Whitehead's problem and condensed mathematics

(Joint and ongoing work with Jeffrey Bergfalk)

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I. Whitehead's problem

Whitehead groups

All groups in this talk are abelian.

Definition

A group A is *Whitehead* if, for every surjective group homomorphism $\pi : B \to A$ with $\ker(\pi) \cong \mathbb{Z}$, there is a homomorphism $\sigma : A \to B$ such that $\pi \circ \sigma = \mathrm{id}_A$.

Equivalently, every short exact sequence of the form

$$0 \to \mathbb{Z} \to B \to A \to 0$$

splits. Equivalently, $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$.

Whitehead's problem

Fact

A group A is free if and only if $\mathrm{Ext}^1(A, C) = 0$ for every group C. Equivalently, for every surjective homomorphism $\pi : B \to A$, there is a homomorphism $\sigma : A \to B$ such that $\pi \circ \sigma = \mathrm{id}_A$.

A free group is therefore manifestly Whitehead. Whitehead's problem, first posed in the 1940s/50s, asks whether the converse holds.

Question (Whitehead)

Is every Whitehead group free?

A solution

Shortly after the problem was posed, Stein proved that all *countable* Whitehead groups are free. Twenty years later, Shelah resolved the full question in a way that was very surprising at the time.

Theorem (Shelah)

Whitehead's problem is independent of ZFC*.*

1. *If $V = L$, then every Whitehead group is free.*
2. *If* $\mathrm{MA}_{\aleph_1}$ *holds, then there is a nonfree Whitehead group of cardinality* $\aleph_1$*.*

The Hom functor

For groups A, G, $\mathrm{Hom}(A, G)$ denotes the group of all homomorphisms from A to G. For a fixed abelian group G, $\mathrm{Hom}(\cdot, G)$ is a *contravariant* functor from Ab to Ab (a homomorphism $\pi : B \to A$ gives rise to a homomorphism $\pi^* : \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G)$, given by $\rho \mapsto \rho \circ \pi$). This functor is *left exact*, i.e., if

$$0 \to C \to B \to A \to 0$$

is exact, then

$$0 \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(C, G)$$

is exact. But

$$0 \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(C, G) \to 0$$

may fail to be exact, i.e., $\mathrm{Hom}(B, G) \to \mathrm{Hom}(C, G)$ may not be surjective.

The Ext functor

The functor $\mathrm{Ext}^1(\cdot, G)$ is the *first right derived functor* of $\mathrm{Hom}(\cdot, G)$, and can be seen as measuring the failure of $\mathrm{Hom}(\cdot, G)$ to be fully exact. Given a short exact sequence

$$0 \to C \to B \to A \to 0,$$

we get a *long exact sequence*

$$\mathbf{0} \longrightarrow \mathrm{Hom}(A, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(C, G) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1(A, G) \longrightarrow \mathrm{Ext}^1(B, G) \longrightarrow \mathrm{Ext}^1(C, G) \longrightarrow 0$$

Free resolutions

Given any group A, we can form a short exact sequence

$$0 \to K \to F \to A \to 0$$

where K is a subgroup of F and F is free (hence K is also free). This induces the exact sequence

$$0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z}) \to \mathrm{Ext}^1(A, \mathbb{Z}) \ldots$$

If A is Whitehead, i.e., $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$, then $\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z})$ is a surjection, i.e., every homomorphism $\varphi : K \to \mathbb{Z}$ extends to a homomorphism $\psi : F \to \mathbb{Z}$. It turns out that these are *equivalent*: A is Whitehead if and only if every element of $\mathrm{Hom}(K, \mathbb{Z})$ extends to an element of $\mathrm{Hom}(F, \mathbb{Z})$.

Condensed mathematics

Motivation

In general, classical categories of objects carrying topologies are badly behaved from an algebraic viewpoint. To take a simple example, in the category of topological abelian groups, the "identity map"

$$(\mathbb{R}, \mathrm{discrete}) \to (\mathbb{R}, \mathrm{Euclidean})$$

is *not* an isomorphism, but this failure is not witnessed by a nontrivial kernel or cokernel. In recent years, Clausen and Scholze have introduced and developed the theory of categories of *condensed* objects to try to address this deficiency. (Barwick and Haine have a similar theory of *pyknotic* objects.)

"Definition"

A condensed set/abelian group/ring/...is a contravariant functor T from CHaus = {compact Hausdorff spaces} (or Prof = {totally disconnected compact Hausdorff spaces} or ED = {extremally disconnected compact Hausdorff spaces}) to Set/Ab/Ring/ ... satisfying

1. $T(\emptyset) = *$;
2. $T(S_0 \sqcup S_1) \cong T(S_0) \times T(S_1)$;
3. whenever $S' \to S$ is a surjection of spaces having fiber product $S' \times_S S'$ with projections π_0, π_1, we have

$$T(S) \cong \{x \in T(S') \mid \pi_0^*(x) = \pi_1^*(x) \in T(S' \times_S S')\}.$$

(Condition (3) is automatic if the domain of our functors is ED.) Formally, T is a sheaf of sets/abelian groups/rings/...on the pro-étale site of the point.

Condensed abelian groups

Let CondAb denote the category of condensed abelian groups. For $T \in$ CondAb, we call $T(*)$ the *underlying group* of T. The category of locally compact topological abelian groups embeds (fully faithfully) into CondAb via the map $A \mapsto \underline{A}$, where

$$\underline{A}(S) = \mathrm{Cont}(S, A)$$

for every $S \in$ CHaus. Note that $\underline{A}(*) \cong A$. CondAb is very well-behaved algebraically; for example, in CondAb our example $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eul}}$ completes to a short exact sequence

$$0 \to \underline{\mathbb{R}}_{\mathrm{disc}} \to \underline{\mathbb{R}}_{\mathrm{eucl}} \to Q \to 0, \text{ where}$$

$Q(S) \cong \{$cont. maps from S to $\mathbb{R}\}/\{$loc. constant maps from S to $\mathbb{R}\}$. (Note that $Q(*) = 0$, the trivial group.)

Internal Hom

For $T_0, T_1 \in \text{CondAb}$, $\mathrm{Hom}(T_0, T_1)$ is an abelian group. CondAb also has a tensor product and an *internal Hom functor*, $\underline{\mathrm{Hom}}(\cdot,\cdot)$, which takes values in CondAb. It satisfies the adjunction

$$\mathrm{Hom}(T_0, \underline{\mathrm{Hom}}(T_1, T_2)) \cong \mathrm{Hom}(T_0 \otimes T_1, T_2).$$

If A and G are locally compact topological abelian groups, then

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{G}) \cong \underline{\mathrm{Hom}(A, G)},$$

where $\mathrm{Hom}(A, G)$ is given the compact-open topology (if A and G are both discrete, then this is just the product topology). Thus, for $S \in \text{CHaus}$, we have $\underline{\mathrm{Hom}}(\underline{A}, \underline{G})(S) \cong \mathrm{Cont}(S, \mathit{Hom}(A, G))$.

Internal Ext

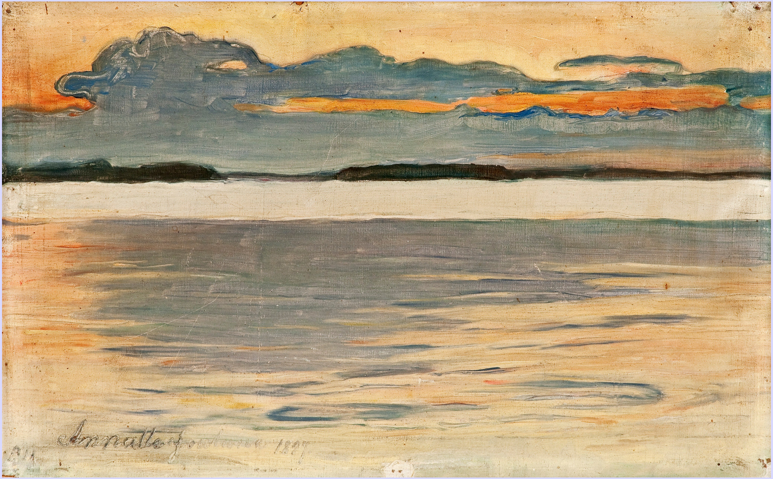
$\underline{\mathrm{Hom}}(\cdot,\cdot)$ has a first derived functor, $\underline{\mathrm{Ext}}^1(\cdot,\cdot)$. When Whitehead's problem is formulated in terms of $\underline{\mathrm{Ext}}^1$ (applied to abelian groups with the discrete topology), it turns out that it is *not* independent of ZFC.

Theorem (Clausen–Scholze)

Suppose that A is an abelian group and $\underline{\mathrm{Ext}}^1(\underline{A},\underline{\mathbb{Z}}) = 0$. Then A is free.

Clausen and Scholze's original proof relies heavily on deep structural facts about the category of condensed abelian groups (and the subcategory of *solid abelian groups*). It is also very inexplicit, e.g., given a nonfree group A, it does not identify a space S for which $\underline{\mathrm{Ext}}^1(\underline{A},\underline{\mathbb{Z}})(S) \neq 0$. This motivated us to find a more explicit, combinatorial proof.

III. Whitehead's problem in the condensed world



A refinement

Theorem (Bergfalk–LH)

Suppose that A is a nonfree abelian group and κ is the least cardinality of a nonfree subgroup of A. Then $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(2^\kappa) \neq 0$.

Sketch of proof. The proof is by induction on κ. We sketch a proof in the case in which $|A| = \aleph_1$ and A is almost free (i.e., all countable subgroups of A are free). Let

$$0 \to K \to F \to A \to 0$$

be exact with $K \subseteq F$ free of size $\aleph_1$. This yields the sequence

$$0 \to \underline{\mathrm{Hom}}(\underline{A}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Hom}}(\underline{K}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}}) \to$$

To show that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(2^{\omega_1}) \neq 0$, it suffices to show that $\underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}})(2^{\omega_1}) \to \underline{\mathrm{Hom}}(\underline{K}, \underline{\mathbb{Z}})(2^{\omega_1})$ is not a surjection.

To show that $\underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}})(2^{\omega_1}) \to \underline{\mathrm{Hom}}(\underline{K}, \underline{\mathbb{Z}})(2^{\omega_1})$ is not a surjection, we must show that there is a continuous map $\varphi : 2^{\omega_1} \to \mathrm{Hom}(K, \mathbb{Z})$ that does not extend pointwise to a continuous map $\psi : 2^{\omega_1} \to \mathrm{Hom}(F, \mathbb{Z})$, i.e., there is no continuous $\psi : 2^{\omega_1} \to \mathrm{Hom}(F, \mathbb{Z})$ such that $\psi(f)$ extends $\varphi(f)$ for all $f \in 2^{\omega_1}$.

Let B_K and B_F be bases for K and F. Let $\vec{M} = \langle M_\alpha \mid \alpha < \omega_1 \rangle$ be an $\in$-increasing, continuous chain of elementary submodels of some large $H(\theta)$, with everything relevant in M_0. Let $A_\alpha := A \cap M_\alpha$, and similarly define F_α, K_α. Let $A^*_\alpha := A_{\alpha+1}/A_\alpha$, and similarly define F^*_α, K^*_α. Since A is almost free but nonfree, by thinning out $\vec{M}$ if necessary, we may assume that the set

$$S := \{\alpha < \omega_1 \mid A^*_\alpha \text{ is not free}\}$$

is stationary in ω_1.

For each $\alpha \in S$, we have a short exact sequence

$$0 \to K_\alpha^* \to F_\alpha^* \to A_\alpha^* \to 0.$$

Note that F_α^* and K_α^* are free groups with bases $\{e + F_\alpha \mid e \in B_F \cap (M_{\alpha+1} \setminus M_\alpha)\}$ and $\{z + K_\alpha \mid z \in B_K \cap (M_{\alpha+1} \setminus M_\alpha)\}$. Since A_α^* is a countable nonfree group, it is not Whitehead, so we can find a homomorphism $\varphi_\alpha : K_\alpha^* \to \mathbb{Z}$ that does not lift to a homomorphism $\psi_\alpha : F_\alpha^* \to \mathbb{Z}$.

To define $\varphi : 2^{\omega_1} \to \mathrm{Hom}(K, \mathbb{Z})$, it suffices to specify $\varphi(f)(z)$ for all $f \in 2^{\omega_1}$ and $z \in B_K$. To do so, if there is $\alpha \in S$ such that $z \in M_{\alpha+1} \setminus M_\alpha$ **and** $f(\alpha) = 1$, then let $\varphi(f)(z) = \varphi_\alpha(z + K_\alpha)$. Otherwise, let $\varphi(f)(z) = 0$. (The idea behind this precise construction is due to Jan Šaroch.) φ is continuous: to determine $\varphi(f)(z)$, one only need inspect $f(\alpha)$ for the unique α such that $z \in M_{\alpha+1} \setminus M_\alpha$ (if it exists).

This works: suppose for sake of contradiction that $\psi : 2^{\omega_1} \to \mathrm{Hom}(F, Z)$ is continuous and extends φ pointwise. By continuity, we can find $\alpha \in S$ such that $\psi(f) \restriction F_\alpha$ is determined by $f \restriction \alpha$, i.e., for all $x \in 2^\alpha$ and all $f, g \in 2^{\omega_1}$ extending x, we have $\psi(f) \restriction F_\alpha = \psi(g) \restriction F_\alpha$. Fix an arbitrary $x \in 2^\alpha$, and let $f_0, f_1 \in 2^{\omega_1}$ extend x with $f_i(\alpha) = i$ for $i < 2$. Define a homomorphism $\psi_\alpha : F_\alpha^* \to \mathbb{Z}$ by letting $\psi_\alpha(e + F_\alpha) = (\psi(f_1) - \psi(f_0))(e)$ for all $e \in B_F \cap (M_{\alpha+1} \setminus M_\alpha)$.

We claim that ψ_α extends φ_α. To see this, fix $z \in B_K \cap (M_{\alpha+1} \setminus M_\alpha)$. Write z as $z^+ + z^-$, where $z^- \in F_\alpha$ and z^+ is a linear combination of elements of $B_F \cap (M_{\alpha+1} \setminus M_\alpha)$. Note that $(\psi(f_1) - \psi(f_0)) \restriction F_\alpha = 0$, so

$$(\psi(f_1) - \psi(f_0))(z) = (\psi(f_1) - \psi(f_0))(z^+) = \psi_\alpha(z^+ + F_\alpha) = \psi_\alpha(z + F_\alpha).$$

Then

$$\psi_\alpha(z + F_\alpha) = \psi(f_1)(z) - \psi(f_0)(z) = \varphi(f_1)(z) - \varphi(f_0)(z) = \varphi_\alpha(z + K_\alpha).$$

This contradicts the fact that φ_α cannot be extended to an element of $\mathrm{Hom}(F_\alpha^*, \mathbb{Z})$. $\square$

Extremally disconnected spaces

Extremally disconnected compact Hausdorff spaces play a central role in the theory of condensed mathematics (and condensed objects are fully determined by the values they take on ED spaces). This is in part due to the fact that ED spaces are precisely the projective objects in the category of compact Hausdorff spaces.

ED spaces are familiar to set theorists largely for another reason: they are precisely the Stone spaces of complete Boolean algebras. This hints at a connection between results in categories of condensed objects and forcing results in set theory.

A variation

A variation on the proof of the previous theorem yields the following.

Theorem (Bergfalk–LH)

Suppose that A is a nonfree group, $0 \to K \to F \to A \to 0$ is a free resolution, and κ is at least the minimal cardinality of a nonfree subgroup of A. Then, in $V[\mathrm{Add}(\omega, \kappa)]$, there is a homomorphism $\sigma : K \to \mathbb{Z}$ that does not extend to an element of $\mathrm{Hom}(F, \mathbb{Z})$. Moreover, given a basis B for K, we can ensure that $\sigma(z) \in \{0, 1\}$ for all $z \in B$.

A translation

Let S_κ denote the Stone space of the Boolean completion of $\mathrm{Add}(\omega, \kappa)$. Given any set B, every $\mathrm{Add}(\omega, \kappa)$-name $\dot{\sigma}$ for a function from B to $\{0, 1\}$ naturally gives rise to a continuous function $\varphi : S_\kappa \to 2^B$. (The point is that, for every $b \in B$, every ultrafilter $U \in S_\kappa$ must contain either $[\![\dot{\sigma}(b) = 0]\!]$ or $[\![\dot{\sigma}(b) = 1]\!]$). Conversely, for any set X, any continuous function $\psi : S_\kappa \to X^B$ gives rise to an $\mathrm{Add}(\omega, \kappa)$-name for a function from B to X. The result from the previous slide can then be translated as follows:

Theorem (Bergfalk–LH)

Suppose that A is a nonfree group and κ is at least the minimal cardinality of a nonfree subgroup of A. Then $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S_\kappa) \neq 0$.

Thank you for your attention