ITERATIONS OF ω_1-COHEN FORCING

CHRIS LAMBIE-HANSON

The purpose of this note is to prove that, over a model of CH, iterations of ω_1-Cohen forcing with ω_1-support are ω_1-proper. In particular, these iterations preserve ω_2. We first recall the relevant definitions.

Definition Let θ be a sufficiently large, regular cardinal, and let $\mathbb{P} \in H(\theta)$ be a poset. We say $N \prec H(\theta)$ is *relevant* for $\mathbb{P}$ if:

- $|N| = \aleph_1$.
- ${}^{\omega}N \subseteq N$.
- $N = \bigcup_{\alpha<\omega_1} N_\alpha$, where $\langle N_\alpha \mid \alpha < \omega_1\rangle$ is an internally approachable chain of countable elementary substructures of $H(\theta)$.

Definition Let $\mathbb{P}$ be a poset, and let N be relevant for $\mathbb{P}$. $q \in \mathbb{P}$ is $(N, \mathbb{P})$-generic if, for all dense, open sets D of $\mathbb{P}$ such that $D \in N$, $q \Vdash$ "$\dot{G}_{\mathbb{P}} \cap D \cap N \neq \emptyset$".

Definition $\mathbb{P}$ is ω_1-proper if, for all sufficiently large, regular θ, for all $N \prec H(\theta)$ relevant for $\mathbb{P}$, and for all $p \in \mathbb{P} \cap N$, there is $q \leq p$ such that q is $(N, \mathbb{P})$-generic.

For us, ω_1-Cohen forcing refers to the poset whose conditions are functions $s : \alpha \to 2$, where $\alpha < \omega_1$. A condition in an iteration of ω_1-Cohen forcing of length γ with ω_1-support is a function whose domain is a subset of γ of size $\leq \omega_1$. We will need the following Lemma.

Lemma 0.1. *Let γ be an ordinal, and let $\mathbb{P}_\gamma$ be an iteration of ω_1-Cohen forcing of length γ with ω_1-support. Let $p \in \mathbb{P}_\gamma$, and let F be a countable subset of* $\mathrm{dom}(p)$. *There is $q \leq p$ such that, for every $\beta \in F$, there is $\alpha_\beta < \omega_1$ and $s_\beta : \alpha_\beta \to 2$ such that $q \restriction \beta \Vdash$ "$q(\beta) = s_\beta$".*

Proof. The proof is by induction on γ. The lemma is trivially true for $\gamma = 1$. Suppose $\gamma = \eta + 1$. We may assume $\eta \in F$. First, extend $p \restriction \eta$ to $r \in \mathbb{P}_\eta$ such that, for some $\alpha_\eta < \omega_1$ and $s_\eta : \alpha_\eta \to 2$, $r \Vdash$ "$p(\eta) = s_\eta$". Then, using the inductive hypothesis, extend r to t such that, for all $\beta \in F \cap \eta$, there is $\alpha_\beta < \omega_1$ and $s_\beta : \alpha_\beta \to 2$ such that $t \restriction \beta \Vdash$ "$t(\beta) = s_\beta$". Then $t ^\frown p(\eta)$ is as desired.

Now suppose γ is a limit ordinal of countable cofinality. Let $\langle \gamma_n \mid n < \omega\rangle$ be an increasing sequence of ordinals cofinal in γ with $\gamma_0 = 0$. We build a sequence $\langle p_n \mid n < \omega\rangle$ such that $p_n \in \mathbb{P}_{\gamma_n}$ and $\langle p_n {}^\frown p \restriction [\gamma_n, \gamma) \mid n < \omega\rangle$ is decreasing in $\mathbb{P}_\gamma$. We ensure that, for every $n < \omega$ and every $\beta \in F \cap \gamma_n$, there is $\alpha_{\beta,n} < \omega_1$ and $s_{\beta,n} : \alpha_{\beta,n} \to 2$ such that $p_n \restriction \beta \Vdash$ "$p_n(\beta) = s_{\beta,n}$". This is easily achieved by the inductive hypothesis. Now let q be the greatest lower bound of the sequence $\langle p_n {}^\frown p \restriction [\gamma_n, \gamma) \mid n < \omega\rangle$. Letting $\alpha_\beta = \sup(\{\alpha_{\beta,n} \mid n < \omega\})$ and $s_\beta = \bigcup_{n<\omega} s_{\beta,n}$ for all $\beta \in F$, it is easily seen that q is as desired.

Finally, suppose γ is a limit ordinal of uncountable cofinality. Then there is $\eta < \gamma$ such that $F \subseteq \eta$. We can then finish by applying the inductive hypothesis to $p \restriction \eta$. $\square$

Theorem 0.2. *Assume CH. Let γ be an ordinal, and let $\mathbb{P} = \mathbb{P}_\gamma$ be an iteration of ω_1-Cohen forcing of length γ with ω_1-support. Then $\mathbb{P}_\gamma$ is ω_1-proper.*

Proof. The proof is an adaptation of an argument of Kanamori in [1]. Assume $\diamondsuit$ holds in V. (Since $\diamondsuit$ is added by ω_1-Cohen forcing, this is not really an additional assumption). Let $\bar{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence guessing subsets of $\omega_1 \times \omega_1$, i.e., for each $\alpha < \omega_1$, $A_\alpha \subseteq (\alpha \times \alpha)$, and for all $X \subseteq (\omega_1 \times \omega_1)$, there are stationarily many $\alpha < \omega_1$ such that $X \cap (\alpha \times \alpha) = A_\alpha$.

Definition If $p \in \mathbb{P}$ and F is a countable subset of $\mathrm{dom}(p)$, we say $q \leq_F p$ if $q \leq p$ and, for all $\beta \in F$, $q(\beta) = p(\beta)$.

Let θ be a sufficiently large, regular cardinal, and let $N \prec H(\theta)$ be relevant for $\mathbb{P}$. Let $p \in \mathbb{P} \cap N$. We will find $q \leq p$ such that q is $(N, \mathbb{P})$-generic.

Let $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ enumerate all dense open subsets of $\mathbb{P}$ that lie in N. We will build a decreasing sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ of conditions in $\mathbb{P} \cap N$. We will also beforehand fix a bookkeeping device that will give us a sequence of countable sets $\langle F_\alpha \mid \alpha < \omega_1 \rangle$, functions $\langle g_\alpha \mid \alpha < \omega_1 \rangle$, and ordinals $\langle \eta_\alpha \mid \alpha < \omega_1 \rangle$ such that:

- $F_\alpha \subseteq \mathrm{dom}(p_\alpha)$.
- $g_\alpha : F_\alpha \to \eta_\alpha$ is a bijection and $\eta_\alpha \geq \alpha$.
- If $\alpha < \beta$, then $F_\alpha \subseteq F_\beta$ and $g_\alpha \subseteq g_\beta$.
- If β is a limit ordinal, then $F_\beta = \bigcup_{\alpha < \beta} F_\alpha$.
- $\bigcup_{\alpha < \omega_1} F_\alpha = \bigcup_{\alpha < \omega_1} \mathrm{dom}(p_\alpha)$.

In our construction, we will ensure that, if $\alpha < \beta < \omega_1$, then $p_\beta \leq_{F_\alpha} p_\alpha$. This will allow us to find a lower bound for the sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$.

Let $p_0 = p$. If $\beta < \omega_1$ is a limit ordinal, let p_β be the greatest lower bound of $\langle p_\alpha \mid \alpha < \beta \rangle$. Now suppose p_α has been defined. Assume that $\eta_\alpha = \alpha$. (This happens for a club of α. If it is not the case, then let $p_{\alpha+1} = p_\alpha$.) Now define a function $\sigma_\alpha : F_\alpha \to {}^\alpha 2$ as follows: if $\beta \in F_\alpha$ and $\delta < \alpha$, then let

$$(\sigma_\alpha(\beta))(\delta) = \begin{cases} 1 & \text{if } (g_\alpha(\beta), \delta) \in A_\alpha \\ 0 & \text{otherwise} \end{cases}$$

Now ask whether there is $r \leq p_\alpha$ such that:

- $r \in \bigcap_{\beta < \alpha} D_\beta$.
- For all $\beta \in F_\alpha$, $r \restriction \beta \Vdash$ "$r(\beta) = \sigma_\alpha(\beta)$".

Let r_α be such an r if it exists (if not, just let $p_{\alpha+1} = p_\alpha$). Note that, by elementarity, exploiting the fact that N is closed under countable sequences, we may assume that $r_\alpha \in N$. We now define $p_{\alpha+1}$ to resemble r_α as closely as possible while requiring that $p_{\alpha+1} \leq_{F_\alpha} p_\alpha$. Namely, we let $\mathrm{dom}(p_{\alpha+1}) = \mathrm{dom}(r_\alpha)$. For $\beta \in F_\alpha$, $p_{\alpha+1}(\beta) = p_\alpha(\beta)$. If $\beta \in \mathrm{dom}(p_\alpha) \setminus F_\alpha$, then $p_{\alpha+1}(\beta)$ is a name such that $r_\alpha \restriction \beta \Vdash$ "$p_{\alpha+1}(\beta) = r_\alpha(\beta)$" and, if $c \leq p_\alpha \restriction \beta$ is incompatible with $r_\alpha \restriction \beta$, $c \Vdash$ "$p_{\alpha+1}(\beta) = p_\alpha(\beta)$". If $\beta \in \mathrm{dom}(r_\alpha) \setminus \mathrm{dom}(p_\alpha)$, then $p_{\alpha+1}(\beta)$ is a name such that $r_\alpha \restriction \beta \Vdash$ "$p_{\alpha+1}(\beta) = r_\alpha(\beta)$" and, if $c \leq p_\alpha \restriction \beta$ is incompatible with $r_\alpha \restriction \beta$, $c \Vdash$ "$p_{\alpha+1}(\beta) = \emptyset$". It is clear that $p_{\alpha+1} \leq_{F_\alpha} p_\alpha$ and, since everything needed to define $p_{\alpha+1}$ is in N, we may assume $p_{\alpha+1} \in N$.

Let $g = \bigcup_{\alpha < \omega_1} g_\alpha$. Then g is a bijection from $\bigcup_{\alpha < \omega_1} F_\alpha$ to ω_1. Let q be a lower bound for $\langle p_\alpha \mid \alpha < \omega_1 \rangle$. We claim that q is $(N, \mathbb{P})$-generic. To prove this, fix $t \leq q$ and $\xi < \omega_1$. We show that t is compatible with an element of $N \cap D_\xi$. To do this, we will construct sequences $\langle t_\alpha \mid \alpha < \omega_1\rangle$, $\langle \rho_\alpha \mid \alpha < \omega_1\rangle$ and $\langle \tau_\alpha \mid \alpha < \omega_1\rangle$ such that:

- $\langle t_\alpha \mid \alpha < \omega_1\rangle$ is a decreasing sequence of conditions from $\mathbb{P}_\gamma$ and t_α is a greatest lower bound of $\langle t_\beta \mid \beta < \alpha\rangle$ if α is a limit ordinal.
- $\langle \rho_\alpha \mid \alpha < \omega_1\rangle$ is an increasing, continuous sequence of countable ordinals, with $\rho_\alpha \geq \alpha$.
- For $\alpha < \alpha' < \omega_1$, $\tau_\alpha : F_\alpha \to {}^{\rho_\alpha}2$ and, for $\beta \in F_\alpha$, $\tau_\alpha(\beta) \subseteq \tau_{\alpha'}(\beta)$.
- For $\alpha < \omega_1$ and $\beta \in F_\alpha$, $t_\alpha \restriction \beta \Vdash$ "$t_\alpha(\beta) = \tau_\alpha(\beta)$".
- For $\alpha < \omega_1$, $t_\alpha \in \bigcap_{\beta<\alpha} D_\beta$.

Let $t_0 = t$. If α is a limit ordinal, it is clear how to proceed. Suppose $\alpha = \zeta + 1$ and t_ζ, ρ_ζ, and τ_ζ have been defined. Let $t^*_\alpha \leq t$ be such that $t^* \in D_\alpha$. Apply Lemma 0.1 to t^*_α and F_α to get $t'_\alpha \leq t^*_\alpha$, $\{\rho_{\alpha,\beta} \mid \beta \in F_\alpha\}$ and $\{s_\beta \mid \beta \in F_\alpha\}$ such that, for all $\beta \in F_\alpha$, $s_\beta : \rho_{\alpha,\beta} \to 2$ and $t'_\alpha \restriction \beta \Vdash$ "$t'_\alpha(\beta) = s_\beta$". We can then find $\rho_\alpha \geq \alpha$ greater than all of the $\rho_{\alpha,\beta}$'s. and arbitrarily extend all of the s_β's to be functions $\tau_\alpha(\beta)$ in ${}^{\rho_\alpha}2$. We can then define $t_{\alpha+1} \leq t'_\alpha$ as desired.

At the end of this construction, let $X = \{(g(\beta), \delta) \mid \beta \in \bigcup_{\alpha<\omega_1} F_\alpha,\ \delta < \omega_1,$ and, for all α such that $\beta \in F_\alpha$ and $\delta < \rho_\alpha$, $(\tau_\alpha(\beta))(\delta) = 1\}$. $X \subseteq \omega_1 \times \omega_1$. Note that the set of limit ordinals $\alpha < \omega_1$ such that $\eta_\alpha = \rho_\alpha = \alpha$ is club. Let $\alpha > \xi$ in this club be such that $X \cap (\alpha \times \alpha) = A_\alpha$. Note that, working through the definitions, this implies that $\tau_\alpha = \sigma_\alpha$. Thus, $t_\alpha \leq q \leq p_\alpha$, $t_\alpha \in \bigcap_{\beta<\alpha} D_\beta$ and, for all $\beta \in F_\alpha$, $t_\alpha \restriction \beta \Vdash$ "$t_\alpha(\beta) = \sigma_\alpha(\beta)$", so, in our construction of $p_{\alpha+1}$, we answered our question positively and thus were in the non-trivial case. Now, noting that $t_\alpha \leq p_{\alpha+1}$, it is easily verified that, in fact, $t_\alpha \leq r_\alpha$ (simply check by induction on $\beta \in \mathrm{dom}(r_\alpha) = \mathrm{dom}(p_{\alpha+1})$ that $t_\alpha \restriction \beta \Vdash$ "$t_\alpha(\beta) \leq r_\alpha(\beta)$"). But $r_\alpha \in N \cap \bigcap_{\beta<\alpha} D_\beta$, so $r_\alpha \in N \cap D_\xi$, so we have demonstrated that t is compatible with an element of $N \cap D_\xi$, thus completing the proof. □

References

[1] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19(1):97–114, 1980.