at rest. Let the origin of S' move along

*Adapted and extended from the Czech version [12].

[1] This can be, in fact, interpreted so that all clocks are synchronized by an infinite speed of signal.

the x axis at a constant speed $v \in (-c, c)$, where c is the speed of light in vacuum[2], see Figure 1.

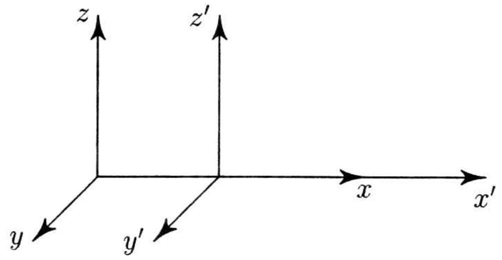

Figure 1: The inertial system S' is moving by speed $v \in (-c, c)$ with respect to the system S.

The Lorentz transformation (see [13]) is a fundamental tool of the STR. The parameter defined by

$$\gamma_v = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \geq 1 \tag{1}$$

is called the *Lorentz factor*. Points of the spacetime $\mathbb{R}^4$ are called *events*. Unless otherwise stated, we will restrict ourselves to one pair of the above described inertial systems, where the event is determined by the encounter of the origins of S and S' determines the beginning of time counting in the first and in the second inertial system, respectively, i.e., $t = 0$ in S and $t' = 0$ in S'. In this special case the *Lorentz transformation*[3] has the form $\mathcal{L}_v \colon \mathbb{R}^4 \to \mathbb{R}^4$,

$$\begin{aligned} x' &= \gamma_v (x - vt), & (2) \\ y' &= y, \\ z' &= z, \\ t' &= \gamma_v \left(t - \frac{v}{c^2} x \right), & (3) \end{aligned}$$

where $x, y, z, t \in (-\infty, \infty)$ and the last equality expresses how to transform a uniformly flowing proper time during transition from S to S'. Events which are simultaneous[4] in S are given by the identity $t \equiv t_0$, where t_0 is a fixed constant. By (3)

[2]The basic postulate of the STR that the speed of light c has the same size in all inertial systems was verified experimentally on the Earth by the well-known Michelson's experiments, see [14].

[3]Albert Einstein uses the transformation (2)–(3) in his pioneering paper [5, p. 902] from 1905, but does not use the term intertial. He also does not cite Lorentz's paper [13, p. 185] from 1892 nor does he mention Hendrik Lorentz himself. Einstein probably knew Lorentz's work [13], since the titles of their two papers are very similar. Moreover, Lorentz was very famous after receiving the Nobel Prize in 1902.

[4]Let us emphasize that any two different events which are simultaneous in S are not causally connected. Thus, one can verify that they were really simultaneous only when their future light cones intersect (cf. Figure 5).

we see that the time t' depends not only on t but also on the position x, i.e., t' is not constant and thus the corresponding events do not have to be simultaneous in S' for $v \neq 0$.

Notice that the right-hand sides of relations (2) and (3) are linear functions in variables x and t for any fixed v. Thus, for $\boldsymbol{x} = (ct, x, y, z)$ and $\boldsymbol{x'} = (ct', x', y', z')$ the Lorentz transformation can be rewritten into the matrix form

$$\boldsymbol{x'} = \boldsymbol{L}_v \boldsymbol{x},$$

where

$$\boldsymbol{L}_v = \begin{pmatrix} \gamma_v & -\frac{v}{c}\gamma_v & 0 & 0 \\ -\frac{v}{c}\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

is a block diagonal symmetric and positive definite matrix. Note that the physical dimension of all entries of the vectors $\boldsymbol{x}$ and $\boldsymbol{x'}$ is one meter.

The inverse matrix $\boldsymbol{L}_v^{-1}$ has a similar form as $\boldsymbol{L}_v$, only the two minus signs in (4) have to be replaced by plus. Therefore, the Lorentz transformation $\mathcal{L}_v$ is a one-to-one mapping from $\mathbb{R}^4$ onto $\mathbb{R}^4$ for $v \in (-c, c)$.

Let us point out that in the limit case $|v| = c$, the matrix (4) becomes singular, since its two first rows are linearly dependent. Consequently, the Lorentz transformation should not be applied to the surface of the light cone. Its inverse does not exist.

2. Time dilatation

The relation (3) is to be understood only as the time which we would record at the moment when the two clocks in S and S' are closely passing each other at one single x-coordinate (e.g. at the origin). So we can compare only time data t and t' of local clocks, because the concept of the present is relative. By definition, all clocks in each inertial system at rest show the same time in the whole infinite three-dimensional space (e.g. at the beginning and at the end of a motionless bar). So when we are exactly in the middle between any two fixed clocks, they will show us the same time.

Consider a fixed time interval

$$\Delta t' = t'_2 - t'_1,$$

where t'_i are space independent coordinates in S'. For an arbitrary fixed point x in S we determine the corresponding t_2 and t_1 from formulae (cf. (3))

$$t'_2 = \gamma_v \left(t_2 - \frac{v}{c^2} x \right), \quad t'_1 = \gamma_v \left(t_1 - \frac{v}{c^2} x \right),$$

and we set $\Delta t = t_2 - t_1$. From this we get the so-called *time dilatation* (see e.g. [10, p. 430])

$$\Delta t' = \gamma_v \left(t_2 - \frac{v}{c^2}x - t_1 + \frac{v}{c^2}x \right) = \gamma_v \Delta t. \tag{5}$$

By (1) we see that $\Delta t' > \Delta t$ for any $v \neq 0$ independently of the sign of v. The relation (5) actually expresses that the time, measured by a clock in a moving system S', runs slower than the time measured by a clock that is at rest with respect to S.

The clock at rest is fastest.

The time dilation is usually theoretically justified as follows: A photon launched from the origin of the system S in the z direction flies obliquely at S' with speed c. Therefore, in terms of an observer in S' this photon needs longer time to reach the plane $z' = z = 1$ than in terms of an observer in the system S.

Remark 1. The experimental verification of time dilation can be demonstrated by means of particles called muons whose mean half-life time at rest is $\tau = 2.2 \cdot 10^{-6}$ s. From observations of cosmic rays we know that if muons move linearly at almost the speed of light, they will travel on average much longer distance than $c\tau = 660$ m. However, it should be emphasized that in the inertial system associated with muons their decay will not slow down. In another experiment [3], the time dilatation is verified by means of the transverse Doppler effect.[5] Lithium ions accelerated to the speed $v = 0.338c$ are used as clocks. Note that the Hafele-Keating experiment [9] with two atomic clocks in airplanes and one on the Earth is not too credible, since none of the corresponding three systems was inertial. □

The non-relativistic longitudinal Doppler effect[6] (see [4]) is described by the relation

$$f_v = \frac{c}{c - v} f, \tag{6}$$

where f is the source frequency at rest, v is the speed of the source approaching an observer along the axis x, f_v is the frequency measured by the observer, and c is the speed of signal. For relativistic speeds this relationship needs to be corrected by time dilation, see [7]. All physical processes including clock speed in S' will run by (3) slower when observed from S. Thus by (6), the new relation will be of the form

$$f_v = \frac{c}{c - v} f', \tag{7}$$

[5]The transverse relativistic Doppler effect was first measured by Ives in [11] already in 1938. In classical mechanics, this transverse effect does not occur, because it is given by time dilation (5) only.

[6]Olaf Rømer (in Journal des Sçavans, 1676) suggested an elegant method to measure the speed of light. When the Earth moved toward Jupiter, the time interval between successive eclipses of Jupiter's moon Io became steadily shorter with respect to the terrestrial time. When the Earth moved away from Jupiter these eclipses became steadily longer, i.e., they were behind the expected values. Rømer thus actually found a phenomenon, which was later named after Christian Doppler.

where c is the speed of light and

$$f' = \gamma_v^{-1} f \tag{8}$$

corresponds to the lower frequency calculated from (5). By (7), (8), and (1) we obtain a relativistic Doppler relation for the frequency detected in S (see [5]),

$$f_v = \frac{1}{1-\dfrac{v}{c}} f' = \frac{\gamma_v^{-1}}{1-\dfrac{v}{c}} f = \frac{\sqrt{1-\left(\dfrac{v}{c}\right)^2}}{1-\dfrac{v}{c}} f = \sqrt{\frac{c+v}{c-v}} f. \tag{9}$$

From this we immediately get the following theorem.

Theorem 1. *For any $v \in (0, c)$ we have that $f_v/f' > f_v/f > 1$. Moreover, $f_v/f \to \infty$ as $v \to c$.*

Consequently, the Doppler effect manifests more than the time dilation itself, whenever the clock approaches the observer. Hence, the higher the speed v, the greater the Doppler effect. Special relativity effects for large v are of higher order than those arising from the Doppler effect.

It is therefore very important to distinguish consistently between reconstructions (calculations by means of the Lorentz transformation) and observations (measurements, photographs, videos). The notion "observer" in the STR is somewhat confusing. It should not be a person who only applies relations (2)–(3). The observer performs real observations and measurements including all effects together as it is usually understood, i.e., the observer measures incoming frequencies.

Example 1. Suppose that a clock will be approaching the origin of S at relativistic speed $v = 0.8c$. Its proper time will pass slower than on clocks fixed in the system S, since by (1) and (5) we have

$$\gamma_v = \frac{1}{\sqrt{1-0.64}} = \frac{5}{3}$$

and

$$\Delta t' = \frac{5}{3}\Delta t.$$

However, substituting $v = 0.8c$ into (9), we find that

$$f_v = 3f \quad \text{and} \quad f_v = 5f', \tag{10}$$

i.e., the observer at the origin of S will detect a 3× higher (blue-shifted) frequency than the same clock has at rest in the system S and even a 5× higher frequency than the time dilatation predicts (see (8)). This may seem to be paradoxical. For a clock receding the origin by the speed $(-v)$, the observer will detect by (9) a 3× lower (red-shifted) frequency than f. So there is a jump in these constant frequencies at the origin and only in this single point the observer can theoretically detect the proper frequency f'. So the Doppler effect plays an essential role. □

Remark 2. The observer usually does not have a possibility to measure directly the speed v of some distant object so that he could immediately use the Lorentz transformation. However, he can measure the frequency f_v (blue-shifted or red-shifted) of some characteristic spectral line of a certain chemical compound and establish the corresponding quiescent frequency f. From this and (9) he can establish the speed v (or its radial component in general case). Then he can determine the factor γ_v^{-1} and find how significant are the corresponding relativistic effects (2)–(3). □

3. The Lorentz transformation does not allow superluminal velocities

First we recall Einstein's formula [5] for a relativistic addition of velocities, see also [17, Chapt. I.6].

Theorem 2 (Einstein). *Let $u \in (-c, c)$ and $w \in (-c, c)$ be constant velocities of a point-like object in the system S and S', respectively, in the direction of the horizontal axis. Then*

$$u = \frac{v + w}{1 + \frac{vw}{c^2}}, \tag{11}$$

where $v \in (-c, c)$ is a constant speed of S' with respect to S.

P r o o f: Velocities u and w are constant in S and S', respectively. Therefore,

$$u = \frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{and} \quad w = \frac{\mathrm{d}x'}{\mathrm{d}t'}. \tag{12}$$

By (3) we get

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \gamma_v \left(1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \gamma_v \left(1 - \frac{uv}{c^2}\right),$$

where the difference in parenthesis is obviously positive. Using this equality, (12), and (2) we find that

$$w = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} = \gamma_v \left(\frac{\mathrm{d}x}{\mathrm{d}t} - v\right) \gamma_v^{-1} \left(1 - \frac{uv}{c^2}\right)^{-1} = \frac{u - v}{1 - \frac{uv}{c^2}}.$$

From this it follows that

$$u - v = w - \frac{uvw}{c^2}.$$

Now it is enough to evaluate u and we obtain (11). □

For example, by Theorem 2 we see that for $v = w = \frac{2}{3}c$ the speed $u = \frac{12}{13}c$ is less than c. In the next theorem we prove that from (11) we can never get the speed of light or faster-than-light speed u, even if $|v|$ and $|w|$ are arbitrarily close to c.

First, let us recall that a *group* G is a set equipped with an associative binary operation $\circ\colon G\times G\to G$ and with the neutral element e such that for any $g\in G$ there exists exactly one inverse element $g^{-1}\in G$ for which

$$g\circ g^{-1}=e=g^{-1}\circ g.$$

The composition $\circ$ of two Lorentz transformations $\mathcal{L}_v$ and $\mathcal{L}_w$ given by relations (2)–(3) for $v,w\in(-c,c)$ is defined as follows

$$\mathcal{L}_u=\mathcal{L}_v\circ\mathcal{L}_w, \tag{13}$$

where u satisfies Einstein's formula (11).

Theorem 3. *Lorentz transformations $\mathcal{L}_v$ for all $v\in(-c,c)$ defined by* (2)–(3) *form an Abelian group.*

P r o o f: If $v,w\in(-c,c)$ then obviously

$$\Big(1+\frac{v}{c}\Big)\Big(1+\frac{w}{c}\Big)>0\quad\text{and}\quad\Big(1-\frac{v}{c}\Big)\Big(1-\frac{w}{c}\Big)>0.$$

From this we find that

$$-\Big(1+\frac{vw}{c^2}\Big)<\frac{v+w}{c}<1+\frac{vw}{c^2},$$

and thus

$$-c<\frac{v+w}{1+\dfrac{vw}{c^2}}<c.$$

Comparing with Einstein's formula (11), we see that $u\in(-c,c)$, i.e., $|u|$ is always less than c.

Using (1) for $v=0$, we find that $\gamma_0=1$ and the corresponding transformation $\mathcal{L}_0$ is the identity, i.e. the neutral element.

From (11) and (13) we immediately get that

$$\mathcal{L}_v\circ\mathcal{L}_{-v}=\mathcal{L}_0=\mathcal{L}_{-v}\circ\mathcal{L}_v,$$

where $\mathcal{L}_{-v}$ is the inverse transformation, i.e., $x=\gamma_v(x'+vt')$, $t=\gamma_v(t'+vx'/c^2)$.

The composition $\circ$ is commutative, since the special block diagonal matrices $\boldsymbol{L}_v$ and $\boldsymbol{L}_w$ defined by (4) are commutative, i.e.

$$\boldsymbol{L}_v\boldsymbol{L}_w=\begin{pmatrix}\gamma_v & -\dfrac{v}{c}\gamma_v & 0 & 0\\ -\dfrac{v}{c}\gamma_v & \gamma_v & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_w & -\dfrac{w}{c}\gamma_w & 0 & 0\\ -\dfrac{w}{c}\gamma_w & \gamma_w & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\gamma_v\gamma_w+\dfrac{vw}{c^2}\gamma_v\gamma_w & -\dfrac{v+w}{c}\gamma_v\gamma_w & 0 & 0\\ -\dfrac{v+w}{c}\gamma_v\gamma_w & \gamma_v\gamma_w+\dfrac{vw}{c^2}\gamma_v\gamma_w & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\boldsymbol{L}_w\boldsymbol{L}_v$$

for all $v, w \in (-c, c)$. The associativity of the operation $\circ$ is due to the fact that matrix multiplication is associative. $\square$

4. Length contraction

Lorentz's length contraction is an immediate consequence of the Lorentz transformation. On the horizontal axis x' consider a fixed bar which is at rest in the system S'. Denote its length by

$$\Delta x' = x'_2 - x'_1, \tag{14}$$

where x'_i are fixed time independent coordinates of its ends in S'. For an arbitrary fixed time instant t in S we determine the corresponding x_2 and x_1 from formulae (cf. (2))

$$x'_2 = \gamma_v(x_2 - vt), \quad x'_1 = \gamma_v(x_1 - vt),$$

and we set $\Delta x = x_2 - x_1$. Substituting this into (14), we get (cf. (5))

$$\Delta x' = \gamma_v(x_2 - vt - x_1 + vt) = \gamma_v \Delta x. \tag{15}$$

Denoting $\ell_0 = \Delta x'$ and $\ell = \Delta x$, we get by (1) the well-known *length contraction*

$$\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}}. \tag{16}$$

The bar at rest has the greatest length.

Remark 3. For the time being there is no direct experimental evidence of length contraction (16). Anyway, we can verify it indirectly by means of muons mentioned in Remark 1. In the system S associated with these muons at rest we will observe their usual mean half-life time $\tau = 2.2 \cdot 10^{-6}$ s. However, in S' connected with Earth's atmosphere they will travel longer distance than $c\tau = 660$ m. This demonstrates the length contraction is S'. $\square$

In 1959, Roger Penrose published a paper [18] (see also [19, p. 431], [20]) describing why we should see a quickly flying non-rotating ball in a photo again like a ball. In the same year, his thoughts were elaborated in more detail by James Terrell [21] using light aberration. Here is a specific example showing the substantial effect of light aberration for relativistic speeds.

Example 2. Consider a bar with length $\ell_0 = 1$ m. Assume that it moves from the left to the right along the axis x by the constant speed $v = 0.8c$ and that its front end just reached the origin of the coordinate system S. By (16) the bar is shortened to

$$\ell = \ell_0 \sqrt{1 - 0.64} = 0.6 \text{ m},$$

and thus the length of the straight line segment AC in Figure 2 is $|AC| = 0.4$ m. We will photograph this bar from the axis z by a fixed nonrotating camera which is placed at the distance

$$d = 0.75 \text{ m} \tag{17}$$

from the origin. Using the similarity of right triangles from Figure 2, we find that $|BC| = |AC|d/\ell_0 = 0.3$ m. From this we have $|AB| = \sqrt{0.4^2 + 0.3^2} = 0.5$ m. The segment on the hypotenuse from B to the camera has the same length in meters as d in (17),

$$\sqrt{1^2 + 0.75^2} - |AB| = 1.25 - 0.5 = d. \tag{18}$$

To avoid blurred photos, we assume that our idealized camera can take pictures within 1 picosecond. During this time period, the light will fly 0.3 mm only and a possible blurring will not play a significant role. For simplicity, we shall analyze only that photo, in which the front end of the bar just reached the coordinate origin of S. However, the rear end of the bar will be on the photo farther than ℓ, since the light from the front end flies along a shorter distance d than the light from the rear end (see Figure 2). That is why there will be recorded photons on the photo from the

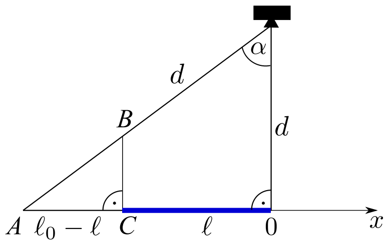

Figure 2: The length of the legs of the larger (or smaller) right triangle is 1 and 0.75 (or 0.4 and 0.3) meters. The ratio between the lengths of sides of the both triangles is 5 : 4 : 3. Due to light aberration the flying bar from the left to the right at the speed $0.8c$ has the same length in the photo as the same bar at rest. A photon emitted to the camera from the rear end of the moving bar will always have the same x coordinate as this rear end in this special case.

rear end of the bar that were emitted earlier than those from the front end. During the time period, when the rear end of the bar moves from A to C, a photon pointing from A to the camera will travel the distance $|AB|$, since $v/c = |AC|/|AB| = 0.8$. Hence, thanks to light aberration and (18) the moving bar will have on the photo the same length as the fixed one meter long bar. $\square$

Let us point out that a photon will travel the distance d from the origin to the camera during the time period $\Delta t = d/c$. During this period, the bar will shift about $v\Delta t = 0.8d = 0.6$ m, i.e., it will be placed entirely to the right of point 0.

Example 3. Let again $\ell_0 = 1$ m and $v = 0.8c$. Hence, $\ell = 0.6$ m. This time, however, we place the camera closer to the axis x, i.e. $d < 0.75$ m. We shall again analyze the image, where the right end of the bar is at the origin. The left end of the bar will shift from the point $A = (-a, 0)$ to the point $(-\ell, 0)$ during the time period $\Delta t = (a - \ell)/v$. During this period, a photon will travel the distance $c\Delta t$ from the point A to the camera. From the relation $a^2 + d^2 = ((a - \ell)c/v + d)^2$ we can derive the following inverse formula

$$d = \frac{a^2\left(\frac{v}{c} - \frac{c}{v}\right) + 2a\ell\frac{c}{v} - \ell^2\frac{c}{v}}{2(a - \ell)}.$$

For instance, when $a = 2$ m we obtain $d = \frac{15}{56} = 0.26\ldots$ m. So if we place the camera on the axis z at a distance of 26 cm from the origin, the one meter flying bar will appear extended in the photo as two meters long. Similarly for $v = 0.9$ and $d = 4$ cm we even get $a = 4$ m. The main reason for these surprising phenomena is that photons, which simultaneously passed through the lens, were not emitted simultaneously in S.

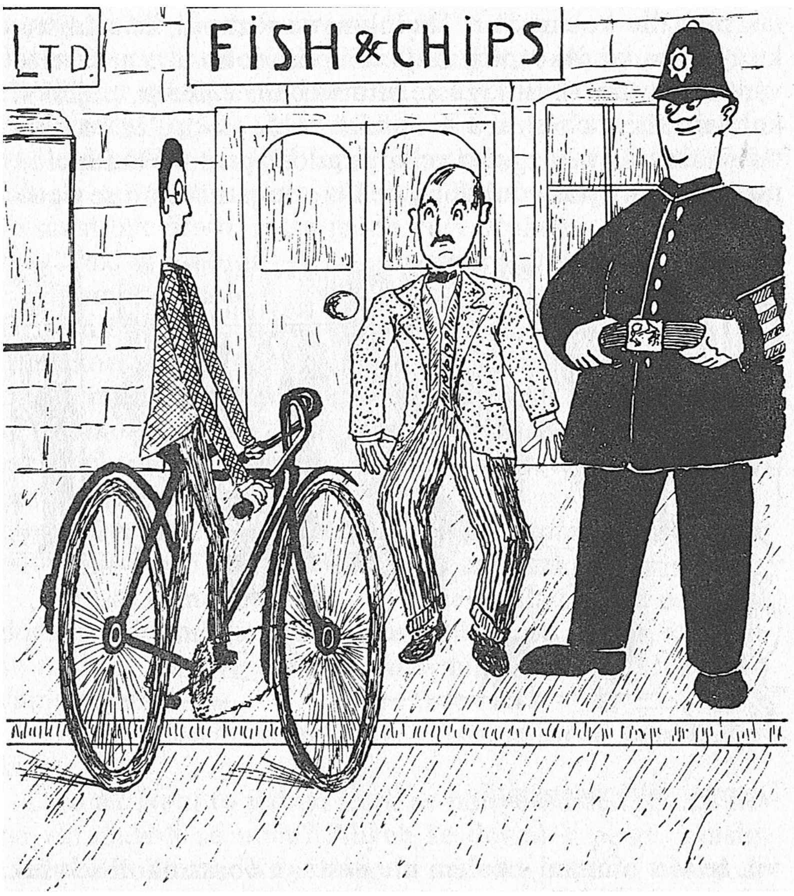

Figure 3: (see [8]). The man in the middle should observe much longer bicycle in the x-direction due to light aberration (compare with the observed bar from Example 3). Moreover, the wheels should not look like ellipses and its wires should be bent.

For $d = 0$ no aberration effect appears. The photon from the end of the bar will travel a distance of 3 m in the x-direction within 1 μs. During this time period, the end of the bar will move $3 \cdot 0.8 = 2.4$ m. Thus the photon just gets to the beginning of the bar, since $2.4 + 0.6 = 3$ m. □

Approaching objects are manifested by blue shift (i.e. shortening the wave length). However, due to aberration they may seem to be prolonged, which is paradoxical. On the other hand, receding objects that are manifested by red shift may seem to be shortened.

For $d > 0.75$ m we shall see the bar in the photo shorter than 1 meter. The same bar will also be shorter than ℓ, if we photograph it so that its left end is at the origin. If it is placed exactly symmetrically with respect to the origin, its length on the photo will be just ℓ, but nonlinearly deformed. In the article [23], Weisskopf describes an apparent deformation of a quickly flying cube on a photo.

Example 4. Due to Example 3, Figures 3 and 4 taken from the popularization book [8, Chapt. 1] are confusing. □

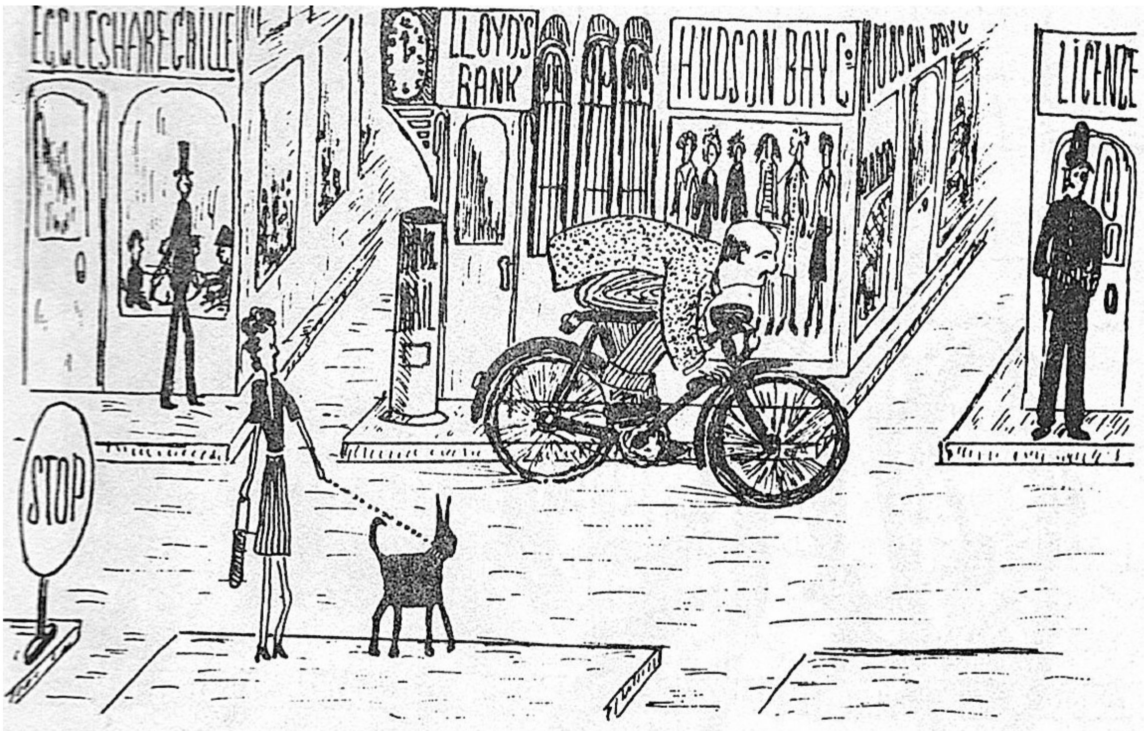

Figure 4: (see [8]). The man on the bicycle should see the policemen on the right thicker due to light aberration.

5. The twin paradox

In a series of publications [6], [8], [15, p. 167], [16], ... the twin paradox (called also the clock paradox) is described by means of time dilation as follows:

One of the twins Adam stays on Earth while the other twin Bob flies in a rocket e.g. to the star Sirius about 8 light years away with constant relativistic speed v. Bob's time runs slower due to time dilatation and his distance to Sirius is less than 8 light years due to length contraction. When Bob returns with velocity $(-v)$ back on Earth, he finds out that his brother Adam is much older than him.

Now let us look at the twin paradox in more detail. First we present a wrong argumentation which is sometimes proposed in the literature (and on internet).

Example 5 (Wrong argumentation). Let again $v = 0.8c$. For simplicity, the speed of light $c = 1$ ly/yr will be not explicitly marked in this example. According to Adam's time, Bob will reach Sirius within 10 years and the same amount of time he will fly back, i.e. 20 years altogether. The first half of Bob's trajectory is defined by the equation $x' = 0$ in the system S' implying by (2) that Bob will follow the line $t = \frac{5}{4}x$ in S (see the lower thick line in Figure 5). Its second half in the inertial system S'' associated with speed $(-v)$ is given by the equation $x'' = 0$ yielding $t = 20 - \frac{5}{4}x$.

Bob's proper time in the first half of his trip is $t' = \gamma_v(t - 0.8x)$ due to (3). From this for a constant time t' we obtain the equation $t = \frac{4}{5}x$ + const which determines in S the space of simultaneous events in S'. Its graph has to pass through the event given by $x = 8$ ly and $t = 10$ yr when Bob reaches Sirius. Hence, the corresponding space of simultaneity is given by the equation $t = \frac{4}{5}x + 3.6$, since $10 = \frac{4}{5} \times 8 + 3.6$ (see the dashed line passing through $t = 3.6$ on vertical axis in Figure 5). Similarly we find that the second space of simultaneous events in S'' is given by $t = -\frac{4}{5}x + 16.4$. During Bob's turn at Sirius, the time on Earth will jump about $20 - 2 \times 3.6 = 12.8$ yr. This time interval is not accounted by Bob and thus he will return 12.8 years younger than his twin Adam (see the left vertical axis t in Figure 5). □

Why is the above argumentation wrong? There are several reasons. Bob's proper time t' (and also t'') was not taken into account correctly as we shall see in Example 6. We saw in Theorem 1 and Example 1 that the Doppler effect plays an important role in the STR. However, it was also not taken into account in Example 5. Bob feels an infinite deceleration[7] during his turnover at Sirius. The reader is juggled that this is the main reason for the twin paradox, since Bob changes its inertial system. However, this is not true, since when Bob reaches Sirius, he can just transmit information about his real age in the system S' to another traveler who is fixed in the system S'' associated with speed $(-v)$. The same applies for Bob's departure and arrival back to the Earth.

Moreover, it is said that Bob will observe a large jump in time on Earth, since no event from the red interval (3.6, 16.4) on the vertical axis in Figure 5 is simultaneous

[7] Roger Penrose [19, p. 421] rounds the corresponding world lines on a small neighborhood around the three critical points to avoid an infinite acceleration or deceleration.

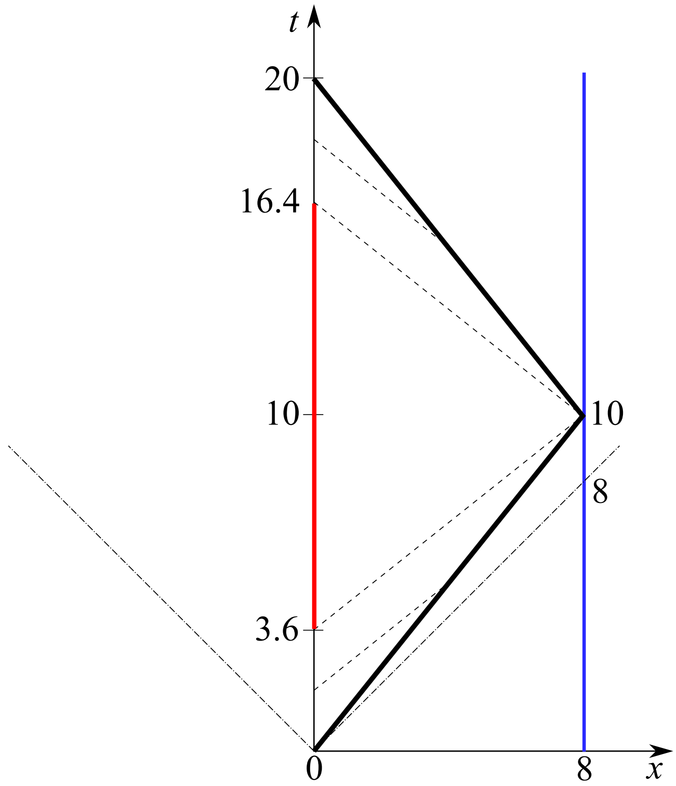

Figure 5: The vertical axis t shows the time in years and the horizontal axis x shows the distance in light years. The left vertical line is the world line of Adam who stays on Earth. The right vertical line corresponds to the world line of Sirius. The world line of flying Bob is marked with a thick piecewise linear line given by equations $t = \frac{5}{4}x$ and $t = 20 - \frac{5}{4}x$. The future light cone $t = |x|$ is marked by the dot-and-dash line and the dashed lines stand for events with simultaneous times in S' and S'' such that t' and t'' are constant.

with Bob. In Example 6, we will show that Bob will see only a jump in frequencies and no jump in time in S during his turnover (see Figure 6).

So further, we shall investigate the twin paradox from another point of view. We will get different values than in Example 5.

Theorem 4. *The difference $(ct)^2 - x^2$ is invariant with respect to the Lorentz transformation.*

P r o o f: From (2)–(3) we see that

$$
\begin{aligned}
(ct')^2 - x'^2 &= (ct' - x')(ct' + x') \\
&= \gamma_v \left(ct - \frac{vx}{c} - x + vt \right) \gamma_v \left(ct - \frac{vx}{c} + x - vt \right) \\
&= \gamma_v^2 \left(1 + \frac{v}{c} \right) (ct - x) \left(1 - \frac{v}{c} \right) (ct + x) = (ct)^2 - x^2. \qquad (19)
\end{aligned}
$$

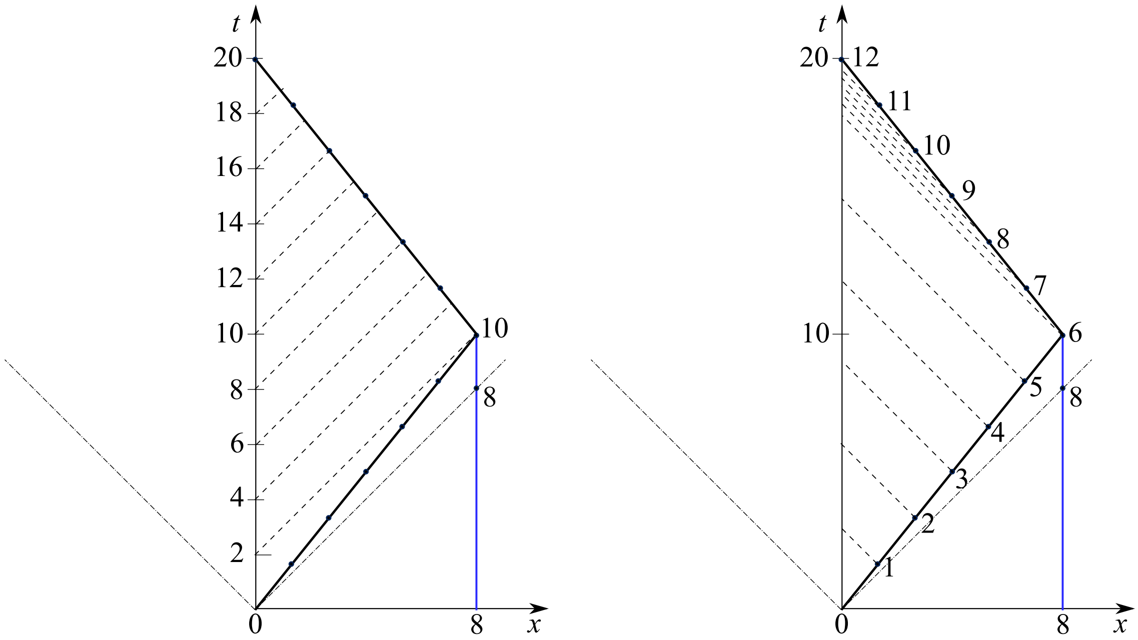

Figure 6: The notation is the same as in Figure 5 except for dashed lines. Here dashed lines indicate trajectories of photons launched every year by Adam and Bob. Left: Adam sends a periodic signal to Bob. Right: Bob sends a period signal to Adam. There is not jump in time — only a jump in received frequencies.

Hence, this difference of squares is invariant with respect to the Lorentz transformation. □

Example 6 (Right argumentation). Let again $v = 0.8c$. Since Bob is at rest in the system S', we have that $x' = 0$. Thus from (19) we get for $\Delta t = 10$ yr and $\Delta x = 8$ ly

$$c\Delta t' = \sqrt{(c\Delta t)^2 - (\Delta x)^2} = \sqrt{100 - 64} = 6 \text{ ly}. \tag{20}$$

Consequently, Bob will fly 6 years to Sirius according to his proper time (see bullets in Figure 6). Thus his clock at Sirius will show 6 years. On the other hand, the corresponding time interval on Earth is only 3.6 years (see Figure 5), since $\Delta t = \gamma_v^{-1}\Delta t' = \frac{3}{5} \cdot 6 = 3.6$ years by (5).

Let f be the same frequency of Adam's and Bob's clock at rest. Due to the relativistic Doppler relation (9), Adam will observe 3× lower frequency from Bob's clock, since

$$f_{-v} = f\sqrt{\frac{c-v}{c+v}} = \frac{1}{3},$$

and after the turnover of Bob, Adam will observe the frequency $f_v = 3f$, see (10) and Figure 6. Adam will receive the same frequencies from Bob due to the relativity principle.

Obviously, Bob will fly after the turnover the same distance back to Earth with velocity $(-v)$ again 6 years according to his proper time. So he will be $20-2\times 6 = 8$ years younger than his twin Adam. This is a different result than in Example 5. Bob will see all instants on Earth, i.e., no time interval will be skipped. □

Example 7. How younger was Niel Armstrong when he returned from the Moon? For simplicity, assume that he was flying there and back by the constant speed $v = 10$ km/s and that $\Delta x = 384\,000$ km is the Earth-Moon distance. Hence, the one-way flight lasted

$$\Delta t = \frac{\Delta x}{v} = 38\,400 \text{ s} \tag{21}$$

with respect to Earth's clock. Then like in (20) we get

$$\Delta t' = \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2}{c^2}} = \sqrt{38\,400^2 - 1.28^2} = \sqrt{1\,474\,559\,998.3616}$$
$$= 38\,399.999\,978\,\overline{666} \text{ s}$$

From this and (21) we find that

$$\Delta t - \Delta t' = 0.000\,021\,344\,\text{s}.$$

Thus, when Niel Armstrong returned, he was approximately 43 μs younger than if he stayed on Earth. □

6. Conclusions

The Special Theory of Relativity has a number of unexpected claims that contradict our intuition. According to the STR, no experiment can be made to decide whether the body is at rest or moving. All inertial systems for describing physical phenomena are equivalent and there is no preferred inertial system. However, at present we know that the cosmic microwave background radiation (CMB) actually determines a certain kind of a fixed reference system in our neighborhood. Thus there arise speculations whether the principle of relativity in the real universe holds.

In the limiting case $w = \pm c$, Einstein's formula (11) for $v \in (-c, c)$ gives $u = c$. Hence, every photon always has the speed of light in any inertial system.

It is often said that the Lorentz transformation for low speeds $|v| \ll c$ changes into the Galileo transformation

$$x' = x - vt,$$
$$y' = y,$$
$$z' = z,$$
$$t' = t.$$

This is not true (see [2], [22]), since for an arbitrarily small fixed $v > 0$ we can always find x such that the term vx/c^2 in (3) will dominate significantly over t. However, from (2)–(3) it follows that the Lorentz transformation changes into the Galileo transformation for a fixed v, if we treat c as a parameter and assume that $c \to \infty$. However, for an infinite speed of light there would be no Doppler effect nor aberration of light.

Remark 4. For $\vec{x} = (x, y, z)$ and a constant velocity vector $\vec{v} \in \mathbb{R}^3$ with length $|\vec{v}| \in (0, c)$ the *general Lorentz transformation* is of the form (see e.g. [10, p. 434])

$$\vec{x}\,' = \vec{x} + \left(\frac{\gamma - 1}{|\vec{v}|^2} \vec{v} \cdot \vec{x} - \gamma t \right) \vec{v}, \tag{22}$$

$$t' = \gamma \left(t - \frac{\vec{v} \cdot \vec{x}}{c^2} \right). \tag{23}$$

Here the Lorentz factor γ is defined similarly as in (1), only v^2 needs to be rewritten as $|\vec{v}|^2$. It is easy to find that for nonzero $\vec{v} = (v, 0, 0)$, where $v \in (-c, c)$, relations (22)–(23) change to (2)–(3). By Wikipedia [24] (see also [1]), the Einstein addition of velocities is neither commutative nor associative, in general. □

We conclude by stating that the longitudinal Doppler effect and aberration of light may cause that we observe completely opposite phenomena than those predicted by the Special Theory of Relativity by meas of (2)–(3). Note that relations (2)–(3) represent only a transformation of spacetime coordinates of points from one inertial systems into spacetime coordinates of the second inertial system. We saw that some other effect than time dilatation and length contraction can manifest stronger and they cannot be shielded in any way. For a visualization of several further accompanying effects (like nonlinear distortion) we refer to `www.spacetimetravel.org`.

Acknowledgements

The author is indebted to J. Brandts, F. Křížek, M. Prunescu, and A. Ženíšek for inspiration and valuable suggestions, and H. Bílková for drawing some figures. Supported by RVO 67985840 of the Czech Republic.

References

[1] Ahmad, M., Alam, M. S.: Non-associativity of Lorentz transformation and associative reciprocal symmetric transformation. Iternat. J. Rec. Sym. Theor. Phys. **1** (2014), 9–19.

[2] Baierlein, R.: Two myths about special relativity. Amer. J. Phys. **74** (2006), 193–195.

[3] Botermann, B., et al.: Test of time dilatation using stored Li+ ions as clocks at relativistic speed. Phys. Rev. Lett. **113** (2014), 120405; errata **114** (2015), 239902.

[4] Doppler, Ch.: Ueber das farbige Licht der Doppelsterne und einiger anderer Gestirne des Himmels. Abh. böhm. Ges. Wiss. **2** (1842), 466–482.

[5] Einstein, A.: Zur Elektrodynamik bewegter Körper. Ann. der Phys. **322** (10) (1905), 891–921.

[6] Feynman, R. P., Leighton, R. B., Sand, M.: *The Feynman lectures of physics I.* Addison-Wesley Publ. Company, 1966.

[7] Foster, J., Nightingale, J. D.: *A short course in general relativity* (3rd edition). Springer, New York, 2006.

[8] Gamow, G.: *Mr. Tompkins in paperback.* Cambridge Univ. Press, London, 1965.

[9] Hafele, J. C., Keating, R. E.: Around-the-world atomic clocks: predicted relativistic time gains. Science **177** (1972), 166–168.

[10] Horák, Z., Krupka, F.: *Fyzika*, sv. 2 (in Czech). SNTL, Praha, 1976.

[11] Ives, H. E., Stilwell, G. R.: An experimental study of the rate of a moving atomic clock. J. Optic. Soc. Amer. **28** (1938), 215–226.

[12] Křížek, M.: O paradoxech ve speciální teorii relativity (in Czech). Obzory Mat. Fyz. Inf. **49**(1) (2020), 31–40.

[13] Lorentz, H. A.: La théorie électromagnetique de Maxwell et son application aux corps mouvants. Arch. Néerl. **25** (1892), 1–190.

[14] Michelson, A. A., Morley, E. W.: On the relative motion of the Earth and the luminiferous ether. Amer. J. Sci. **34** (1887), 333–345.

[15] Misner, C. W., Thorne, K. S., Wheeler, J. A.: *Gravitation*, 20th edition. W. H. Freeman, New York, 1997.

[16] Mixson, J. S.: *The twin paradox of special relativity is solved.* Infinity Publishing, 2014.

[17] Pauli, W.: *Theory of relativity.* Dover Publ., Inc., New York, 1981.

[18] Penrose, R.: The apparent shape of a relativistically moving sphere. Proc. Cambridge Phil. Soc. **55** (1959), 137–139.

[19] Penrose, R.: *The road to reality.* Vintage Books, London, 2005.

[20] Synge, J. L.: *Relativity: the special theory.* North-Holland Publ. Co., Amsterdam, 1956.

[21] Terrell, T.: Invisibility of Lorentz contraction. Phys. Rev. **116** (1959), 1041–1045.

[22] Vankov, A.: On controversies in special relativity. ArXiv: physics/0606130v1, 2006, 1–23.

[23] Weisskopf, V. F.: The visual appearance of rapidly moving objects. Physics Today **13**(9) (1960), 24–27.

[24] `https://en.wikipedia.org/wiki/Velocity-addition_formula`

[25] Ženíšek, A.: *Relativita do kapsy* (in Czech). Masaryk Univ., Brno, 2015.