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ABSTRACT CONVEX STRUCTURES IN TOPOLOGY
AND SET THEORY

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Introduction

Abstract convexity theory is a branch of mathematics dealing with set-theoretic structures satisfying axioms similar to that usual convex sets fulfill. Here, by "usual convex sets" we mean convex sets in real linear spaces. First papers on axiomatic convexities come from early fifties, a complete bibliography can be found in a book of V.P. Soltan [**40**] and in a more recent monograph of M. van de Vel [**49**]. In a general setting, the axioms of convexity are the following:

(1) The empty set and the whole space are convex;
(2) The intersection of a nonempty collection of convex sets is convex;
(3) The union of a chain of convex sets is convex.

Clearly, usual convex sets have properties (1)-(3) but there are many other collections of sets, coming from various types of mathematical objects, that satisfy conditions (1)-(3). Probably the most interesting are: convexities in lattices and in Boolean algebras (see Varlet [**46**] or van de Vel [**47**]) convexities in metric spaces and graphs (see e.g. Menger [**30**], Lassak [**28**] and Soltan [**39**]). Also, convex structures appeared naturally in topology, especially in the theory of supercompact spaces (see e.g. van Mill [**31**]). Combinatorial properties of convexities (such as relations between some natural invariants) were investigated first by Levi [**29**]. There are also considered morphisms between convexities, called *convexity preserving mappings*. This notion allows us to compare different convexities and to define isomorphisms, embeddings, products, etc.

The dissertation deals with various properties related to separation and extending of convexity preserving maps. By "separation" we mean the property that certain two maps from fixed two classes can be separated (by means of a suitable partial order) by a third map which belongs to both classes simultaneously. It appears that complete Boolean algebras (as convexity spaces) play a fundamental role in developing these properties. We also investigate topological convexity spaces, specifically *compact median spaces*. We are interested in a problem of continuous convexity preserving extending maps. Our results are applied for finding some linearly ordered subspaces of compact median spaces.

Chapter 1 collects definitions and some basic facts from abstract convexity theory which will be needed later. All results stated there are known and can be found in the first two chapters of van de Vel's book [**49**]. One can find there also detailed references.

In Chapter 2 we characterize the Kakutani separation property (the property of separating convex sets by half-spaces) in the language of pairs of convexities on a fixed set. The results are related to a paper of Ellis [**7**], where a characterization of the Kakutani property for pairs of join-hull commutative convexities is given, and to a paper of Chepoi [**4**], where the Kakutani property for single convexities is discussed. In fact we generalize both the result of Ellis and that of Chepoi.

Chapter 3 contains a "sandwich" type theorem for maps defined on bi-convexity spaces (sets with two convexities) with values in complete Boolean algebras. The main result says that certain two maps can be separated (in terms of the partial order on a Boolean algebra) by a convexity preserving map. Identifying sets with their characteristic functions, we can look at this as a far generalization of the Kakutani property. As an application we obtain some extension theorems for convexity preserving maps and a theorem on separating of meet and join homomorphisms of lattices.

In Chapter 4 we discuss the problem of convexity preserving extending maps. We state a short proof of an extension theorem for maps with values into complete Boolean algebras. As a corollary we get the extension theorem of Sikorski for homomorphisms of Boolean algebras. Finally we characterize complete Boolean algebras as the only S_3 point-convex convexity spaces satisfying the mentioned extension theorem.

Chapters 5 and 6 deal with topological convexity spaces (i.e. topological spaces endowed with a convexity) and, in particular, with compact median spaces. It appears that this class plays a similar role in convexity to the role of compact Hausdorff spaces in topology. A structure of a compact median space in a topological space is induced by a normal binary T_1-subbase for closed sets (such a topological space is called *normally supercompact*, see [**31**]). In Chapter 5 we state a criterion for extending continuous convexity preserving maps with values into a compact median space. This result can be viewed as a convex version of Taĭmanov's theorem [**45**] in topology. We apply our criterion for obtaining a known extension theorem due to Verbeek and van Mill, van de Vel in the theory of supercompactifications. Another application is the extension criterion of Sikorski for maps of Boolean algebras (finite Boolean algebra with discrete topology is a compact median space). We also use our extension criterion for investigating linearly ordered subspaces of compact median spaces.

Chapter 6 contains a joint work with A. Kucharski. The main result is that every topological retract of a Cantor cube has a convexity structure that makes it a compact median space. As a consequence, a topological retract of a Cantor cube has a binary subbase which is closed under the complements. This is a strengthening of a result of Heindorf [**13**] who proved that every such a space has a binary subbase consisting of closed-open sets. Our proof is simpler than Heindorf's one and does not require any algebraic or lattice structures.

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Contents

Notation and terminology

We shall use the standard notation and terminology of set theory and topology. We treat ordinal numbers as sets of ordinals, e.g. $0 = \emptyset$, $1 = \{0\}$ and so on. The first infinite cardinal number is the set of natural numbers $\omega = \{0, 1, 2, \dots\}$. The cardinality (or the size) of a set A will be denoted by $|A|$. We use the notation $[X]^{\kappa} = \{A \subset X : |A| = \kappa\}$ and related symbols $[X]^{\leqslant \kappa}, [X]^{< \kappa}$ with obvious meaning. If $f \colon A \to B$ is a function then *the domain of f* is $\mathrm{dom}(f) = A$. If $A' \subset A$ then $f|A'$ denotes *the restriction of f to A'*, i.e. $\mathrm{dom}(f|A') = A'$ and $(f|A')(x) = f(x)$ for $x \in A'$. For a subset A of a fixed set X we denote by χ_A *the characteristic function of A*, i.e. $\chi_A \colon X \to \{0, 1\}$ is defined by $\chi_A(x) = 1$ iff $x \in A$. Given two mappings $f \colon A \to B$ and $g \colon B \to C$ we denote by gf *the composition of f* and g. Specifically $gf \colon A \to C$ is defined by $(gf)(x) = g(f(x))$ for $x \in A$.

Let $(P, \leqslant)$ be a partially ordered set. A subset $A \subset P$ is called *up-directed* if $a, b \in A$ implies that there exists $c \in A$ with $a \leqslant c$ and $b \leqslant c$. A *chain* is a linearly ordered subset of P. A subset $A \subset P$ is *order-convex* provided $a, b \in A$ and $a \leqslant x \leqslant b$ imply $x \in A$. We shall use these notions for $P = \mathcal{P}(X)$ (the power set of X) with $\subset$ as the partial order.

By a lattice we mean a partially ordered set $(L, \leqslant)$ such that each two elements $a, b \in L$ have the supremum (the least upper bound) $a \vee b$ and the infimum (the greatest lower bound) $a \wedge b$. The operations $\wedge$ and $\vee$ are called *meet* and *join* respectively. The supremum (infimum) of a subset $A \subset L$ (if it exists) will be denoted by $\sup A$ ($\inf A$). A lattice L is *distributive* provided $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every $a, b, c \in L$. *A lattice of sets* is a collection of sets $\mathcal{L}$ which forms a lattice with respect to the inclusion and $a \wedge b = a \cap b$, $a \vee b = a \cup b$ for all $a, b \in \mathcal{L}$. By Stone's representation theorem, every distributive lattice is isomorphic to a lattice of sets. Lattices can be defined in terms of their algebraic operations $\wedge, \vee$; then the partial order is defined by $x \leqslant y$ iff $x \wedge y = x$ (or equivalently $x \vee y = y$). Thus one can treat lattices as triples $(L, \wedge, \vee)$ where $\wedge$ and $\vee$ are two binary operations satisfying suitable conditions. If K, L are two lattices then a map $f \colon K \to L$ is *a meet homomorphism* provided $f(a \wedge b) = f(a) \wedge f(b)$ for every $a, b \in K$. Similarly we define the notion of *a join homomorphism*. The map f is *a lattice homomorphism* if it is simultaneously a meet and join homomorphism. For the details on lattice theory we refer to the book of Grätzer [**10**].

Let L be a lattice. A subset $A \subset L$ is *a filter* if $a, b \in A$ and $x \geqslant a \wedge b$ imply $x \in A$. Dually (by replacing $\wedge$ with $\vee$ and $\geqslant$ with $\leqslant$) we define the notion of *an ideal*. A filter (ideal) is *prime* if its complement is an ideal (a filter). *An ultrafilter* is a maximal (with respect to the inclusion) filter not equal to L. By *an ultrafilter on a set X* is meant an ultrafilter in the lattice of its all subsets $\mathcal{P}(X)$.

A Boolean algebra is defined as a bounded complemented distributive lattice $\mathbb{B}$, i.e. a distributive lattice $\mathbb{B}$ which has the least element $0_{\mathbb{B}}$ and the greatest element $1_{\mathbb{B}}$ and for each $x \in \mathbb{B}$ there exists $y \in \mathbb{B}$ with $x \wedge y = 0_{\mathbb{B}}$ and $x \vee y = 1_{\mathbb{B}}$; the element y is unique and is called *the complement of x* and denoted by $\neg x$. This allows us to define *the symmetrical difference*

of $a, b \in \mathbb{B}$ as $x \div y = (x \wedge \neg y) \vee (\neg x \wedge y)$. It is well known that $(\mathbb{B}, \div, \wedge, 0_\mathbb{B}, 1_\mathbb{B})$ is a ring. Conversely, if $(R, +, \cdot, 0, 1)$ is a ring satisfying the equation $x^2 = x$ for all $x \in R$ then R is a Boolean algebra with the operations $a \vee b = a + b + ab$ and $a \wedge b = ab$. In a Boolean algebra, every prime filter is an ultrafilter.

Let R be a ring. *A module over R (an R-module)* is an Abelian group $(M, +)$ together with an operation $R \times M \ni (r, x) \mapsto rx \in M$ satisfying the conditions $1x = x$, $(rs)x = r(sx)$, $(r + s)x = (rx) + (sx)$ and $r(x + y) = (rx) + (ry)$ for all $r, s \in R$, $x, y \in M$. In the case when R is a field, an R-module is called *a vector space over R*.

We denote by $\mathbb{Q}$ and $\mathbb{R}$ the set of all rational and real numbers respectively.

CHAPTER 1

Preliminaries

We collect some preliminary and basic facts from convexity theory which will be used later. Most of them can be found in van de Vel's book [**49**]. For completeness, we present all the proofs.

1. Definitions

Following van de Vel's monograph [**49**], by *a convexity* on a set X we mean a collection $\mathcal{G} \subset \mathcal{P}(X)$ satisfying the conditions

(1) $\emptyset, X \in \mathcal{G}$;
(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for nonempty $\mathcal{A} \subset \mathcal{G}$;
(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$ is a chain with respect to the inclusion.

Members of $\mathcal{G}$ are called *convex sets* and the pair $(X, \mathcal{G})$ is called *a convexity space*. Also, for simplicity, the set X alone will be called the convexity space. *A half-space* is a convex set with the convex complement. The complements of convex sets are called *concave. The convex hull* of a set $A \subset X$ is

$$\operatorname{conv} A = \bigcap \{G \in \mathcal{G} : \ A \subset G\}.$$

The convex hull of a finite set $\{x_1, \ldots, x_n\}$ is called an *n-polytope* and is denoted by $[x_1, \ldots, x_n]$. A 2-polytope $[a, b]$ is called *the segment joining* a and b. Denoting a convex hull or a segment we shall use sometimes the letter $\mathcal{G}$ as a subscript, to avoid any misunderstandings about the convexity under consideration. We shall also consider *bi-convexity spaces*, i.e. triples of the form $(X, \mathcal{L}, \mathcal{U})$ where $\mathcal{L}, \mathcal{U}$ are two convexities on a set X, called *the lower* and *the upper* convexity. The adjectives "lower" and "upper" will be used also for all notions connected with these convexities. The lower/upper convex hull will be denoted also by $\operatorname{conv}_L / \operatorname{conv}_U$.

PROPOSITION 1.1. *For every subset A of a convexity space X the following holds:*

$$\operatorname{conv} A = \bigcup_{F \in [A]^{<\omega}} \operatorname{conv} F.$$

Consequently, the union of an up-directed collection of convex sets is convex.

PROOF. We use induction on $|A|$. The statement is clear when $|A| < \omega$. Suppose $|A| = \kappa \geqslant \omega$ and let $A = \bigcup_{\xi < \kappa} A_\xi$, where $A_\xi \subset A_\eta$ for $\xi < \eta < \kappa$ and $|A_\xi| < \kappa$ for all $\xi < \kappa$. Set $B = \bigcup_{\xi < \kappa} \operatorname{conv} A_\xi$. Then B is convex, being the union of a chain of convex sets. By induction hypothesis

$$B = \bigcup_{\xi < \kappa} \bigcup_{F \in [A_\xi]^{<\omega}} \operatorname{conv} F = \bigcup_{F \in [A]^{<\omega}} \operatorname{conv} F$$

and $A \subset B \subset \operatorname{conv} A$, hence $B = \operatorname{conv} A$.

If $\mathcal{A}$ is an up-directed family of convex sets then for a finite set $F \subset \bigcup \mathcal{A}$ we have $F \subset A$ for some $A \in \mathcal{A}$ and consequently $\operatorname{conv} F \subset \bigcup \mathcal{A}$. Thus $\bigcup \mathcal{A}$ is convex. $\qquad \square$

An easy application of the Kuratowski-Zorn Lemma gives

PROPOSITION 1.2. *If A, B are two disjoint sets and A is convex then there exists a maximal, with respect to the inclusion, convex set G with $A \subset G$ and $G \cap B = \emptyset$.*

Let $N \in \omega$. A convexity $\mathcal{G}$ is called *N-ary* (or *of arity N*) if $A \in \mathcal{G}$ whenever $\operatorname{conv} F \subset A$ for all $F \in [A]^{\leqslant N}$. A 2-ary convexity is called *an interval convexity* (see Calder [**3**]), a space with 2-ary convexity will be called briefly *geometrical*.

A convexity space X is called *join-hull commutative* [**19**] (*JHC* for short) provided for each finite set $F \subset X$ and for each $x \in X$ we have

$$\operatorname{conv}(F \cup \{x\}) = \bigcup_{y \in \operatorname{conv} F} [x, y].$$

By Proposition 1.1 this is equivalent to the fact that for every convex set $G \subset X$ and for every $x \in X$ we have

$$\operatorname{conv}(G \cup \{x\}) = \bigcup_{g \in G} [g, x].$$

PROPOSITION 1.3 (Kay, Womble [**19**]). *Every join-hull commutative convexity space is geometrical.*

PROOF. Suppose that X is JHC and $A \subset X$ is such that $[x, y] \subset A$ for all $x, y \in A$. By induction we show that for each $n \in \omega$ and for every $x_1, \ldots, x_n \in A$ it holds $[x_1, \ldots, x_n] \subset A$. Indeed, if this is true for some n then for $x_1, \ldots, x_{n+1} \in A$ we have

$$[x_1, \ldots, x_{n+1}] = \bigcup_{y \in [x_1, \ldots, x_n]} [y, x_{n+1}] \subset A.$$

Now, by Proposition 1.1, the set A is convex. $\qquad \square$

2. Finitary set operators

We shall describe some natural ways of defining a convexity on a set.

Let X be a set and let $D = [X]^{\leqslant N}$ or $D = [X]^{<\omega}$. A map $r \colon D \to \mathcal{P}(X)$ such that $F \subset r(F)$ for all $F \in D$ will be called *a finitary set operator* (*FS-operator* for short) on X. An FS-operator $r \colon D \to \mathcal{P}(X)$ is called *N-ary* provided $D \subset [X]^{\leqslant N}$. We say that a set $G \subset X$ is *r-convex* if for every $F \in D \cap \mathcal{P}(G)$ it holds $r(F) \subset G$. Denote by $\mathcal{G}_r$ the collection of all r-convex subsets of X. Clearly, this is a convexity on X, $\mathcal{G}_r$ is N-ary if r is N-ary. We shall say that a convexity $\mathcal{G}$ is *defined by r* if $\mathcal{G} = \mathcal{G}_r$. By Proposition 1.1, every convexity is defined by its *polytope operator* $I = \operatorname{conv} |[X]^{<\omega}$. An N-ary convexity is defined by its *N-polytope operator* $I_N = \operatorname{conv} |[X]^{\leqslant N}$.

A set operator r is *transitive* provided for every $y \in r(\{x\} \cup F)$ it holds $r(\{y\} \cup F) \subset r(\{x\} \cup F)$. A transitive 2-ary operator is called *an interval operator* (see [**4**]).

Let r be an FS-operator. We set

$$rA = \bigcup \{r(F) : \; F \in \operatorname{dom}(r) \cap \mathcal{P}(A)\},$$

$$r^{n+1}A = r(r^n A) \quad \text{and} \quad r^0 A = A.$$

We shall write $r(x_1, \ldots, x_N)$ instead of $r(\{x_1, \ldots, x_N\})$.

PROPOSITION 2.1. *If r is a finitary set operator on X then for every $A \subset X$ it holds*

$$\mathrm{conv}\, A = \bigcup_{n \in \omega} r^n A,$$

where conv *is the convex hull operator associated with the convexity $\mathcal{G}_r$.*

PROOF. Let $D = \mathrm{dom}(r)$. Denote by B the set on the right hand side. Clearly $A \subset B$ and $B \subset \mathrm{conv}\, A$. Let $F = \{a_1, \ldots, a_k\} \in D \cap \mathcal{P}(B)$ and $a_i \in r^{n_i} A$. If we set $n = \max\{n_1, \ldots, n_k\}$ then $F \subset r^n A$ and thus $r(F) \subset r^{n+1} A \subset B$. Hence B is r-convex. $\square$

Let X be a geometrical space. We define a relation $\mathrm{B} \subset X \times X \times X$ by the formula

$$\mathrm{B}(a, c, b) \iff c \in [a, b].$$

Observe that B has the following properties:

(B1) $\mathrm{B}(a, a, b)$ & $\mathrm{B}(a, b, b)$,
(B2) $\mathrm{B}(a, x, b)$ & $\mathrm{B}(a, y, b)$ & $\mathrm{B}(x, z, y) \implies \mathrm{B}(a, z, b)$.

Every ternary relation B satisfying (B1), (B2) will be called *a betweenness relation*, the relation B defined above will be referred to as *the natural betweenness relation* in a geometrical space X (see [**49**, p. 32] for a more general case).

PROPOSITION 2.2. *For every betweenness relation B on a set X there exists a unique interval convexity $\mathcal{G}$ such that $\mathrm{B}(a, c, b) \iff c \in [a, b]_{\mathcal{G}}$ for $a, b, c \in X$.*

PROOF. Conditions (B1), (B2) imply that $\mathrm{B}(a, c, b) \iff \mathrm{B}(b, c, a)$. Indeed, if $\mathrm{B}(a, c, b)$ then also $\mathrm{B}(b, a, a)$ and $\mathrm{B}(b, b, a)$ whence by (B2) we get $\mathrm{B}(b, c, a)$. Thus we can define a 2-ary set operator

$$r(\{a, b\}) = \{x \in X : \mathrm{B}(a, x, b)\}.$$

Let $\mathcal{G} = \mathcal{G}_r$. By Proposition 2.1 and (B2) we have $r(a, b) = [a, b]_{\mathcal{G}}$. Hence B is the natural betweenness relation in $(X, \mathcal{G})$. The uniqueness of $\mathcal{G}$ follows from the fact that every interval convexity is determined by its segments. $\square$

The theory of betweenness relations (ternary relations satisfying similar axioms to (B1), (B2)) was studied by several authors, see e.g. Moszyńska [**35**], Szmielew [**43**] or Cibulskis [**5**].

3. Subbases and subspaces

We describe a useful way of defining a convexity by means of subbases. Observe that the intersection of any collection of convexities on a given set X is also a convexity. The same applies to interval convexities or, more generally, to N-ary convexities. Moreover, $\mathcal{P}(X)$ is the largest convexity on X, called *the discrete convexity*. It follows that for every collection of sets $\mathcal{B} \subset \mathcal{P}(X)$ there exists the smallest convexity $\mathcal{G}$ such that $\mathcal{B} \subset \mathcal{G}$. We shall say then that $\mathcal{G}$ is *generated by $\mathcal{B}$* and $\mathcal{B}$ is *a subbase for $\mathcal{B}$*, see [**49**, p. 10]. Similarly, there exists an interval convexity $\mathcal{G}'$ generated by $\mathcal{B}$, clearly $\mathcal{G} \subset \mathcal{G}'$ and $\mathcal{G} \neq \mathcal{G}'$ unless $\mathcal{G}$ is 2-ary.

PROPOSITION 3.1. *Let $\mathcal{G}$ be the convexity generated by a collection $\mathcal{B} \subset \mathcal{P}(X)$. Then for every finite set $F \subset X$ the following holds:*

$$\mathrm{conv}_{\mathcal{G}}\, F = \bigcap \{B \in \mathcal{B} \cup \{X\} : F \subset B\}.$$

PROOF. Denote by $r(F)$ the right-hand side of the above equality. Then r is an FS-operator and $r^2 F = rF$. In view of Proposition 2.1, we have $r(F) = \operatorname{conv}_r F$ for finite $F \subset X$. As $\mathcal{B}$ consists of r-convex sets, we get $\mathcal{G} \subset \mathcal{G}_r$. On the other hand $r(F) \in \mathcal{G}$ for every $F \subset X$. It follows that $\operatorname{conv}_{\mathcal{G}} F \subset \operatorname{conv}_r F$ for $F \in [X]^{<\omega}$ and hence $\mathcal{G}_r \subset \mathcal{G}$. Thus $\mathcal{G} = \mathcal{G}_r$. $\square$

COROLLARY 3.2. *A collection $\mathcal{B}$ of convex sets in a convexity space $(X, \mathcal{G})$ is a subbase for $\mathcal{G}$ iff for every finite set $F \subset X$ and $x \notin \operatorname{conv} F$ there exists $B \in \mathcal{B}$ with $x \notin B$ and $F \subset B$.*

Similar statements are valid for interval convexities but then "finite set" must be replaced with "at most 2-element set".

Let M be a subset of a given convexity space $(X, \mathcal{G})$. Define

$$\mathcal{G}_M = \{G \cap M : G \in \mathcal{G}\}.$$

Clearly, $\mathcal{G}_M$ is stable under intersections and $\emptyset, M \in \mathcal{G}_M$. If $\{G_\alpha \cap M\}_{\alpha \in \Gamma}$ is a chain in $\mathcal{G}_M$ then setting $G'_\alpha = \operatorname{conv}(G_\alpha \cap M)$ we see that $\{G'_\alpha\}_{\alpha \in \Gamma}$ is a chain in $\mathcal{G}$ and $G'_\alpha \cap M = G_\alpha \cap M$. It follows that $\bigcup_{\alpha \in \Gamma} G_\alpha \cap M \in \mathcal{G}_M$. Thus $\mathcal{G}_M$ is a convexity on M, called *the subspace convexity* (see [**49**, p. 13]); $(M, \mathcal{G}_M)$ is called *a subspace of $(X, \mathcal{G})$*. This is an analogy to the notion of a subspace in topology. Observe that $\operatorname{conv}_M A = M \cap \operatorname{conv} A$ for $A \subset M$, where conv_M denotes the convex hull with respect to $\mathcal{G}_M$. If $(X, \mathcal{G})$ is geometrical then we may also consider an interval convexity $\mathcal{G}'_M$ on M generated by $\mathcal{G}_M$. In this case $(M, \mathcal{G}'_M)$ is called *a geometrical subspace of $(X, \mathcal{G})$*. Observe that the betweenness relation on M is just the restriction to $M \times M \times M$ of the betweenness relation of X.

4. Convexity spaces as a category

The class of convexity spaces can be viewed as a category with convexity preserving mappings as morphisms. A map of convexity spaces $f \colon X \to Y$ is *convexity preserving* (briefly *CP*) [**49**, p. 15] provided $f^{-1}(G)$ is convex in X whenever G is convex in Y. Similar notions are defined for maps of bi-convexity spaces, using the adjectives "lower" and "upper" and the abbreviations *LCP, UCP*.

PROPOSITION 4.1 ([**49**, Prop. I.1.12(1)]). *A map of convexity spaces $f \colon X \to Y$ is convexity preserving iff $f(\operatorname{conv} S) \subset \operatorname{conv} f(S)$ for every finite set $S \subset X$; if X is of arity N then it is enough to assume $|S| \leqslant N$.*

PROOF. Suppose first that f is CP and fix $S \in [X]^{<\omega}$. Then $S \subset f^{-1}(\operatorname{conv} f(S))$ so $\operatorname{conv} S \subset f^{-1}(\operatorname{conv} f(S))$. Hence $f(\operatorname{conv} S) \subset \operatorname{conv} f(S)$.

Suppose now that f satisfies the above condition and fix a convex set $G \subset Y$. If $S \subset f^{-1}(G)$ is finite then $f(\operatorname{conv} S) \subset \operatorname{conv} f(S) \subset G$. Thus $\operatorname{conv} S \subset f^{-1}(G)$. It follows that $f^{-1}(G)$ is convex. The same applies for X of arity N and $|S| \leqslant N$. $\square$

COROLLARY 4.2. *A map of geometrical spaces $f \colon X \to Y$ is convexity preserving iff it preserves the betweenness relations, that is*

$$\mathrm{B}(a, c, b) \implies \mathrm{B}(f(a), f(c), f(b))$$

for $a, b, c \in X$.

The class of convexity spaces together with CP maps forms a category. In view of Proposition 4.1 a bijective map $f\colon X \to Y$ is an isomorphism iff $f(\operatorname{conv} S) = \operatorname{conv} f(S)$ for all $S \in [X]^{<\omega}$. If X, Y are geometrical this is equivalent to $f([a,b]) = [f(a), f(b)]$ for all $a, b \in X$. We say that $f\colon X \to Y$ is *a CP embedding* (see [**49**, p. 15]) if f is injective and the restricted map $f\colon X \to f(X)$ is an isomorphism, where $f(X)$ is equipped with the subspace convexity. Clearly, if M is a subspace of X then the natural injection $i_M\colon M \to X$ is a CP embedding. We can define a similar notion of CP embeddings in the (sub)category of geometrical spaces.

PROPOSITION 4.3 ([**49**, Prop. I.1.12(3)]). *Let X, Y be two convexity spaces, let $\mathcal{B}$ be a subbase of Y and let $f\colon X \to Y$ be a map. If $f^{-1}(B)$ is convex in X for every $B \in \mathcal{B}$ then f is convexity preserving. The same holds for geometrical spaces.*

PROOF. Set $\mathcal{H} = \{G \subset Y\colon f^{-1}(G) \text{ is convex}\}$. Then $\mathcal{H}$ is a convexity on Y. If X is geometrical then $\mathcal{H}$ is an interval convexity. In both cases $\mathcal{H}$ contains the original convexity of Y and hence f is CP. $\square$

Let $\{X_\alpha\}_{\alpha \in \Gamma}$ be a collection of convexity spaces. Set $X = \prod_{\alpha \in \Gamma} X_\alpha$ and denote by $\mathcal{G}$ the convexity on X, called *the product convexity* which is generated by all sets of the form $\operatorname{pr}_\alpha^{-1}(G)$ where G is convex in X_α and $\operatorname{pr}_\alpha\colon X \to X_\alpha$ is the α's projection (see [**49**, p. 14]). Then $(X, \mathcal{G})$ with the standard projections is the product of $\{X_\alpha\}_{\alpha \in \Gamma}$ in the category of convexity spaces. Indeed, if Y is a convexity space together with a collection of CP maps $\{g_\alpha\colon Y \to X_\alpha\}_{\alpha \in \Gamma}$ then there exists a unique mapping $h\colon Y \to X$ such that $g_\alpha = \operatorname{pr}_\alpha h$ for every $\alpha \in \Gamma$. By Proposition 4.3 the map h is CP, since $h^{-1}(\operatorname{pr}^{-1}(G)) = g_\alpha^{-1}(G)$ for $G \subset X_\alpha$ and $\alpha \in \Gamma$.

PROPOSITION 4.4 ([**49**, I.1.10.3]). *If S is a finite set in a product $\prod_{\alpha \in \Gamma} X_\alpha$ then $\operatorname{conv} S = \prod_{\alpha \in \Gamma} \operatorname{conv} \operatorname{pr}_\alpha(S)$.*

PROOF. It follows from Proposition 3.1 that $\operatorname{conv} S$ is a product set. On the other hand, $\prod_{\alpha \in \Gamma} \operatorname{conv} \operatorname{pr}_\alpha(S)$ is a minimal product set with convex factors containing S. $\square$

PROPOSITION 4.5 ([**49**, Prop. I.1.10.2]). *If $X = X_1 \times \cdots \times X_n$ then every convex set in X has the form $G_1 \times \cdots \times G_n$ where each G_i is convex in X_i.*

PROOF. It is enough to check that the collection
$$\mathcal{A} = \{G_1 \times \cdots \times G_n\colon G_i \text{ is convex in } X_i \text{ for } i = 1, \dots, n\},$$
forms a convexity on X. Clearly, $\mathcal{A}$ is stable under intersections and $\emptyset, X \in \mathcal{A}$. If $\{A_\alpha\}_{\alpha \in \Gamma}$ is a chain in $\mathcal{A}$ and $A_\alpha = G_1^\alpha \times \cdots \times G_n^\alpha$ then $\bigcup_{\alpha \in \Gamma} A_\alpha = \prod_{i=1}^n (\bigcup_{\alpha \in \Gamma} G_i^\alpha)$. $\square$

The above result is not valid for infinite products. For example consider the two-element discrete space $2 = \{0, 1\}$ and its κ-th power 2^κ, which is called *a Cantor cube*. If p is a non-principal ultrafilter on κ then the set $G_p = \{\chi_A\colon A \in p\}$ is convex in 2^κ but $\operatorname{pr}_\alpha(G_p) = \{0, 1\}$ for every $\alpha \in \kappa$. It follows in particular that there are 2^{2^κ} convex sets in 2^κ for an infinite cardinal κ.

PROPOSITION 4.6 ([**49**, I.4.3.1]). *A finite product of geometrical spaces is a geometrical space.*

PROOF. It is enough to check that $X \times Y$ is a geometrical space provided X, Y are geometrical. Consider a set $A \subset X \times Y$ such that $[a, b] \subset A$ whenever $a, b \in A$. Let $B = \operatorname{pr}_X(A) \times \operatorname{pr}_Y(A)$. Clearly $A \subset B$. If $(x, y) \in B$ then there exist $a^1 = (a_1^1, a_2^1), a^2 = (a_1^2, a_2^2) \in A$ such that $x = a_1^1$ and $y = a_2^2$. By Proposition 4.4 $(x, y) \in [a^1, a^2] \subset A$. Hence $A = B$. Clearly $\operatorname{pr}_X(A), \operatorname{pr}_Y(A)$ are convex, since X, Y are geometrical. $\square$

This result is not valid for infinite products (see [**49**, p. 87] for a counterexample), however some properties stronger than "being a geometrical space" are productive. We have for instance the following:

PROPOSITION 4.7 ([**49**, p. 46]). *A product of JHC spaces is JHC.*

PROOF. Let $X = \prod_{\alpha \in \Gamma} X_\alpha$ and assume that each X_α is JHC. Consider $p \in \mathrm{conv}(S \cup \{x\})$ where $S \in [X]^{<\omega}$ and $x \in X$. Using Proposition 4.4 we get $p(\alpha) \in \mathrm{conv}(\mathrm{pr}_\alpha(S) \cup \{x(\alpha)\})$ whence $p(\alpha) \in [y_\alpha, x(\alpha)]$ for some $y_\alpha \in \mathrm{conv}\,\mathrm{pr}_\alpha(S)$, since X_α is JHC. It follows that $p \in [y, x]$ where $y(\alpha) = y_\alpha$ and $y \in \mathrm{conv}\,S$. Thus $\mathrm{conv}(S \cup \{x\}) \subset \bigcup_{y \in \mathrm{conv}\,S}[y, x]$. The reverse inclusion always holds. $\square$

5. Separation and screening axioms

We shall consider the following separation axioms for convexity spaces ([**49**, p. 53]):

S_0: For every two distinct points there exists a convex set which contains exactly one of them.

S_1: Every one-point subset is convex.

S_2: Distinct points are separated by half-spaces.

S_3: Every convex set is an intersection of half-spaces.

S_4: Disjoint convex sets are separated by half-spaces.

Here, two sets are *separated by* a set H if one of them is contained in H and the other one is disjoint from H. We say that two sets $A, B \subset X$ are *screened with* C, D if $A \cap D = \emptyset = B \cap C$ and $C \cup D = X$. There are also "screening" analogues of some separation axioms (see also [**49**, p. 344]):

C_2: Every two distinct points can be screened with convex sets.

C_3: A convex set and a point in its complement can be screened with convex sets.

C_4: Every two disjoint convex sets can be screened with convex sets.

It is clear that $S_4 \implies S_3$, $S_2 \implies S_1 \implies S_0$ and $S_3 + S_1 \implies S_2$. Also $S_i \implies C_i$ for $i = 2, 3, 4$. In Example 8.1 below there is described an S_3 space which is not S_4. Example 6.3(a) below gives an S_2 space which is not S_3. Also we shall show below that in fact $S_3 \iff C_3$ and $S_4 \iff C_4$. An S_1 convexity space is called *point-convex*. In Chapter 2 we give some characterizations of axiom S_4 which is also called *the Kakutani separation property*.

The screening axioms are quite similar to T_2, T_3, T_4 in topology. Observe that every subset of an S_i or C_i space also satisfies S_i or C_i provided $i = 0, 1, 2, 3$. Every convex subset of an S_4 $(= C_4)$ space satisfies S_4 $(= C_4)$. One can easily check that a product of S_i spaces is also S_i whenever $i = 0, 1, 2$. The same is true for C_2. It follows from Theorem 5.2 below and Proposition 4.4 that the class of S_3 $(= C_3)$ spaces is also productive. The same result is valid for S_4 $(= C_4)$ spaces, it is a consequence of Theorem 1.1 from Chapter 2 which implies that S_4 is equivalent to the property that every two disjoint polytopes can be screened with convex sets.

THEOREM 5.1 ([**49**, Thm. I.3.8(2)]). *Every C_4 convexity space is S_4.*

PROOF. Let X be C_4 and consider disjoint convex sets $A, B \subset X$. By Proposition 1.2 there exists a maximal convex set H such that $B \subset H$ and $A \cap H = \emptyset$. Similarly, there exists a maximal convex set G with $A \subset G$ and $G \cap H = \emptyset$. Let G', H' be a convex screening of G, H that is $G \subset G' \setminus H'$, $H \subset H' \setminus G'$ and $G' \cup H' = X$. As G is maximal convex with

the property $G \cap H = \emptyset$ we infer that $G = G'$. Similarly $H = H'$. It follows that G, H are complementary half-spaces. $\qquad\square$

THEOREM 5.2. *For every convexity space X the following conditions are equivalent:*

(a) *X satisfies C_3.*
(b) *If $S \subset X$ is finite and $x \notin \mathrm{conv}\, S$ then the pair $S, \{x\}$ can be screened with convex sets.*
(c) *For each two finite sets $S, T \subset X$ and $x \in X$ it holds*

$$S \cap \mathrm{conv}\, T \neq \emptyset \wedge (\forall\, t \in T,\ x \in \mathrm{conv}(S \cup \{t\})) \implies x \in \mathrm{conv}\, S.$$

(d) *X satisfies S_3.*

PROOF. The implications (a) $\implies$ (b) and (d) $\implies$ (a) are obvious. It remains to show that (b) $\implies$ (c) $\implies$ (d).

(b) $\implies$ (c) Suppose $x \notin \mathrm{conv}\, S$. There exist convex sets C, D with $C \cup D = X$ and $x \notin C$, $S \cap D = \emptyset$. Consequently $T \subset D$ and therefore $S \cap \mathrm{conv}\, T = \emptyset$.

(c) $\implies$ (d) Let G be a maximal convex set not containing x (see Proposition 1.2). Suppose that $X \setminus G$ is not convex. There exists $T \in [X \setminus G]^{<\omega}$ and $g \in G$ with $g \in \mathrm{conv}\, T$ (Proposition 1.1). By the maximality of G, for each $t \in T$ there exists $S_t \in [G]^{<\omega}$ such that $x \in \mathrm{conv}(S_t \cup \{x\})$. Set $S = \{g\} \cup \bigcup_{t \in T} S_t$. Then $S \in [G]^{<\omega}$, $S \cap \mathrm{conv}\, T \neq \emptyset$ and $x \in \mathrm{conv}(S \cup \{t\})$ for $t \in T$. By (b) we have $x \in \mathrm{conv}\, S$, a contradiction. $\qquad\square$

The equivalence of (a), (b) and (d) is contained in [**49**, Thm. I.3.8(1)]. For JHC spaces the formula in (c) is equivalent to the so-called *sand-glass property* which can be formulated in terms of betweenness relation, see [**49**, Thm. I.4.13].

COROLLARY 5.3. *If each finite subset of a given convexity space X is contained in a subspace satisfying S_3 then X also satisfies S_3.*

The same result holds for S_4 spaces. It is a consequence of Theorem 1.1 from Chapter 2. In the class of spaces of arity N, an interesting stronger result is valid: if each at most $(2N+1)$-element subset is contained in an S_4 subspace then the whole space satisfies S_4 (see Theorem 3.2 of Chapter 2).

The following problem seems to be open.

QUESTION 5.4. Does there exist a C_2 space which is not S_2 ?

6. Inner transitive and antisymmetric spaces

We describe two classes of interval operators and related geometrical spaces which will play an important role later.

Let $r \colon [X]^{\leqslant 2} \to \mathcal{P}(X)$ be an interval operator. We say that r is *inner transitive* [**43**] (or *geometric* [**51**]) provided

$$(\forall\, a, b, c, d)\ d \in r(a, b) \ \& \ c \in r(a, d) \implies d \in r(c, b).$$

We say that r is *antisymmetric* if

$$(\forall\, a, b, c)\ c \in r(a, b) \ \& \ b \in r(a, c) \implies b = c.$$

A convexity space is called *inner transitive* (*antisymmetric*) if its segment operator is inner transitive (antisymmetric). Observe that every convexity space defined by antisymmetric

interval operator is S_1. Indeed, if r is antisymmetric and $y \in r(x,x)$ then also $x \in r(x,y)$, whence $x = y$. Thus $r(x,x) = \{x\}$ which implies that $\mathrm{conv}_r\{x\} = \{x\}$.

Note that a product of inner transitive (antisymmetric) spaces is inner transitive (antisymmetric). The class of spaces defined by a geometric interval operator was investigated by van de Vel [**49**] and Verheul [**51**].

PROPOSITION 6.1. *An S_0 inner transitive space is antisymmetric and S_1.*

PROOF. Fix $a, b, c \in X$ with $b \in [a,c]$ and $c \in [a,b]$. By inner transitivity we have $c \in [b,b]$ and $b \in [c,c]$. Applying S_0 we get $b = c$.   □

PROPOSITION 6.2. *Every C_2 space is antisymmetric and every S_3 space is inner transitive.*

PROOF. Suppose X is C_2 and $[a,b] = [a,c]$ while $b \neq c$. If B, C is a convex screening of b, c then $a \in B$ or $a \in C$. Both cases yield a contradiction. Suppose now that X is S_3 and $d \in [a,b]$, $c \in [a,d]$. If $d \notin [c,b]$ then there exists a half-space H with $d \notin H$ and $c, b \in H$. Now $a \notin H$ since otherwise $d \in [a,b] \subset H$. Hence $c \in [a,d] \subset X \setminus H$, a contradiction.   □

Next we give three examples of geometrical spaces distinguishing the classes of inner transitive, antisymmetric and S_2, S_3 spaces.

EXAMPLES 6.3. (a) Let $\mathbb{K}_0$ be a geometrical S_1 space consisting of four distinct points a, b, c, d and with the relations $c, d \in [a,b]$, $c \in [a,d]$. This means that neither $d \in [c,b]$ nor $c \in [d,b]$ nor $b \in [c,d]$. Then $\mathbb{K}_0$ is antisymmetric and S_2 but it is not inner transitive (and not S_3).

(b) Let $\mathbb{K}_1$ be a four-element S_1 geometrical space consisting of points a, b, c, d with the relations $c, d \in [a,b]$, $c \in [a,d]$ and $c \in [d,b]$. Then $\mathbb{K}_1$ is antisymmetric, neither inner transitive nor C_2. (since c, d cannot be screened with convex sets).

(c) Consider a geometrical S_1 space Y consisting of five points a, b, c, d_1, d_2 with the relations $d_1, d_2 \in [a,b] \cap [a,c] \cap [b,c]$. Observe that Y is inner transitive but not C_2 since there is no convex screening of d_1, d_2.

PROPOSITION 6.4. *An antisymmetric space is inner transitive iff it does not contain a subspace isomorphic to $\mathbb{K}_0$ or $\mathbb{K}_1$.*

PROOF. Let X be antisymmetric and suppose that X fails to be inner transitive, i.e. there are $a, b, c, d \in X$ with $d \in [a,b]$, $c \in [a,d]$ but $d \notin [c,b]$. As X is antisymmetric, we infer that $Y = \{a,b,c,d\}$ consists of pairwise distinct points, $d \notin [a,c]$ and $b \notin [c,d]$. Now we have two cases: either $d \notin [c,b]$ and Y is isomorphic to $\mathbb{K}_0$, or $d \in [c,b]$ and Y is isomorphic to $\mathbb{K}_1$.   □

7. Helly number

A classical theorem of Helly says that every finite collection of convex (in the usual sense) subsets of $\mathbb{R}^N$ with empty intersection contains $N+1$ sets $A_0, \ldots, A_N$ with $A_0 \cap \cdots \cap A_N = \emptyset$; see Helly [**14**]. We shall discuss a natural invariant of convexity spaces related to Helly's theorem.

We say that a collection of sets $\mathcal{A}$ is *n-centered* if each its at most n-element subcollection has nonempty intersection. A 2-centered family is called *linked*.

THEOREM 7.1 (cf. [**49**], p. 167]). *Let X be a convexity space and let $n > 0$ be a natural number. The following conditions are equivalent:*

(a) *If $G_0, \ldots, G_n$ are convex subsets of X and $\bigcap_{j \neq i} G_j \neq \emptyset$ for all $i \leqslant n$ then*

$$G_0 \cap \cdots \cap G_n \neq \emptyset.$$

(b) *Every finite n-centered family of convex subsets of X has nonempty intersection.*

(c) *For every finite nonempty set $S \subset X$ and for natural $k > \frac{n-1}{n}|S|$ it holds*

$$\bigcap_{T \in [S]^k} \operatorname{conv} T \neq \emptyset.$$

(d) *For each $x_0, \ldots, x_n \in X$ the set*

$$\bigcap_{i \leqslant n} \operatorname{conv}\{x_j : \; j \leqslant n, \; j \neq i\}$$

is nonempty.

PROOF. (a) $\Longrightarrow$ (b) Induction on $m = |\mathcal{A}|$. If $m \leqslant n + 1$ then the statement follows easily from (a). Assume $m > n + 1$ and $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$. Consider the collection $\mathcal{A}' = \{A_0 \cap A_1, \ldots, A_0 \cap A_{m-1}\}$. Clearly $\mathcal{A}'$ consists of convex sets, $|\mathcal{A}'| < m$ and by (a) $\mathcal{A}'$ is n-centered. It follows that $\bigcap \mathcal{A} = \bigcap \mathcal{A}' \neq \emptyset$.

(b) $\Longrightarrow$ (c) It is enough to show that $T_1 \cap \cdots \cap T_n \neq \emptyset$ for $T_1, \ldots, T_n \in [S]^k$. Suppose, if possible, otherwise. We have $S = \bigcup_{i=1}^n (S \setminus T_i)$ and $|S \setminus T_i| = |S| - k$. Hence $|S| \leqslant n(|S| - k)$ and $k \leqslant \frac{n-1}{n}|S|$ which gives a contradiction.

(c) $\Longrightarrow$ (d) If $x_k = x_l = x$ for some $k < l \leqslant n$ then $x \in \bigcap_{i \leqslant n} \operatorname{conv}\{x_j : \; j \leqslant n, \; j \neq i\}$. Thus we can assume that the set $S = \{x_0, \ldots, x_n\}$ has cardinality $n + 1$. Now we can apply (c) to $k = n > \frac{n-1}{n}|S|$.

(d) $\Longrightarrow$ (a) For $i \leqslant n$ choose an $x_i \in \bigcap_{j \neq i} G_j$. By (d) we have

$$\emptyset \neq \bigcap_{i \leqslant n} \operatorname{conv}\{x_j : \; j \leqslant n, \; j \neq i\} \subset \bigcap_{i \leqslant n} G_i.$$

This completes the proof. $\square$

Let X be a nonempty convexity space. The least natural number n such that any n-centered finite family of convex subsets of X has nonempty intersection, is called *the Helly number* of X and denoted by $\mathrm{h}(X)$. If such number does not exist then we set $\mathrm{h}(X) = \omega$. For some convenience we set $\mathrm{h}(\emptyset) = 0$.

Let us state some basic properties of Helly number (see also [**49**, Chapter II]).

PROPOSITION 7.2. *For any convex subspace M of a space X it holds $\mathrm{h}(M) \leqslant \mathrm{h}(X)$. If $f \colon X \to Y$ is a CP map onto Y then $\mathrm{h}(Y) \leqslant \mathrm{h}(X)$.*

PROPOSITION 7.3 ([**49**, Thm. II.2.1]). *For each family $\{X_\alpha\}_{\alpha \in \Gamma}$ of convexity spaces it holds $\mathrm{h}(\prod_{\alpha \in \Gamma} X_\alpha) = \sup_{\alpha \in \Gamma} \mathrm{h}(X_\alpha)$.*

PROOF. Let $X = \prod_{\alpha \in \Gamma} X_\alpha$ and $n = \sup_{\alpha \in \Gamma} \mathrm{h}(X_\alpha)$. Since each X_α is a CP image of X, we see that $\mathrm{h}(X) \geqslant n$. Suppose n is finite and consider $a_0, \ldots, a_n \in X$. For each $\alpha \in \Gamma$ there exists $x_\alpha \in \bigcap_{j \leqslant n} \operatorname{conv}\{a_i(\alpha) : i \neq j\}$. If $x \in X$ is such that $x(\alpha) = x_\alpha$ for $\alpha \in \Gamma$ then by Proposition 4.4 we get $x \in \bigcap_{j \leqslant n} \operatorname{conv}\{a_i : i \neq j\}$. By Theorem 7.1 we have $\mathrm{h}(X) \leqslant n$. $\square$

The last proposition implies in particular that $h(2^\kappa) = 2$ for every cardinal $\kappa > 0$, where 2^κ is equipped with the product convexity. A convexity space of Helly number $\leqslant 2$ is called *binary*, see [**47**].

8. Some examples

We now present some examples and classes of convexity spaces, which will be used later. First of all note that every real vector space together with the collection of all convex sets in the usual meaning, is a geometrical space. The standard convexity on a real vector space will be called *the Euclidean convexity*. If we consider a finite-dimensional space $\mathbb{R}^n$ then we may also consider the product convexity (induced by $\mathbb{R}$), clearly the product convexity is weaker than the Euclidean one.

Consider the real line $\mathbb{R}$. The usual convexity on $\mathbb{R}$ can be defined in terms of ordering as follows: a set G is convex iff $a, b \in G$ and $a \leqslant x \leqslant b$ implies $x \in G$. We can define in the same way a convexity on a partially ordered set (see [**49**, p. 6]). Such a convexity is called *the order convexity*.

Here we give an example of a subspace of the plane $\mathbb{R}^2$ which is S_3 but not S_4.

EXAMPLE 8.1. Let $X = \{a, b, c, a_1, b_1\} \subset \mathbb{R}^2$ with the Euclidean convexity, where a, b, c are the vertices of a non-degenerated triangle and a_1, b_1 lie in the middle of $[a, c]$ and $[b, c]$ respectively. Then, relatively to X, the segments $[a, b_1], [a_1, b]$ are disjoint but cannot be separated by a half-space. Thus X is not S_4. One can easily check that X is S_3 (it is also a consequence of the fact that $\mathbb{R}^2$ is S_3).

Next we describe three interesting classes of convexity spaces: lattices, median spaces and geometrical modules (which includes real vector spaces).

Lattices. Let $(L, \wedge, \vee)$ be a lattice. Denote by $\mathcal{L}$ and $\mathcal{U}$ the collections of all ideals and all filters respectively (the empty set and the whole lattice are treated as (non-proper) ideals and filters). Since the union of a chain of filters (ideals) is a filter (ideal), these are two convexities on L that will be called *the lower* and *the upper lattice convexity* respectively (cf. [**7**]). Also there exists a convexity $\mathcal{G}$ generated by $\mathcal{L} \cup \mathcal{U}$, the least convexity containing all ideals and filters. This convexity will be called *the lattice convexity* on L (see [**46, 47**] or [**49**, p. 7]). Note that if L is linearly ordered then $\mathcal{G}$ equals the order convexity. The convexity of the dual lattice is the same as the original one. In the sequel, we shall consider lattices as convexity spaces (with the lattice convexity) as well as bi-convexity spaces (with the lower and upper lattice convexities).

PROPOSITION 8.2. *Let L be a lattice equipped with the lattice convexity. Then*

(a) *A subset $G \subset L$ is convex iff $G = I \cap F$ where I is an ideal and F is a filter on L.*
(b) *If $S \subset L$ is finite then*

$$\operatorname{conv} S = \{x \in L \colon \inf S \leqslant x \leqslant \sup S\}.$$

(c) *L is a geometrical space and*

$$[a, b] = \{x \in L \colon a \wedge b \leqslant x \leqslant a \vee b\}$$

for every $a, b \in L$.

PROOF. Let $r\colon [L]^{\leqslant 2} \to \mathcal{P}(L)$ be defined as $r(a,b) = \{x \in L\colon a \wedge b \leqslant x \leqslant a \vee b\}$. Then r is an FS-operator and $r(x,y) \subset r(a,b)$ whenever $x,y \in r(a,b)$. It follows that r is a segment operator defining an interval convexity $\mathcal{G}_r$. Observe that (b) holds for conv replaced with conv_r. Hence, if $x \notin \mathrm{conv}\, S$ and $S \in [L]^{<\omega}$ then $\inf S \not\leqslant x$ or $x \not\leqslant \sup S$. Thus we have $x \notin G \supset S$ where G is either the principal filter generated by $\inf S$ or the principal ideal generated by $\sup S$. On the other hand, all ideals and filters are r-convex. By Corollary 3.2, $\mathcal{G}_r$ equals the lattice convexity. This shows (b) and (c). Now, if G is r-convex then, by statement (b), G is a sublattice and $G = I(G) \cap F(G)$ where $I(G)$ and $F(G)$ denote the ideal and the filter generated by G. This shows (a). □

Observe that the lattice convexity on a sublattice equals the subspace convexity. Proposition 4.4 implies that the product and the lattice convexity on a product of lattices are equal. A map of lattices will be called *lower (upper) convexity preserving* if it is CP with respect to the lower (upper) convexities.

PROPOSITION 8.3. *A map $f\colon K \to L$ of a lattice K into a lattice L is lower convexity preserving if and only if it is a join homomorphism. Dually, f is upper convexity preserving iff it is a meet homomorphism.*

PROOF. Let f be LCP. Denote by I_p the principal ideal generated by a point $p \in L$. If $x,y \in K$ and $x \leqslant y$ then $y \in f^{-1}(I_{f(y)})$, hence $x \in f^{-1}(I_{f(y)})$ and consequently $f(x) \in I_{f(y)}$ which means $f(x) \leqslant f(y)$. Thus f is order preserving. Now, for $x,y \in K$ we have $x,y \in f^{-1}(I_{f(x) \vee f(y)})$; hence also $x \vee y \in f^{-1}(I_{f(x) \vee f(y)})$ and consequently $f(x \vee y) \leqslant f(x) \vee f(y)$. As f is order preserving, we get $f(x) \vee f(y) = f(x \vee y)$.

Now let f be a join homomorphism and consider an ideal $I \subset L$. If $x,y \in f^{-1}(I)$ and $z \leqslant x \vee y$ then $f(z) \leqslant f(x \vee y) = f(x) \vee f(y) \in I$; hence $z \in f^{-1}(I)$. Thus $f^{-1}(I)$ is an ideal in K and therefore f is LCP. □

PROPOSITION 8.4. *Let K, L be two lattices and let $f\colon K \to L$ be a CP map. Then f is a lattice homomorphism in each of the following situations:*

(a) *f is order preserving.*
(b) *K, L have least elements $0_K, 0_L$ and $f(0_K) = 0_L$.*

PROOF. Suppose f is order preserving and fix an ideal $I \subset L$. If $x \leqslant y$ and $y \in f^{-1}(I)$ then $x \in f^{-1}(I)$ since $f(x) \leqslant f(y)$. Hence $f^{-1}(I)$ is an ideal. The same argument can be applied for filters. By Proposition 8.3 f is a lattice homomorphism. The second statement follows from the first one, since $f(0_K) = 0_L$ implies that f is order preserving. Indeed, if $x \leqslant y$ then $x \in [0_K, y]$ so $f(x) \in [0_L, f(y)]$ which means $f(x) \leqslant f(y)$. □

PROPOSITION 8.5. *Every lattice is binary, i.e. has Helly number at most two.*

PROOF. Let a, b, c be three points in a lattice and consider $x = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$. We have $a \wedge b \leqslant x$ and $x \leqslant (a \wedge b) \vee a \vee b = a \vee b$. Hence $x \in [a,b]$. Similarly $x \in [a,c]$ and $x \in [b,c]$. □

Particular examples of lattices are Boolean algebras. Observe that a Cantor cube 2^κ with the product convexity is isomorphic to the algebra of all subsets of a set of cardinality κ endowed with the lattice convexity. Later on, we shall identify 2^κ with the power set $\mathcal{P}(\kappa)$. The class of complete Boolean algebras will play an important role in Chapters 3 and 4.

Median spaces. *A median space* is, by definition, a C_2 convexity space X with binary convexity (see Section 7). By C_2 for each $a, b, c \in X$ the point in $[a, b] \cap [a, c] \cap [b, c]$ is unique. We call it *the median of* a, b, c and denote by $m(a, b, c)$. This defines a map $m\colon X \times X \times X \to X$, called *the median operator* on X. Generally, in any convexity space, every point in $[a, b] \cap [a, c] \cap [b, c]$ is called *a median of* a, b, c. There is a natural way to define the structure of a median space by means of the median operator, this is done e.g. in van de Vel [**49**] and Verheul [**51**], where the name *median algebra* is used (see [**49**] for further references). The notion of a median appeared first, in the context of lattices, in the paper of Birkhoff and Kiss [**2**]. Every distributive lattice is a median space with $m(a, b, c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ (a distributive lattice is S_4 by Theorem 4.1 from Chapter 2). A subset M of a median space X is called *median-stable* provided $m(M \times M \times M) \subset M$. The collection of all median-stable subsets forms a convexity (which is of arity 3), the convex hull with respect to this convexity will be denoted by med; med A will be called *the median stabilization* of A. Notice that the union of two convex sets is an example of (not necessarily convex) median-stable set.

Note that a product of median spaces is a median space. A median-stable subset of a median space is again a median space, since it is binary by Theorem 7.1.

PROPOSITION 8.6 (cf. [**49**, Thm. I.6.10]). *Every median space is join-hull commutative and geometrical.*

PROOF. Let X be a median space. Fix a finite set $F \subset X$ and $x, c \in X$ with $x \in \operatorname{conv}(F \cup \{c\})$. As X is binary, there exists a point

$$y \in \operatorname{conv} F \cap \bigcap_{p \in F} [p, x].$$

Suppose that $x \notin [y, c]$. Then there exists a convex screening, say A, B, of the pair $x, m(x, y, c)$. We have $x \in A \setminus B$ and $m(x, y, c) \in B$, whence $y, c \in B$. Hence $F \subset B$. It follows that $\operatorname{conv}(F \cup \{c\}) \subset B$; a contradiction. Thus $x \in \bigcup_{p \in F} [p, c]$. Hence X is JHC. By Proposition 1.3, X is geometrical. $\square$

PROPOSITION 8.7. *Let* $f\colon X \to Y$ *be a CP map from a median space onto a* C_2 *convexity space. Then* Y *is a median space and the image of every convex set under* f *is convex in* Y.

PROOF. The first part follows from the fact that $h(Y) \leqslant h(X)$; see Proposition 7.2. Fix a convex set $G \subset X$ and consider $a, b \in G$ and $y \in [f(a), f(b)]$. Let $y = f(x)$ and set $x' = m(x, a, b)$. Then $f(x') = m(f(a), f(b), y) = y$. On the other hand $x' \in G$, so $y \in f(G)$. It follows that $f(G)$ is convex, since by Proposition 8.6, Y is geometrical. $\square$

Applying the above proposition for the identity map we see that the convexity of a median space is minimal in the collection of all C_2 convexities on a given set. Specifically:

COROLLARY 8.8. *If* $(X, \mathcal{G})$ *is a median space,* $\mathcal{G}'$ *is a* C_2 *convexity on* X *with* $\mathcal{G}' \subset \mathcal{G}$ *then* $\mathcal{G}' = \mathcal{G}$.

The last assertion suggests that the class of median spaces in convexity theory plays a similar role to the class of Hausdorff compact spaces in topology. Let us present an example of a binary convexity space with unique medians which is not C_2.

EXAMPLE 8.9. Consider a geometrical subspace

$$X = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}$$

of $\mathbb{R}^2$ with the product convexity. Let $\mathcal{G}$ be the convexity of X. Clearly $(X, \mathcal{G})$ is a median space and each segment in $(X, \mathcal{G})$ has cardinality at most 3. Now set $\mathcal{G}' = \{X\} \cup (\mathcal{G} \cap [X]^{\leqslant 3})$. Observe that $\mathcal{G}'$ is a convexity on X and $(X, \mathcal{G}')$ has the same segments as $(X, \mathcal{G})$. Thus $(X, \mathcal{G}')$ is binary and $|[a,b] \cap [a,c] \cap [b,c]| = 1$ for every $a, b, c \in X$. On the other hand, $(X, \mathcal{G}')$ is not C_2 (and not geometrical).

Geometrical modules. Let us consider a ring R (with unity, but not necessarily commutative). Following Jamison [**17**] we say that a subset J of R is *an algebraic interval* provided

(i) $0, 1 \in J$ and
(ii) $\alpha, \beta, \gamma \in J$ implies $\gamma\alpha + (1-\gamma)\beta \in J$.

Fix an R-module M. For $a, b \in M$ we set

$$[a,b]_J = \{\lambda a + (1-\lambda)b : \lambda \in J\}.$$

This defines a 2-ary convexity $\mathcal{G}_J$ in M; elements of $\mathcal{G}_J$ will be called *J-convex*. Observe that (i) and (ii) imply that the set $[a,b]_J$ defined above is in fact the segment joining a, b with respect to $\mathcal{G}_J$. Any module over a ring R with such defined convexity will be called *a geometrical module over* (R, J). In particular R alone is a geometrical module over itself. Observe that by (ii) J is a convex subset of R.

Let us present some examples of geometrical modules.

EXAMPLES 8.10. (a) Let $\mathbb{R}$ denote the field of reals. If $J = [0,1]$ then the convexity $\mathcal{G}_J$ in a real vector space is the usual one. If $J = [0,1] \cap \mathbb{Q}$ then it is called *the rational* convexity. Finally, if $J = \{\frac{k}{2^n} \in [0,1] : k, n \in \omega\}$ it is called *the Jensen* convexity (see also Eggleston [**6**], Green and Gustin [**11**]).

(b) Let B be a Boolean algebra and $J = B$. Denote by $\div$ the symmetrical difference in B. Then $(B, \div, \wedge, 0, 1)$ is a (Boolean) ring. One can easily check that $[a,b]_J = \{x \in B : a \wedge b \leqslant x \leqslant a \vee b\}$. Hence the convexity $\mathcal{G}_J$ is the same as the convexity defined by the lattice structure in B.

(c) Let R be the ring of all real measurable functions defined on a measurable space $(T, \mathfrak{M})$. We set $J = \{\chi_A : A \in \mathfrak{M}\}$. It is easily seen that $\chi_C \chi_A + (1 - \chi_C)\chi_B = \chi_G$ where $G = (A \cap C) \cup (B \setminus C)$, hence J is an algebraic interval. A J-convex set is called *decomposable*; decomposable subsets of $L^1(\mu)$ were introduced in [**15**].

The following proposition provides examples of CP maps between geometrical modules.

PROPOSITION 8.11. *Let M, N be two geometrical modules over (R, J) and let $D \subset M$ be a J-convex set. Then*

(a) *Every map $f : D \to N$ satisfying the condition $f(\lambda a + (1-\lambda)b) = \lambda f(a) + (1-\lambda)f(b)$ for all $a, b \in D$, $\lambda \in J$, is convexity preserving.*
(b) *Every R-homomorphism between M, N is convexity preserving.*
(c) *For every $u \in M$ the translation $t_u : M \to M$, given as $t_u(x) = x + u$, is a CP isomorphism.*

It follows from (c) of the above proposition that every geometrical module is CP homogeneous, i.e. for each two points a, b there exists a CP automorphism f with $f(a) = b$.

9. Embedding theorem

As we have already mentioned, for every set S the power set $\mathcal{P}(S)$ can be considered as a convexity space with the lattice convexity. Clearly $\mathcal{P}(S)$ is S_3 and binary (in fact it is S_4, see Theorem 4.1 from Chapter 2). Recall that for $a_1, \ldots, a_n \in \mathcal{P}(S)$ we have $[a_1, \ldots, a_n] = \{x \in \mathcal{P}(S) : a_1 \cap \cdots \cap a_n \subset x \subset a_1 \cup \cdots \cup a_n\}$. Below we show that power sets (or Cantor cubes) are in some sense universal objects for point-convex S_3 spaces.

THEOREM 9.1 (cf. [**49**, Lemma I.3.16]). *Let X be an S_3 space and assume that $\mathcal{H}$ is a collection of half-spaces in X. Then the mapping $\varphi \colon X \to \mathcal{P}(\mathcal{H})$ given by the formula*

$$\varphi(x) = \{H \in \mathcal{H} : x \in H\},$$

is convexity preserving. If, moreover, X is point-convex and for each $x \in X$ and $F \in [X]^{<\omega}$ with $x \notin \operatorname{conv} F$ there exists an $H \in \mathcal{H}$ which separates x from F (i.e. either $x \in H$ and $F \subset X \setminus H$ or $x \notin H$ and $F \subset H$) then φ is a CP embedding.

PROOF. For $H \in \mathcal{H}$ set $H^+ = \{p \in \mathcal{P}(\mathcal{H}) : H \in p\}$. Then H^+ is a half-space in $\mathcal{P}(\mathcal{H})$ (it is a principal ultrafilter) and the collection $\{H^+ : H \in \mathcal{H}\} \cup \{\mathcal{P}(\mathcal{H}) \setminus H^+ : H \in \mathcal{H}\}$ forms a subbase of $\mathcal{P}(\mathcal{H})$. We have $\varphi^{-1}(H^+) = H$ which implies that φ is CP. Suppose now that $\mathcal{H}$ separates points from polytopes and consider $x \notin \operatorname{conv} F$. If $H \in \mathcal{H}$ separates x from F then either $H \in \varphi(x) \setminus \bigcup_{y \in F} \varphi(y)$ or $H \in \bigcap_{y \in F} \varphi(y) \setminus \varphi(x)$. It follows that $\varphi(x) \notin \operatorname{conv} \varphi(F)$. Thus φ is a CP embedding, since φ is one-to-one. $\qquad\square$

The last theorem implies that every point-convex S_3 space embeds into $\mathcal{P}(\kappa)$ for a sufficiently large cardinal κ. More specifically, it is not hard to see that a point-convex S_3 space X can be embedded into $\mathcal{P}(\kappa)$ for $\kappa = |X|$ when X is infinite and for $\kappa = 2^{|X|}$ otherwise. In connection with the last theorem the following question seems to be open.

QUESTION 9.2. Let X be an infinite point-convex S_3 space having a subbase of cardinality κ. Can X be embedded into $\mathcal{P}(\kappa)$?

CHAPTER 2

Kakutani separation property

A classical theorem of Kakutani [**18**], says that each two disjoint convex sets in a real vector space can be separated by a half-space (i.e. a convex set with the convex complement). This theorem is also known as a geometric version of Hahn-Banach theorem.

In 1952, Ellis [**7**] showed an abstract version of Kakutani's theorem (in terms of a pair of join-hull commutative convexities). Recently, in 1993, Chepoi [**4**] proved a similar result for a wider class of convexities. It is worth to notice that the mentioned results can be applied to the lattice theory, obtaining the theorem of Stone-Birkhoff [**41**] on separation by prime filters in distributive lattices (see also [**47**]).

Our purpose is to characterize the Kakutani separation property for spaces with convexities defined by finitary set operators. We state a general separation theorem which is a common generalization of results of Ellis and Chepoi.

The results of this chapter are contained in our paper [**22**].

1. The Pasch axiom

We start with a general separation theorem concerning two arbitrary convexities on a set.

THEOREM 1.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be two convexities on a set X. The following conditions are equivalent:*

  (a) *For every $x, y, z \in X$ and finite sets $S, T \subset X$ such that $x \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup S)$ and $y \in \mathrm{conv}_{\mathcal{H}}(\{z\} \cup T)$ it holds $\mathrm{conv}_{\mathcal{G}}(\{y\} \cup S) \cap \mathrm{conv}_{\mathcal{H}}(\{x\} \cup T) \neq \emptyset$.*
  (b) *If $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are disjoint then there exist disjoint sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$.*
  (c) *If $S, T \subset X$ are finite and $\mathrm{conv}_{\mathcal{G}} S \cap \mathrm{conv}_{\mathcal{H}} T = \emptyset$ then there exist $G \in \mathcal{G}$, $H \in \mathcal{H}$ such that $S \cap H = \emptyset = T \cap G$ and $G \cup H = X$.*

PROOF. (a) $\Longrightarrow$ (b) Let G be a maximal $\mathcal{G}$-convex set containing A disjoint from B and let H be a maximal $\mathcal{H}$-convex set containing B disjoint from G. We show that $G \cup H = X$. Suppose otherwise, i.e. there is some $z \in X \setminus (G \cup H)$. By the maximality of G we get $\mathrm{conv}_{\mathcal{G}}(G \cup \{z\}) \cap H \neq \emptyset$ and therefore there exists an $h \in H$ and a finite set $S \subset G$ with $h \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup S)$. By the same argument there is a $g \in G$ and a finite set $T \subset H$ with $g \in \mathrm{conv}_{\mathcal{H}}(\{z\} \cup T)$. By (a) we get $\mathrm{conv}_{\mathcal{G}}(\{g\} \cup S) \cap \mathrm{conv}_{\mathcal{H}}(\{h\} \cup T) \neq \emptyset$ which means that $G \cap H \neq \emptyset$; a contradiction.

(b) $\Longrightarrow$ (c) Trivial.

(c) $\Longrightarrow$ (a) Suppose that $\mathrm{conv}_{\mathcal{G}}(\{y\} \cup S) \cap \mathrm{conv}_{\mathcal{H}}(\{x\} \cup T) = \emptyset$. By (c) there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ with $(\{y\} \cup S) \cap H = \emptyset = (\{x\} \cup T) \cap G$ and $G \cup H = X$. Now, if e.g. $z \in G$ then $x \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup S) \subset G$, a contradiction. $\qquad\square$

In virtue of the theorem above we introduce the following definitions. A pair (r, s) of FS-operators on X is said to satisfy *the Pasch axiom* provided for each $N, K < \omega$ with $[X]^{\leqslant N} \subset$ $\mathrm{dom}(r)$, $[X]^{\leqslant K} \subset \mathrm{dom}(s)$ and for every $c, a_1, \ldots, a_{N-1}, b_1, \ldots, b_{K-1}, x, y \in X$ the following implication holds:

$$x \in r(c, a_1, \ldots, a_{N-1}) \ \& \ y \in s(c, b_1, \ldots, b_{K-1}) \implies$$

(P) $$\implies r(y, a_1, \ldots, a_{N-1}) \cap s(x, b_1, \ldots, b_{K-1}) \neq \emptyset.$$

If $r = s$ then we simply say that r satisfies *the Pasch axiom*.

A pair of convexities $(\mathcal{G}, \mathcal{H})$ on a set X has *the Kakutani separation property* provided for each two disjoint sets $A \in \mathcal{G}$ and $B \in \mathcal{H}$ there exist disjoint sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$. In this case we also say that the bi-convexity space $(X, \mathcal{G}, \mathcal{H})$ is S_4. This is in accordance with axiom S_4 for a convexity space.

Theorem 1.1 now says that the Kakutani separation property for two convexities is equivalent to the Pasch axiom for its polytope operators. A natural question arises whether the Kakutani property holds for convexities defined by FS-operators satisfying the Pasch axiom. As we shall show, the answer is affirmative for transitive set operators. We give also a similar result for arbitrary FS-operators but the Pasch axiom is then replaced with a more complicated formula.

We say that a pair (r, s) of FS-operators on X satisfies *axiom (Q)* provided for each natural numbers N, K with $[X]^{\leqslant N} \subset \mathrm{dom}(r)$, $[X]^{\leqslant K} \subset \mathrm{dom}(s)$ and for each $b, y_1, \ldots, y_K, h_1, \ldots, h_K$, $a_1, \ldots, a_{N-1}$ the following implication holds:

$$b \in s(y_1, \ldots, y_K) \ \& \ \forall i \leqslant K \ h_i \in r(y_i, a_1, \ldots, a_{N-1}) \implies$$

(Q) $$\implies r(b, a_1, \ldots, a_{N-1}) \cap s(h_1, \ldots, h_K) \neq \emptyset.$$

Setting $h_i = y_i$ for $i > 1$ we see that (Q) implies the Pasch axiom.

PROPOSITION 1.2. *If (r, s) is a pair of FS-operators satisfying the Pasch axiom and r is transitive then (r, s) satisfy (Q).*

PROOF. Assume $b \in s(y_1, \ldots, y_K)$ and $h_i \in r(y_i, a_1, \ldots, a_{N-1})$ for $i \leqslant K$. By the Pasch axiom there exists $x_1 \in r(b, a_1, \ldots, a_{N-1}) \cap s(h_1, y_2, \ldots, y_K)$ and by the transitivity of r we have $r(x_1, a_1, \ldots, a_{N-1}) \subset r(b, a_1, \ldots, a_{N-1})$. Inductively, we can find $x_2, \ldots, x_N$ such that $x_i \in s(h_1, \ldots, h_i, y_{i+1} \ldots, y_K)$ and $r(x_i, a_1, \ldots, a_{N-1}) \subset r(b, a_1, \ldots, a_{N-1})$. For $i = K$ we get $r(b, a_1, \ldots, a_{N-1}) \cap s(h_1, \ldots, h_K) \neq \emptyset$. $\square$

2. A characterization of the Kakutani property

We are going to show that axiom (Q) implies the Kakutani separation property. From now on, assume that (r, s) is a pair of FS-operators satisfying axiom (Q).

For some technical reasons we use the following abbreviations: $A_r[x] = \mathrm{conv}_r(A \cup \{x\})$ and $A_r^n[x] = r^n(A \cup \{x\})$ (the same for s). By Proposition 1.2.1 we have $A_r[x] = \bigcup_{n \in \omega} A_r^n[x]$ (and the same for s).

LEMMA 2.1. *If H is s-convex and $r(x, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$, then for every $y \in H_s[x]$ it holds that $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$.*

PROOF. The statement is clear if $y \in H_s^0[x] = H \cup \{x\}$, so assume that $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$ whenever $y \in H_s^n[x]$ and consider a $y \in H_s^{n+1}[x]$. There are $y_1, \ldots, y_K \in H_s^n[x]$ with $y \in s(y_1, \ldots, y_K)$, where K is such that $[X]^{\leqslant K} \subset \mathrm{dom}(s)$. Now, by induction hypothesis,

there exist $h_i \in r(y_i, a_1, \ldots, a_{N-1}) \cap H$, $i \leqslant K$. Applying (Q) we have $r(y, a_1, \ldots, a_{N-1}) \cap s(h_1, \ldots, h_K) \neq \emptyset$. Since H is s-convex, we get $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$. $\square$

LEMMA 2.2. *If G, H are such subsets of X that G is r-convex, H is s-convex and $G_r[x] \cap H \neq \emptyset \neq G \cap H_s[x]$ for some $x \in X$, then $G \cap H \neq \emptyset$.*

PROOF. We shall proceed by induction. Suppose that $G \cap H \neq \emptyset$ whenever G is r-convex, H is s-convex and $G_r^n[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. Assume $G_r^{n+1}[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. It means in particular that $r(u_1, \ldots, u_N) \cap H \neq \emptyset$ for some $u_1, \ldots, u_N \in G_r^n[x]$, where N is such that $[X]^{\leqslant N} \subset \mathrm{dom}(r)$. Now, for $i \leqslant N$, we have

$$G_r^n[x] \cap H_s[u_i] \neq \emptyset \neq G \cap (H_s[u_i])_s[x].$$

By induction hypothesis there are $g_i \in G \cap H_s[u_i]$ for $i \leqslant N$. Hence, as $r(u_1, \ldots, u_N) \cap H \neq \emptyset$ and $g_1 \in H_s[u_1]$, applying Lemma 2.1 we see that $H \cap r(g_1, u_2, \ldots, u_N) \neq \emptyset$. Inductively, using Lemma 2.1, we infer that $H \cap r(g_1, \ldots, g_i, u_{i+1}, \ldots, u_N) \neq \emptyset$ for $i = 1, \ldots, N$. In particular we get $H \cap r(g_1, \ldots, g_N) \neq \emptyset$ which means that G and H intersect, since G is r-convex. $\square$

Now we can state the main result of this chapter; see [**22**].

THEOREM 2.3. *If (r, s) is a pair of finitary set operators satisfying axiom (Q) then the pair of convexities $(\mathcal{G}_r, \mathcal{G}_s)$ has the Kakutani separation property.*

PROOF. Let G be a maximal r-convex set containing A, disjoint from B and let H be a maximal s-convex set containing B disjoint from G. It is enough to show that $G \cup H = X$. Suppose otherwise. Then there exists an $x \in X \setminus (G \cup H)$, by the maximality of G and H we obtain $G_r[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. Hence, in view of Lemma 2.2, the sets G, H intersect; a contradiction. $\square$

3. Consequences

THEOREM 3.1. *Let $\mathcal{G}, \mathcal{H}$ be two 2-ary convexities on a set X. The following conditions are equivalent:*

  (a) *If $a_1 \in [c, a]_\mathcal{G}$ and $b_1 \in [c, b]_\mathcal{H}$ then $[a, b_1]_\mathcal{G} \cap [b, a_1]_\mathcal{H} \neq \emptyset$.*
  (b) *$(X, \mathcal{G}, \mathcal{H})$ has the Kakutani property.*

PROOF. The implication (a) $\Longrightarrow$ (b) follows from Theorem 2.3 and Proposition 1.2 by setting $r(a, b) = [a, b]_\mathcal{G}$ and $s(a, b) = [a, b]_\mathcal{H}$ since polytope maps are transitive. The implication (b) $\Longrightarrow$ (a) follows from Theorem 1.1. $\square$

J.W. Ellis proved in [**7**] that the Kakutani separation property holds for each two join-hull commutative convexities such that its segments satisfy condition (a) above. By Proposition 1.1.3 each join-hull commutative convexity is 2-ary. Thus Theorem 3.1 extends the result of Ellis.

THEOREM 3.2. *Let X be a space with N-ary convexity. The following conditions are equivalent:*

  (i) *X has the Kakutani property.*
  (ii) *Every two disjoint N-polytopes can be separated by a half-space.*
  (iii) *Every two disjoint N-polytopes can be screened with convex sets.*

(iv) *If $x \in [c, a_1, \ldots, a_{N-1}]$, $y \in [c, b_1, \ldots, b_{N-1}]$ then*

$$[y, a_1, \ldots, a_{N-1}] \cap [x, b_1, \ldots, b_{N-1}] \neq \emptyset.$$

PROOF. The implication (iv) $\implies$ (i) follows from Theorem 2.3 by setting $r = s = \text{conv} \,|[X]^{\leqslant N}$. Since the implications (i) $\implies$ (ii) $\implies$ (iii) are trivial, it remains to show that (iii) $\implies$ (iv).

We use the same argument as in the proof of Theorem 1.1. Suppose (iv) fails for some x, y, c, $a_1, \ldots, a_{N-1}$ and $b_1, \ldots, b_{N-1}$. Let C, D be two convex sets screening $[y, a_1, \ldots, a_{N-1}]$ and $[x, b_1, \ldots, b_{N-1}]$. If $c \in C$ then $x \in [c, a_1, \ldots, a_{N-1}] \subset C$, since C is convex. Hence $c \notin C$; by the same argument $c \notin D$. It follows that $C \cup D \neq X$; a contradiction. $\square$

V. Chepoi proved in [**4**] the equivalence of (i), (ii) and a property of N-polytopes which is in fact a reformulation of axiom (Q). Thus Theorem 3.2 generalizes the result of Chepoi.

THEOREM 3.3 (Chepoi [**4**]). *Let X be a geometrical space defined by an interval operator $I \colon [X]^{\leqslant 2} \to \mathcal{P}(X)$. If I satisfies the Pasch axiom then X has the Kakutani separation property.*

PROOF. This folllows immediately from Theorem 2.3 and Proposition 1.2 by setting $r = s = I$. $\square$

4. Applications

Here we discuss the Kakutani property for lattices, median spaces and geometrical modules.

Lattices. Let L be a lattice and let $H \subset L$ be a half-space. We have $\text{conv}_L H \cap \text{conv}_U H \cap \text{conv}_L(L \backslash H) \cap \text{conv}_U(L \backslash H) = \emptyset$. Hence, since L is binary, we get e.g. $\text{conv}_L H \cap \text{conv}_U(L \backslash H) = \emptyset$. It follows that $H = \text{conv}_L H$, $L \backslash H = \text{conv}_U(L \backslash H)$ and hence H is a prime ideal. Thus each half-space in L is either a prime ideal or a prime filter.

We give here a "convex" version of Stone-Birkhoff's theorem.

THEOREM 4.1. *A lattice (considered with the lattice convexity) has the Kakutani property iff it is distributive.*

PROOF. Let L be a lattice. Assume first that L has the Kakutani property. Suppose that there are $a, b, c \in L$ with $x = (a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c) = y$ (always we have $x \leqslant y$). Thus $y \notin [a \wedge b, a \wedge c]$ so there exists a half-space $H \subset L$ with $y \notin H \supset [a \wedge b, a \wedge c]$. Observe that $a \in L \backslash H$ since $x \in H$ and $y \in [x, a] \cap (L \backslash H)$. Hence $b, c \in H$ since $a \wedge b \in H \cap [a, b]$ and $a \wedge c \in H \cap [a, c]$. Now $b \vee c \in H$ and $y \in [x, b \vee c] \subset H$ which gives a contradiction.

Now assume that L is distributive and consider $a, a_1, b, b_1, c \in L$ with $a_1 \in [c, a]$ and $b_1 \in [c, b]$. We have $a \wedge c \leqslant a_1 \leqslant a \vee c$ and $b \wedge c \leqslant b_1 \leqslant b \vee c$, whence

$$(1) \qquad\qquad a_1 \wedge b \leqslant (a \vee c) \wedge b = (a \wedge b) \vee (c \wedge b) \leqslant a \vee b_1.$$

Similarly $a \wedge b_1 \leqslant a_1 \vee b$. We set $x = (a \vee b_1) \wedge (a_1 \vee b)$. Clearly $x \leqslant a_1 \vee b$ and by the condition (1) we get $x \geqslant a_1 \wedge b \wedge (a_1 \vee b) = a_1 \wedge b$. Hence $x \in [a_1, b]$. By the same argument $x \in [a, b_1]$. Now Theorem 3.3 implies that L has the Kakutani property. $\square$

It follows that every distributive lattice has the Kakutani property as a bi-convexity space. The same proof as above shows that the reverse also holds. Every Boolean algebra is a distributive lattice, hence it has the Kakutani property.

It follows from the embedding theorem that every S_3 point-convex space can be embedded into an S_4 space, namely into an algebra of sets. In particular, for every distributive lattice L there exists a CP embedding $\varphi\colon L \to \mathcal{P}(\mathcal{H})$ where $\mathcal{H}$ is the collection of all prime filters on L. Observe that φ is order preserving, whence φ is a lattice homomorphism by Proposition 1.8.4. Thus we have obtained the representation theorem of Stone which says that every distributive lattice is isomorphic to a lattice of sets.

Median spaces. We show that every median space has the Kakutani property.

THEOREM 4.2 (cf. [**49**, Thm. I.6.10]). *Let X be a geometrical space defined by an interval operator r which satisfies the condition $|r(a,b)\cap r(a,c)\cap r(b,c)| = 1$ for every $a,b,c \in X$. Then X has the Kakutani property and r is the segment operator. Consequently, X is a median space.*

PROOF. (cf. Chepoi [**4**]) Denote by $\mathrm{m}_r(a,b,c)$ the unique point in $r(a,b)\cap r(a,c)\cap r(b,c)$. We first verify the Pasch axiom for r. Let $a_1 \in r(a,c)$, $b_1 \in r(b,c)$. Set $a_2 = \mathrm{m}_r(a_1,a,b)$ and $b_2 = \mathrm{m}_r(b_1,a,b)$. Clearly $a_2 \in r(a,c)$ and $b_2 \in r(b,c)$, since r is transitive. Let $x = \mathrm{m}_r(a_2,b,c)$. We have $x \in r(a_1,b)$ and $x \in r(a,b)\cap r(a,c)\cap r(b,c)$. Hence $x = \mathrm{m}_r(a,b,c)$. Similarly we get $\mathrm{m}_r(b_2,a,c) = \mathrm{m}_r(a,b,c)$. Hence $\mathrm{m}_r(a_2,b,c) = \mathrm{m}_r(b_2,a,c) \in r(a_1,b)\cap r(a,b_1)$.

Thus X is S_4. We have $\mathrm{m}_r(a,b,c) \in [a,b]\cap[a,c]\cap[b,c]$. On the other hand, X has unique medians by C_2. In particular, $r(a,b) = [a,b]$ since $x \in [a,b]$ implies $x = \mathrm{m}_r(a,b,x) \in r(a,b)$. □

COROLLARY 4.3. *Every median space satisfies axiom S_4.*

COROLLARY 4.4 (cf. [**49**, Thm. I.6.10]). *Every inner transitive S_0 binary geometrical space is a median space.*

PROOF. Let X be S_0, geometrical, inner transitive and binary. Then $[a,b]\cap[a,c]\cap[b,c] \neq \emptyset$ for all $a,b,c \in X$. Suppose $x,y \in [a,b]\cap[a,c]\cap[b,c]$. Pick $z \in [x,y]\cap[a,x]\cap[a,y]$. By inner transitivity we get $x \in [z,c]$ and $x \in [z,b]$. Furthermore $z \in [b,c]$ and again, by inner transitivity, we have $z \in [x,b]$. Thus $x = z$ since X is antisymmetric. The same argument gives $z = y$. Now X is a median space by Theorem 4.2. □

Combining Theorem 4.2, Corollary 4.4 and Proposition 1.8.6 we get the following characterization of median spaces.

THEOREM 4.5. *For a binary convexity space X the following conditions are equivalent:*
  (a) *X is a median space.*
  (b) *X has the Kakutani property.*
  (c) *X is an inner transitive geometrical space satisfying axiom S_0.*
  (d) *X is JHC and has unique medians.*

Geometrical modules. We shall formulate an algebraic condition for geometrical modules equivalent to the Kakutani property.

THEOREM 4.6. *Let R be a ring and $J \subset R$. The following conditions are equivalent:*

(a) *Every geometrical module over (R, J) has the Kakutani property.*
(b) *J (with the relative convexity) is inner transitive.*
(c) *For all $\alpha, \beta \in J$ there exists $\gamma \in J$ with $(1 - \alpha)\beta = \gamma(1 - \alpha\beta)$.*

PROOF. (a) $\Longrightarrow$ (b) This follows from the fact that every point-convex S_4 space is inner transitive (see Proposition 1.6.2).

(b) $\Longrightarrow$ (c) Fix $\alpha, \beta \in J$. We have $\alpha\beta \in [0, \beta]_J \subset J$ and by inner transitivity $\beta \in [\alpha\beta, 1]_J$, i.e. there exists a $\gamma \in J$ with $\beta = (1 - \gamma)\alpha\beta + \gamma$. Modifying this expression we get $(1 - \alpha)\beta = \gamma(1 - \alpha\beta)$.

(c) $\Longrightarrow$ (a) In view of Theorem 3.3 we must verify the Pasch axiom. Fix a geometrical module M over (R, J) and $a, a_1, b, b_1, c \in M$ such that $a_1 \in [c, a]_J$ and $b_1 \in [c, b]_J$. We have

$$a_1 = \alpha a + (1 - \alpha)c, \qquad b_1 = \beta b + (1 - \beta)c,$$

for some $\alpha, \beta \in J$. Let $\gamma \in J$ be such as in condition (c), i.e.

$$(1) \qquad\qquad \beta = \alpha\beta + \gamma(1 - \alpha\beta) = (1 - \gamma)\alpha\beta + \gamma.$$

We set

$$(2) \qquad\qquad\qquad\qquad \delta = (1 - \gamma)\alpha.$$

Observe that $\delta \in [0, \alpha]_J \subset J$ and $\gamma = (1 - \delta)\beta$. Now take

$$x_1 = (1 - \gamma)a_1 + \gamma b, \qquad x_2 = (1 - \delta)b_1 + \delta a.$$

Notice that $x_1 \in [a_1, b]_J$ and $x_2 \in [a, b_1]_J$. It remains to check that $x_1 = x_2$. Let us compute the difference $x_1 - x_2$. We have

$$
\begin{aligned}
x_1 - x_2 &= (1 - \gamma)\Big(\alpha a + (1 - \alpha)c\Big) + \gamma b - (1 - \delta)\Big(\beta b + (1 - \beta)c\Big) - \delta a \\
&= \delta a + (1 - \gamma)(1 - \alpha)c + \gamma b - \gamma b - (1 - \delta)(1 - \beta)c - \delta a \\
&= \Big((1 - \gamma)(1 - \alpha) - (1 - \delta)(1 - \beta)\Big)c.
\end{aligned}
$$

Now, applying (1) and (2) we see that

$$
\begin{aligned}
(1 - \delta)(1 - \beta) &= 1 - \delta - \beta + (1 - \gamma)\alpha\beta = 1 - \delta - \beta + \beta - \gamma \\
&= 1 - \gamma - (1 - \gamma)\alpha = (1 - \gamma)(1 - \alpha).
\end{aligned}
$$

Hence $x_1 = x_2$, which completes the proof. $\qquad\square$

COROLLARY 4.7. *(a) If $J = F \cap [0, 1]$ where F is a subfield of reals then every real vector space with J-convexity has the Kakutani property.*

(b) (Palés [36]) Jensen convexity on $\mathbb{R}$ does not have the Kakutani property.

(c) For any measure μ, the ring of all μ-measurable real functions with the convexity of decomposable sets has the Kakutani property.

PROOF. (a) This follows immediately from condition (c) of Theorem 4.6.

(b) This follows from the equality $(1 - \frac{1}{2})\frac{1}{2}\Big/(1 - \frac{1}{2}\frac{1}{2}) = \frac{1}{3}$.

(c) The equality $(1 - \chi_A)\chi_B = f(1 - \chi_A\chi_B)$ is valid for $f = \chi_B$. $\qquad\square$

The space $L^1(\mu)$ is a decomposable subset of the ring of all μ-measurable functions. Thus it has the Kakutani separation property.

CHAPTER 3

A sandwich theorem

We state a "sandwich" type theorem for maps of bi-convexity spaces. In a special case, this yields a result on separation of meet and join homomorphisms of distributive lattices. We also derive some extension theorems for convexity preserving maps and, in particular, the theorem of Sikorski on extending of homomorphisms of Boolean algebras. Here we shall consider lattices and Boolean algebras as bi-convexity spaces.

The results of this chapter are contained in our paper [**24**].

1. Main result

We say that a lattice L has *κ-separation property*, where κ is an uncountable cardinal, if for any $\lambda < \kappa$ and for any two sequences $\{a_\alpha\}_{\alpha<\lambda}, \{b_\alpha\}_{\alpha<\lambda} \subset L$ such that $a_\alpha \leqslant b_\beta$ for $\alpha, \beta < \lambda$, there exists a point $x \in L$ with $a_\alpha \leqslant x \leqslant b_\alpha$ for all $\alpha < \lambda$. This is equivalent to the following: if $\lambda < \kappa$ and $\{S_\alpha\}_{\alpha<\lambda}, \{T_\alpha\}_{\alpha<\lambda}$ are two sequences of finite sets such that $\mathrm{conv}_L S_\alpha \cap \mathrm{conv}_U T_\beta \neq \emptyset$ for every $\alpha, \beta < \lambda$ then

$$\bigcap_{\alpha<\lambda} (\mathrm{conv}_L S_\alpha \cap \mathrm{conv}_U T_\alpha) \neq \emptyset.$$

Indeed, $\mathrm{conv}_L S_\alpha \cap \mathrm{conv}_U T_\beta \neq \emptyset$ is equivalent to $\inf T_\beta \leqslant \sup S_\alpha$ and $\bigcap_{\alpha<\lambda}(\mathrm{conv}_L S_\alpha \cap \mathrm{conv}_U T_\alpha) \neq \emptyset$ means exactly that there exists an x with $\inf T_\alpha \leqslant x \leqslant \sup S_\alpha$ for every $\alpha < \lambda$.

In case of Boolean algebras, κ-separation property is also known as *weak κ-completeness*. Clearly, every κ-complete lattice has κ-separation property. The converse does not hold; an example of a non ω_1-complete Boolean algebra which has ω_1-separation property is the quotient algebra $\mathcal{P}(\omega)/\mathcal{F}$, where $\mathcal{F}$ denotes the ideal of finite subsets of ω; see [**34**].

Let X be a bi-convexity space and let $\mathbb{B}$ be a Boolean algebra. Furthermore, let $f, g \colon X \to \mathbb{B}$ be two maps such that f is UCP, g is LCP and $f \leqslant g$. We say that a map $h \colon M \to \mathbb{B}$, where $M \subset X$, is *well-placed* (*between f, g*) provided for each $S, T \in [M]^{<\omega}$ and $F, G \in [X]^{<\omega}$ the following implication holds:

$$\mathrm{conv}_L(S \cup G) \cap \mathrm{conv}_U(T \cup F) \neq \emptyset \implies \mathrm{conv}_L(h(S) \cup g(G)) \cap \mathrm{conv}_U(h(T) \cup f(F)) \neq \emptyset.$$

Observe that if $h \colon X \to \mathbb{B}$ is well-placed then $f \leqslant h \leqslant g$ and h is CP. Indeed, setting $S = F = \{p\}$ and $T = G = \emptyset$ above, we get $f(p) \leqslant h(p)$. Similarly $h(p) \leqslant g(p)$. If S is finite and $p \in \mathrm{conv}_L S$ then $\mathrm{conv}_L S \cap \mathrm{conv}_U \{p\} \neq \emptyset$ whence $\mathrm{conv}_L h(S) \cap \mathrm{conv}_U \{h(p)\} \neq \emptyset$. This means $h(p) \in \mathrm{conv}_L h(S)$. Thus h is LCP. By the dual argument, h is also UCP.

LEMMA 1.1. *In every Boolean algebra $\mathbb{B}$, for any $A, B, C, D \subset \mathbb{B}$ the following equivalence holds:*

$$\mathrm{conv}_L(A \cup B) \cap \mathrm{conv}_U(C \cup D) \neq \emptyset \iff \mathrm{conv}_L(A \cup \neg D) \cap \mathrm{conv}_U(C \cup \neg B) \neq \emptyset,$$

where $\neg X = \{\neg x : x \in X\}$ and $\neg x$ denotes the complement of x.

PROOF. Suppose that $\mathrm{conv}_L(A \cup \neg D) \cap \mathrm{conv}_U(C \cup \neg B) = \emptyset$. Then there exists an ultrafilter P with $C \cup \neg B \subset P$ and $(A \cup \neg D) \cap P = \emptyset$. Now $D \subset P$ and B is disjoint from P. It follows that P separates $C \cup D$ from $A \cup B$ and consequently $\mathrm{conv}_L(A \cup B) \cap \mathrm{conv}_U(C \cup D) = \emptyset$. $\quad\square$

THEOREM 1.2. *Let $\mathbb{B}$ be a Boolean algebra with κ-separation property and let X be an S_4 bi-convexity space of size $< \kappa$ and let $f, g \colon X \to \mathbb{B}$ be such two maps that f is UCP, g is LCP and $f \leqslant g$. If $M \subset X$ then every well-placed map $h \colon M \to \mathbb{B}$ can be extended to a convexity preserving map $\overline{h} \colon X \to \mathbb{B}$ such that $f \leqslant \overline{h} \leqslant g$.*

PROOF. We show that h can be extended to a well-placed map $\overline{h} \colon X \to \mathbb{B}$. The union of a chain of well-placed maps is also well-placed. Thus we should only show that for a fixed point $a \in X \setminus M$, h can be extended to a well-placed map $h' \colon M \cup \{a\} \to \mathbb{B}$. Consider two collections of intervals:

$$\mathcal{A}_U = \{\mathrm{conv}_U(h(T) \cup f(F) \cup \neg h(S) \cup \neg g(G)) : S, T \in [M]^{<\omega}, F, G \in [X]^{<\omega},$$
$$\mathrm{conv}_L(S \cup \{a\} \cup G) \cap \mathrm{conv}_U(T \cup F) \neq \emptyset\},$$
$$\mathcal{A}_L = \{\mathrm{conv}_L(h(S) \cup g(G) \cup \neg h(T) \cup \neg f(F)) : S, T \in [M]^{<\omega}, F, G \in [X]^{<\omega},$$
$$\mathrm{conv}_L(S \cup G) \cap \mathrm{conv}_U(T \cup \{a\} \cup F) \neq \emptyset\}.$$

We shall show that every element of $\mathcal{A}_U$ meets every element of $\mathcal{A}_L$. For this goal fix $S_1, S_2, T_1, T_2 \in [M]^{<\omega}$ and $F_1, F_2, G_1, G_2 \in [X]^{<\omega}$ such that

$$\mathrm{conv}_L(S_1 \cup \{a\} \cup G_1) \cap \mathrm{conv}_U(T_1 \cup F_1) \neq \emptyset \neq \mathrm{conv}_L(S_2 \cup G_2) \cap \mathrm{conv}_U(T_2 \cup \{a\} \cup F_2).$$

Applying the Pasch axiom for X we get $\mathrm{conv}_L(S_1 \cup G_1 \cup S_2 \cup G_2) \cap \mathrm{conv}_U(T_1 \cup F_1 \cup T_2 \cup F_2) \neq \emptyset$. Since h is well-placed, we obtain

$$\mathrm{conv}_L(h(S_1) \cup g(G_1) \cup h(S_2) \cup g(G_2)) \cap \mathrm{conv}_U(h(T_1) \cup f(F_1) \cup h(T_2) \cup f(F_2)) \neq \emptyset.$$

Applying Lemma 1.1 for $A = h(S_2) \cup g(G_2)$, $B = h(S_1) \cup g(G_1)$, $C = h(T_1) \cup f(F_1)$ and $D = h(T_2) \cup f(F_2)$ we get

$$\mathrm{conv}_L(h(S_2) \cup g(G_2) \cup \neg h(T_2) \cup \neg f(F_2)) \cap \mathrm{conv}_U(h(T_1) \cup f(F_1) \cup \neg h(S_1) \cup \neg g(G_1)) \neq \emptyset.$$

Now, since $|\mathcal{A}_U \cup \mathcal{A}_L| < \kappa$, we can apply κ-separation property to find a point

$$b \in \bigcap \mathcal{A}_U \cap \bigcap \mathcal{A}_L.$$

Define $h' \colon M \cup \{a\} \to \mathbb{B}$ by setting $h'(a) = b$ and $h'|M = h$. It remains to show that h' is well-placed. For let $S', T' \in [M \cup \{a\}]^{<\omega}$ and $F, G \in [X]^{<\omega}$ be such that $\mathrm{conv}_L(S' \cup G) \cap \mathrm{conv}_U(T' \cup F) \neq \emptyset$. We have to check that

$$(*) \qquad\qquad \mathrm{conv}_L(h'(S') \cup g(G)) \cap \mathrm{conv}_U(h'(T') \cup f(F)) \neq \emptyset.$$

If $S', T' \subset M$ or $a \in S' \cap T'$ then we are done, so suppose that e.g. $S' = S \cup \{a\}$ and $T' = T \subset M$. By the construction of $\mathcal{A}_U$, $b \in \mathrm{conv}_U(h(T) \cup f(F) \cup \neg h(S) \cup \neg g(G))$. Applying Lemma 1.1 for $A = \{b\}$, $B = h(S) \cup g(G)$, $C = h(T) \cup f(F)$ and $D = \emptyset$ we get $(*)$. This completes the proof. $\quad\square$

COROLLARY 1.3. *Under the above assumptions, there exists a convexity preserving map $h \colon X \to \mathbb{B}$ with $f \leqslant h \leqslant g$.*

PROOF. It is enough to show that the empty map is well-placed. Suppose that $p \in \operatorname{conv}_L G \cap \operatorname{conv}_U F \neq \emptyset$, where $F, G \in [X]^{<\omega}$. Since f is UCP and g is LCP, we have $f(p) \in \operatorname{conv}_U f(F)$ and $g(p) \in \operatorname{conv}_L g(G)$. Now, since $f(p) \leqslant g(p)$, we get $f(p), g(p) \in \operatorname{conv}_L g(G) \cap \operatorname{conv}_U f(F)$. $\qquad\square$

REMARKS 1.4. (a) Observe that the statement of Corollary 1.3 for $\mathbb{B} = \{0, 1\}$ is in fact a reformulation of axiom S_4, since lower/upper convex sets can be identified with LCP/UCP characteristic functions.

(b) If f, g are as in Theorem 1.2 and h can be extended to a CP map $\overline{h}$ with $f \leqslant \overline{h} \leqslant g$ then h is well-placed between f, g. Indeed, if $x \in \operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup F)$ in X then $\overline{h}(x) \in \operatorname{conv}_L(\overline{h}(S \cup G)) \cap \operatorname{conv}_U(\overline{h}(T \cup F)) \subset \operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F))$.

2. Applications

We can derive from Theorem 1.2 some extension theorems for CP maps and lattice homomorphisms.

THEOREM 2.1 (see [23]). *Let $(X, \mathcal{L}, \mathcal{U})$ be an S_4 bi-convexity space, let $M \subset X$ and let $\mathbb{B}$ be a Boolean algebra satisfying κ-separation property. If $|X| < \kappa$ then every map $h \colon M \to \mathbb{B}$ satisfying the condition*

(I) $\qquad\qquad \operatorname{conv}_L S \cap \operatorname{conv}_U T \neq \emptyset \implies \operatorname{conv}_L h(S) \cap \operatorname{conv}_U h(T) \neq \emptyset$

for all $S, T \in [M]^{<\omega}$, can be extended to a convexity preserving map $\overline{h} \colon X \to \mathbb{B}$.

PROOF. Set $f = 0_{\mathbb{B}}$ and $g = 1_{\mathbb{B}}$. Clearly, f is UCP, g is LCP and condition (I) implies that h is well-placed between f, g. Applying Theorem 1.2 we obtain the desired extension. $\quad\square$

The next result is similar to the Tietze-Urysohn extension theorem in topology.

COROLLARY 2.2 ([23]). *Let G be a convex subset of an S_4 convexity space $(X, \mathcal{G})$ and let $\mathbb{B}$ be a complete Boolean algebra. Then every CP map $h \colon G \to \mathbb{B}$ can be extended to a CP map $\overline{h} \colon X \to \mathbb{B}$.*

PROOF. It is enough to observe that a CP map defined on a convex set satisfies condition (I) from Theorem 2.1, with $\mathcal{L} = \mathcal{U} = \mathcal{G}$. $\qquad\square$

THEOREM 2.3. *Let L be a distributive lattice, let $\mathbb{B}$ be a Boolean algebra with κ-separation property, where $\kappa > |L|$. Furthermore, let $f, g \colon L \to \mathbb{B}$ be such two maps that f is a meet homomorphism, g is a join homomorphism and $f \leqslant g$. If K is a sublattice of L and $h \colon K \to \mathbb{B}$ is a lattice homomorphism satisfying*

(1) $\qquad \forall\, s, t \in K \ \forall\, a, b \in L \Big(t \wedge a \leqslant s \vee b \implies h(t) \wedge f(a) \leqslant h(s) \vee g(b) \Big),$

then there exists a lattice homomorphism $\overline{h} \colon L \to \mathbb{B}$ with $f \leqslant \overline{h} \leqslant g$ and $\overline{h}|K = h$. In particular, there exists a lattice homomorphism between f and g.

PROOF. We should only check that h is well-placed. Let $S, T \in [K]^{<\omega}$ and $F, G \in [L]^{<\omega}$ be such that $\operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup F) \neq \emptyset$. This is equivalent to $(\inf T) \wedge (\inf F) \leqslant (\sup S) \vee (\sup G)$. Applying condition (1) we get $h(\inf T) \wedge f(\inf F) \leqslant h(\sup S) \vee g(\sup G)$ which means $\operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F)) \neq \emptyset$. $\qquad\square$

COROLLARY 2.4. *Let M be a subset of a distributive lattice L, let $\mathbb{B}$ be a Boolean algebra with κ-separation property. If $|L| \leqslant \kappa$ and $|M| < \kappa$ then every map $h \colon M \to \mathbb{B}$ satisfying the condition*

$$\forall\, a_1, \ldots, a_n, b_1 \ldots, b_m \in M \Big(a_1 \wedge \cdots \wedge a_n \leqslant b_1 \vee \cdots \vee b_m \implies$$

$$(2) \qquad\qquad\qquad \implies\ h(a_1) \wedge \cdots \wedge h(a_n) \leqslant h(b_1) \vee \cdots \vee h(b_m) \Big),$$

can be extended to a lattice homomorphism $\overline{h} \colon L \to \mathbb{B}$.

PROOF. Observe that condition (2) is equivalent to the one in Theorem 2.1 and every partial homomorphism satisfies (2). Thus, using Theorem 2.1, we can define inductively a chain of partial homomorphisms $\{ h_\alpha \colon L_\alpha \to \mathbb{B} \}_{\alpha < \kappa}$ where $h \subset h_0$, each L_α is a sublattice of size $< \kappa$ and $L = \bigcup_{\alpha < \kappa} L_\alpha$. Finally we can set $\overline{h} = \bigcup_{\alpha < \kappa} h_\alpha$. $\qquad\square$

The last corollary is due to Sikorski [**38**]. Originally, it was formulated for complete Boolean algebras as two extension theorems: Sikorski's Extension Criterion and Sikorski's Extension Theorem on injectivity of complete Boolean algebras.

CHAPTER 4

Convexity absolute extensors

In this chapter we shall give a short and straightforward proof of a version of Theorem 3.2.1 for maps of spaces with single convexity.

The extension theorem mentioned above implies in particular that every complete Boolean algebra $\mathbb{B}$ has the following property: for every S_4 convexity space X, every CP map $f\colon G \to \mathbb{B}$ defined on a convex subset of X, can be extended to a CP map $\overline{f}\colon X \to \mathbb{B}$ (Corollary 3.2.2). We shall say that a convexity space Y is *a convexity absolute extensor* if it has the above extension property. We prove that every point-convex S_3 convexity absolute extensor is isomorphic to a complete Boolean algebra. In particular, we obtain an external characterization of complete Boolean algebras in the category of point-convex S_3 convexity spaces.

The results of this chapter are contained in our paper [**26**].

1. Extension theorem

We shall use an auxiliary lemma. The proof is the same as that of Lemma 1.1 from Chapter 3.

LEMMA 1.1. *In every Boolean algebra, the following equivalence holds:*

$$\operatorname{conv}(A \cup B) \cap \operatorname{conv}(C \cup D) \neq \emptyset \iff \operatorname{conv}(A \cup \neg D) \cap \operatorname{conv}(C \cup \neg B) \neq \emptyset,$$

where $\neg X = \{\neg x : x \in X\}$ and $\neg x$ denotes the complement of x.

THEOREM 1.2. *Let $\mathbb{B}$ be a complete Boolean algebra and let X be an S_4 convexity space. If $M \subset X$ then every map $f\colon M \to \mathbb{B}$ satisfying the condition*

(I) $$\forall\, S, T \in [M]^{<\omega} \Big(\operatorname{conv} S \cap \operatorname{conv} T \neq \emptyset \implies \operatorname{conv} f(S) \cap \operatorname{conv} f(T) \neq \emptyset \Big),$$

can be extended to a convexity preserving map $\overline{f}\colon X \to \mathbb{B}$.

PROOF. Observe that the union of a chain of maps satisfying condition (I) also satisfies (I) and every map satisfying condition (I) is convexity preserving. Thus, it is enough to show that for a fixed $x \in X \setminus M$ there exists a map $g\colon M \cup \{x\} \to \mathbb{B}$ satisfying condition (I) and extending f. Consider the collection of intervals

$$\mathcal{A} = \{\operatorname{conv}(f(S) \cup \neg f(T)) : \ S, T \in [M]^{<\omega} \ \& \ \operatorname{conv} S \cap \operatorname{conv}(T \cup \{x\}) \neq \emptyset\}.$$

Let $S_i, T_i \in [M]^{<\omega}$ be such that $\operatorname{conv} S_i \cap \operatorname{conv}(T_i \cup \{x\}) \neq \emptyset$, where $i = 0, 1$. Observe that $\operatorname{conv}(S_0 \cup T_1) \cap \operatorname{conv}(S_1 \cup T_0) \neq \emptyset$. Indeed, otherwise, by S_4, there exists a half-space $H \subset X$ with $S_1 \cup T_0 \subset H$ and $S_0 \cup T_1 \subset X \setminus H$. Consequently, if $x \in H$ then $\operatorname{conv} S_0 \cap \operatorname{conv}(T_0 \cup \{x\}) = \emptyset$; a contradiction. Now, condition (I) applied for f gives

$$\operatorname{conv}(f(S_0) \cup f(T_1)) \cap \operatorname{conv}(f(S_1) \cup f(T_0)) = \emptyset.$$

By Lemma 1.1 we get

$$\mathrm{conv}(f(S_0) \cup \neg f(T_0)) \cap \mathrm{conv}(f(S_1) \cup \neg f(T_1)) \neq \emptyset.$$

Thus we have shown that the collection $\mathcal{A}$ is linked.

As $\mathbb{B}$ is complete, we can find a point $y \in \bigcap \mathcal{A}$. Define $g \colon M \cup \{x\} \to \mathbb{B}$ by setting $g|M = f$ and $g(x) = y$. It remains to check that g satisfies condition (I). Let $S, T \in [M]^{<\omega}$ be such that $\mathrm{conv}\, S \cap \mathrm{conv}(T \cup \{x\}) \neq \emptyset$. Then $y \in \mathrm{conv}(f(S) \cup \neg f(T))$ and applying Lemma 1.1 for $A = \{y\}$, $B = f(T)$, $C = f(S)$, $D = \emptyset$, we get $\mathrm{conv}\, g(T \cup \{x\}) \cap \mathrm{conv}\, g(S) \neq \emptyset$. This completes the proof. $\qquad\square$

2. A characterization

Here we characterize complete Boolean algebras as the only point-convex S_3 convexity absolute extensors. We shall use the following characterization of Boolean algebras, which is an immediate consequence of a theorem of van de Vel [**47**, Thm. 3.5].

LEMMA 2.1. *A convexity space Y is isomorphic to a Boolean algebra iff it is C_2, binary and complemented, i.e. for every $a \in Y$ there exists $b \in Y$ with $[a, b] = Y$.*

PROOF. Let Y satisfy the above assumptions. Then Y is point-convex and S_3, so, applying the embedding theorem, we may assume that $Y \subset \mathcal{P}(S)$ and $\emptyset \in Y$. Now $a \cap b$ is the unique element of $[a, b] \cap [a, \emptyset] \cap [b, \emptyset]$. If T is a complement of $\emptyset$, that is $Y = [\emptyset, T]_Y$, then $Y \subset \mathcal{P}(T)$ and $a \cup b$ is the unique element of $[a, b] \cap [a, T] \cap [b, T]$. Moreover, the complement of a equals exactly $T \setminus a$. thus Y is a subalgebra of $\mathcal{P}(T)$. $\qquad\square$

THEOREM 2.2. *Every point-convex S_3 convexity absolute extensor is isomorphic to a complete Boolean algebra.*

PROOF. Let Y be a point-convex S_3 convexity absolute extensor. We first check the assumptions of Lemma 2.1.

Fix $a \in Y$ and consider a space $X = Y \cup \{p\}$, where $p \notin Y$, with the convexity $\mathcal{G} = \{A \subset X :$ either $|\{a, p\}| = 1$ or $A = X\}$. It is easy to check that $(X, \mathcal{G})$ is S_4 and $Y \in \mathcal{G}$. Let $f \colon Y \to Y$ be the identity map. Then f is CP. Let $\overline{f} \colon X \to Y$ be a CP extension of f. Since $Y \subset [a, p]$ in X we get $Y \subset [a, f(p)]$ in Y. Thus $f(p)$ is a complement of a.

We check that Y is binary. Fix $a, b, c \in Y$. Consider a subspace $X = G \cup \{x\}$ of $\mathbb{R} \times \mathbb{R}$ with the lattice convexity (with coordinate-wise order), where $G = \{(0, 0), (2, 0), (1, 1), (3, 1)\}$, $x = (4, 0)$. It is easy to check that X is S_4 and G is convex in X. Now define $f \colon G \to Y$ by setting $f(0, 0) = \neg c$, $f(2, 0) = b$, $f(1, 1) = \neg a$, $f(3, 1) = c$, where $\neg a, \neg c$ denote the complements of a, c (which are unique by antisymmetry). One can easily observe that f is CP. If $\overline{f} \colon X \to Y$ is an extension of f then setting $y = \overline{f}(x)$ we get $b, c \in [\neg a, y]$ and $b \in [\neg c, y]$. Applying inner transitivity we obtain $y \in [a, b] \cap [a, c] \cap [b, c]$.

Thus we see that Y is isomorphic to a Boolean algebra. Fix a partial order $\leqslant$ on Y induced by a given isomorphism. We show that every maximal linearly ordered subset $L \subset Y$ is complete. Consider $A, B \subset L$ such that $a < b$ for all $a \in A, b \in B$. Let $X = A \cup B \cup \{p\}$ where $p \notin L$

and define a linear order $\leqslant^*$ on X by letting

$$x \leqslant^* y \text{ iff } \begin{cases} x = p & \text{or} \\ x \in B \ \& \ y \in A & \text{or} \\ x, y \in A \ \& \ x \leqslant y & \text{or} \\ x, y \in B \ \& \ y \leqslant x. \end{cases}$$

Every linearly ordered set is an S_4 convexity space (being a distributive lattice). Define $f \colon A \cup B \to Y$ by setting $f(a) = \neg a$ for $a \in A$ and $f(b) = b$ for $b \in B$. Clearly, f is CP; if $\overline{f} \colon X \to Y$ is a CP extension of f then by inner transitivity we get $a \leqslant f(x) \leqslant b$ for all $a \in A$, $b \in B$. Now, if B is the set of all upper bounds of A then $f(x) = \sup A$ in L.

If all maximal chains in a Boolean algebra are complete then the algebra alone is complete, so this finishes the proof. $\qquad \square$

The proof above combined with Theorem 2.4.5 shows also that every S_0 inner transitive geometrical space which is a convexity absolute extensor is isomorphic to a complete Boolean algebra.

CHAPTER 5

Compact median spaces

A classical theorem of Taĭmanov [**45**] states that a map f defined on a dense subset of a topological space X with values in a compact Hausdorff space Y can be extended to a continuous map $F\colon X \to Y$ if and only if f satisfies the following condition: if $A, B \subset Y$ are closed and disjoint then $\operatorname{cl} f^{-1}(A) \cap \operatorname{cl} f^{-1}(B) = \emptyset$, where "cl" denotes the closure in X. We present an analogue of Taĭmanov's Theorem for maps of topological convexity spaces with values in compact median spaces. The class of compact median spaces corresponds to the class of *normally supercompact spaces*, see van Mill [**31**]. We apply the mentioned result to obtain the extension criterion of Sikorski for maps of lattices and to find some linearly ordered subspaces of compact median spaces with infinite Radon independent sets.

The results of Sections 2,3 are contained in our paper [**25**].

1. Convexity and topology

We shall consider *topological convexity spaces*, i.e. triples $(X, \mathcal{T}, \mathcal{G})$ where $\mathcal{T}$ is a topology and $\mathcal{G}$ is a convexity on X. For our purpose, we do not assume any compatibility conditions on $\mathcal{T}$ and $\mathcal{G}$, as it was done in van de Vel's book [**49**]. However, some natural conditions imply that the topology and the convexity are *compatible* in the sense that all polytopes are closed (equivalently, the collection of closed convex sets generates the convexity).

Let X be a topological convexity space. For $A \subset X$ we define its *closed convex hull* $\operatorname{clco} A$ as the intersection of all closed convex sets containing A. Note that $\operatorname{clco} A$ is not necessarily equal to $\operatorname{cl conv} A$ (see [**49**, p. 271]). Every collection of sets $\mathcal{A} \subset \mathcal{P}(X)$ generates a topology $\mathcal{T}$ (as a closed subbase) and a convexity $\mathcal{G}$. In this case we say that $\mathcal{A}$ is *a subbase* for $(X, \mathcal{T}, \mathcal{G})$.

A collection $\mathcal{A}$ of subsets of a set X is called *normal* [**49**, p. 62] provided for each two disjoint sets $A, B \in \mathcal{A}$ there exist $A', B' \in \mathcal{A}$ with $A \cap B' = \emptyset = B \cap A'$ and $A' \cup B' = X$, i.e. A, B are screened with A', B'.

The next result is a generalization of Urysohn's Lemma. For the proof see e.g. van de Vel's book [**49**, p. 331].

THEOREM 1.1 (Frink [**9**]). *Let $\mathcal{A}$ be a normal family of closed convex subsets of a topological convexity space X. Then for each two disjoint sets $A, B \in \mathcal{A}$ there exists a continuous CP map $h\colon X \to [0,1]$ with $A \subset h^{-1}(0)$ and $B \subset h^{-1}(1)$.*

A topological convexity space is called *normal* if the collection of its closed convex sets is normal. *A compact median space* is, by definition, a compact topological space $(X, \mathcal{T})$ with a binary convexity $\mathcal{G}$ satisfying the condition

CC_2: Each two distinct points can be screened with closed convex sets.

By compactness, the collection of all closed convex sets $\mathcal{C}$ is a closed subbase for the topology $\mathcal{T}$ (since it generates a Hausdorff topology). Also, $\mathcal{C}$ is a subbase for the convexity $\mathcal{G}$, since it generates a C_2 convexity (see Corollary 1.8.8).

There is also a notion of *a topological median space*, i.e. a median space with such a topology that the median operator is continuous, see [**49**]. Our definition corresponds to the notion of a compact locally convex median algebra, in the sense of van de Vel [**49**]. Clearly, a compact median space is also a median convexity space, since CC_2 is stronger than C_2.

PROPOSITION 1.2. *Every compact median space is normal.*

PROOF. Let X be a compact median space. Using condition CC_2 we see that one-point subsets are closed and convex. Fix two disjoint closed convex sets $A, B \subset X$.

Suppose first that $A = \{a\}$. Using CC_2 we can find for each $b \in B$ two closed convex sets G_b, F_b with $a \notin F_b$, $b \notin G_b$ and $F_b \cup G_b = X$. Now the collection of closed convex sets $\{B\} \cup \{G_b : b \in B\}$ has empty intersection. By the compactness and binarity of X there exists a $b_0 \in B$ such that $B \cap G_{b_0} = \emptyset$, since $a \in G_b$ for every $b \in B$. Setting $F = F_{b_0}$, $G = G_{b_0}$ we obtain two closed convex sets such that $B \cap G = \emptyset = \{a\} \cap F$ and $F \cup G = X$.

Now let A be an arbitrary closed convex set. Using the first part of our proof we can find for each $a \in A$ two closed convex sets C_a, D_a such that $a \notin C_a$, $B \cap D_a = \emptyset$ and $C_a \cup D_a = X$. The same argument as above gives us an $a_0 \in A$ with $A \cap C_{a_0} = \emptyset$ and setting $C = C_{a_0}$, $D = D_{a_0}$ we get $A \cap C = \emptyset = B \cap D$ and $C \cup D = X$.   □

One can prove that every median space with compact segments is normal, see [**49**, Proposition III.4.13.3]. For our purpose only the normality of compact median spaces will be needed.

COROLLARY 1.3. *Let X be a compact median space. Then the collection of all closed half-spaces in X is a subbase both for the closed sets and for the convexity. The median operator* m$: X \times X \times X \to X$ *is continuous.*

PROOF. If $x \neq y$ then by Proposition 1.2 and Theorem 1.1 there exists a continuous CP map $h: X \to [0, 1]$ with $h(x) = 0$ and $h(y) = 1$. Then the sets $h^{-1}([0, 1/2]), h^{-1}([1/2, 1])$ are closed half-spaces screening x, y. It follows that the collection of closed half-spaces $\mathcal{H}$ generates a Hausdorff topology weaker than the original one; by compactness $\mathcal{H}$ is a subbase for the closed sets. Similarly, $\mathcal{H}$ generates a C_2 binary convexity weaker than the original one; by Corollary 1.8.8 $\mathcal{H}$ is a subbase for the original convexity.

To prove the second statement, consider a closed half-space $H \subset X$. We have

$$\text{m}^{-1}(H) = (H \times H \times X) \cup (H \times X \times H) \cup (X \times H \times H),$$

since m$(a, b, c) \in H$ iff at least two points of a, b, c are in H. Hence m is continuous.   □

The next corollary says that the topology of a compact median spaces is determined by its convexity.

COROLLARY 1.4. *Let $(X, \mathcal{G})$ be a median convexity space. Then there exists at most one topology $\mathcal{T}$ on X such that $(X, \mathcal{T}, \mathcal{G})$ is a compact median space.*

PROOF. Suppose that $\mathcal{T}'$ is a topology such that $(X, \mathcal{T}', \mathcal{G})$ is a compact median space. Fix a proper nonempty half-space H which is $\mathcal{T}'$-closed and fix $p \in X \setminus H$. Let $q \in H \cap \bigcap_{x \in H}[x, p]$ (such a point exists by the fact that $(X, \mathcal{T}', \mathcal{G})$ is compact and binary). Now $H = \{x \in$

$X \colon q = \mathrm{m}(p, q, x)\}$. As $(X, \mathcal{T})$ is T_1 and the median operator is $\mathcal{T}$-continuous, we see that H is $\mathcal{T}$-closed. Thus $\mathcal{T}' \subset \mathcal{T}$, by Corollary 1.3. By symmetry we get $\mathcal{T} = \mathcal{T}'$. □

Let us note that there are median convexity spaces which are not compact median spaces with any topology. The most basic example is the set of natural numbers endowed with the convexity induced by the linear order.

The notion of a compact median space appeared in a natural way in the theory of supercompact spaces, see van Mill [**31**]. A topological space $(X, \mathcal{T})$ is *supercompact*, if it possesses *a binary closed subbase* $\mathcal{B}$, i.e. a subbase for the closed sets which has the property that every its linked subcollection has nonempty intersection. By Alexander Subbase Lemma [**8**, p. 257] the space $(X, \mathcal{T})$ is compact. If $\mathcal{G}$ is the interval convexity generated by $\mathcal{B}$ then $(X, \mathcal{G})$ is a binary geometrical space. If, moreover, each two distinct points of X are screened with sets from $\mathcal{B}$ then $(X, \mathcal{T}, \mathcal{G})$ is a compact median space. In this case $(X, \mathcal{T})$ is called *normally supercompact*.

The class of normally supercompact spaces has many interesting properties, see e.g. van Mill [**31**]. Let us mention that every compact metric space is supercompact (the theorem of Strok-Szymański [**42**]) but a normally supercompact connected metric space is an absolute retract, see [**31**].

2. Extension criterion

We describe a condition for maps, an analogue to what has appeared in Taĭmanov's Theorem.

PROPOSITION 2.1. *Let X, Y be two topological convexity spaces, $M \subset X$ and let $f \colon M \to Y$ be a map. The following conditions are equivalent:*

(a) $\mathrm{clco}\, f(A) \cap \mathrm{clco}\, f(B) \neq \emptyset$ *holds for each $A, B \subset M$ with $\mathrm{clco}\, A \cap \mathrm{clco}\, B \neq \emptyset$.*
(b) *If $C, D \subset Y$ are closed convex and disjoint then $\mathrm{clco}\, f^{-1}(C) \cap \mathrm{clco}\, f^{-1}(D) = \emptyset$.*

PROOF. (a) $\implies$ (b) If $\mathrm{clco}\, f^{-1}(C) \cap \mathrm{clco}\, f^{-1}(D) \neq \emptyset$ then, by condition (a), we have $\emptyset \neq \mathrm{clco}\, f(f^{-1}(C)) \cap \mathrm{clco}\, f(f^{-1}(D)) \subset C \cap D$.
(b) $\implies$ (a) If $\mathrm{clco}\, f(A) \cap \mathrm{clco}\, f(B) = \emptyset$ then, by condition (b), we get

$$\emptyset = \mathrm{clco}\, f^{-1}(\mathrm{clco}\, f(A)) \cap \mathrm{clco}\, f^{-1}(\mathrm{clco}\, f(B)) \supset \mathrm{clco}\, A \cap \mathrm{clco}\, B.$$

□

We say that a map *f satisfies condition (T)* if f fulfills (a) (or (b)) above. Observe that if X and Y are topological spaces considered with discrete convexity then condition (T) is exactly the condition of Taĭmanov.

Let X be a topological convexity space (or, in particular, a median space). A subset $M \subset X$ is *median-stable* provided for each $a, b, c \in M$

$$[a, b] \cap [a, c] \cap [b, c] \subset M.$$

We say that a subset $M \subset X$ is *geometrically dense* if X is the least closed median-stable subset containing M. Notice that every topologically dense set is geometrically dense (see also [**49**, Lemma I.6.20] for other examples).

Now we can state our main result. The proof is quite similar to the proof of Taĭmanov's Theorem in [**8**, p. 164].

THEOREM 2.2. *Let M be a geometrically dense subset of a topological convexity space X, let Y be a compact median space and let $f\colon M \to Y$ be a map satisfying condition (T). Then there exists a unique continuous convexity preserving map $F\colon X \to Y$ such that $F|M = f$.*

PROOF. (i) For $x \in X$ denote by $\operatorname{nbd}(x)$ the collection of all open concave sets containing x and set
$$\mathcal{F}(x) = \{\operatorname{clco} f(M \cap U) : \ U \in \operatorname{nbd}(x)\}.$$
Since M is geometrically dense, if $U_1, U_2 \in \operatorname{nbd}(x)$ then $U_1 \cap U_2 \cap M \neq \emptyset$ (otherwise M would be contained in a proper closed median-stable set $(X \setminus U_1) \cup (X \setminus U_2)$). Hence $\operatorname{clco} f(U_1 \cap M) \cap \operatorname{clco} f(U_2 \cap M) \supset \operatorname{clco} f(U_1 \cap U_2 \cap M) \neq \emptyset$. The compactness and binarity of Y imply that $\bigcap \mathcal{F}(x) \neq \emptyset$.

(ii) Suppose that there exist distinct $y_1, y_2 \in \bigcap \mathcal{F}(x)$. Since Y is normal (Proposition 2.1), there exist two open concave sets $U_1, U_2 \subset Y$ with $y_i \in U_i$ and $\operatorname{clco} U_1 \cap \operatorname{clco} U_2 = \emptyset$. Condition (T) implies that $\operatorname{clco} f^{-1}(U_1) \cap \operatorname{clco} f^{-1}(U_2) = \emptyset$. Assume that $x \notin \operatorname{clco} f^{-1}(U_1)$. Setting $W = X \setminus \operatorname{clco} f^{-1}(U_1)$ we have $W \in \operatorname{nbd}(x)$ and hence $\operatorname{clco} f(M \cap W) \in \mathcal{F}(x)$. On the other hand
$$\operatorname{clco} f(M \cap W) = \operatorname{clco} f(M \setminus \operatorname{clco} f^{-1}(U_1))$$
$$\subset \operatorname{clco} f(M \setminus f^{-1}(U_1)) \subset \operatorname{clco}(Y \setminus U_1) = Y \setminus U_1,$$
which gives a contradiction, since $y_1 \in \operatorname{clco} f(M \cap W)$.

(iii) Thus we have proved that $|\bigcap \mathcal{F}(x)| = 1$ for every $x \in X$. Define $F\colon X \to Y$ by letting $F(x) \in \bigcap \mathcal{F}(x)$. If $x \in M$ then $f(x) \in \bigcap \mathcal{F}(x)$, consequently $F(x) = f(x)$. It remains to check that F is continuous and CP.

(iv) Let $U \in \operatorname{nbd}(F(x))$. As $\bigcap \mathcal{F}(x) = \{F(x)\}$, we have
$$(Y \setminus U) \cap \bigcap \mathcal{F}(x) = \emptyset.$$
Now the binarity and the compactness of Y give a $W \in \operatorname{nbd}(x)$ with $\operatorname{clco} f(M \cap W) \cap (Y \setminus U) = \emptyset$. It follows that for each $x' \in W$ we have $F(x') \in \operatorname{clco} f(M \cap W) \subset U$. Hence $F(W) \subset U$. In view of Lemma 2.1, F is continuous and convexity preserving.

(v) If $F_1, F_2\colon X \to Y$ are two continuous CP extensions of f then the set
$$G = \{x \in X : \ F_1(x) = F_2(x)\}$$
is closed median-stable and contains M; hence $G = X$ and $F_1 = F_2$. This completes the proof. $\qquad\square$

REMARKS 2.3. Condition (T) is necessary for the existence of a continuous CP extension. Indeed, if f can be extended to a continuous CP map $F\colon X \to Y$ then for two disjoint closed convex sets $C, D \subset Y$ we have $\operatorname{clco} f^{-1}(C) \cap \operatorname{clco} f^{-1}(D) \subset F^{-1}(C) \cap F^{-1}(D) = \emptyset$.

3. Applications

One can easily observe that Theorem 2.2 implies the classical Taĭmanov's theorem. Indeed, let $f\colon M \to Y$ be a map satisfying the condition of Taĭmanov, where M is a (topologically) dense subset of a topological space X and Y is a compact Hausdorff space. Embed Y into a Hilbert cube $H = [0,1]^\kappa$ and consider X as a topological convexity space with discrete convexity. Now H is a compact median space and $f\colon M \to H$ satisfies (T). Applying Theorem 2.2 we obtain a unique continuous map $F\colon X \to H$ which extends f. Finally, $F(X) = F(\operatorname{cl} M) \subset \operatorname{cl} F(M) = \operatorname{cl} f(M) \subset Y$, since Y is closed in H.

We now give an application of Theorem 2.2 to the theory of superextensions.

A collection $\mathcal{P}$ is called a T_1-*subbase* for a topological convexity space $(X, \mathcal{T}, \mathcal{G})$ provided:

(i) $\mathcal{P}$ is a closed subbase for the topology $\mathcal{T}$ and $\mathcal{P}$ generates the convexity $\mathcal{G}$;
(ii) for every $x \in X$ there exists a $P \in \mathcal{P}$ with $x \notin P$;
(iii) if $x \notin P \in \mathcal{P}$ then there exists a $Q \in \mathcal{P}$ with $x \in Q$ and $P \cap Q = \emptyset$.

Let $\mathcal{P}$ be a T_1-subbase of a space X. Then there exists a topological convexity space $\lambda(X, \mathcal{P})$, called *the superextension of X with respect to $\mathcal{P}$*, with the following properties:

(1) X is continuously CP embedded into $\lambda(X, \mathcal{P})$ and X is geometrically dense in $\lambda(X, \mathcal{P})$.
(2) If $P, Q \in \mathcal{P}$ are disjoint then their closed convex hulls in $\lambda(X, \mathcal{P})$ are disjoint as well.
(3) If $\mathcal{A}$ is any collection of closed convex subsets of $\lambda(X, \mathcal{P})$ with $\bigcap \mathcal{A} = \emptyset$ then there exist $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$.

The details of the construction one can find in van de Vel's monograph [**49**, pp. 13, 279] or, in a different language, in van Mill's book [**31**]. Condition (3) says that the collection of all closed convex subsets of $\lambda(X, \mathcal{P})$ is binary. If a T_1-subbase $\mathcal{P}$ is normal then $\lambda(X, \mathcal{P})$ satisfies condition CC_2 and consequently it is a compact median space.

Applying Theorem 2.2 and condition (2) above, we obtain the following result due to Verbeek [**50**] and van Mill & van de Vel [**32**] (see also [**49**, Corollary III.4.17]).

THEOREM 3.1. *Let $\mathcal{P}$ be a T_1-subbase of a topological convexity space X, let Y be a compact median space and let $f \colon X \to Y$ be such a map that $f^{-1}(G) \in \mathcal{P}$ whenever $G \subset Y$ is closed convex. Then there exists a unique continuous convexity preserving map $F \colon \lambda(X, \mathcal{P}) \to Y$ such that $F|X = f$.*

REMARKS 3.2. Let Y be a topological convexity space satisfying the condition CC_2 and suppose that Y fulfills the statement of Theorem 2.2. Then Y is a compact median space.

Indeed, taking the collection $\mathcal{P}(Y)$ of all subsets of Y we see that $\mathcal{P}(Y)$ is a normal T_1-subbase for the discrete topology and the discrete convexity on Y. Therefore, by Theorem 3.1, the identity map $\mathrm{id}_Y \colon Y \to Y$ can be extended to a continuous CP map $F \colon \lambda(Y, \mathcal{P}(Y)) \to Y$ which is onto. It follows that Y is a compact median space, as a continuous CP image of such a space.

Let us now present a discrete version of Theorem 2.2. We need an auxiliary result on median stabilization.

LEMMA 3.3. *In every median space, the median stabilization of a finite set is finite.*

PROOF. Let Y be a median space. As Y is S_4, applying the embedding theorem 1.9.1 we may assume that $Y \subset \mathcal{P}(\mathcal{H})$, where $\mathcal{H}$ is the collection of all half-spaces in Y. Now the median of $a, b, c \in Y$ is equal to $(a \cap b) \cup (a \cap c) \cup (b \cap c)$. It follows that the median stabilization of a finite set $S \subset Y$ is contained in the lattice of sets generated by S and therefore is finite. $\quad\square$

THEOREM 3.4 ([**23**]). *Let Y be a median convexity space and let $f \colon M \to Y$ be a map defined on a geometrically dense subset of a convexity space X. If f satisfies the condition*

(I) $\mathrm{conv}\, f(S) \cap \mathrm{conv}\, f(T) \neq \emptyset$ *whenever $S, T \subset M$ are finite and $\mathrm{conv}\, S \cap \mathrm{conv}\, T \neq \emptyset$,*

then there exists a unique CP map $F \colon X \to Y$ such that $F|M = f$.

PROOF. If $S \in [M]^{<\omega}$ then, according to Lemma 3.3, med $f(S)$ is finite and can be viewed as a compact median space (with the discrete topology). Moreover $f|S$ satisfies condition (T). Applying Theorem 2.2 we obtain a unique CP extension $F_S \colon \mathrm{med}\, S \to Y$ of $f|S$. By uniqueness we have $F_S \subset F_T$ whenever $S \subset T$. Then setting $F = \bigcup_{S \in [M]^{<\omega}} F_S$ we obtain a convexity preserving map with the domain $\bigcup_{S \in [M]^{<\omega}} \mathrm{med}\, S = \mathrm{med}\, M = X$ and $F|M = f$. □

Finally, we apply the last result to obtain an extension criterion for maps of lattices which, in the case of Boolean algebras, is known as Sikorski's Extension Criterion [**38**].

THEOREM 3.5. *Let L be a distributive lattice and let K be a lattice generated by its subset M. If $f \colon M \to L$ is a map satisfying the implication*

(S) $a_1 \wedge \cdots \wedge a_n \leqslant b_1 \vee \cdots \vee b_m \implies f(a_1) \wedge \cdots \wedge f(a_n) \leqslant f(b_1) \vee \cdots \vee f(b_m).$

for all $a_1, \ldots, a_n, b_1, \ldots, b_m \in M$, then f can be uniquely extended to a lattice homomorphism $F \colon K \to L$.

PROOF. First, add to K two elements $0_K, 1_K$ in such a way that $0_K < x < 1_K$ for all $x \in K$ and set $K' = K \cup \{0_K, 1_K\}$. Let us make the same operation for L and set $L' = L \cup \{0_L, 1_L\}$. Then K' is a lattice and L' is a distributive lattice. Now set $M' = M \cup \{0_K, 1_K\}$ and extend f to a map $f' \colon M' \to L'$ by letting $f'(0_K) = 0_L$, $f'(1_K) = 1_L$. It is easy to see that f' satisfies the condition (S) above. If G is a median-stable subset of K' containing M' then G is a sublattice, since for $x, y \in G$ we have $x \wedge y \in [x, y] \cap [x, 0_K] \cap [y, 0_K]$ and $x \vee y \in [x, y] \cap [x, 1_K] \cap [y, 1_K]$. It follows that M' is geometrically dense in K'.
In any lattice, the convex hull of a finite set P is equal to the segment $[\inf P, \sup P]$. Let S, T be two finite subsets of M'. If there exists a point $x \in \mathrm{conv}\, S \cap \mathrm{conv}\, T$ then $\inf S \leqslant x \leqslant \sup T$ and $\inf T \leqslant x \leqslant \sup S$. By condition (S) we get $\inf f'(S) \leqslant \sup f'(T)$ and $\inf f'(T) \leqslant \sup f'(S)$. Hence, setting $y = \inf f'(S) \vee \inf f'(T)$, we have $y \in \mathrm{conv}\, f'(S) \cap \mathrm{conv}\, f'(T)$. It follows that f' satisfies the condition (I) of Theorem 3.4.
Since every distributive lattice is a median convexity space, we can apply Theorem 3.4 to obtain a unique CP map $F \colon K' \to L'$ with $F|M' = f'$. By Proposition 1.8.4, F is a lattice homomorphism. Finally, $F(K)$ is a sublattice generated by $f(M)$ and hence $F(K) \subset L$. This completes the proof.   □

Let us finally mention that the last theorem is no longer true when we drop the assumption of the distributivity of the lattice L. Indeed, if L satisfies the above statement then taking K equal to the free distributive lattice generated by the set L we see that L is a homomorphic image of a distributive lattice; consequently L is distributive.

4. Radon independent sets

Using Theorem 2.2, we shall show that every compact median space having an infinite Radon independent set contains a linearly ordered subspace of size continuum, a CP continuous image of the compact linearly ordered space $\mathbb{S}$ (see the definition below). The space $\mathbb{S}$ is universal in the following sense: every compact linearly ordered space with a dense subset isomorphic to the set of rationals $\mathbb{Q}$ is a continuous CP image of $\mathbb{S}$ (see Lemma 4.2 below).
A subset M of a convexity space X is called *Radon independent* [**49**] provided for every two disjoint finite sets $S, T \subset M$ it holds $\mathrm{conv}\, S \cap \mathrm{conv}\, T = \emptyset$. Then the same is true for arbitrary two disjoint subsets of M. For a cardinal κ denote by $\lambda\kappa$ the superextension of the discrete space κ with respect to the collection of all its subsets (see the previous section). Then,

identifying κ with a subset of $\lambda\kappa$ we see that κ is Radon independent, by condition (2) in the previous section.

The notion of Radon independence was inspired by the classical theorem of Radon [**37**] which states that every at least $N+2$ element subset of the Euclidean space $\mathbb{R}^N$ (with the Euclidean convexity) contains two disjoint subsets S, T with $\operatorname{conv} S \cap \operatorname{conv} T \neq \emptyset$.

In this section we denote by $\mathbb{Q}_I$ the set of all rationals from the closed unit inerval in $\mathbb{R}$.

THEOREM 4.1. *Let X be a median space with an infinite Radon independent subset M. Then for each two distinct points $a, b \in M$ there exists a convexity preserving embedding $h\colon \mathbb{Q}_I \to X$ such that $h(0) = a$ and $h(1) = b$.*

PROOF. Denote by $\leqslant$ the, so called, *base-point order* on X induced by a, i.e. $x \leqslant y$ iff $x \in [a, y]$ (cf. [**49**, p. 91]). Observe that every order preserving map from a linearly ordered space into X is convexity preserving, since $x \leqslant z \leqslant y$ in X implies, by inner transitivity, that $z \in [x, y]$. Let $\mathbb{Q}_I = \{q_n \colon n \in \omega\}$ where $q_0 = 0$ and $q_1 = 1$. Let $\{s_n\}_{n \in \omega} \subset M$ be a one-to-one sequence such that $s_0 = a$ and $s_1 = b$. We construct inductively a sequence $\{x_n\}_{n \in \omega} \subset [a, b]$ such that $x_0 = a$, $x_1 = b$ and

    (i) $q_i < q_j$ implies $x_i \leqslant x_j$,
    (ii) if $i < j$ then there exist disjoint sets $S, T \subset \{s_0, \ldots, s_j\}$ such that $x_i \in \operatorname{conv} S$ and $x_j \in \operatorname{conv} T$.

Suppose that $n > 0$ and $x_0, \ldots, x_{n-1}$ are already defined. Consider such $k, l < n$ that $q_k < q_n < q_l$ and q_k, q_l are nearest to q_n. Set $x_n = m(x_k, x_l, s_n)$. Then $x_n \in [x_k, x_l] \subset [a, b]$, hence $x_k \leqslant x_n \leqslant x_l$ and consequently condition (i) is satisfied. Fix $i < n$ and assume that e.g. $i \neq k$. By induction hypothesis, there exist disjoint sets $S, T \subset \{s_0, \ldots, s_{n-1}\}$ with $x_i \in \operatorname{conv} S$ and $x_k \in \operatorname{conv} T$. Set $T_1 = T \cup \{s_n\}$. Then $x_n \in [x_k, s_n] \subset \operatorname{conv} T_1$ and $S \cap T_1 = \emptyset$. This shows condition (ii).

Now define $h(q_n) = x_n$ for $n \in \omega$. By condition (i), h is convexity preserving. If $n < m$ then by condition (ii), the points x_n, x_m lie in the convex hulls of disjoint subsets of M. In particular, $x_n \neq x_m$ since M is Radon independent. It follows that h is a CP embedding. $\square$

Denote by $\mathbb{S}$ the compact linearly ordered space $((0, 1] \times \{-1\}) \cup (\mathbb{Q}_I \times \{0\}) \cup ([0, 1) \times \{1\})$ with lexicographic ordering, i.e. $(x, i) < (y, j)$ iff $x < y$ or $x = y$ and $i < j$. Observe that $\mathbb{Q}_I \times \{0\}$ is a topologically dense open and discrete subset of $\mathbb{S}$, CP isomorphic to $\mathbb{Q}_I$. We shall identify $\mathbb{Q}_I$ with $\mathbb{Q}_I \times \{0\}$. The subspace $\mathbb{S} \setminus \mathbb{Q}_I$ is also known under the name of *the double arrow line.*

LEMMA 4.2. *Every convexity preserving map $h\colon \mathbb{Q}_I \to X$ into a compact median space X can be uniquely extended to a continuous CP map $\overline{h}\colon \mathbb{S} \to X$.*

PROOF. In view of Theorem 2.2 it is enough to check that every CP map $h\colon \mathbb{Q}_I \to X$ satisfies condition (T). We use Proposition 2.1(b). Let $C, D \subset X$ be closed convex and disjoint. Then $A = f^{-1}(C)$ and $B = f^{-1}(D)$ are two disjoint convex subsets of $\mathbb{Q}_I$. Thus there exists an $r \in \mathbb{R}$ between A, B. Assume that $A \subset [0, r]$ and $B \subset [r, 1]$. Define $U_r^j = \{(x, i) \in \mathbb{S} \colon (r, j) \leqslant (x, i)\}$ and $L_r^j = \{(x, i) \in \mathbb{S} \colon (x, i) \leqslant (r, j)\}$. Then U_r^j, L_r^j are closed half-spaces in $\mathbb{S}$ and there are $j, k \in \{-1, 0, 1\}$ with $j < k$ and $A \subset L_r^j$ and $B \subset U_r^k$. It follows that $\operatorname{clco} A \cap \operatorname{clco} B = \emptyset$. $\square$

In terms of superextensions, the above lemma says that $\mathbb{S}$ is the superextension of $\mathbb{Q}_I$ with respect to the collection of all its convex subsets.

THEOREM 4.3. *Let X be a compact median space with an infinite Radon independent set M. Then for each two distinct points $a, b \in M$ there exists a continuous CP map $h\colon \mathbb{S} \to X$ such that $h|\mathbb{Q}_I$ is one-to-one and $h(0) = a$, $h(1) = b$. In particular, $h(\mathbb{S})$ is a compact linearly ordered subspace of $[a, b]$ of size continuum.*

PROOF. The existence of a map h follows from Theorem 4.1 and Lemma 4.2. Clearly $h(\mathbb{S})$ is compact and linearly ordered. For $p \in X$ the pre-image $P = h^{-1}(p)$ is convex in $\mathbb{S}$, thus $|P \cap \mathbb{Q}_I| \leqslant 1$, so $|P| \leqslant 3$. It follows that $h(\mathbb{S})$ has size continuum. $\qquad\square$

The last result describes situations when $\mathbb{S}$ can be topologically CP embedded into a compact median space.

THEOREM 4.4. *Let M be an infinite subset of a compact median space X satisfying the condition $\mathrm{clco}\, A \cap \mathrm{clco}\, B = \emptyset$ whenever $A, B \subset M$ are disjoint. Then for every two distinct points $a, b \in M$ there exists a continuous CP embedding $\varphi\colon \mathbb{S} \to X$ with $\varphi(0) = a$ and $\varphi(1) = b$.*

PROOF. Set $Z = \mathrm{med}\, M$. As M is Radon independent, we can apply Theorem 4.1 to get a CP embedding $h\colon \mathbb{Q}_I \to Z$ with $h(0) = a$ and $h(1) = b$. Now we set $L = h(\mathbb{Q}_I)$ and $Y = \mathrm{cl}_X L$. Clearly, Y is a compact median space and L is geometrically dense in Y. We shall show that every CP map defined on L which has values in a compact median space can be uniquely extended to a continuous CP map onto Y. Then, using arguments from category theory, we can deduce that Y is CP isomorphic and homeomorphic to $\mathbb{S}$. We show that every CP map defined on L satisfies condition (T). By Proposition 2.1(b), it is enough to check that each two disjoint sets $A, B \subset L$, which are convex in L, have disjoint closed convex hulls in X.

Fix two disjoint, convex in L, sets $A, B \subset L$. Then $\mathrm{conv}_X A \cap \mathrm{conv}_X B = \emptyset$. Indeed, for $S \in [A]^{<\omega}, T \in [B]^{<\omega}$ we have either $\sup S < \inf T$ or $\sup T < \inf S$. If $\sup S < \inf T$ then every half-space separating these two points separates also S, T. Thus $\mathrm{conv}_X S \cap \mathrm{conv}_X T = \emptyset$ and the same holds for A, B. Now, as X is S_4, there exists a half-space $H \subset X$ with $A \subset H$ and $B \cap H = \emptyset$. By the assumption on M and Theorem 1.1, there exists a continuous CP map $f\colon X \to [0, 1]$ with $M \cap H \subset f^{-1}(1)$ and $M \setminus H \subset f^{-1}(0)$. Observe that the set $Z' = \{x \in Z \colon f(x) = \chi_H(x)\}$ is median-stable and contains M. Thus $Z' = Z$ and $A \subset f^{-1}(1)$, $B \subset f^{-1}(0)$. It follows that $\mathrm{clco}\, A \cap \mathrm{clco}\, B = \emptyset$. $\qquad\square$

The above result can be applied for the space $\lambda\kappa$, where κ is infinite. Theorem 4.4 says that each two distinct points $\alpha, \beta \in \kappa$ are contained in a subspace of $\lambda\kappa$ which is CP isomorphic and homeomorphic to $\mathbb{S}$.

CHAPTER 6

Topological retracts of Cantor cubes

In this chapter we shall show that every topological retract of a Cantor cube, called *a zero-dimensional Dugundji space*, has a binary subbase which is closed under the complements. This strengthens the result of Heindorf [**13**] which says that zero-dimensional Dugundji spaces are supercompact and admit binary subbases consisting of clopen sets. Introducing a suitable convexity we construct a binary subbase consisting of convex sets, which is closed under the complements. We use a result due to Haydon [**12**] and Koppelberg [**21**] which says that every zero-dimensional Dugundji space can be represented as the inverse limit of a suitable, simple inverse system. The proof of our main result is simpler than Heindorf's one and does not require algebraic or lattice structures.

The contents of this chapter is a joint work with A. Kucharski [**27**].

1. Boolean median spaces

A Boolean median space is, by definition, a zero-dimensional compact median space. It follows from the results of van de Vel [**48**] on dimension of topological convex structures that every Boolean median space has a subbase for closed sets consisting of clopen (i.e. closed and open) half-spaces. More precisely:

LEMMA 1.1. *Let A, B be two disjoint closed convex subsets of a Boolean median space X. Then there exists a clopen half-space $H \subset X$ such that $A \cap H = \emptyset$ and $B \subset H$.*

PROOF. Fix $a_1 \in A$ and take a $b_0 \in B \cap \bigcap_{y \in B}[a_1, y]$. By binarity, such a point exists (in fact, by axiom C_2, it is unique). Similarly, let $a_0 \in A \cap \bigcap_{x \in A}[x, b_0]$. Then, by inner transitivity, we have $a_0 \in [x, b_0]$ and $b_0 \in [a_0, y]$ for $x \in A, y \in B$. It follows that if H is such a half-space that $b_0 \in H$ and $a_0 \notin H$ then $B \subset H$ and $A \cap H = \emptyset$. Thus we have to find a clopen half-space separating a_0 from b_0. The pair (a_0, b_0) is called *a pair of gates between A, B*; see [**49**, p. 98]). Consider the segment $L = [a_0, b_0]$. Applying the embedding theorem we see that L can be viewed as a (distributive) lattice with the smallest element a_0 and the greatest one b_0. The partial order is given by $x \leqslant y$ iff $x \in [a_0, y]$; see also [**49**, p. 91] where $\leqslant$ is called *the base-point order* induced by a_0. Let D be a maximal linearly ordered subset of L. Then $a_0, b_0 \in D$ and D is closed (since the closure of a linearly ordered set is still linearly ordered). Thus D is a compact linearly ordered subspace of L. Since D is not connected, there exist $a_1, b_1 \in D$ such that $a_1 < b_1$ and $[a_1, b_1] = \{a_1, b_1\}$. Now define

$$H = \{x \in X : b_1 = \mathrm{m}(a_1, b_1, x)\}.$$

Clearly, by Corollary 1.3, H is closed. If $x, y \in H$ and $z \in [x, y]$ then $z \in H$. Indeed, otherwise there exists a half-space K with $b_1 \notin K$ and $a_1, z \in K$ and then $x \in K$ or $y \in K$; a contradiction. Thus H is convex. Set $G = \{x \in X : a_1 = \mathrm{m}(a_1, b_1, x)\}$. By the same argument as above, G is closed convex and disjoint from H. Moreover $G \cup H = X$ since for every $x \in X$

we have $m(a_1, b_1, x) \in [a_1, b_1]$ and consequently $m(a_1, b_1, x) \in \{a_1, b_1\}$. It follows that H is a clopen half-space. Clearly $b_0 \in H$ and $a_0 \notin H$. $\square$

The above lemma says topologically that each normally supercompact topological space has a binary subbase closed under the complements, in particular such a subbase consists of clopen sets. A similar result for zero-dimensional supercompact topological spaces is not valid. Namely, there exists a supercompact zero-dimensional space which has no binary subbase consisting of clopen sets, see Bell & Ginsburg [1].

2. Inverse systems

Let Σ be a directed partially ordered set and let $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$ be an inverse system of sets such that each X_σ is a topological convexity space and p_σ^τ's are continuous CP-maps, i.e. $p_\sigma^\tau \colon X_\tau \to X_\sigma$ and $p_\sigma^\tau p_\tau^\mu = p_\sigma^\mu$ whenever $\sigma \leqslant \tau \leqslant \mu$. The system S will be called *an inverse system of topological convexity spaces.* Let $\varprojlim S$ be the inverse limit of S in the category of sets, i.e. the set consisting of all points $x \in \prod_{\sigma \in \Sigma} X_\sigma$ such that $p_\sigma^\tau(x(\tau)) = x(\sigma)$ for all $\sigma \leqslant \tau$. Denote by p_σ the projection of $\varprojlim S$ into X_σ. The set $\varprojlim S$ is a topological space, with the topology $\mathcal{T}$ generated by sets $p_\sigma^{-1}(U)$ where U is open in X_σ, $\sigma \in \Sigma$. It is also equipped with a convexity $\mathcal{G}$ inherited from the product $\prod_{\sigma \in \Sigma} X_\sigma$. It is easy to check that $(\varprojlim S, \mathcal{T}, \mathcal{G})$ is the inverse limit in the category of topological convexity spaces. In other words, if Y is a topological convexity space and $\{f_\sigma \colon Y \to X_\sigma\}_{\sigma \in \Sigma}$ is a collection of continuous CP maps such that $p_\sigma^\tau f_\tau = f_\sigma$ for $\sigma \leqslant \tau$ then there exists a unique continuous CP map $h \colon Y \to X$ with the property $p_\sigma h = f_\sigma$ for all $\sigma \in \Sigma$. Observe that the equality

$$[a, b]_\mathcal{G} = (\varprojlim S) \cap \prod_{\sigma \in \Sigma} [a(\sigma), b(\sigma)]$$

holds for each $a, b \in \varprojlim S$.

PROPOSITION 2.1. *The inverse limit of a system of Boolean median spaces is a Boolean median space.*

PROOF. Let $S = \{X_\sigma, p_\sigma^\tau, \Sigma\}$ be an inverse system of Boolean median spaces and let $X = \varprojlim S$. Clearly X is a compact space. If $a, b \in X$ and $a \neq b$ then there exists $\sigma \in \Sigma$ with $a(\sigma) \neq b(\sigma)$. Thus, by Lemma 1.1, there exists a clopen half-space $H \subset X_\sigma$ with $a(\sigma) \notin H$ and $b(\sigma) \in H$. Now $p_\sigma^{-1}(H)$ is a clopen half-space separating a, b.

Fix $a, b, c \in X$. Let $x_\sigma = m(a(\sigma), b(\sigma), c(\sigma))$. For $\sigma \leqslant \tau$ we have $p_\sigma^\tau(x_\tau) = x_\sigma$ since p_σ^τ preserves medians. Thus there exists $x \in X$ with $x(\sigma) = x_\sigma$ for each $\sigma \in \Sigma$. Clearly $x = m(a, b, c)$. It follows that X is a Boolean median space. $\square$

3. Main result

We start with an auxiliary lemma, which can be derived from The Normal Form Theorem [20] in the theory of Boolean algebras.

LEMMA 3.1. *Let $\mathcal{P}$ be a subbase of a zero-dimensional compact topological space X, which is closed under the complements, and let $\mathcal{B}$ be the collection of all finite intersections of sets from $\mathcal{P}$. Then every clopen subset of X can be partitioned into a finite number of members of $\mathcal{B}$.*

PROOF. Set $\mathfrak{M} = \{M_1 \cup \cdots \cup M_n : \; M_i \in \mathcal{B}, M_i \cap M_j = \emptyset$ for $i \neq j, n \in \omega\}$. Since $\mathcal{B} \subset \mathfrak{M}$ and $\mathcal{B}$ is an open base of X, it is enough to show that $\mathfrak{M}$ is an algebra of sets. Fix $B, C \in \mathfrak{M}$ and let B and C have partitions $M_1 \cup \cdots \cup M_n$ and $N_1 \cup \cdots \cup N_k$ respectively, where $M_i, N_j \in \mathcal{B}$. We have $B \cap C = \bigcup_{i,j} M_i \cap N_j$ and $M_i \cap N_j$'s are pairwise disjoint. It follows that $B \cap C \in \mathfrak{M}$. Now observe that $X \setminus M \in \mathfrak{M}$ for $M \in \mathcal{B}$. Indeed, if $M = H_1 \cap \cdots \cap H_n$, where $H_i \in \mathcal{P}$ for $i \leqslant n$, then $X \setminus M$ has a partition into sets of the form $H_1^{\varepsilon(1)} \cap \cdots \cap H_n^{\varepsilon(n)}$, where $\varepsilon \colon \{1, \ldots, n\} \to \{-1, 1\}$ is a function not equal constantly to 1 and $H_i^1 = H_i$, $H_i^{-1} = X \setminus H_i$. Finally, if $B = M_1 \cup \cdots \cup M_n \in \mathfrak{M}$, where $M_i \in \mathcal{B}$, then the set $X \setminus B = \bigcap_{i \leqslant n} (X \setminus M_i)$ does belong to $\mathfrak{M}$. This completes the proof. $\qquad\square$

THEOREM 3.2. *If X is a retract of a Cantor cube then there exists a convexity on X such that X is a Boolean median space.*

PROOF. According to Haydon [12] and Koppelberg [21, Thm. 2.7] we can represent X as the limit of an inverse system $S = \{X_\alpha, p_\alpha^\beta, \alpha < \beta < \tau\}$ with the following properties:

(1) $|X_0| = 1$,
(2) $X_\gamma = \varprojlim\{X_\alpha, \alpha < \gamma\}$ for limit ordinals $\gamma < \tau$,
(3) $X_{\alpha+1} = (X_\alpha \times \{0\}) \cup (U_\alpha \times \{1\})$ where U_α is clopen in X_α and $p_\alpha^{\alpha+1} \colon X_{\alpha+1} \to X_\alpha$ is the projection.

We define inductively suitable convexities in X_α's in such a way that each X_α becomes a Boolean median space and each $p_\alpha^{\alpha+1}$ becomes a CP map. Suppose that this is already done for all $\xi < \gamma$ and assume that $\gamma = \alpha + 1$.

By Lemma 3.1, $U_\alpha = G_1 \cup \cdots \cup G_n$ where G_i's are pairwise disjoint clopen and convex. Hence $X_{\alpha+1} = X_\alpha \oplus G_1 \oplus \cdots \oplus G_n$ and $p_\alpha^{\alpha+1}$ is the superposition of n projections of the form $X_\alpha \oplus G_1 \oplus \cdots \oplus G_{i+1} \to X_\alpha \oplus G_1 \oplus \cdots \oplus G_i$. Thus we may assume that U_α is convex. Now $X_{\alpha+1}$ is a median-stable subset (the union of two convex sets) of the product $X_\alpha \times \{0, 1\}$. It follows that $X_{\alpha+1}$ endowed with the subspace convexity is a Boolean median space. Clearly $p_\alpha^{\alpha+1}$ is a CP map.

If γ is a limit ordinal and convexities $\mathcal{G}_\alpha$ are already defined for $\alpha < \gamma$ then, by Proposition 2.1, X_γ is a Boolean median space. This completes the proof. $\qquad\square$

REMARKS 3.3. Actually, we have proved that if X is a zero-dimensional Dugundji space then there exists a convexity $\mathcal{G}$ in X and an inverse system of Boolean median spaces $S = \{(X_\alpha, \mathcal{G}_\alpha), p_\alpha^\beta, \alpha < \beta < \tau\}$ such that $(X, \mathcal{G}) = \varprojlim S$ and S has properties $(1)-(3)$ above, with $X_{\alpha+1} = X_\alpha \oplus U_\alpha$, where U_α is clopen and convex in X_α. On the other hand, by Haydon's Theorem [12], the inverse limit of such a system S is a topological zero-dimensional Dugundji space.

COROLLARY 3.4. *Every retract of a Cantor cube has a binary subbase closed under the complements.*

The example below shows that the converse assertion does not hold.

EXAMPLE 3.5. Consider the one-point compactification $\alpha\kappa$ of the discrete space of cardinality κ. Let $\mathcal{P}$ be the collection of all one-element subsets of κ and all their complements. One can check that $\mathcal{P}$ is a binary subbase. On the other hand, if $\kappa > \omega$ then $\alpha\kappa$ is not a continuous image of a Cantor cube (since it does not have the Suslin property) and therefore cannot be a Dugundji space.

Let us finally mention that topological retracts of Tikhonov cubes need not be normally supercompact, see Szymański [**44**], so Theorem 3.2 is not valid for all compact median spaces. The following question is open.

QUESTION 3.6. Does there exist a zero-dimensional compact space with a binary subbase consisting of clopen sets, which is not normally supercompact?

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