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Michal Křížek and Yurii Dumin (Eds.)
Institute of Mathematics CAS, Prague

VISUALIZATION OF A CLOSED 3-SPHERE UNIVERSE

Petri Lievonen[1], Michal Křížek[2]

[1] Physics Foundation Society, Espoo, Finland
petri.lievonen@gmail.com

[2] Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic
krizek@math.cas.cz

Abstract: We describe how to imagine a three-dimensional closed universe. To do this, we consider six orthogonally intersecting great circles of a three-dimensional sphere. Then we apply a stereographic projection of these circles into three-dimensional Euclidean space. This projection maps circles on circles and preserves angles. The resulting model can be printed on a standard 3D printer. This paper is an upgrade and large extension of our previous 2026 paper [5].

Keywords: 3-sphere, stereographic projection, cosmological principle, Copernicus principle, great circle, Ptolemy theorem

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1. Introduction

According to Copernicus' principle, people on Earth are not privileged observers of the universe. We are not in some special place – the center of the world, like the tip of an egg. According to Einstein's cosmological principle (as named by Milne [9]), the physical universe is homogeneous and isotropic on large scales at any fixed time instant. Roughly speaking, its curvature is constant at every point and in all directions.

More precisely, homogeneity is the assumed property of the universe that at any given moment in time and on large spatial scales, the universe appears the same to all observers, wherever they are. In other words, translational symmetry of the universe is required at all times, which is actually what Copernicus' principle asserts. However, this only applies to spatial coordinates. Homogeneity in time apparently does not apply, because the average density of the universe is gradually decreasing.

Similarly, isotropy is a presumed property of the universe, in which the universe appears the same in all directions to an observer at any point on large spatial scales, i.e., at any given time, rotational symmetry of the universe is required.

There are a number of tests that confirm the validity of isotropy, because the relevant astrophysical objects (microwave background radiation, active galactic nuclei, quasars, galaxies in the Hubble Deep Field, gamma-ray bursts, etc.) are distributed almost uniformly across the entire celestial sphere. The Hubble test is used to prove homogeneity, see [7, p. 108].

Denote by $\mathbb{E}^n$ the n-dimensional Euclidean space. Around 1900, the renowned German physicist and astronomer Karl Schwarzschild [10] came up with the idea that the geometry of the universe may not correspond to infinite three-dimensional Euclidean space $\mathbb{E}^3$, as Giordano Bruno had assumed at the end of the 16th century [8], but that instead, it could be described as a closed three-dimensional sphere for a given time, i.e., the three-dimensional surface of a four-dimensional ball of radius $r > 0$, see e.g. [1, 2, 4, 12, 13]:

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{E}^4 \,|\, x^2 + y^2 + z^2 + w^2 = r^2\}.$$

This geometry cannot change over time, i.e., the bounded sphere $\mathbb{S}^3$ cannot suddenly change into the unbounded Euclidean space $\mathbb{E}^3$. For simplicity, we shall assume that the radius $r = 1$ in what follows.

Remark. The term "closed universe" has a completely different meaning in cosmology from what it has in mathematics. For example, the unbounded interval $[0, \infty)$ is a closed set in $\mathbb{E}^1$, while a closed universe is bounded (closed within itself). On the other hand, the bounded interval $(0, 1)$ is open in $\mathbb{E}^1$, but the open universe is unbounded.

Since $\mathbb{S}^3$ is a curved three-dimensional manifold, it is difficult to visualize it. In [7, p. 112], ten ways are presented to imagine it. One of them is a stereographic projection.

2. Stereographic projection

The one-dimensional sphere $\mathbb{S}^1$ and the two-dimensional sphere $\mathbb{S}^2$ are easy to imagine, because they can be embedded (see Fig. 1) in three-dimensional Euclidean space, whose geometry is familiar to us.

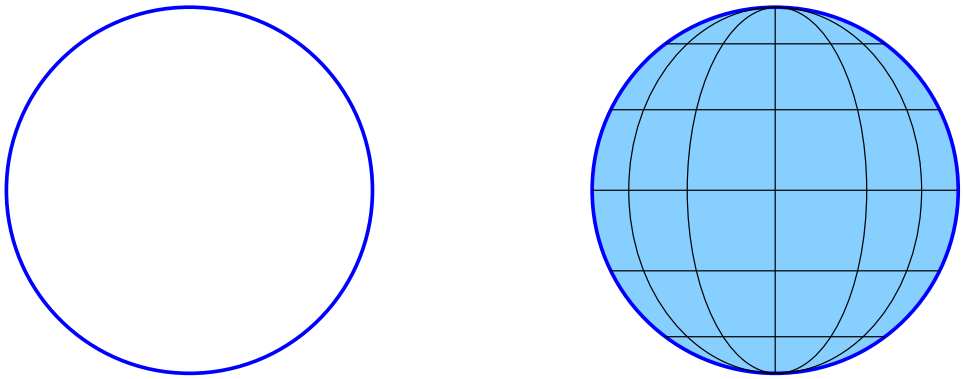


Fig. 1. Sphere $\mathbb{S}^1$ on the left and sphere $\mathbb{S}^2$ on the right can be embedded in three-dimensional Euclidean space in such a way that angles and distances are preserved.

However, sphere $\mathbb{S}^3$ cannot be embedded in $\mathbb{E}^3$ in order to preserve distances. Similarly, the spherical surface $\mathbb{S}^2$ (or its two-dimensional part) cannot be flattened onto the plane $\mathbb{E}^2$. One option is to use a stereographic projection, which preserves angles but not distances. Therefore, all planar maps of curved surfaces are necessarily distorted.

Stereographic projection from the North Pole N of the celestial sphere onto a plane tangent to the celestial sphere at the South Pole S was used, for example, in the construction of the astronomical dial of the Prague Astronomical Clock, see [6, p. 246]. It is defined as follows. Let $M \in \mathbb{S}^2$ be any point on a unit two-dimensional sphere such that $M \neq N$. Consider the meridian section MNS, where $N = (0, 1)$ is the North Pole and $S = (0, -1)$ is the South Pole, i.e., only two coordinates are enough. Let us denote (x, z) as the coordinates of point M in the meridian section. From the projection point N, let us draw a ray through point M. Let us denote its intersection with the projection plane $z = -1$ as $M' = (x', -1)$. Point M' is called the *stereographic projection* of point M. From the similarity of the right triangles in Fig. 2, we see that for the unknown coordinate x', the following applies: $x' : 2 = x : (1 - z)$. Consequently,

$$x' = \frac{2x}{1 - z} \quad \text{for } z \in [-1, 1).$$

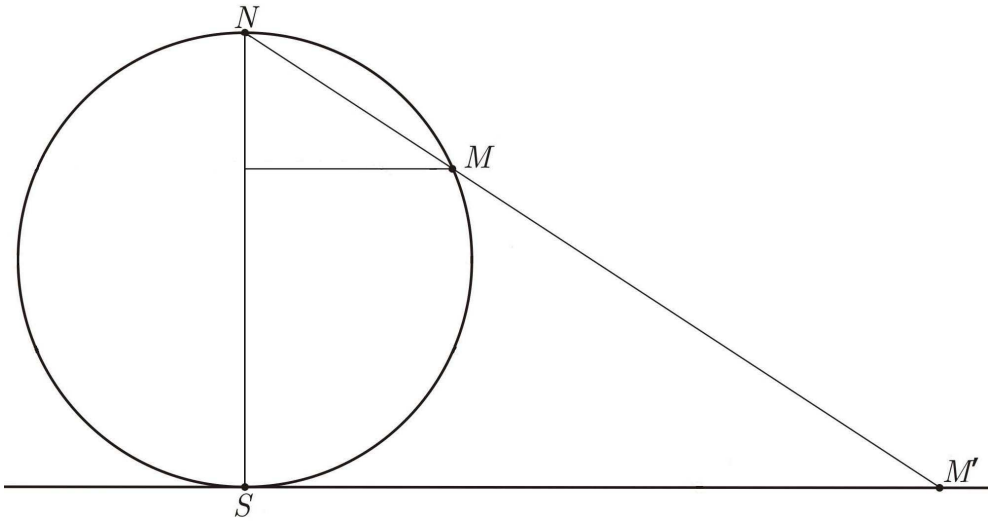


Fig. 2. Stereographic projection from the North Pole N of point M onto M' which lies in the plane tangent to the spherical surface at the South Pole S.

Let us recall the beautiful Ptolemy's theorem, almost 2000 years old, formulated by the famous ancient mathematician and astronomer Claudius Ptolemy. Its detailed proof is given, for example, in [6, p. 247].

Theorem 1 (Ptolemy's). *Every circle lying on the sphere $\mathbb{S}^2$ and not passing through the center of projection is again projected onto a circle in stereographic projection.*

For example, on the astronomical dial of the Prague Astronomical Clock, which is a projection plane of the celestial sphere, there are images of the celestial equator, the ecliptic (zodiac), the Prague horizon, and the tropics of Cancer and Capricorn are again circles. Another important property is given by the following statement, see [1, Chapt. 36].

Theorem 2. *A stereographic projection preserves angles.*

Let us recall that a *great circle* on the spherical surface $\mathbb{S}^2$ has the same radius and thus also the same center as $\mathbb{S}^2$. It is actually a geodesic that is the shortest connecting curve between two points on $\mathbb{S}^2$. Next, we will consider three great circles

$$x^2 + y^2 = 1, \quad x^2 + z^2 = 1 \quad \text{and} \quad y^2 + z^2 = 1$$

of the sphere $\mathbb{S}^2$, which lie in mutually perpendicular planes $z = 0$, $y = 0$ and $x = 0$. Hence, they intersect at right angles at six distinct points

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad \text{and} \quad (0, 0, \pm 1),$$

where the last two are the North and South Poles. On the globe, these circles correspond to the equator and the meridians $(0°, 180°)$ and $(90°, 270°)$, see Fig. 3.

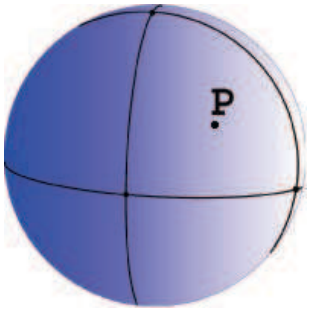


Fig. 3. Three great circles of the spherical sphere $\mathbb{S}^2$ intersect at right angles. The projection point is marked with the letter **P**.

Now we will show how these three great circles are displayed in a stereographic projection onto a plane. When projecting from the North Pole, the two circles formed by the meridians would be displayed as straight lines, because the conditions of Theorem 1 are not met. Consequently, this time we will not project from the North Pole (as in the case of the Prague Astronomical Clock) but from the point $\mathbf{P} = (p, p, p) \in \mathbb{S}^2$, where $p = \sqrt{3}/3$, which is symmetrically positioned with respect to the three great circles under consideration, see Fig. 3. Hence, the proposed stereographic projection will be a one-to-one mapping $\mathbb{S}^2 \backslash \mathbf{P} \rightarrow \mathbb{E}^2$. More precisely, we will project $\mathbb{S}^2 \backslash \mathbf{P}$ onto a plane in $\mathbb{E}^3$ that is tangent to $\mathbb{S}^2$ at the antipodal point $\mathbf{Q} = (-p, -p, -p)$. According to Ptolemy's Theorem 1, the images of the circles under consideration will again be circles that intersect at six right angles, see Theorem 2. The stereographic image of the equator is located at the bottom of Fig. 4, the projections of the North

and South poles are denoted by N' and S' in Fig. 4, the projection point $\mathbf{P}$ is actually "projected to infinity" and its antipodal point $\mathbf{Q} \in \mathbb{S}^2$ is projected to the origin $(0,0)$ of the (x',y')-coordinates.

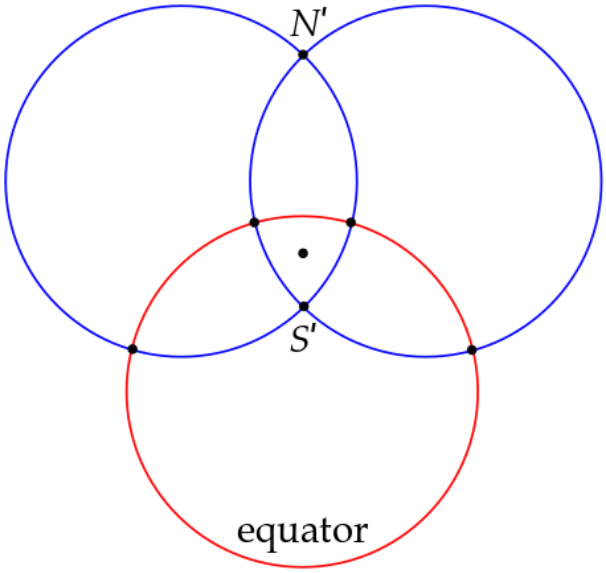


Fig. 4. The images of the three great circles of the sphere from Fig. 3 are again circles that intersect at right angles by Theorem 2.

The North pole is projected on $N' = (0, 2\sqrt{3})$ and the South pole on $S' = (0, -2\sqrt{3}/3)$. The other cross-section points are mapped on $(\pm 3, -\sqrt{3})$ and $(\pm 1, \sqrt{3}/3)$. These points are vertices of two equilateral triangles. The center of the equator $x'^2 + (y' - s')^2 = r'^2$ is given by $(0, s') = (0, -4\sqrt{3}/3)$ and its radius $r' = 2\sqrt{21}/3$.

3. Printing a rounded model of a closed universe on a 3D printer

Now consider six great circles on the unit sphere $\mathbb{S}^3$ that intersect perpendicularly at eight points with coordinates

$$(\pm 1, 0, 0, 0), \quad (0, \pm 1, 0, 0), \quad (0, 0, \pm 1, 0), \text{ and } (0, 0, 0, \pm 1):$$

$$x^2 + y^2 = 1, \quad x^2 + z^2 = 1, \quad y^2 + z^2 = 1, \tag{1}$$
$$x^2 + w^2 = 1, \quad y^2 + w^2 = 1, \quad z^2 + w^2 = 1. \tag{2}$$

For instance, let us denote by $N = (0, 0, 0, 1)$ the North Pole and by $S = (0, 0, 0, -1)$ the South Pole of the sphere $\mathbb{S}^3$. Then the equator of $\mathbb{S}^3$ (belonging to the hyperplane $w = 0$) is a two-dimensional sphere $\{(x, y, z, w) \in \mathbb{E}^4 \,|\, x^2 + y^2 + z^2 = 1\}$ on which the first three circles (1) lie (see Fig. 3). The next three circles (2) intersect perpendicularly at the North Pole N and the South Pole S.

The stereographic projection of the above six circles into three-dimensional Euclidean space $\mathbb{E}^3$ can be constructed in the same way as in the case of two-dimensional

case. This time, however, we will project from the point $\mathbf{P} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbb{S}^3$ which is symmetrically positioned with respect to all six great circles (1)–(2) which do not contain $\mathbf{P}$. Their centers do not belong to the sphere $\mathbb{S}^3$ and therefore are not mapped anywhere in the stereographic projection.

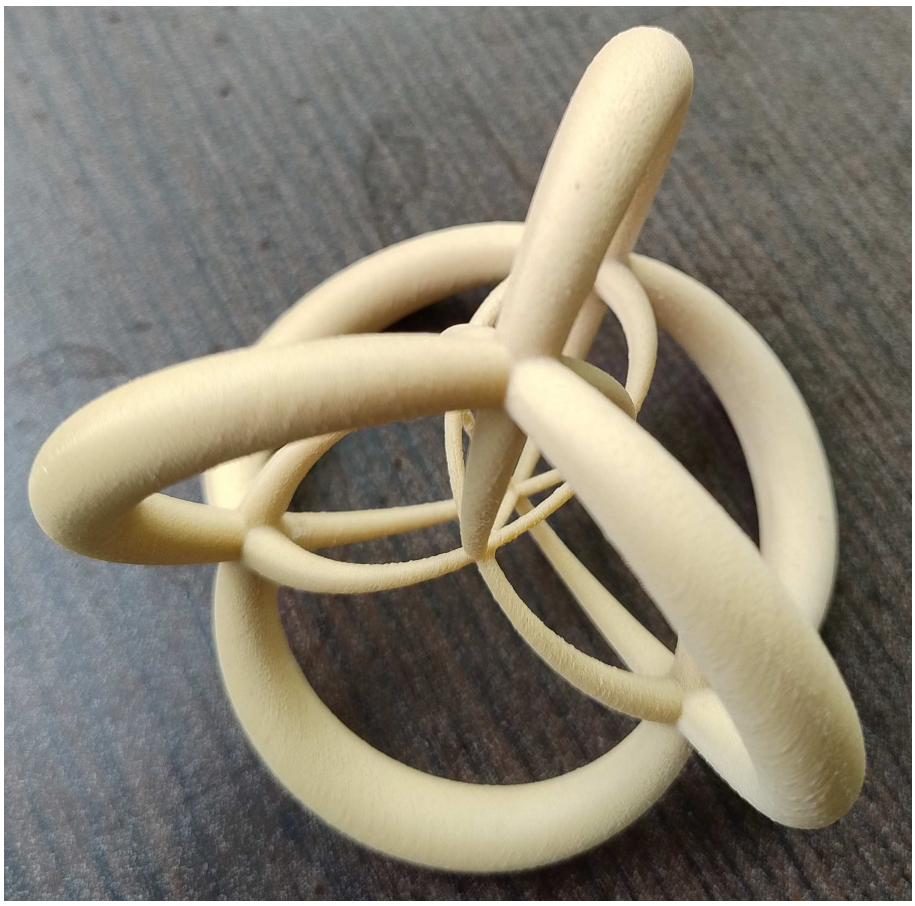


Fig. 5. A rounded model of a closed universe in which distances are not preserved, but angles remain the same.

It is known that Ptolemy's Theorem 1 and Theorem 2 can be generalized to any dimension, see e.g. [8]. Therefore, the six great circles (1)–(2) of $\mathbb{S}^3$ in stereographic projection into three-dimensional Euclidean space $\mathbb{E}^3$ are again mapped to circles that intersect at right angles. The proposed stereographic projection will be a one-to-one mapping $\mathbb{S}^3 \backslash \mathbf{P} \rightarrow \mathbb{E}^3$. More precisely, we will project $\mathbb{S}^3 \backslash \mathbf{P}$ onto a three-dimensional subspace in $\mathbb{E}^4$ that is tangent to $\mathbb{S}^3$ at the antipodal point $\mathbf{Q} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ which will be projected to the origin $(0, 0, 0)$ of the (x', y', z')-coordinates.

The resulting model can be printed on a standard 3D printer, see Fig. 5 and [11, p. 53]. The relevant program is available at the following web address:

`https://www.thingiverse.com/thing:1570212`

Using this model, we can demonstrate that if we start at the North Pole N and

travel along any great circle in any direction, we will end up back at the North Pole N via the South Pole S. So, we will cross the equator twice. This situation is similar to what we encounter on Earth's surface, where the two main directions at the North Pole are perpendicular to each other, see Figs 3 and 4, However, at the North Pole N of the sphere $\mathbb{S}^3$, there are three directions mutually perpendicular: up–down, right–left, and forward–backward, see [15].

If we assume that the radius $r = r(t)$ of a three-dimensional sphere grows linearly as a function of time t, then it is theoretically possible to orbit the entire universe along an expanding geodesic in a finite amount of travel time, see [7, p. 166]. The resulting trajectory is a logarithmic spiral, see also [12, pp. 253, 277].

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References

[1] Di Valentino, E. et al.: Planck evidence for a closed Universe and a possible crisis for cosmology. Nature Astronomy **4** (2020), 196–203.

[2] Friedman, A.: Über die Krümmung des Raumes. Z. Phys. **10** (1922), 377–386; English translation: On the curvature of space. General Relativity and Gravitation **31** (1999), 1991–2000.

[3] Hilbert, D., Cohn-Vossen, S.: Geometry and the imagination. Chelsea Publ., New York, 1990.

[4] Křížek, M.: Infrared measurements of the JWST suggest that our dynamic universe is spatially closed. J. Phys.: Conf. Ser. **2948** (2025), 012007.

[5] Křížek, M., Lievonen, P.: How to imagine a closed universe? Math. Intelligencer (2026), 1–2 ; see also Učitel matematiky, **34** (2025), 58–66 for a Czech version.

[6] Křížek, M., Somer, L., Šolcová, A.: From great discoveries in number theory to applications. Springer, Cham, 2021.

[7] Křížek, M., Somer, L.: Mathematical aspects of paradoxes in cosmology. Can mathematics explain the contemporary cosmological crisis? Springer, Cham, 2023.

[8] Lanzendörferová, M.: Giordano Bruno — the founder of modern cosmology. In: Cosmology on small scales: Dark matter problem and other cosmological puzzles (M. Křížek and Y. V. Dumin Eds), Inst. of Math., Czech Acad. Sci., Prague, 2026, 49–54.

[9] Milne, E. A.: Relativity, gravitation and world structure. Clarendon Press, Oxford, 1935.

[10] Schwarzschild, K.: Über das zulässige Krümmungsmaaß des Raumes. Vierteljahrsschift der Astronomischen Gesellschaft **35** (1900), 337–347; English translation: On the permissible numerical value of the curvature of space. Abraham Zelmanov J. **1** (2008), 64–73.

[11] Segerman, H.: Visualizing mathematics with 3D printing. Johns Hopkins Univ. Press, Baltimore, 2016.

[12] Suntola, T.: The dynamic universe. Toward a unified picture of physical reality. Phys. Foundation Soc., The Finnish Soc. Natur. Phil., Drukatava Ltd., Latvia, 2018.

[13] Suntola, T.: The dynamic universe: Space as a spherically closed zero-energy system. Frontiers in Astron. and Space Sci. **13** (2026), Article no. 1827522.

[14] Whittaker, E. J. W.: The hyper-stereographic projection of the four-dimensional hyper-sphere. Acta Cryst. **A29** (1973), 673–678.

[15] `https://youtu.be/5aWSyYDp_Y`