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The rôle of the protractor in understanding the universe

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Abstract: We present several examples to show how such an ordinary and simple instrument as the usual protractor helped to create the modern view of the universe.

Súhrn: Na niekoľkých príkladoch ukážeme, ako taký obyčajný a jednoduchý prístroj, ktorým je uhlomer, pomohol vytvoriť moderný pohľad na náš vesmír.

Introduction

It took thousands of years utill humankind attained our current imagination and knowledge about the structure and functioning of the universe. In the following nine examples (taken from [1, 2, 3], and [4]) we show that an ordinary protractor played an essential rôle in this process.

1 Measurement of relative distances in the solar system

The following measurement is usually attributed to the Greek astronomer Aristarchus of Samos (3rd century BC) who first proposed a heliocentric model to explain the seasons. He had several really ingenious perceptions and showed that seemingly complicated cosmic problems can be solved by elementary geometrical tools. When the Moon was in the first or last quarter (see Figure 1), he realized that the angle SME is right, where S, M, E stand for the Sun, Moon, and Earth, respectively. Using an ancient protractor, he found that the angle SEM is roughly $\alpha = 87°$ (in today's degrees). Since the Sun and Moon in the sky have approximately the same angular size, he deduced that the Sun is 19 times further from the Earth than the Moon, i.e.,

$$\cos\alpha = \frac{1}{19}.$$

Note that it was very difficult to establish the instant of the first quarter exactly and measure the angle α by the instruments of the day. At present we know that the Sun is approximately 389 times further from the Earth than the Moon, which corresponds to an almost right angle $\alpha = 89.852°$. The big discrepancy in distances

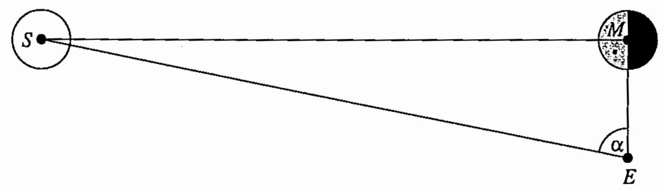

Fig. **1:** *When the Moon is in the first or last quarter, the angle SME is right (S = Sun, M = Moon, E = Earth) and we have* $|ES| = |EM|/\cos\alpha$.

is due to the fact that $\cos^{-1} 87° \ll \cos^{-1} 89.852°$ even though the associated angles are almost the same (cf. Figure 1).

Aristotle (cca 384–322 BC) in his treatise *On the heaven* [5] argued that the Earth is a sphere, because its shadow on the Moon during lunar eclipses is always circular (see Figure 2).

Later Aristarchus measured the angular size of this shadow ($\approx 1.5°$). He stated that the Earth freely hovers in space and its radius is 3 times larger than the radius of the Moon (we know now that it is 3.67 times). From this he calculated that the Moon is 70 Earth's radii from the Earth, whereas the actual value is about 60 Earth's radii.[1] Moreover, he formulated the groundbreaking hypothesis that the Earth moves round the Sun and not vice versa, since the Sun is much bigger than the Earth (cf. [6]).

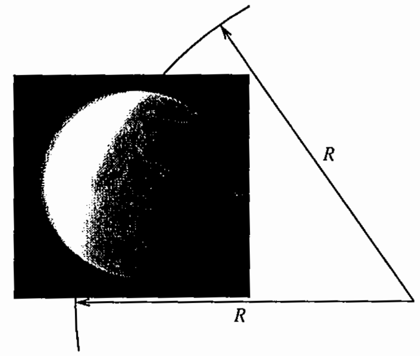

Fig. **2:** *The shadow of the Earth on the Moon during a lunar eclipse is circular. Its radius R is more than 3 times larger than the radius of the Moon.*

The measurements of the size of the solar system as deduced by the ancient Greek astronomers appears also in [7].

2 Establishment of absolute distances

Aristarchus' conception of the determination of relative distances in the solar system was supplemented more sophisticatedly by another Greek astronomer and scholar,

[1] Note that $\tan 1.5° \approx \frac{2R}{70R}$, where R is radius of the Earth. Since the angular size of the Moon is about 31.1′, we get $\tan(3.67 \times 31.1') \doteq \frac{2R}{60R}$.

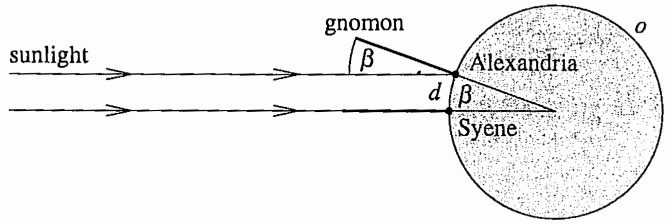

Fig. 3: The Earth's circumference o was calculated from the known distance d between Alexandria and Syene and the angle β was determined by gnomon at the midday of the summer solstice in Alexandria.

Eratosthenes of Cyrene (cca 276–194 BC). He is famous not only for his prime number sieve, but also for the first scientific calculation of the Earth's circumference (see [8] for details). At that time it was known that the Sun's zenith varies at distinct latitudes. Eratosthenes used the simplest astronomical instrument – the gnomon – which is just a straight stick perpendicularly raised to the Earth's surface. He knew that the Sun shines on the bottom of deep wells in Syene (at the tropic of Cancer near to today's Asuan) at midday of the summer solstice. This means that the gnomon does not throw any shadow here. At the same time, in Alexandria (which is on almost the same meridian as Syene) Eratosthenes measured the length of the gnomon and the length of its shadow on the ground. The ratio of these two values gave him the angle $\beta = 7\frac{1}{5}^\circ$ between the vertical and the rays of sunlight (see Figure 3), i.e., $\frac{1}{50}$ of the angle 360°. Then from the relation

$$\frac{d}{o} = \frac{\beta}{360^\circ}$$

he derived that the Earth's circumference is $o = 250\,000$ stadia[2] $\doteq 46\,000$ km, where the value $d = 5\,000$ stadia $\doteq 920$ km was found by travelling on cart as traditionally referred. At present we know that $o = 40\,000$ km.

3 Substantial improvement of accuracy

Due to the above results by Aristarchus and Eratosthenes it was believed in the Middle Ages that the distance between the Sun and Earth is about $19 \cdot 70 \cdot 46\,000/(2\pi)$

[2] It is not known how large the Greek distance unit "stadium" was exactly (most probably in the interval 165–210 m).

km, i.e., less than 10 million kilometers (in today's units). This estimate was dramatically increased in 1672, when G. D. Cassini measured the distance to Mars by using a protractor. In Paris he measured the position of Mars on the celestial sphere when Mars was at its nearest point to Earth (see [9] Chapt. 1). At the same instant, his colleague J. F. Richer in French Guyana also measured the position of Mars on the celestial sphere. From the corresponding parallax of 18″ and the known distance between Paris and French Guyana it was found by standard trigonometric methods that Mars is 73 million km far-away from the Earth (this result was obtained, in fact, in French miles). Then Kepler's third law was applied

$$\frac{T_i^2}{T_j^2} = \frac{a_i^3}{a_j^3}, \quad i,j = 1,2,3,\ldots, \tag{1}$$

where T_i is the sidereal period of ith planet and a_i is the length of the semimajor axis of its elliptical orbit. For the Earth and Mars we have $T_3 = 1$ and $T_4 = 1.88$ years. Hence, $a_4 = (1.88)^{2/3}a_3$. The second equation for the unknowns a_3 and a_4 comes from the fact that planetary orbits are almost circular and the above-mentioned angular measurement, i.e., $a_4 - a_3 = 73 \cdot 10^6$ km. From this we get $a_3 \doteq 140 \cdot 10^6$ km which is a quite good estimate of the modern value $a_3 = 149.6 \cdot 10^6$ km. The distances a_i of all the other known planets were then calculated from Kepler's third law (1) and the observed periods T_i.

The distances a_1 and a_2 of the inner planets were also estimated by the relation $a_i = a_3 \sin \alpha_i$, where α_i is the maximum angle of elongation (which is about 28° for Mercury and 47° for Venus). For instance, Nicholas Copernicus established that the radius of Venus' orbit is about 72% that of the Earth's by measuring the maximum angle of separation from the Sun (see [10], p. 39 and 44).

4 Further improvement of accuracy

Another ingenious geometric method (see [11, 12]) was suggested by the famous astronomer E. Halley (1656–1742). He developed the idea of using transits of Venus from different places of known latitude into a practical method, which was applied later in 1769 (also in 1761) when the planet Venus was passing over the Sun's disc. This phenomenon is very rare, since it happens only several times per millennium (recently in June 8, 2004). According to [10], p. 133, more than 120 astronomers made observations from about 60 stations. In particular, one group of astronomers headed by Maximilian Hell was at the island Vardø (at present in Norway) and another one with Captain James Cook and Charles Green travelled to Tahiti (see [1], p. 267). In Figure 4 we see a sketch of two trajectories *AB* and *CD* of Venus observed

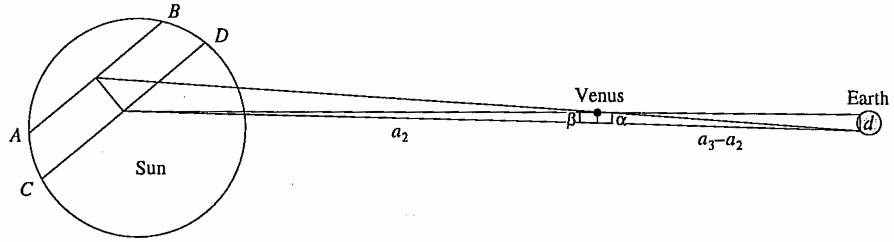

Fig. 4: Schematic illustration of two different trajectories AB and CD of the Venus passing over the Sun's disc observed from Vardø and Tahiti in 1769. The real angular distance between AB and CD is much smaller than in this figure.

from these two places. The angular distance between the two line segments AB and CD was found to be approximately $\alpha = 40''$. Note that the angular diameter of the Sun is about $32'$, i.e., it is almost fifty times larger.

From (1) and the fact that $T_2 = 0.615$ years we get $a_2 = 0.723 a_3$. Moreover, from Figure 4 we find that $a_2 \tan\beta = (a_3 - a_2)\tan\alpha$. Since the Earth-Sun line was perpendicular to the segment Vardø-Tahiti at certain instant during the time of the transit, we obtain

$$a_3 = \frac{d}{\tan\beta} = \frac{a_2}{a_3 - a_2}\cdot\frac{d}{\tan\alpha} = \frac{0.723\,d}{(1-0.723)\tan\alpha},$$

where $d = 11425$ km is the distance between Vardø and Tahiti (it can be calculated from the latitudes and longitudes of these two places by means of ellipsoidal coordinates). In this way the calculation of the distance a_3 between the Earth and Sun was improved to $153\cdot 10^6$ km.

There exist, of course, more sophisticated methods (see [10]) which take into account the motion of the Earth and Venus during the time of the transit and other circumstances.

5 Measurement of mass density and mass

A great breakthrough in understanding the behavior of the solar system came from Newton's laws. First, we show how to calculate the mean mass density of the Sun by means of Newton's laws and the protractor.

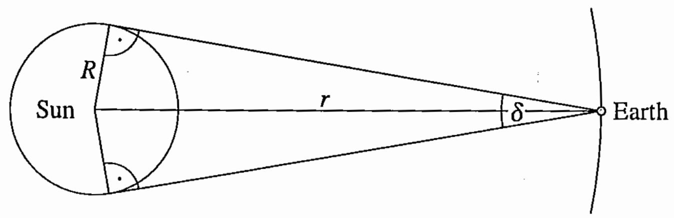

Fig. 5: The mean mass density of the Sun can be determined from its viewing angle δ and the Earth's orbital period (see (3)).

To solve this seemingly absurd problem assume for simplicity that the Earth's orbit is circular. By Newton's law of gravitation, the second and third law (of action and reaction), we get

$$G\frac{Mm}{r^2} = \frac{mv^2}{r}, \qquad (2)$$

where M is the mass of the Sun, m is the mass of the Earth, r is their mutual distance, v is the speed of the Earth, and $G = 6.67\cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ is the gravitational constant (whose approximate value was found in 1798 by H. Cavendish by means of a precise torsion balance and large lead spheres). Using the protractor, we can find that the viewing angle of the Sun is $\delta = 32'$. Then $R = r\sin\frac{1}{2}\delta$ is the radius of the Sun (see Figure 5). Clearly, $v = 2\pi r/T$, where $T = 31\,558\,149.5$ s (=365.25636 days), is the Earth's orbital period (sidereal year). Denoting by V the volume of the Sun, we get by (2) that the mean mass density is

$$\rho = \frac{M}{V} = \frac{v^2 r}{GV} = \frac{(2\pi r)^2\cdot r}{T^2 G\cdot\frac{4}{3}\pi(r\sin\frac{1}{2}\delta)^3} = \frac{3\pi}{T^2 G\sin^3\frac{1}{2}\delta} = 1409 \quad [\mathrm{kg/m^3}], \qquad (3)$$

i.e., slightly higher than the density of water.

The total mass of the Sun can be now calculated as follows. From the real mean distance $r = 149.6\cdot 10^9$ m and the measured angle $\delta/2$, we find the radius R and volume V of the Sun. Then we get $M = \rho V = 1.99\cdot 10^{30}$ kg.

6 Slowing-down of Earth's rotation

During the last 2700 years the Earth's rotation slowed-down so that the length of a day increased by $1.7\cdot 10^{-3}$ s per century (see [13], p. 270). This value has been obtained by a thorough data analysis of ancient Babylonian records of angular measurement of solar eclipses. We restrict ourselves only to a simple example to illustrate it.

According to [14], p. 340, the Babylonians observed an entirely total eclipse of the Sun on April 15, 136 BC.[3] At that time, a day was about $\tau = 0.036312$ seconds ($\doteq 21.42$ century $\times$ 1.7 ms/century) shorter than in 2006. This period contains approximately $N = 783\,000$ days. Due to cumulative effects the rotation of the Earth was slowed down about 4 hours more than if it would have rotated uniformly. This corresponds to an angle of $60°$ $(= 360° \cdot 4/24)$. To check this, assume that the length of the ith day increased linearly about the value

$$\Delta t_i = \tau \frac{i}{N}, \quad i = 1, \ldots, N.$$

Thus, the entire growth during N days is

$$\Delta T = \sum_{i=1}^{N} \Delta t_i = \tau \frac{N+1}{2} = 14216\,[\mathrm{s}] \doteq 4\,[\mathrm{hours}]. \tag{4}$$

The value τ in [13] was, in fact, calculated by the reverse procedure. If the Earth's rotation would be constant, then the ancient Babylonians could not observe the total eclipse at the place where they actually describe it, but 4 time zones shifted to the west of Babylon. Now we can exactly establish their local time during the eclipse from the height of the Sun over the horizon, which was measured by protractor and carefully recorded. From the shift $\Delta T = 4$ hours and the known number of days N we can calculate the corresponding τ due to (4) and thus also the reduction of the Earth's rotation $\tau/21.36 = 1.7$ ms per century. Late Babylonian astronomical tablet containing a record of the total solar eclipse of 15 April in 136 BC is preserved in the British Museum (see [15]).

7 Annual parallax of the nearest stars

The Earth's orbital motion around the Sun causes stars to circumscribe very small ellipses on the celestial sphere. This enables us to find distances of the nearest stars by means of the measurement of the so-called annual parallax.

Let C stand for a relatively nearby star. Suppose for simplicity that the Earth's orbit is circular and let r be its radius. Then there exist two opposite points A and

[3] This date was derived with respect to the present calendar.

B on the Earth's orbit such that the triangle ABC is isosceles with the base AB (see Figure 6). Then the distance from C to AB is

$$d = \frac{r}{\tan\gamma},$$

where one half of the angle ACB is called the *annual parallax* γ.

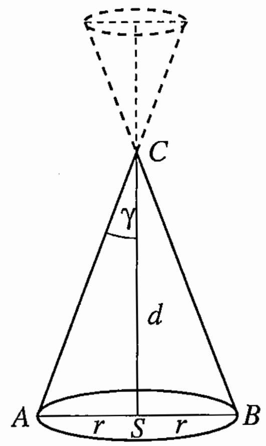

Fig. 6: The distance d of a close star at the point C can be determined from the annual parallax γ and the radius r of the Earth's orbit.

The first measurements of annual parallaxes of the nearest stars were carried out by F. W. Bessel in 1838. At present we know that the nearest star is Proxima Centauri. Its annual parallax is $0.76''$ and the corresponding distance about $d = 4 \cdot 10^{13}$ km ($\doteq 4.22$ light years).

8. Deflection of light. In 1911 A. Einstein in his pioneering paper [2] derived that light (which "traveles" on shortest joins) deviates from rectilinear motion near massive objects due to gravitation (cf. Figure 7). This effect was first photographed during the total eclipse in 1919, when the light rays of stars in a close neighbourhood of the Sun's disc were deflected from the original direction. By comparison of this photo with pictures of the same part of the celestial sphere, it was found in good agreement with the value $1.75''$ predicted by Einstein. In this way the angular measurements of the deflection contributed to confirm the validity of Einstein's general theory of relativity.

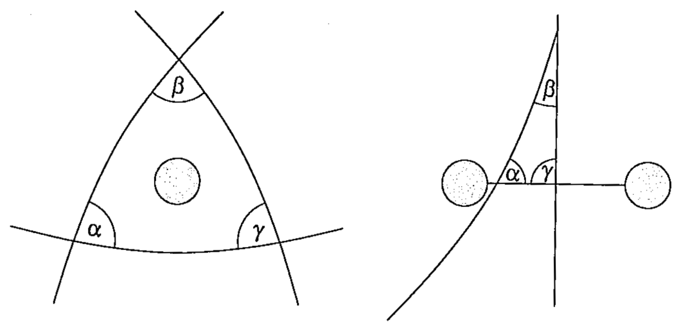

Fig. 8: Bended trajectories of light near massive objects show that the geometry of the universe can be locally a) Riemannian and also b) Lobachevskian.

Fig. 7: The discovery of deflection of light near the Sun is illustrated on Einstein's plaque in the Old Town Square of Prague. The curved ray of light is slightly above the sketch of the Charles bridge (bottom right).

Each mass object thus makes the universe locally curved and this causes a deflection of light (and is also the basis of famous gravitational lenses). In Figure 8, we observe two examples of bending of light in a close neighbourhood of stars. The three trajectories in Figure 8a) form a curved triangle. Notice that the sum of its angles satisfies the inequality

$$\alpha+\beta+\gamma > 180°,$$

which reminds one of a famous assertion from Riemannian geometry. On the other hand, in Figure 8b) we see two stars of equal masses and three trajectories forming another curved triangle with $\alpha+\beta+\gamma < 180°$. This case is reminiscent of Lobachevskian geometry.

These two examples show that the universe locally has different kinds of geometries with various curvatures. However, to find a "global curvature" of the universe for a fixed time, we have to consider very large scales, on which all local curvatures are averaged. This is like Earth's surface, whose curvature locally changes very much (due to mountains, valleys, saddle-points, etc.), but whose global curvature is positive, and almost constant. According to Einstein's cosmological principle, our

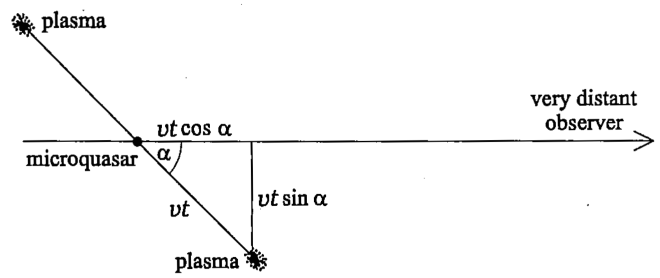

Fig. 9: The observed time period t^* of the phenomenon is shorter that the real time period t, since light has to cover the distance $vt\cos\alpha$.

universe on each isochrone is homogeneous and isotropic on large scales, i.e., its curvature is constant at any point and in any direction. This assumption has been confirmed by astronomers (e.g. from pictures of the Hubble Deep Field and Hubble Deep Field-South, from the homogenity and isotropy of the cosmic microwave background radiation and γ-ray bursts). It yields a very restrictive conditions on the global topology of the universe (see [16]).

8 Observation of superluminal speeds

According to Einstein's theory of relativity, no mass can move faster than the speed of light $c = 299\,792$ km/s in vacuum. However, astronomers observe by angular measurements hundreds of superluminal plasma jets in the whole universe. For instance, by [17], one of the two jets ejected from the microquasar GRS1915+105 that is in our Galaxy seemingly traveled $s^* = 6250$ AU (1 AU= 149 597 870 km) during $t^* = 29$ days in 1994. The value 6250 AU has been obtained by protractor and from the known distance (40 000 light years). The associated illusory speed of the jet

$$v^* = \frac{s^*}{t^*} = \frac{6250 \cdot 149\,597\,870}{29 \cdot 24 \cdot 3600} \doteq 373\,159\text{km/s} \tag{5}$$

is thus greater than c.

To explain this paradox, consider the situation of Figure 9. For simplicity, assume that the microquasar does not move and that the real speed v of its jets is constant. Let $\alpha \leq 90°$ be the angle between the line microquasar – observer and the line approximating the jets (see Figure 9). Then $v\cos\alpha$ ($v\sin\alpha$) is the radial (tangential) component of v with respect to the observer. During the time period t each jet covers the distance vt from the microquasar.

For $\alpha < 90°$ one jet moves towards the observer and the second one away. Therefore, the illusory observed time period t^* is shorter than the real time period t, and thus,

$$t^* = t - \frac{v}{c} t \cos\alpha, \tag{6}$$

where $(vt \cos\alpha)/c$ is the time interval that is necessary for light to cover the distance $vt \cos\alpha$. This time interval plays a key role in the paradox, since the illusory speed by (6) is then

$$v^* = \frac{s^*}{t^*} = \frac{vt \sin\alpha}{t - \frac{v}{c} t \cos\alpha} = \frac{v \sin\alpha}{1 - \frac{v}{c} \cos\alpha}, \tag{7}$$

which can be easily greater than c. For instance, by [17], $\alpha = 71°$ and $v = 0.92c$. Substituting this into (7), we get $v^* = 1.24c$, which is in good agreement with (5). The plasma jet travelled $t = 41.4$ days and the light covered the distance $vt \cos\alpha$ in 12.4 days. Therefore, the time period of the phenomenon observed from the Earth was only $t^* = 41.4 - 12.4 = 29$ days.

By a suitable choice of the angle α and the real speed $v < c$, we can obtain an arbitrarily large illusory speed v^* in (7). For example, taking $\alpha = 8°$ and $v = 0.99c$, we get $v^* = 7c$.

The intrinsic expansion of the universe, moreover, magnifies the velocities of jets in galaxies at cosmological distances. The younger the objects which are observed, the larger the magnification effect appears. We call (see [3]) this effect the *time-lens principle*. An example of an enormous magnification due to the time-lens principle is the Big Bang itself, which appeared $13.7 \cdot 10^9$ years ago. Although it probably happened in a minimal volume, its current position is on the possibly greatest sphere (the so-called horizon) with a very large radius. Now the Big Bang is, in fact, on the entire horizon, i.e., $13.7 \cdot 10^9$ light years away in every direction.

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