WHITEHEAD'S PROBLEM AND CONDENSED MATHEMATICS

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Abstract. One of the better-known independence results in general mathematics is Shelah's solution to Whitehead's problem of whether $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ implies that an abelian group A is free. The point of departure for the present work is Clausen and Scholze's proof that, in contrast, one natural interpretation of Whitehead's problem within their recently-developed framework of condensed mathematics has an affirmative answer in ZFC. We record two alternative proofs of this result, as well as several original variations on it, both for their intrinsic interest and as a springboard for a broader study of the relations between condensed mathematics and set theoretic forcing. We show more particularly how the condensation $\underline{X}$ of any locally compact Hausdorff space X may be viewed as an organized presentation of the forcing names for the points of canonical interpretations of X in all possible set-forcing extensions of the universe, and we argue our main result by way of this fact. We also show that when interpreted within the category of light condensed abelian groups, Whitehead's problem is again independent of the ZFC axioms. In fact we show that it is consistent that Whitehead's problem has a negative solution within the category of κ-condensed abelian groups for every uncountable cardinal κ, but that this scenario, in turn, is inconsistent with the existence of a strongly compact cardinal.

1. Introduction

Recall that an abelian group A is *Whitehead* if $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$. Around 1950, J.H.C. Whitehead asked whether every such group is free. The converse is easily seen to be true (every free abelian group A satisfies $\mathrm{Ext}^1(A, G) = 0$ for *every* abelian group G, and hence for $G = \mathbb{Z}$), and Stein showed soon after the question was posed that every *countable* Whitehead group is free [25]. Progress on the general question was slow, though, and the latter soon assumed the status of a major open problem in homological algebra. Shelah's solution to the question 20 years later was completely unexpected, for Whitehead's problem thereby became one of the first originating outside of logic or set theory to be proven independent of the ZFC axioms.

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Theorem 1.1 (Shelah, [21])**.** *Whitehead's problem is independent of* ZFC*. In particular:*

(1) If $V = L$, then every Whitehead group is free.
(2) If MA_{ω_1} holds, then there is a nonfree Whitehead group of size $\aleph_1$.

Shelah's solution of Whitehead's problem initiated a program of research applying set theoretic methods to module theory that continues to bear fruit today. This program has uncovered deep connections between Whitehead's problem and its variations and purely combinatorial set theoretic phenomena, and has touched on additional topics including the study of almost free modules and of the approximation theory of modules. For more on such developments in the intervening years, see, e.g., [6, 8, 12].

Recently, Clausen and Scholze have developed a framework, known as *condensed mathematics*, for applying algebraic tools to algebraic structures endowed with nontrivial topologies (cf. [20], [19], [3]). Here, we will be particularly interested in the category CondAb of *condensed abelian groups*, where the objects are, roughly speaking, certain contravariant functors from the category of Stone spaces to the category of abelian groups. Every T1 topological abelian group A has a corresponding condensed abelian group $\underline{A}$ (if an abelian group is given without specifying its topology, then we interpret it as a topological abelian group with the discrete topology). In addition, CondAb has an internal Hom functor $\underline{\mathrm{Hom}} : \mathsf{CondAb}^{\mathrm{op}} \times \mathsf{CondAb} \to \mathsf{CondAb}$, and corresponding internal Ext functors $\underline{\mathrm{Ext}}^n$ for $n > 0$. It turns out that, when interpreted appropriately in CondAb, Whitehead's problem is no longer independent of ZFC. In particular, Clausen and Scholze proved the following theorem (cf. [2]).

Theorem 1.2 (Clausen–Scholze, 2020)**.** *It follows from the* ZFC *axioms that, for every abelian group A, if*

$$\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}}) = 0,$$

then A is free.

Clausen and Scholze's proof of Theorem 1.2 is quite slick and relies on a structural analysis of the abelian subcategory of CondAb spanned by what are known as *solid* abelian groups. In this paper, we present a more elementary, combinatorial proof of Theorem 1.2 that is also more *constructive* in the sense that, given a nonfree abelian group A, we specify a highly natural associated Stone space S and construct a nonzero element of $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S)$. Our motivation for doing this is twofold. First, we seek to excavate some of the combinatorial and set theoretic ideas underlying Theorem 1.2 and thereby to improve our understanding of the mathematics surrounding Whitehead's problem, in both general and condensed contexts. To this end, we prove the following theorem.

Theorem 1.3. *Suppose that A is a nonfree abelian group and κ is the least cardinality of a nonfree subgroup of A. Let $\mathbb{B}$ be the Boolean completion of the forcing to add κ-many Cohen reals, and let S be the Stone space of $\mathbb{B}$. Then:*

(1) $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(2^\kappa) \neq 0$;
(2) in $V^{\mathbb{B}}$, A is not Whitehead;
(3) $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) \neq 0$.

Second, using Theorem 1.3, particularly clauses (2) and (3) thereof, as a starting point, we begin to investigate a more general connection between condensed mathematics and set theoretic forcing. We isolate a precise sense in which, given a locally compact Hausdorff space X, the condensed set $\underline{X}$ is simply an organized presentation of all forcing names for points in the canonical interpretations of X in all possible set forcing extensions. We then exhibit a correspondence between satisfaction of certain simple formulas concerning a topological space X in two settings: first, in a Boolean-valued model $V^{\mathbb{B}}$, and second in the context of the space $C(S, X)$ of continuous functions from the Stone space S of $\mathbb{B}$ to X. This correspondence, stated in Theorem 9.1, allows us, e.g., to immediately deduce clause (3) of Theorem 1.3 from clause (2) of the same theorem.

It is natural, in light of the preceding paragraph, to wonder whether there are provable implications between instances of clauses (2) and (3) of Theorem 1.3 for more general complete Boolean algebras $\mathbb{B}$. We record several partial answers to this question. In particular, in Theorem 9.2 we show that if $\mathbb{B}$ is a complete Boolean algebra, S is its Stone space, and $V^{\mathbb{B}}$ satisfies a certain technical strengthening of the failure of A to be Whitehead, then $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) \neq 0$. In the ensuing discussion, we show that the converse fails in general. The question of whether clause (2) more generally implies clause (3) (i.e., of whether the aforementioned technical strengthening can be dropped) remains open and is recorded as Conjecture 13.1.

We also investigate some variations of Theorem 1.2. First, we show that, if we restrict our attention to the subcategory CondAb_κ of CondAb for certain uncountable cardinals κ, then the classical independence of Whitehead's problem remains, while it again vanishes when κ is a sufficiently large cardinal. More precisely, we prove the following theorem. (In what follows, given an uncountable cardinal κ, CondAb_κ denotes the category of κ-condensed sets, with the particular case of $\mathsf{CondAb}_{\omega_1}$ denoting the category of *light* condensed abelian groups, the ambient framework for the lecture series "Analytic Stacks" delivered by Clausen and Scholze in 2023–24 [4]).

Theorem 1.4.

(1) *If MA_{ω_1} holds, then there is a nonfree abelian group of size $\aleph_1$ such that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_{\omega_1}}(\underline{A}, \underline{\mathbb{Z}}) = 0$.*

(2) *It is consistent with the axioms of ZFC that for every uncountable cardinal κ, there is a nonfree abelian group A such that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_\kappa}(\underline{A}, \underline{\mathbb{Z}}) = 0$.*

(3) *If κ is a strongly compact cardinal, A is an abelian group, and*
$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_\kappa}(\underline{A}, \underline{\mathbb{Z}}) = 0,$$
then A is free.

Second, we extend Theorem 1.2 from the class of discrete abelian groups to the class of locally compact abelian groups and show that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}}) = 0$ for a locally compact abelian group A if and only if A is projective in the category of locally compact abelian groups.

The structure of the paper is as follows. In Section 2, we introduce the basic definitions and facts regarding condensed mathematics that we will need. In Section 3, we recall some classical algebraic facts pertinent to Whitehead's problem and introduce their analogues in the condensed setting. In Section 4, we prove a slight strengthening of Stein's aforementioned classical result [25] that all countable Whitehead groups are free. In Section 5, we prove clause (1) of Theorem 1.3, and, in Section 6, we adapt that proof to establish clause (2) of Theorem 1.3. From here we present two paths to clause (3) of Theorem 1.3. The first, recorded in Section 7, is by far the shorter and more direct of the two. The second, comprising Sections 8 and 9, passes through a more general investigation into the interplay between condensed mathematics and forcing. In Section 8, a robust correspondence is established between

- $\mathbb{B}$-names for points in a certain extension of a topological space X; and
- continuous functions from the Stone space S of $\mathbb{B}$ to X.

Section 9 then establishes the aforementioned equivalence between the settings of $C(S, X)$ and $V^{\mathbb{B}}$ with respect to interpretations of certain formulas. We expect the results of this section to be applicable in a wide variety of settings, and derive clause (3) of Theorem 1.3 as an immediate corollary. In Section 10, we prove Theorem 1.4(1). In Section 11, we record an algebraic lemma converting the groups $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S)$ into a more tractable, and suggestive, form. This lemma yields an alternative proof of clauses (1) and (3) of Theorem 1.3 (and hence of Theorem 1.2), as well as clauses (2) and (3) of Theorem 1.4. In Section 12 we prove an extension of Theorem 1.2 to the class of locally compact abelian groups. We close out our work in Section 13 with a few concluding remarks and main open questions.

We have pursued wherever possible an account of our results accessible to readers with no more than a basic knowledge of either set theory or homological algebra. With this in mind, let us record here the places in the paper where more background knowledge may be required. First, some of the arguments in Section 3 presume some knowledge of derived functors and derived categories. That said, at the end of Section 3 we isolate an elementary combinatorial statement equivalent to the assertion that $\underline{A}$ is not Whitehead for a given abelian group A. It is this combinatorial statement that we work with for most of the remainder of the paper, so the reader may safely take the calculations of Section 3 as a black box. Sections 6–9 require some basic knowledge of forcing. Both [10] and [11] contain more than is needed here; we also direct the reader to [14], which provides a concise introduction to forcing aimed at non-specialists. The final sections are slightly more demanding, with Section 10 requiring a somewhat more sophisticated understanding of forcing and Sections 11 and 12 requiring some knowledge of homological algebra and condensed mathematics.

1.1. **Notation and conventions.** Our standard references for undefined notions and notations are [10] for set theory and [28] for homological algebra.

Given a group G, we will sometimes slightly abuse notation and write G to refer to its underlying set. We let $|G|$ denote the cardinality of the underlying set of G. If f is a function and $X \subseteq \mathrm{dom}(f)$, then $f[X]$ denotes the image of X under f, i.e., $f[X] = \{f(x) \mid x \in X\}$.

Given a Boolean algebra $\mathbb{B}$, we let $S(\mathbb{B})$ denote the Stone space of $\mathbb{B}$. Concretely, $S(\mathbb{B})$ is the space of ultrafilters on $\mathbb{B}$, with basic clopen sets given by

$$N_b := \{U \in S(\mathbb{B}) \mid b \in U\}$$

for $b \in \mathbb{B}$.

2. Condensed abelian groups

Here we briefly introduce the category CondAb of condensed abelian groups, as well as the category Cond of condensed sets. For more details and proofs of many of the statements contained in this section, we refer the reader to [20]. We let CHaus,

Prof, and ED denote the categories of compact Hausdorff spaces, profinite sets (i.e., totally disconnected compact Hausdorff spaces, also known as Stone spaces), and extremally disconnected compact Hausdorff spaces, respectively, noting that $\mathsf{ED} \subseteq \mathsf{Prof} \subseteq \mathsf{CHaus}$. Recall that profinite sets are precisely those spaces S that can be represented as inverse limits $S = \varprojlim_{i \in I} S_i$, where I is a directed partial order and each S_i is a finite (discrete) space. Given an uncountable cardinal κ, we say that a profinite set S is *κ-small* if it can be represented as an inverse limit as above with $|I| < \kappa$. Equivalently, a profinite set S is κ-small if the set of all of its clopen subsets (or, equivalently, its topological weight) has cardinality less than κ.[1] We let Prof_κ denote the full subcategory of Prof consisting of all κ-small profinite sets, and define ED_κ in the same way. The category of sets is denoted Set, and the category of abelian groups is denoted Ab. In any category with a terminal object, we let $*$ denote that terminal object (so, e.g., $*$ is the one-point space in CHaus, the one-point set in Set, and the one-element group in Ab).

Definition 2.1. Let κ be an uncountable cardinal. A *κ-condensed set* (resp. *κ-condensed abelian group*) is a contravariant functor $T : \mathsf{Prof}_\kappa \to \mathsf{Set}$ (resp. $T : \mathsf{Prof}_\kappa \to \mathsf{Ab}$) such that the following conditions hold.

(1) $T(\emptyset) = *$.
(2) For all $S_0, S_1 \in \mathsf{Prof}_\kappa$, the natural map
$$T(S_0 \sqcup S_1) \to T(S_0) \times T(S_1)$$
is a bijection.
(3) Suppose that $S, S' \in \mathsf{Prof}_\kappa$ and $f : S' \to S$ is a surjective continuous map. Let p_0 and p_1 denote the two projections from $S' \times_S S'$ to S', and p_0^* and p_1^* their images under T. Then the natural map
$$T(S) \to \{x \in T(S') \mid p_0^*(x) = p_1^*(x)\}$$
is a bijection.

More succinctly, a κ-condensed set (resp. κ-condensed abelian group) is a sheaf of sets (resp. sheaf of abelian groups) on the site $*_{\kappa\text{-proét}}$. The category of κ-condensed sets is denoted Cond_κ, and the category of κ-condensed abelian groups is denoted CondAb_κ.

Three sorts of choices of κ in Definition 2.1 have predominated so far:

- strong limit cardinals κ, valued for their closure properties in [20, 19, 3], with the associated categories Cond_κ assembling to form Cond in the fashion described just below;
- inaccessible cardinals κ, valued for their even stronger closure properties in the closely related and contemporaneous *pyknotic* framework of Barwick and Haine [1];
- $\kappa = \omega_1$, valued for its relative minimality in [4], wherein the associated condensed sets are termed *light*. We will return to this setting in Section 10. Note that Prof_{ω_1} is precisely the category of totally disconnected *metrizable* compact Hausdorff spaces.

[1]There is some discrepancy in the terminology around this notion in various presentations of condensed mathematics. In [20], κ-small profinite sets are defined to be those profinite sets of *cardinality* less than κ, while in the later [3], κ-small profinite sets are defined to be those profinite sets with fewer than κ-many clopen subsets. This distinction is irrelevant for strong limit cardinals κ, but in general we find the latter convention more natural and thus adopt it here.

Given strong limit cardinals $\kappa < \kappa'$, the forgetful functors $\mathsf{Cond}_{\kappa'} \to \mathsf{Cond}_\kappa$ and $\mathsf{CondAb}_{\kappa'} \to \mathsf{CondAb}_\kappa$ have natural left adjoints; by way of these adjoints, the categories Cond_κ form a direct system, where κ ranges over the strong limit cardinals κ; one then defines the category Cond of condensed sets as $\mathsf{Cond} = \varinjlim_\kappa \mathsf{Cond}_\kappa$. Similarly, define the category CondAb of condensed abelian groups as $\mathsf{CondAb} = \varinjlim_\kappa \mathsf{CondAb}_\kappa$. In practice, we will slightly abuse notation and think of condensed sets or condensed abelian groups as contravariant functors defined on the entire category Prof. Given a condensed set/abelian group T, we call $T(*)$ the *underlying set/abelian group* of T. We note that an element T of Cond or CondAb is fully determined by its restriction to ED (cf. [20, Proposition 2.7]).

Given a Hausdorff space X, one can define a condensed set $\underline{X}$ by letting $\underline{X}(S) = C(S, X)$, i.e., $X(S)$ is the set of continuous functions from S to X for all $S \in \mathsf{Prof}$; we will sometimes term this $\underline{X}$ the *condensation* of X. This operation defines a functor from the category Haus of Hausdorff topological spaces to Cond. When restricted to the full subcategory of Haus spanned by the compactly generated Hausdorff spaces, it is a fully faithful embedding (cf. [20, Proposition 1.7]).

Similarly, given a topological abelian group A, one can define a condensed abelian group $\underline{A}$ by letting $\underline{A}(S) = C(S, A)$ (considered as an abelian group with pointwise addition) for all $S \in \mathsf{Prof}$. When restricted to the class LCA of locally compact abelian groups, this again describes a fully faithful embedding of LCA into CondAb.

For each uncountable cardinal κ, there is an obvious forgetful functor from CondAb_κ to Cond_κ, and this functor admits a left adjoint that sends a κ-condensed set T to a κ-condensed abelian group denoted $\mathbb{Z}[T]$. Concretely, as noted in [20, §2], $\mathbb{Z}[T]$ is the sheafification of the functor from ED_κ to Ab that sends each $S \in \mathsf{ED}_\kappa$ to the free group on $T(S)$. Moreover, $\{\mathbb{Z}[\underline{S}] \mid S \in \mathsf{ED}_\kappa\}$ is, for any strong limit cardinal κ, a set of compact projective generators for CondAb_κ. These facts straightforwardly propagate upward to the category CondAb of all condensed abelian groups.

3. Algebraic preliminaries

Given any abelian group A, a *free resolution* of A is a short exact sequence of the form

$$0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$$

in which both F and K are free abelian groups. In such resolutions, we will often implicitly assume that K is a subgroup of F and ι is the inclusion map. We note that every infinite abelian group A has a canonical free resolution, in which F is the free abelian group generated by the underlying set of A and K is the kernel of the induced surjection from F onto A. The elements of K are therefore the formal linear combinations of elements of F that evaluate to 0 in A.

By applying the functor $\mathrm{Hom}(\cdot, \mathbb{Z})$ to a free resolution of A and using the fact that F and K are free, and therefore Whitehead, we obtain an exact sequence of the form

$$0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z}) \to \mathrm{Ext}^1(A, \mathbb{Z}) \to 0$$

We thus see that A is Whitehead if and only if the map $\mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z})$ is surjective, i.e., if and only if every group homomorphism $\varphi : K \to \mathbb{Z}$ extends to a homomorphism $\psi : F \to \mathbb{Z}$.

We now turn our attention to the situation in the category of condensed abelian groups. CondAb is a symmetric monoidal category with an internal Hom-functor, which we will denote by $\underline{\mathrm{Hom}} : \mathsf{CondAb}^{\mathrm{op}} \times \mathsf{CondAb} \to \mathsf{CondAb}$. If A and B are sufficiently nice topological groups, then $\underline{\mathrm{Hom}}(\underline{A}, \underline{B})$ has a particularly nice description:

Proposition 3.1 (Clausen–Scholze, [20, Proposition 4.2]). *Suppose that A and B are Hausdorff topological abelian groups and A is compactly generated. Then there is a natural isomorphism of condensed abelian groups*

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \cong \underline{\mathrm{Hom}(A, B)},$$

where, on the right-hand side, $\mathrm{Hom}(A, B)$ is computed in the category of topological abelian groups and endowed with the compact-open topology.

$\underline{\mathrm{Hom}}$ has corresponding derived functors, the internal Ext-functors

$$\underline{\mathrm{Ext}}^n : \mathsf{CondAb}^{\mathrm{op}} \times \mathsf{CondAb} \to \mathsf{CondAb}$$

for $n \geq 1$ and, at the level of the derived category, an internal $R\mathrm{Hom}$-functor

$$R\underline{\mathrm{Hom}} : D(\mathsf{CondAb})^{\mathrm{op}} \times D(\mathsf{CondAb}) \to D(\mathsf{CondAb});$$

The former arise, of course, as the n^{th} cohomology groups of the latter. The following lemma shows that these functors behave as expected in relation to their classical analogues in Ab; namely, the classical functors can be recovered by evaluating their condensed analogues at a point. When we need to be careful about the category in which certain functors are being considered, we will write the name of the category as a subscript on the functor, i.e., $\mathrm{Ext}^1_{\mathsf{Ab}}$ is the Ext-functor in the category of abelian groups.

Lemma 3.2. *For any abelian groups A and B (with the discrete topology),*

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{B})(*) = \mathrm{Ext}^1_{\mathsf{Ab}}(A, B).$$

More generally, for any condensed abelian groups T and M, we have

$$R\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(T, M)(*) = R\mathrm{Hom}_{\mathsf{CondAb}}(T, M),$$

and, for discrete abelian groups A and B, we in turn have

$$R\mathrm{Hom}_{\mathsf{CondAb}}(\underline{A}, \underline{B}) = R\mathrm{Hom}_{\mathsf{Ab}}(A, B).$$

Proof. As noted in [20, proof of Corollary 4.8] (and as follows from items (ii) and (iii) of its page 13),

$$R\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(T, M)(S) = R\mathrm{Hom}_{\mathsf{CondAb}}(T \otimes \mathbb{Z}[S], M).$$

for any condensed abelian groups T, M and any extremally disconnected profinite set S. When $S = *$, our general claim follows from the facts that $\mathbb{Z}[*] = \underline{\mathbb{Z}}$ and that $\underline{\mathbb{Z}}$ is the unit object for the condensed tensor product $\otimes$. To see the assertion for discrete abelian groups, recall that their category fully and faithfully embeds into CondAb, and observe by [13, Props. 2.18 and 2.19] that if $\cdots \to P^1 \to P^0 \to A$ is a projective (hence free) resolution of A then $\cdots \to \underline{P}^1 \to \underline{P}^0 \to \underline{A}$ is a projective resolution of $\underline{A}$. Writing $\mathcal{P}$ and $\underline{\mathcal{P}}$ for $\cdots \to P^1 \to P^0 \to 0$ and $\cdots \to \underline{P}^1 \to \underline{P}^0 \to 0$, respectively, we then have

$$R\mathrm{Hom}_{\mathsf{CondAb}}(\underline{A}, \underline{B}) = \mathrm{Hom}_{\mathsf{CondAb}}(\underline{\mathcal{P}}, \underline{B}) = \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{P}, B) = R\mathrm{Hom}_{\mathsf{Ab}}(A, B),$$

as desired. □

It is natural at this point in the discussion to declare a *condensed* abelian group T to be Whitehead if $\underline{\mathrm{Ext}}^1(T, \underline{\mathbb{Z}}) = 0$. Observe next that, in close analogy with the classical situation, if F is a free (discrete) abelian group, then $\underline{F}$ is Whitehead in CondAb. To see this, let κ be a cardinal, and let $F = \bigoplus_\kappa \mathbb{Z}$ be the free group on κ generators; it follows then that $\underline{F} \cong \bigoplus_\kappa \underline{\mathbb{Z}}$, where the direct sum on the right is computed in CondAb, from [13, Proposition 2.19]. By [13, Remark 3.10 and Lemma 4.2], we then have

$$R\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(\underline{F}, \underline{\mathbb{Z}}) = \prod_\kappa R\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) = \prod_\kappa \underline{\mathbb{Z}}[0],$$

where $\underline{\mathbb{Z}}[0]$ denotes the chain complex that is $\underline{\mathbb{Z}}$ in degree 0 and 0 elsewhere. In particular, we have $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{F}, \underline{\mathbb{Z}}) = 0$.

One reasonable translation (though, as we will discuss later, not the only possible one) of Whitehead's problem to the condensed setting is then the following:

> Suppose that A is a discrete abelian group and $\underline{A}$ is Whitehead in CondAb. Must A be free?

Note that, by Lemma 3.2, the assertion that $\underline{A}$ is Whitehead in CondAb is a strengthening of the assertion that A is Whitehead in Ab; the latter is equivalent to asserting that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(*) = 0$, whereas the former is equivalent to asserting that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) = 0$ for *every* profinite set S. And, indeed, Theorem 1.2 implies that, unlike the classical Whitehead problem, this translation to the condensed setting has a positive answer in ZFC.

In the coming sections, we will give a more concrete, combinatorial proof of Theorem 1.2. In particular, given a nonfree discrete abelian group A, we will identify a natural profinite set S such that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) \neq 0$. The process by which this will be done is analogous to that described in the classical setting at the beginning of this section, so in the interest of symmetry, we end this section by giving an outline of the structure of the argument.

Suppose that A is a nonfree discrete abelian group, and let $0 \to K \to F \to A \to 0$ be a free resolution of A. Applying $\underline{\mathrm{Hom}}(\cdot, \underline{\mathbb{Z}})$ to the image of this short exact sequence in CondAb yields the exact sequence

$$0 \to \underline{\mathrm{Hom}}(\underline{A}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Hom}}(\underline{F}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Hom}}(\underline{K}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}}) \to 0.$$

Now, evaluating this sequence at a profinite set S and recalling Proposition 3.1, we obtain

$$0 \to \underline{\mathrm{Hom}(A, \mathbb{Z})}(S) \to \underline{\mathrm{Hom}(F, \mathbb{Z})}(S) \to \underline{\mathrm{Hom}(K, \mathbb{Z})}(S) \to \underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) \to 0.$$

It follows that $\underline{\mathrm{Ext}}^1(\underline{A}, \underline{\mathbb{Z}})(S) = 0$ if and only if the map

$$\underline{\mathrm{Hom}(F, \mathbb{Z})}(S) \to \underline{\mathrm{Hom}(K, \mathbb{Z})}(S)$$

is a surjection. Recall that $\underline{\mathrm{Hom}(F, \mathbb{Z})}(S)$ consists of all continuous functions $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ (and similarly for K), where $\mathrm{Hom}(F, \mathbb{Z})$ is given the compact-open topology (which, since F and $\mathbb{Z}$ are discrete, is simply the topology $\mathrm{Hom}(F, \mathbb{Z})$ inherits as a subspace of $\prod_F \mathbb{Z}$ endowed with the product topology). Moreover, the map $\underline{\mathrm{Hom}(F, \mathbb{Z})}(S) \to \underline{\mathrm{Hom}(K, \mathbb{Z})}(S)$ sends a continuous $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ to the map $\phi : S \to \mathrm{Hom}(K, \mathbb{Z})$ defined by letting $\phi(s) = \psi(s) \restriction K$ for all $s \in S$.

Therefore, to show that $\underline{A}$ is not Whitehead, it will suffice to find a profinite set S and a continuous $\varphi : S \to \mathrm{Hom}(K, \mathbb{Z})$ that does not "lift pointwise" to a map from S to $\mathrm{Hom}(F, \mathbb{Z})$ (i.e., for which there is no continuous $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ for which $\psi(s) \restriction K = \varphi(s)$ for all $s \in S$). The construction of such S and φ will be the subject of the next sections.

4. Countable groups

Recall that, if A is a torsion-free abelian group and X is a subgroup of A, we say that X is a *pure* subgroup of A if A/X is torsion-free. Given any subgroup X of A, the *pure closure* of X in A is the smallest pure subgroup of A containing X; concretely, this can be seen to be equal to $\{a \in A \mid \exists n > 0\ [na \in X]\}$. The following theorem gives a useful characterization of countable free abelian groups (recall that a finitely generated abelian group is free if and only if it is torsion-free).

Theorem 4.1 ([17] (cf. [5, Theorem 4.2]))**.** *Suppose that A is a countable torsion-free abelian group. Then A is free if and only if every finitely generated subgroup of A is contained in a finitely generated pure subgroup of A.*

As mentioned above, Stein [25] proved that every countable Whitehead group is free. The following theorem gives one way of seeing this (and slightly more).

Theorem 4.2. *Suppose that A is a countable, torsion-free, nonfree abelian group,*

$$0 \to K \xrightarrow{\subseteq} F \xrightarrow{\pi} A \to 0$$

is a free resolution of A, and B_K is a basis for K. Then there is a homomorphism $\varphi : K \to \mathbb{Z}$ such that

(1) φ does not extend to a homomorphism $\psi : F \to \mathbb{Z}$; and
(2) $\varphi[B_K] \subseteq \{0, 1\}$.

Proof. First, fix any basis B_F for the free group F and notice that B_K is infinite: if not, then K would be included in a subgroup G of F generated by a finite $S \subset B_F$; then $F = G \oplus H$ where H is generated by $B_F \setminus S$, and we get $A \cong F/K \cong (G/K) \oplus H$ where H is free and G/K is finitely generated torsion-free, hence free, in contradiction with the nonfreeness of A.

Next, let us argue that, without loss of generality, we may assume that F (and hence also K) is countable. To see this, suppose that K is uncountable. Let χ be a sufficiently large regular cardinal, and let N be a countable elementary submodel of $(H(\chi), \in, A, B_F, B_K)$. Let $F_0 := F \cap N$ and $K_0 := K \cap N$. By elementarity, F_0 and K_0 are free, with bases $B_F \cap N$ and $B_K \cap N$, respectively, and

$$0 \to K_0 \xrightarrow{\subseteq} F_0 \xrightarrow{\pi \restriction F_0} A \to 0$$

is a free resolution of A by countable free groups. Suppose that we can find a homomorphism $\varphi_0 : K_0 \to \mathbb{Z}$ such that $\varphi_0[B_K \cap N] \subseteq \{0, 1\}$ and φ_0 does not extend to a homomorphism $\psi_0 : F_0 \to \mathbb{Z}$. Then we can extend φ_0 to a homomorphism $\varphi : K \to \mathbb{Z}$ by setting $\varphi(t) = 0$ for all $t \in B_K \setminus N$. The fact that φ_0 does not extend to a homomorphism from F_0 to $\mathbb{Z}$ implies that φ does not extend to a homomorphism from F to $\mathbb{Z}$. Therefore, we may assume that F is countable.

We can thus injectively enumerate B_K as $\langle t_n \mid n < \omega \rangle$ and, for each $n < \omega$, let K_n denote the subgroup of K generated by $\{t_k \mid k < n\}$. We will define our homomorphism $\varphi : K \to \mathbb{Z}$ by recursion, requiring that $\varphi(t_n) \in \{0, 1\}$ for all $n < \omega$. We will do this by constructing a strictly increasing sequence $\langle n_i \mid i < \omega \rangle$ of natural numbers and, for each $i < \omega$, a homomorphism $\varphi_i : K_{n_i} \to \mathbb{Z}$ in such a way that $\varphi_i[\{t_k \mid k < n_i\}] \subseteq \{0, 1\}$ and, for all $i < j < \omega$, $\varphi_j \restriction K_{n_i} = \varphi_i$.

By Theorem 4.1, since A is countable, torsion-free, and nonfree, there is a finitely generated subgroup X_0 of A such that its pure closure X is not finitely generated. Let Y_0 be a finite subset of B_F such that $X_0 \subseteq \pi[F_0]$ where F_0 denotes the subgroup of F generated by Y_0. Put $n_0 = 0$ and enumerate $\mathrm{Hom}(F_0, \mathbb{Z}) = \{\psi_i \mid i < \omega\}$. Finally, let φ_0 be the zero homomorphism.

Assume that we have already defined Y_i, F_i, n_i and $\varphi_i : K_{n_i} \to \mathbb{Z}$, for some $i < \omega$, in such a way that $K_{n_i} \subseteq F_i$. Consider any $z_i \in F$ such that $x_i := \pi(z_i) \in X \setminus \pi[F_i]$.

Since X is the pure closure of X_0, there is a positive integer $j_i > 1$ such that $j_i x_i = \pi(j_i z_i) \in X_0 \subseteq \pi[F_0]$. It follows that there exists $y_i \in F_0$ such that $\pi(y_i) = \pi(j_i z_i)$, whence $s_i := j_i z_i - y_i \in K = \mathrm{Ker}(\pi)$. We can thus express s_i as $a_i + \sum_{n_i \le k < n_{i+1}} b_k t_k$ where $a_i \in K_{n_i}$, $n_i < n_{i+1} < \omega$ and $b_k \in \mathbb{Z}$ for each $k \in [n_i, n_{i+1})$. Let Y_{i+1} be any finite subset of B_F such that $Y_i \subseteq Y_{i+1}$ and $K_{n_{i+1}} \subseteq F_{i+1}$ where F_{i+1} denotes the subgroup of F generated by Y_{i+1}.

Claim 4.3. *There is $k \in [n_i, n_{i+1})$ such that $j_i \nmid b_k$.*

Proof. Suppose not, and set

$$s_i^* := s_i - a_i = \sum_{k=n_i}^{n_{i+1}-1} b_k t_k.$$

Then $s_i^* \in K$ is of the form $j_i z_i - r$, where $r \in F_i$. Moreover, since all of the coefficients b_k for $k \in [n_i, n_{i+1})$ are divisible by j_i, it follows that s_i^* is divisible by j_i (in K). Since $r = j_i z_i - s_i^*$, this means that there exists $r^* \in F_i$ such that $r = j_i r^*$. We get $s_i^* = j_i(z_i - r^*) \in K$ which implies $z_i - r^* \in K$ using the fact that A is torsion-free. In particular, $\pi(z_i) = \pi(r^*)$, so $\pi(z_i) \in \pi[F_i]$, contradicting the fact that $\pi(z_i) = x_i \notin \pi[F_i]$. □

Fix a $k_i \in [n_i, n_{i+1})$ as given by the claim. To construct φ_{i+1}, it suffices to define $\varphi_{i+1}(t_k)$ for all $k \in [n_i, n_{i+1})$. For $k \in [n_i, n_{i+1}) \setminus \{k_i\}$, simply let $\varphi_{i+1}(t_k) = 0$. To define $\varphi_{i+1}(t_{k_i})$, consider the homomorphisms $\psi_i : F_0 \to \mathbb{Z}$ and $\varphi_i : K_{n_i} \to \mathbb{Z}$ and the number

$$d_i := \varphi_i(a_i) + \psi_i(y_i).$$

If d_i is divisible by j_i, then let $\varphi_{i+1}(t_{k_i}) = 1$. Otherwise, let $\varphi_{i+1}(t_{k_i}) = 0$.

Now let $\varphi := \bigcup_{i<\omega} \varphi_i$. We claim that this is as desired. Clearly, $\varphi : K \to \mathbb{Z}$ is a homomorphism and $\varphi[B_K] \subseteq \{0, 1\}$. We must therefore show that every homomorphism $\psi : F \to \mathbb{Z}$ fails to extend φ.

To this end, fix a homomorphism $\psi : F \to \mathbb{Z}$. Find $i < \omega$ such that $\psi \restriction F_0 = \psi_i$. We claim that $\psi(s_i) \neq \varphi(s_i)$. Note that

$$\psi(s_i) = j_i \psi(z_i) - \psi_i(y_i),$$

and

$$\varphi(s_i) = b_{k_i} \varphi_{i+1}(t_{k_i}) + \varphi_i(a_i).$$

Therefore, if it were the case that $\psi(s_i) = \varphi(s_i)$, we would have

$$\begin{aligned} j_i \psi(z_i) &= b_{k_i} \varphi_{i+1}(t_{k_i}) + \varphi_i(a_i) + \psi_i(y_i) \\ &= b_{k_i} \varphi_{i+1}(t_{k_i}) + d_i. \end{aligned}$$

Since $j_i \psi(z_i)$ is clearly divisible by j_i, it follows that the right hand side of this equation must also be divisible by j_i. Suppose first that d_i is divisible by j_i. Then $\varphi_{i+1}(t_{k_i}) = 1$, so, since k_i was chosen so that b_{k_i} is not divisible by j_i, it follows that the right hand side of the above equation is not divisible by j_i, which is a contradiction. If, on the other hand, d_i is not divisible by j_i, then $\varphi_{i+1}(t_{k_i}) = 0$, so the right hand side of the above equation equals d_i and is hence not divisible by j_i, which is again a contradiction. Therefore, it must be the case that $\psi(s_i) \neq \varphi(s_i)$. $\square$

It is also well-known that all Whitehead groups are torsion-free. We now record a more general result, analogous to Theorem 4.2, implying this fact.

Proposition 4.4. *Let*

$$0 \to K \xrightarrow{\subseteq} F \xrightarrow{\pi} A \to 0$$

be a short exact sequence of groups where F is torsion-free and K is free with a basis B_K. Assume that A is not torsion-free (i.e. that K is not a pure subgroup of F). Then there is a homomorphism $\varphi : K \to \mathbb{Z}$ such that

(1) φ does not extend to a homomorphism $\psi : F \to \mathbb{Z}$; and
(2) there is a unique $t^* \in B_K$ such that
- *$\varphi(t^*) = 1$;*
- *$\varphi(t) = 0$ for all $t \in B_K \setminus \{t^*\}$.*

Proof. Being free, K is isomorphic to the direct sum of copies of $\mathbb{Z}$ indexed by B_K. We consider the canonical embedding $\nu : K \to \mathbb{Z}^{B_K}$, i.e. for every $t, s \in B_K$, $\nu(t)(t) = 1$ and $\nu(t)(s) = 0$ if $s \neq t$. Then ν is pure, i.e. $\nu[K]$ is pure in $\mathbb{Z}^{B_K}$ (in fact, ν is even an elementary embedding). Since K is not pure in F, it follows that there does not exist any homomorphism $\mu : F \to \mathbb{Z}^{B_K}$ such that $\mu \restriction K = \nu$.

For each $t \in B_K$, let us denote by $p_t : \mathbb{Z}^{B_K} \to \mathbb{Z}$ the t-th canonical projection. Since any homomorphism into direct product is uniquely determined by specifying its compositions with all the canonical projections, there has to be some $t^* \in B_K$ such that $\varphi := p_{t^*}\nu$ cannot be extended to a homomorphism from F into $\mathbb{Z}$. One readily checks that t^* and φ possess the desired properties. $\square$

Remark 4.5. In the formulation of Theorem 4.2, we could not have hoped to find φ with $(\varphi \restriction B_K)^{-1}\{1\}$ finite, let alone a φ satisfying the stronger condition (2) from Proposition 4.4. The reason is that any given countable torsion-free group A, being the union of a countable chain of finitely generated free subgroups, is isomorphic to a (pure) subgroup of $\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}$, cf. [27, Proposition 3.1] or [18, Theorem 3.3.2]. Here, $\mathbb{Z}^{(\omega)}$ denotes the direct sum of countably many copies of $\mathbb{Z}$, i.e., the subgroup of $\mathbb{Z}^\omega$ consisting of elements with finite support.

Let $B = \{e_n \mid n < \omega\}$ denote the canonical basis of the free group $\mathbb{Z}^{(\omega)}$. Then for A as above, there exists a subgroup F of $\mathbb{Z}^\omega$ such that the following diagram with exact rows, where μ and ν denote the respective identity embeddings, commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{(\omega)} \arrow[r, "\nu"] & \mathbb{Z}^\omega \arrow[r] & \mathbb{Z}^\omega/\mathbb{Z}^{(\omega)} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}^{(\omega)} \arrow[r, "\subseteq"] \arrow[u, equal] & F \arrow[r] \arrow[u, "\mu"] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

It follows that F is countable and thus free since all countable subgroups of $\mathbb{Z}^\omega$ are such, cf. [6, Theorems IV.2.3 and IV.2.8]. Since μ extends the map ν, we see, in particular, that $p\mu$ extends the map $p\nu$ for any $p \in \mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z})$. This applies specifically for any p which is a $\mathbb{Z}$-linear combination of the canonical projections

$p_n : \mathbb{Z}^\omega \to \mathbb{Z}$, $n < \omega$. It remains to notice that any $\varphi \in \mathrm{Hom}(\mathbb{Z}^{(\omega)}, \mathbb{Z})$ which is nonzero only on finitely many elements from B is of the form $p\nu$ for p as above.

On the other hand, since $\mathbb{Z}$ is a slender group ([6, Corollary III.2.4]), no homomorphism $\varphi : \mathbb{Z}^{(\omega)} \to \mathbb{Z}$ which is nonzero on infinitely many elements from B can be extended to a homomorphism from $\mathbb{Z}^\omega$ into $\mathbb{Z}$. Of course, it can happen that a particular such φ can be extended to a homomorphism from F into $\mathbb{Z}$.

5. Uncountable groups

In what follows, given a cardinal κ, 2^κ denotes the topological space consisting of all functions $f : \kappa \to 2$, equipped with the product topology. In particular, basic clopen subsets of 2^κ are of the form $U_\sigma := \{f \in 2^\kappa \mid \sigma \subseteq f\}$, where σ is a finite partial function from κ to 2. Note that 2^κ is a profinite set.

Theorem 5.1. *Suppose that A is a nonfree abelian group, and let κ be the minimal cardinality of a nonfree subgroup of A. Suppose moreover that $0 \to K \xrightarrow{\subseteq} F \to A \to 0$ is a free resolution of A. Then there is a continuous map $\varphi : 2^\kappa \to \mathrm{Hom}(K, \mathbb{Z})$ that cannot be lifted pointwise to a continuous map $\psi : 2^\kappa \to \mathrm{Hom}(F, \mathbb{Z})$.*

Moreover, given a basis B_K for K, we can require that, for all $f \in 2^\kappa$ and all $t \in B_K$, we have $\varphi(f)(t) \in \{0, 1\}$.

Proof. If A has nonzero torsion elements, then Proposition 4.4 provides us with a homomorphism $\tau \in \mathrm{Hom}(K, \mathbb{Z})$ such that $\tau[B_K] \subseteq \{0, 1\}$ and τ cannot be lifted to an element of $\mathrm{Hom}(F, \mathbb{Z})$. We can in this case let $\varphi : 2^\kappa \to \mathrm{Hom}(K, \mathbb{Z})$ be the constant map taking value τ. Thus, assume for the rest of the proof that A is torsion-free.

The proof will be by induction on κ. Note that, by Shelah's singular compactness theorem [22], κ must be regular. Let B_F and B_K be bases for F and K, respectively, and note that $|B_K| \leq |B_F|$. Let χ be a sufficiently large regular cardinal, so that all objects of interest are in $H(\chi)$, and let $\lhd$ be a fixed well-ordering of $H(\chi)$.

We first claim that we may assume both that A is *almost free*, i.e., that $\kappa = |A|$, and that $|F| = \kappa$. To see this, let N be an elementary submodel of $(H(\chi), \in, \lhd, A, B_F, B_K)$ with $|N| = \kappa \subseteq N$. Let $A_0 := A \cap N$. By elementarity, N contains as an element a nonfree subgroup of A of cardinality κ. Since $\kappa \subseteq N$, this subgroup is a subset of A_0; therefore, A_0 is nonfree, and, by the minimality of κ, almost free. Let $F_0 := F \cap N$ and $K_0 := K \cap N$; by elementarity, $B_{F,0} := B_F \cap N$ and $B_{K,0} := B_K \cap N$ are bases for F_0 and K_0, respectively, and $0 \to K_0 \to F_0 \to A_0 \to 0$ is a free resolution of A_0. Moreover, we clearly have $|F_0| = \kappa$.

Suppose that we can find a continuous $\varphi_0 : 2^\kappa \to \mathrm{Hom}(K_0, \mathbb{Z})$ that does not lift pointwise to a continuous map $\psi_0 : 2^\kappa \to \mathrm{Hom}(F_0, \mathbb{Z})$, and assume moreover that, for all $f \in 2^\kappa$ and all $t \in B_{K,0}$, we have $\varphi_0(f)(t) \in \{0, 1\}$. Then φ_0 can be extended pointwise to a continuous map $\varphi : 2^\kappa \to \mathrm{Hom}(K, \mathbb{Z})$ by setting

$$\varphi(f)(t) = \begin{cases} \varphi_0(f)(t) & \text{if } t \in B_{K,0} \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in B_K$ and extending linearly. The continuity of φ follows from the continuity of φ_0, and our construction ensures that $\varphi(f)(t) \in \{0, 1\}$ for all $f \in 2^\kappa$ and all $t \in B_K$. Moreover, φ does not lift pointwise to a continuous $\psi : 2^\kappa \to \mathrm{Hom}(F, \mathbb{Z})$, as pointwise restriction of such a ψ to F_0 would induce a continuous pointwise extension $\psi_0 : 2^\kappa \to \mathrm{Hom}(F_0, \mathbb{Z})$ of φ_0. We can therefore assume without loss of generality that A is almost free.

Thus, for the rest of the proof we may assume that A is almost free and $|F| = \kappa$. The case of $\kappa = \aleph_0$ is covered by Theorem 4.2, so we may assume that $\kappa > \aleph_0$. Let $\vec{M} = \langle M_\alpha \mid \alpha < \kappa \rangle$ be a $\in$-increasing, continuous sequence of elementary submodels of $(H(\chi), \in, \lhd, A, F, K)$ such that $|M_\alpha| < \kappa$ and $\delta_\alpha := M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. For each $\alpha < \kappa$, let $A_\alpha := A \cap M_\alpha$, $F_\alpha := F \cap M_\alpha$, and $K_\alpha := K \cap M_\alpha$. Since A, F, and K each have size at most κ and $\vec{M}$ is $\in$-increasing, we have $A = \bigcup_{\alpha<\kappa} A_\alpha$, $F = \bigcup_{\alpha<\kappa} F_\alpha$, and $K = \bigcup_{\alpha<\kappa} K_\alpha$.

Let E be the set of $\alpha < \kappa$ such that there exists β such that $\alpha < \beta < \kappa$ and A_β/A_α is nonfree. Note that, if $\alpha < \beta < \beta' < \kappa$ and A_β/A_α is nonfree, then also $A_{\beta'}/A_\alpha$ is nonfree, since it contains A_β/A_α as a subgroup. We claim that E is stationary. If not, then there exists a club $D \subseteq \kappa$ such that, for all $\alpha < \beta$, both in D, the group A_β/A_α is free. Moreover, since A is almost free, we know that $A_{\min(D)}$ is free. Therefore, it is straightforward to build a basis for A by recursion along D (cf. [5, Theorem 5.3]), contradicting the assumption that A is not free.

For all $\alpha \in E$, let $\alpha < \beta_\alpha < \kappa$ be such that $A_{\beta_\alpha}/A_\alpha$ is not free. Let $D := \{\gamma < \kappa \mid \forall \alpha < \gamma\ [\beta_\alpha < \gamma]\}$. Then D is a club in κ. Letting $\langle \gamma_\eta \mid \eta < \kappa \rangle$ be the increasing enumeration of D, the following two facts are immediate:

(1) $\{\eta < \kappa \mid \gamma_\eta \in E\}$ is stationary in κ;
(2) for all $\eta < \kappa$ for which $\gamma_\eta \in E$, the set $A_{\gamma_{\eta+1}}/A_\gamma$ is not free.

Therefore, by thinning out $\vec{M}$ to only include those models indexed by elements of D and then reindexing, we may assume that the set

$$S := \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha \text{ is nonfree}\}$$

is stationary in κ.

For each $\alpha \in S$, let $A^*_\alpha := A_{\alpha+1}/A_\alpha$, $F^*_\alpha := F_{\alpha+1}/F_\alpha$, and $K^*_\alpha := K_{\alpha+1}/K_\alpha$. Similarly, let $B^*_{F,\alpha} := B_F \cap (M_{\alpha+1} \setminus M_\alpha)$ and $B^*_{K,\alpha} := B_K \cap (M_{\alpha+1} \setminus M_\alpha)$. Note that each of F^*_α and K^*_α is free, with bases given by $\{z + F_\alpha \mid z \in B^*_{F,\alpha}\}$ and $\{t + K_\alpha \mid t \in B^*_{K,\alpha}\}$, respectively. Also, our free resolution of A induces a free resolution

$$0 \to K^*_\alpha \to F^*_\alpha \to A^*_\alpha \to 0$$

of A^*_α. Since $\alpha \in S$, we know that A^*_α is nonfree. Let $\kappa_\alpha := |A^*_\alpha| < \kappa$, and apply the inductive hypothesis to find a continuous $\varphi_\alpha : 2^{\kappa_\alpha} \to \mathrm{Hom}(K^*_\alpha, \mathbb{Z})$ that does not lift pointwise to a continuous $\psi_\alpha : 2^{\kappa_\alpha} \to \mathrm{Hom}(F^*_\alpha, \mathbb{Z})$. Note that we are slightly abusing terminology here, since K^*_α is not formally a subgroup of F^*_α. Nonetheless, we identify K^*_α with the subgroup $\{r + F_\alpha \mid r \in K_{\alpha+1}\}$, so our assumption is formally that there is no continuous $\psi_\alpha : 2^{\kappa_\alpha} \to \mathrm{Hom}(F^*_\alpha, \mathbb{Z})$ such that, for all $f \in 2^{\kappa_\alpha}$ and all $r \in K_{\alpha+1}$, we have $\psi_\alpha(f)(r + F_\alpha) = \varphi_\alpha(f)(r + K_\alpha)$.

We are now ready to describe the construction of our desired continuous map $\varphi : 2^\kappa \to \mathrm{Hom}(K, \mathbb{Z})$. We first need a bit of notation: given $f \in 2^\kappa$ and $\alpha \in S$, define a map $f_\alpha \in 2^{\kappa_\alpha}$ by letting $f_\alpha(\eta) = f(\alpha + 1 + \eta)$ for all $\eta < \kappa_\alpha$. Now, to define φ, it suffices to specify $\varphi(f)(t)$ for all $f \in 2^\kappa$ and $t \in B_K$, and then extend each $\varphi(f)$ linearly to all of K. To do this, suppose we are given $f \in 2^\kappa$ and $t \in B_K$. If there is $\alpha \in S$ such that $f(\alpha) = 1$ and $t \in B^*_{K,\alpha}$,[2] then let

$$\varphi(f)(t) = \varphi_\alpha(f_\alpha)(t + K_\alpha).$$

[2]Note that such an α must be unique, if it exists.

In all other cases, let $\varphi(f)(t) = 0$.

The continuity of $\varphi : 2^\kappa \to \mathrm{Hom}(K, \mathbb{Z})$ follows immediately from the continuity of φ_α for each $\alpha \in S$. Moreover, our construction, and the analogous fact about each φ_α, ensures that $\varphi(f)(t) \in \{0, 1\}$ for each $f \in 2^\kappa$ and $t \in B_K$. It thus remains to show that φ cannot be lifted pointwise to a continuous $\psi : 2^\kappa \to \mathrm{Hom}(F, \mathbb{Z})$.

Suppose for the sake of contradiction that $\psi : 2^\kappa \to \mathrm{Hom}(F, \mathbb{Z})$ is a continuous map such that $\psi(f) \restriction K = \varphi(f)$ for all $f \in 2^\kappa$. Let $\{z_\gamma \mid \gamma < \kappa\}$ be the $\lhd$-least injective enumeration of B_F, and note that, for all $\alpha < \kappa$, $\{z_\gamma \mid \gamma < \delta_\alpha\}$ is a basis for F_α. Temporarily fix $\gamma < \kappa$. By the continuity of ψ, for every $f \in 2^\kappa$ there exists a finite $v(f, \gamma) \subseteq \kappa$ such that $\psi(g)(z_\gamma) = \psi(f)(z_\gamma)$ for all $g \in U_{f \restriction v(f,\gamma)}$. By the compactness of 2^κ, there is a finite $\mathcal{F}_\gamma \subseteq 2^\kappa$ such that $2^\kappa = \bigcup_{f \in \mathcal{F}_\gamma} U_{f \restriction v(f,\gamma)}$. Then, letting $v(\gamma) := \bigcup_{f \in \mathcal{F}_\gamma} v(f, \gamma)$, it follows that, for all $f, g \in 2^\kappa$, if $f \restriction v(\gamma) = g \restriction v(\gamma)$, then $\varphi(f)(z_\gamma) = \varphi(g)(z_\gamma)$.

By the stationarity of S and the fact that κ is a regular, uncountable cardinal, we can fix $\alpha \in S$ such that $\delta_\alpha = \alpha$ and $v(\gamma) \subseteq \alpha$ for all $\gamma < \alpha$. We will complete the proof by constructing a continuous map $\psi_\alpha : 2^{\kappa_\alpha} \to \mathrm{Hom}(F^*_\alpha, \mathbb{Z})$ that lifts φ_α pointwise, contradicting our choice of φ_α.

We first need a bit more notation. Given a function $g \in 2^{\kappa_\alpha}$ and an $i < 2$, define a function $g_i^+ \in 2^\kappa$ as follows. First, let $g_i^+(\alpha) = i$. Then, for all $\eta < \kappa_\alpha$, let $g_i^+(\alpha + 1 + \eta) = g(\eta)$. Finally, let $g_i^+(\beta) = 0$ for all $\beta \in \kappa \setminus [\alpha, \alpha + \kappa_\alpha)$. Note that, for $i < 2$, we have $(g_i^+)_\alpha = g$.

As usual, to define ψ_α, it suffices to specify $\psi_\alpha(g)(z + F_\alpha)$ for all $g \in 2^{\kappa_\alpha}$ and $z \in B^*_{F,\alpha}$. To this end, given such g and z, let

$$\psi_\alpha(g)(z + F_\alpha) = \psi(g_1^+)(z) - \psi(g_0^+)(z).$$

We will be done if we show that $\psi_\alpha(g)(t + F_\alpha) = \varphi_\alpha(g)(t + K_\alpha)$ for all $g \in 2^{\kappa_\alpha}$ and $t \in B^*_{K,\alpha}$. To this end, fix such g and t. Let $B_{F,\alpha} := \{z_\gamma \mid \gamma < \alpha\} = B_F \cap M_\alpha$, and note that $B_{F,\alpha} \cup B^*_{F,\alpha}$ is a basis for $F_{\alpha+1}$. Since $t \in K_{\alpha+1} \subseteq F_{\alpha+1}$, we can find finite subsets $I^- \subseteq B_{F,\alpha}$ and $I^+ \subseteq B^*_{F,\alpha}$ together with integer coefficients $\{a_z \mid z \in I^- \cup I^+\}$ such that, in $F_{\alpha+1}$, we have

$$t = \sum_{z \in I^- \cup I^+} a_z z.$$

Let

$$t^- := \sum_{z \in I^-} a_z z \quad \text{and} \quad t^+ := \sum_{z \in I^+} a_z z.$$

The following observations are immediate, but crucial:

- $t = t^- + t^+$;
- $t^- \in F_\alpha$;
- t^+ is a linear combination of elements of $B^*_{F,\alpha}$; in consequence, we have $\psi_\alpha(g)(t^+ + F_\alpha) = \psi(g_1^+)(t^+) - \psi(g_0^+)(t^+)$.

Since $g_1^+ \restriction \alpha = g_0^+ \restriction \alpha$, our choice of α ensures that $\psi(g_1^+) \restriction F_\alpha = \psi(g_0^+) \restriction F_\alpha$, and hence in particular $\psi(g_1^+)(t^-) = \psi(g_0^+)(t^-)$. But now, putting all of this together we obtain

$$\begin{aligned}
\psi_\alpha(g)(t+F_\alpha) &= \psi_\alpha(g)(t^+ + F_\alpha) \\
&= \psi(g_1^+)(t^+) - \psi(g_0^+)(t^+) \\
&= \psi(g_1^+)(t^-) + \psi(g_1^+)(t^+) - (\psi(g_0^+)(t^-) + \psi(g_0^+)(t^+)) \\
&= \psi(g_1^+)(t) - \psi(g_0^+)(t) \\
&= \varphi(g_1^+)(t) - \varphi(g_0^+)(t) \\
&= \varphi_\alpha(g)(t+K_\alpha) - 0 = \varphi_\alpha(g)(t+K_\alpha).
\end{aligned}$$

In the above equation, the first equality holds because $t = t^- + t^+$ and $t^- \in F_\alpha$, and hence $t + F_\alpha = t^+ + F_\alpha$. The second follows from the definition of ψ, the third from the observation above that $\psi(g_1^+)(t^-) = \psi(g_0^+)(t^-)$, the fourth by linearity, the fifth from the assumption that ψ lifts φ pointwise, and the sixth from the definition of φ, noting that $g_i^+(\alpha) = i$ for $i < 2$. This shows that ψ_α lifts φ_α pointwise, furnishing the desired contradiction and completing the proof. □

6. Cohen forcing

Given an infinite set X, we let $\mathrm{Add}(\omega, X)$ denote the forcing to add $|X|$-many Cohen reals. Concretely, $\mathrm{Add}(\omega, X)$ consists of all finite partial functions from X to $\{0,1\}$, ordered by reverse inclusion.

In this section, we will modify the arguments of the previous section to prove the following theorem:

Theorem 6.1. *Suppose that A is a nonfree abelian group and $\kappa = |A|$. Then, in $V^{\mathrm{Add}(\omega,\kappa)}$, A is not Whitehead.*

Before proving Theorem 6.1, we make the following observation.

Lemma 6.2. *Suppose that $\mathbb{P}$ is a ccc forcing notion and A is an abelian group. Then A is free in V if and only if A is free in $V^{\mathbb{P}}$.*

Proof. If A is free in V, then it clearly remains free in $V^{\mathbb{P}}$, and if A is not torsion-free, then it is nonfree in both models. Therefore, we may assume that A is torsion-free and nonfree in V, and we will show that A remains nonfree in $V^{\mathbb{P}}$.

The proof is by induction on $|A|$. Suppose first that A is countable. Then, by Theorem 4.1, A has a finitely generated subgroup X whose pure closure is not finitely generated. Since A has the same finitely generated subgroups in V and in $V^{\mathbb{P}}$, and since a subgroup of A is pure in V if and only if it is pure in $V^{\mathbb{P}}$, X retains these properties in $V^{\mathbb{P}}$, so, again by Theorem 4.1, A is nonfree in $V^{\mathbb{P}}$.

Now suppose that $|A| = \kappa > \aleph_0$. First, note that we may assume that A is almost free. Indeed, if A_0 is a nonfree subgroup of A and $|A_0| < \kappa$, then we can apply the inductive hypothesis to conclude that A_0, and hence also A, remains nonfree in $V^{\mathbb{P}}$.

Thus, assume that A is almost free, and hence κ is regular. Since A is nonfree, we can fix a $\subseteq$-increasing, continuous sequence $\vec{A} := \langle A_\alpha \mid \alpha < \kappa \rangle$ such that

(1) each A_α is a subgroup of A of cardinality less than κ;
(2) $A = \bigcup_{\alpha<\kappa} A_\alpha$;
(3) the set $S := \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha \text{ is nonfree}\}$ is stationary in κ.

Items (1) and (2) are clearly upward absolute to $V^{\mathbb{P}}$. By the inductive hypothesis, for all $\alpha \in S$, the group $A_{\alpha+1}/A_\alpha$ remains nonfree in $V^{\mathbb{P}}$. Moreover, since $\mathbb{P}$ has the ccc, S remains stationary in κ, so (3) is also upward absolute. Thus, $\vec{A}$ continues to witness that A nonfree in $V^{\mathbb{P}}$. □

Given infinite cardinals $\kappa \leq \lambda$, note that $\mathrm{Add}(\omega, \kappa)$ is a regular suborder of $\mathrm{Add}(\omega, \lambda)$, and $\mathrm{Add}(\omega, \lambda) \cong \mathrm{Add}(\omega, \kappa) \times \mathrm{Add}(\omega, [\kappa, \lambda))$. To establish Theorem 6.1, we will actually prove the following formally stronger statement.

Theorem 6.3. *Suppose that A is a nonfree abelian group, κ is the least cardinality of a nonfree subgroup of A,*

$$0 \to K \to F \to A \to 0$$

is a free resolution of A, and B_K is a basis for K. Then, in $V^{\mathrm{Add}(\omega,\kappa)}$, there is a $\varphi \in \mathrm{Hom}(K, \mathbb{Z})$ such that

(1) $\varphi[B_K] \subseteq \{0, 1\}$;
(2) in every ccc forcing extension of $V^{\mathrm{Add}(\omega,\kappa)}$, there is no $\psi \in \mathrm{Hom}(F, \mathbb{Z})$ extending φ.

Proof. The proof is by induction on $|A|$; since the proof resembles that of Theorem 5.1, we will omit some details and frequently refer the reader back to that earlier proof. If A has nonzero torsion elements or is torsion-free and countable, then the homomorphism $\varphi \in \mathrm{Hom}(K, \mathbb{Z})$ constructed (in V) in the proof of Proposition 4.4 or Theorem 4.2, respectively, satisfies the conclusion of the theorem, since, as the reader may readily verify in either case, the argument that φ cannot be extended to an element of $\mathrm{Hom}(F, \mathbb{Z})$ can be carried out equally well in any outer model of V.

We may thus assume that $\kappa > \aleph_0$ and A is torsion-free. As in the proof of Theorem 5.1, we may assume that A is almost free and $|F| = \kappa$. To see this, let N, A_0, F_0, K_0, $B_{F,0}$, and $B_{K,0}$ be as in the analogous argument in the earlier proof. Suppose that we are able to obtain, in $V^{\mathrm{Add}(\omega,\kappa)}$, a $\varphi_0 \in \mathrm{Hom}(K_0, \mathbb{Z})$ such that $\varphi_0[B_{K,0}] \subseteq \{0, 1\}$ and φ_0 cannot be extended to a map $\psi_0 \in \mathrm{Hom}(F_0, \mathbb{Z})$ in any ccc forcing extension of $V^{\mathrm{Add}(\omega,\kappa)}$. Extend φ_0 to a map $\varphi \in \mathrm{Hom}(K, \mathbb{Z})$ by letting $\varphi(t) = 0$ for all $t \in B_K \setminus B_{K,0}$. Then $\varphi \in V^{\mathrm{Add}(\omega,\kappa)}$. Moreover, if $\dot{\mathbb{P}}$ is an $\mathrm{Add}(\omega, \kappa)$-name for a ccc forcing notion, then φ does not extend in $V^{\mathrm{Add}(\omega,\kappa)*\dot{\mathbb{P}}}$ to an element of $\mathrm{Hom}(F, \mathbb{Z})$, as the restriction of such an extension to F_0 would extend φ_0, contradicting our choice of φ_0.

Assume now that A is almost free and $|F| = \kappa$. For $\alpha < \kappa$, let M_α, δ_α, A_α, F_α, and K_α be as in the proof of Theorem 5.1. Let

$$S := \{\alpha < \kappa \mid A_{\alpha+1}/A_\alpha \text{ is nonfree}\};$$

again, since A is almost free and nonfree, we may assume that S is stationary in κ. For each $\alpha \in S$, let κ_α, A^*_α, F^*_α, K^*_α, $B^*_{F,\alpha}$, and $B^*_{K,\alpha}$ be as in the proof of Theorem 5.1.

Note that $\mathrm{Add}(\omega, \kappa) \cong \mathrm{Add}(\omega, \kappa) \times \mathrm{Add}(\omega, \kappa)$, and hence we can assume that we are in fact working in an extension V_1 of the ground model V by (the first copy of) $\mathrm{Add}(\omega, \kappa)$. In particular, by the inductive hypothesis, we can assume that, for every $\alpha \in S$, we have a $\varphi_\alpha \in \mathrm{Hom}(K^*_\alpha, \mathbb{Z})$ such that

- $\varphi_\alpha[\{t + K_\alpha \mid t \in B^*_{K,\alpha}\}] \subseteq \{0, 1\}$;

- in every ccc forcing extension of V_1, there is no $\psi_\alpha \in \mathrm{Hom}(F^*_\alpha, \mathbb{Z})$ extending φ_α.

We now describe a method for producing elements of $\mathrm{Hom}(K, \mathbb{Z})$. Given a function $f : \kappa \to \{0, 1\}$, define $\varphi(f) \in \mathrm{Hom}(K, \mathbb{Z})$ as follows. For every $t \in B_K$, if there is $\alpha \in S$ such that $t \in B^*_{K,\alpha}$ and $f(\alpha) = 1$, then let $\varphi(f)(t) = \varphi_\alpha(t + K_\alpha)$. In all other cases, let $\varphi(f)(t) = 0$. Then extend linearly to all of K. By construction, we have $\varphi(f)[B_K] \subseteq \{0, 1\}$ for all $f : \kappa \to \{0, 1\}$.

Note that this construction continues to make sense in any outer model of V_1. Let $\dot{g}$ be the canonical $\mathrm{Add}(\omega, \kappa)$-name for the union of the generic filter, and note that $\dot{g}$ is a name for a function from κ to $\{0, 1\}$. Then let $\dot{\varphi}$ be an $\mathrm{Add}(\omega, \kappa)$-name for $\varphi(\dot{g})$. We claim that $\dot{\varphi}$ is forced to be as desired. To see this, let $\dot{\mathbb{P}}$ be an $\mathrm{Add}(\omega, \kappa)$-name for a ccc forcing notion, and suppose for sake of contradiction that $\dot{\psi}$ is an $\mathrm{Add}(\omega, \kappa) * \dot{\mathbb{P}}$-name that is forced to be an element of $\mathrm{Hom}(F, \mathbb{Z})$ extending $\dot{\varphi}$.

Let $G * H$ be $\mathrm{Add}(\omega, \kappa) * \dot{\mathbb{P}}$-generic over V_1, and let $g = \bigcup G$. Let $\varphi = \dot{\varphi}^G = \varphi(g)$, and let $\psi = \dot{\psi}^{G*H}$. Let $\{z_\gamma \mid \gamma < \kappa\}$ be the $\lhd$-least injective enumeration of B_F. For each $\gamma < \kappa$, find $(p_\gamma, q_\gamma) \in G * H$ deciding the value of $\dot{\psi}(z_\gamma)$. Since the domain of each p_γ is a finite subset of κ, we can find $\alpha \in S$ such that

- $\delta_\alpha = \alpha$;
- for all $\gamma < \alpha$, we have $\mathrm{dom}(p_\gamma, q_\gamma) \subseteq \alpha$.

For each $p \in \mathrm{Add}(\omega, \kappa)$, define $\hat{p} \in \mathrm{Add}(\omega, \kappa)$ by letting $\mathrm{dom}(\hat{p}) = \mathrm{dom}(p)$ and, for all $\beta \in \mathrm{dom}(p)$, letting

$$\hat{p}(\beta) = \begin{cases} p(\beta) & \text{if } \beta \neq \alpha \\ 1 - p(\beta) & \text{if } \beta = \alpha. \end{cases}$$

Since the map $p \mapsto \hat{p}$ is an automorphism of $\mathrm{Add}(\omega, \kappa)$, the filter $\hat{G} = \{\hat{p} \mid p \in G\}$ is generic over V_1, and the function $\hat{g} = \bigcup \hat{G}$ agrees with g everywhere except α, where it takes the opposite value. Without loss of generality, assume that $g(\alpha) = 0$ and $\hat{g}(\alpha) = 1$. For ease of comprehension, we then denote (G, g) by (G_0, g_0) and $(\hat{G}, \hat{g})$ by (G_1, g_1). Note also that $V[G_0] = V[G_1]$, and hence H is $\mathbb{P}$-generic over both. For $i < 2$, let $\varphi_i := \varphi(g_i) = \dot{\varphi}^{G_i}$, and let $\psi_i := \dot{\psi}^{G_i * H}$.

By our assumptions about $\dot{\psi}$, we know that, for each $i < 2$, $\psi_i \in \mathrm{Hom}(F, \mathbb{Z})$ extends φ_i. Moreover, by our choice of α and the fact that $g_0 \restriction \alpha = g_1 \restriction \alpha$, we know that $\psi_0 \restriction F_\alpha = \psi_1 \restriction F_\alpha$. Now define a homomorphism $\psi_\alpha \in \mathrm{Hom}(F^*_\alpha, \mathbb{Z})$ as follows. For all $z \in B^*_{F,\alpha}$, let $\psi_\alpha(z + F_\alpha) = \psi_1(z) - \psi_0(z)$, and then extend linearly to the rest of F^*_α. We claim that ψ_α extends φ_α. The proof of this is exactly the same as that of the analogous fact in the proof of Theorem 5.1, so we leave it to the reader. But then, in $V_1[G * H]$, we have an element of $\mathrm{Hom}(F^*_\alpha, \mathbb{Z})$ extending φ_α, contradicting our choice of φ_α and the fact that $V_1[G * H]$ is a ccc extension of V_1. □

7. Stone spaces

In this section, we take a direct route to the proof of clause (3) of Theorem 1.3. A more indirect route, involving a broader investigation into more general connections between condensed mathematics and forcing, occupies Sections 8 and 9, which can be read independently of this one.

Given an infinite cardinal κ, let $\mathbb{B}_\kappa$ denote the Boolean completion of $\mathrm{Add}(\omega,\kappa)$, and let S_κ denote the Stone space $S(\mathbb{B}_\kappa)$. We will sometimes slightly abuse notation and think of $\mathrm{Add}(\omega,\kappa)$ as a subset of $\mathbb{B}_\kappa$. Given sets X and Y, we now highlight a translation between $\mathbb{B}_\kappa$-names for functions from X to Y and continuous functions from S_κ to the product space Y^X. Recall that a clopen basis for S_κ is given by all sets of the form

$$N_b := \{U \in S_\kappa \mid b \in U\}$$

for $b \in \mathbb{B}_\kappa$.

Definition 7.1. Suppose that κ is an infinite cardinal and X and Y are nonempty sets.

(1) Suppose that $\varphi : S_\kappa \to Y^X$ is continuous, where Y is discrete and Y^X is given the product topology. Define a $\mathbb{B}_\kappa$-name $\dot{f}_\varphi$ for a function from X to Y as follows. Given $x \in X$ and $y \in Y$, let $U_{x,y} := \{h \in Y^X \mid h(x) = y\}$, and set

$$[\![\dot{f}_\varphi(x) = y]\!]_{\mathbb{B}_x} := \bigvee\{b \in \mathbb{B}_\kappa \mid N_b \subseteq \varphi^{-1}[U_{x,y}]\}.$$

(2) Suppose now that Y is a *finite* set and $\dot{f}$ is a $\mathbb{B}_\kappa$-name for a function from X to Y. Define a function $\varphi_{\dot{f}} : S_\kappa \to Y^X$ as follows: for each $U \in S_\kappa$ and $x \in X$, let $\varphi_{\dot{f}}(x)$ be the unique $y \in Y$ such that $[\![\dot{f}(x) = y]\!]_{\mathbb{B}_\kappa} \in U$. Note that such a y must exist, since $\{[\![\dot{f}(x) = y]\!]_{\mathbb{B}_\kappa} \mid y \in Y\}$ is a finite maximal antichain in $\mathbb{B}_\kappa$ and therefore must intersect U.

It is routine to verify that $\dot{f}_\varphi$ as defined in Definition 7.1(1) is indeed a $\mathbb{B}_\kappa$ name for a function from X to Y and that $\varphi_{\dot{f}}$ as defined in Definition 7.1(2) is a continuous function from S_κ to Y^X. (These facts will also follow from more general arguments appearing below in Section 8.) This translation, together with Theorem 6.3, allows us to replace the space 2^κ in Theorem 5.1 with the Stone space S_κ.

Theorem 7.2. *Suppose that A is a nonfree abelian group and $\kappa = |A|$. Suppose moreover that $0 \to K \to F \to A \to 0$ is a free resolution of A. Then there is a continuous map $\varphi : S_\kappa \to \mathrm{Hom}(K,\mathbb{Z})$ that cannot be lifted pointwise to a continuous map $\psi : S_\kappa \to \mathrm{Hom}(F,\mathbb{Z})$.*

Moreover, given a basis B_K for K, we can require that, for all $U \in S_\kappa$ and all $t \in B_K$, we have $\varphi(U)(t) \in \{0,1\}$.

Proof. By Theorem 6.3, we can fix an $\mathrm{Add}(\omega,\kappa)$-name $\dot{\tau}$ for an element of $\mathrm{Hom}(K,\mathbb{Z})$ for which it is forced to be the case that

- $\dot{\tau}[B_K] \subseteq \{0,1\}$; and
- there is no $\sigma \in \mathrm{Hom}(F,\mathbb{Z})$ that extends $\dot{\tau}$.

We think of $\dot{\tau}$ as a $\mathbb{B}_\kappa$-name. Let $\dot{f}$ be the $\mathbb{B}_\kappa$-name for the restriction of τ to B_K; $\dot{f}$ is thus a name for a function from B_K to 2. Let $\varphi_{\dot{f}}$ be the continuous function from S_κ to 2^{B_K} given Definition 7.1(2), and let $\varphi : S_\kappa \to \mathrm{Hom}(K,\mathbb{Z})$ be defined by letting $\varphi(U)$ be the linear extension of $\varphi_{\dot{f}}(U)$ to all of K. Then φ is also a continuous function.

We claim that φ is as desired. We clearly have $\varphi(U)[B_K] \subseteq \{0,1\}$ for all $U \in S_\kappa$. Suppose for sake of contradiction that $\psi : S_\kappa \to \mathrm{Hom}(F,\mathbb{Z})$ is a continuous map and extends φ pointwise. Let $\dot{f}_\psi$ be the $\mathbb{B}_\kappa$-name for a function from F to $\mathbb{Z}$ given by Definition 7.1(1).

Claim 7.3. $\Vdash_{\mathbb{B}_\kappa} \dot{f}_\psi \in \mathrm{Hom}(F, \mathbb{Z})$.

Proof. Suppose for sake of contradiction that there are $y, z \in F$ and $b \in \mathbb{B}_\kappa^+$ such that $b \Vdash_{\mathbb{B}_\kappa} \dot{f}_\psi(y) + \dot{f}_\psi(z) \neq \dot{f}_\psi(y+z)$. By extending b, we may assume that there are k_y, k_z, and k_{yz} such that $b \Vdash_{\mathbb{B}_\kappa} (\dot{f}_\psi(y), \dot{f}_\psi(z), \dot{f}_\psi(y+z)) = (k_y, k_z, k_{yz})$ and $k_y + k_z \neq k_{yz}$. Let $U \in S_\kappa$ be such that $b \in U$. Then by the definition of $\dot{f}_\psi$, we have $\psi(U)(y) = k_y$, $\psi(U)(z) = k_z$, and $\psi(U)(y+z) = k_{yz}$, contradicting the fact that $\psi(U) \in \mathrm{Hom}(F, \mathbb{Z})$. □

Claim 7.4. $\Vdash_{\mathbb{B}_\kappa} \dot{f}_\psi \restriction K = \dot{\tau}$.

Proof. Suppose for sake of contradiction that there is $t \in B_K$ and $b \in \mathbb{B}_\kappa^+$ such that $b \Vdash_{\mathbb{B}_\kappa} \dot{f}_\psi(t) \neq \dot{\tau}(t)$. By extending b if necessary, we may assume without loss of generality that there are $k_\psi, k_\tau \in \mathbb{Z}$ such that $b \Vdash_{\mathbb{B}_\kappa} (\dot{f}_\psi(t), \dot{\tau}(t)) = (k_\psi, k_\tau)$. Let $U \in S_\kappa$ be such that $b \in U$. Then by the definition of $\varphi_{\dot{f}}$, we have $\varphi_{\dot{f}}(U)(t) = \varphi(U)(t) = k_\psi$, and by the definition of $\dot{f}_\psi$, we have $\psi(U)(t) = k_\tau$, contradicting the fact that ψ extends φ pointwise. □

But now we have shown that there is forced by $\mathbb{B}_\kappa$ to be a $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$ (namely, $\dot{f}_\psi$) that extends $\dot{\tau}$, which is a contradiction, thus completing the proof. □

8. Topological spaces in forcing extensions

In this section and the next, we begin a general investigation into connections between condensed mathematics and forcing, the results of which will yield clause (3) of Theorem 1.3 as a corollary.

Suppose that (X, τ) is a compact Hausdorff space and W is an outer model of V. Then there is a canonical way to interpret (X, τ) as a compact Hausdorff space $(\hat{X}, \hat{\tau})$ in W. First, let τ_c denote the collection of all closed subsets of X (in V). In W, let $\hat{X}$ denote the collection of all maximal filters on $\tau_c \setminus \{\emptyset\}$. In other words, $\hat{X}$ is the collection of all maximal collections of elements of τ_c with the finite intersection property.

There is a natural map $\pi : X \to \hat{X}$ defined by letting $\pi(x) := \{D \in \tau_c \mid x \in D\}$ for all $x \in X$. We also define $\pi : \tau \to \mathscr{P}(\hat{X})$ by letting $\pi(U) := \{F \in \hat{X} \mid (X \setminus U) \notin F\}$. Then let $\hat{\tau}$ be the topology on $\hat{X}$ generated by $\pi[\tau]$.

Remark 8.1. The space $(\hat{X}, \hat{\tau})$ identified above, together with the associated maps $\pi : X \to \hat{X}$ and $\pi : \tau \to \hat{\tau}$ is precisely the *interpretation* of (X, τ) in W, in the sense of [29]. In particular, this implies the following properties, which can also readily be directly verified:

- $(\hat{X}, \hat{\tau})$ is compact in W;
- $\pi[\tau]$ is a basis for $\hat{\tau}$;
- for all $x \in X$ and $U \in \tau$, we have $x \in U$ if and only if $\pi(x) \in \pi(U)$.

Now suppose that $\mathbb{B}$ is a complete Boolean algebra and $S = S(\mathbb{B})$ is its Stone space. We now show how to identify the space $(\hat{X}, \hat{\tau})$, as computed in the forcing extension by $\mathbb{B}$, with an appropriate quotient of the space $C(S, X)^V$, equipped with the compact-open topology. In fact, we will prove something more general.

We first establish a correspondence between elements of $C(S,X)$ and $\mathbb{B}$-names for points in $\hat{X}$.[3] In the forward direction, fix a function $f \in C(S,X)$, and define a $\mathbb{B}$-name $\dot{F}_f$ for a subset of τ_c as follows. For every $D \in \tau_c$ and every $b \in \mathbb{B}$, put $(\check{D}, b)$ in $\dot{F}_f$ if and only if $N_b \subseteq f^{-1}[D]$.

Lemma 8.2. *$\dot{F}_f$ is a $\mathbb{B}$-name for a maximal filter on τ_c.*

Proof. The fact that $\dot{F}_f$ is forced to be a filter is immediate from the definition. To show that it is forced to be maximal, fix an arbitrary $C \in \tau_c$. It suffices to show that the following set is dense in $\mathbb{B}$:

$$E_C := \{b \in \mathbb{B} \mid N_b \subseteq f^{-1}[C] \textbf{ or } \exists D \in \tau_c[D \cap C = \emptyset \wedge N_b \subseteq f^{-1}[D]]\}.$$

To this end, fix an arbitrary $b_0 \in \mathbb{B}$, and assume that $N_{b_0} \not\subseteq f^{-1}[C]$. Fix $\mathcal{U} \in N_{b_0}$ such that $f(\mathcal{U}) \in X \setminus C$. Since X is compact, it is regular, so we can find an open set $O \in \tau$ such that $\mathcal{U} \in O \subseteq \mathrm{cl}(O) \subseteq X \setminus C$. By the continuity of f, we can find $b \leq b_0$ such that $\mathcal{U} \in N_b$ and $N_b \subseteq f^{-1}[O]$. Then $b \in E_C$, as desired. □

In the other direction, fix a $\mathbb{B}$-name $\dot{F}$ for a maximal filter on τ_c. For each $D \in \tau_c$, let $b_D := [\![D \in \dot{F}]\!]_{\mathbb{B}}$. Given $\mathcal{U} \in S$, let $\dot{F}_{\mathcal{U}} := \{D \in \tau_c \mid b_D \in \mathcal{U}\}$.

Lemma 8.3. *For all $\mathcal{U} \in S$, $\bigcap \dot{F}_{\mathcal{U}}$ contains exactly one point.*

Proof. Fix $\mathcal{U} \in S$. We first show that $\dot{F}_{\mathcal{U}}$ is a filter. To this end, fix $D_0, D_1 \in \dot{F}_{\mathcal{U}}$. Since $\dot{F}$ is forced to be a filter, we have $[\![D_0 \cap D_1 \in \dot{F}]\!]_{\mathbb{B}} = [\![D_0 \in \dot{F}]\!]_{\mathbb{B}} \wedge [\![D_1 \in \dot{F}]\!]_{\mathbb{B}}$. Then, since $\mathcal{U}$ is a filter and $[\![D_i \in \dot{F}]\!]_{\mathbb{B}} \in \mathcal{U}$ for $i < 2$, it follows that $[\![D_0 \cap D_1 \in \dot{F}]\!]_{\mathbb{B}} \in \mathcal{U}$, so $D_0 \cap D_1 \in \dot{F}_{\mathcal{U}}$.

Thus, $\dot{F}_{\mathcal{U}}$ has the finite intersection property; since X is compact, it follows that $\bigcap \dot{F}_{\mathcal{U}}$ is nonempty. Now suppose for sake of contradiction that x_0 and x_1 are distinct elements of $\dot{F}_{\mathcal{U}}$. Since X is Hausdorff, we can find disjoint open neighborhoods U_0 and U_1 of x_0 and x_1, respectively. For $i < 2$, let $D_i := X \setminus U_i$. Since $D_0 \cup D_1 = X$ and $\dot{F}$ is forced to be a maximal filter on $\tau_c \setminus \{\emptyset\}$, a routine argument shows that $[\![D_0 \in \dot{F}]\!]_{\mathbb{B}} \vee [\![D_1 \in \dot{F}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$. Therefore, since $\mathcal{U}$ is an ultrafilter, there must be $i < 2$ such that $[\![D_i \in \dot{F}]\!]_{\mathbb{B}} \in \mathcal{U}$, and hence $D_i \in \dot{F}_{\mathcal{U}}$. But then $x_{1-i} \notin D_i$ and $\bigcap \dot{F}_{\mathcal{U}} \subseteq D_i$, so $x_{1-i} \notin \bigcap \dot{F}_{\mathcal{U}}$, which is a contradiction. It follows that $\bigcap \dot{F}_{\mathcal{U}}$ contains precisely one point. □

Using the claim, define a function $f_{\dot{F}} : S \to X$ by letting, for all $\mathcal{U} \in S$, $f_{\dot{F}}(\mathcal{U})$ be the unique $x \in \bigcap \dot{F}_{\mathcal{U}}$.

Lemma 8.4. *$f_{\dot{F}}$ is continuous.*

Proof. Fix $U \in \tau$ and $\mathcal{U} \in f_{\dot{F}}^{-1}[U]$. For every $y \in X \setminus U$, we can find $D_y \in \tau_c$ such that $y \notin D_y$ and $b_{D_y} \in \mathcal{U}$. Then $\{X \setminus D_y \mid y \in X \setminus U\}$ is an open cover of $X \setminus U$, so we can find a finite subset $a \subseteq X \setminus U$ such that $D := \bigcap\{D_y \mid y \in a\} \subseteq U$. Then $b_D = \bigwedge\{b_{d_y} \mid y \in a\} \in \mathcal{U}$ and $N_{b_D} \subseteq f_{\dot{F}}^{-1}[U]$. □

It is routine to verify that this is a reflexive duality: Given $g \in C(S,X)$, we have $f_{\dot{F}_g} = g$ and, given a $\mathbb{B}$-name $\dot{E}$ for a maximal filter on τ_c, we have $\Vdash_{\mathbb{B}} \dot{E} = \dot{F}_{f_{\dot{E}}}$. Also, this story can clearly be relativized below a given condition $b \in \mathbb{B}$: functions in $C(N_b, X)$ correspond to $\mathbb{B}$-names $\dot{F}$ such that $b \Vdash_{\mathbb{B}}$ "$\dot{F}$ is a maximal filter on τ_c".

[3] Recent work in this direction, focusing especially on the case in which X is a (compact) Polish space, can be found in [26]; many of the basic results we present here are implicit in that work.

Now suppose that X is an arbitrary Hausdorff space, not necessarily compact. Now there may or may not be a well-behaved interpretation $\hat{X}$ of X in the extension by $\mathbb{B}$. However, there is still a natural extension of X in $V^{\mathbb{B}}$ that will correspond to a quotient of the space $C(S,X)^V$ in $V^{\mathbb{B}}$. We will denote this extension by $\hat{X}_{\mathcal{K}}$; it can be thought of as the extension of X spanned by ground model compact sets. We now give a definition of this space.

Let $\mathcal{K}(X)$ denote the collection of compact subsets of X (in V). Then $(\mathcal{K}(X), \subseteq)$ is a directed partial order. For each $K \in \mathcal{K}(X)$, let τ_K denote the subspace topology on K. For each $K \in \mathcal{K}(X)$, there is a canonical interpretation $(\hat{K}, \hat{\tau}_K)$ of (K, τ_K) in $V^{\mathbb{B}}$, as above. Namely, $\hat{K}$ is the collection of all maximal filters on $(\tau_K)_c$. Moreover, if $K \subseteq K'$ are both in $\mathcal{K}(X)$, then the inclusion map $\iota_{KK'} : K \to K'$ induces an inclusion map $\hat{\iota}_{KK'} : \hat{K} \to \hat{K}'$. We now define $(X_{\mathcal{K}}, \tau_{\mathcal{K}})$ in $V^{\mathbb{B}}$.

First, we let the underlying set, $X_{\mathcal{K}}$, be the direct limit (i.e., colimit) of the system $\langle \hat{K}, \hat{\iota}_{KK'} \mid K, K' \in \mathcal{K}(X),\ K \subseteq K' \rangle$. Formally, $X_{\mathcal{K}}$ consists of equivalence classes of the form $[(x, K)]$, where $K \in \mathcal{K}(X)$ and $x \in \hat{K}$. Given an open set $U \in \tau$ and $K \in \mathcal{K}(X)$, let $\hat{U}_K$ denote its interpretation in $\hat{\tau}_K$. We then define $\hat{U}_{\mathcal{K}} \subseteq \hat{X}_{\mathcal{K}}$ as follows: given $K \in \mathcal{K}(X)$ and $x \in \hat{K}$, put $[(x, K)] \in \hat{U}_{\mathcal{K}}$ if and only if $x \in \hat{U}_K$. It is routine to check that this is well-defined. Now let $\hat{\tau}_{\mathcal{K}}$ be the topology generated by $\{\hat{U}_{\mathcal{K}} \mid U \in \tau\}$.

We take a slight detour to note that this is not the only, nor arguably the most natural, topology we could have placed on $\hat{X}_{\mathcal{K}}$. We alternatively could have taken the direct limit topology, which we will denote by $\hat{\tau}^{\lim}_{\mathcal{K}}$, i.e., the finest topology that makes all of the limit maps $\hat{\iota}_K : \hat{K} \to \hat{X}_{\mathcal{K}}$ continuous. We always have $\hat{\tau}_{\mathcal{K}} \subseteq \hat{\tau}^{\lim}_{\mathcal{K}}$, and if X is locally compact, then the two topologies coincide (moreover, if X is locally compact, then $\hat{X}_{\mathcal{K}}$ is precisely the interpretation of X in $V^{\mathbb{B}}$ in the sense of [29]). However, in general the direct limit topology can be strictly finer, even for such relatively nice spaces as ${}^{\omega}\omega$, as the following example shows.

Proposition 8.5. *Suppose that $X = {}^{\omega}\omega$ and $\mathbb{B}$ adds a dominating real. Then $\hat{\tau}^{\lim}_{\mathcal{K}} \supsetneq \hat{\tau}_{\mathcal{K}}$ in $V^{\mathbb{B}}$.*

Proof. Let W denote the extension by $\mathbb{B}$. In V, for every $x \in {}^{\omega}\omega$ let $K_x := \{y \in {}^{\omega}\omega \mid y < x\}$. Then $\{K_x \mid x \in {}^{\omega}\omega\}$ is a $\subseteq$-cofinal subset of $\mathcal{K}(X)$. Moreover, it is routine to show that, for every $x \in {}^{\omega}\omega$, we have $\hat{K}_x = (\prod_{n<\omega} x(n))^W$. Therefore, the underlying set of $\hat{X}_{\mathcal{K}}$ can be identified with $\{y \in ({}^{\omega}\omega)^W \mid \exists x \in ({}^{\omega}\omega)^V\ y < x\}$.

Let $d \in ({}^{\omega}\omega)^W$ dominate every real in V, and let

$$O = \{y \in \hat{X}_{\mathcal{K}} \mid \forall n < \omega\ y(n) \neq d(n)\}.$$

Claim 8.6. $O \in \hat{\tau}^{\lim}_{\mathcal{K}}$.

Proof. It suffices to show that, for every $x \in ({}^{\omega}\omega)^V$, $O \cap \hat{K}_x \in \hat{\tau}_{K_x}$. Fix such an x, and let $A := \{n < \omega \mid d(n) < x(n)\}$. By choice of d, A is finite. Then, in V, the set $U = \{y \in {}^{\omega}\omega \mid \forall n \in A\ y(n) \neq d(n)\}$ is in τ, and we have $O \cap \hat{K}_x = \hat{U}_{K_x}$. □

Claim 8.7. *There is no $U \in \tau$ such that $\hat{U}_{\mathcal{K}} \subseteq O$.*

Proof. Fix $U \in \tau$. By shrinking U if necessary, we may assume that U is a basic open set, i.e., there is a finite set $A \subseteq \omega$ and a function $\sigma : A \to \omega$ such that

$U = \{y \in {}^\omega\omega \mid y \restriction A = \sigma\}$. Choose $n^* \in \omega \setminus A$ and define $y \in {}^\omega\omega$ by

$$y(n) = \begin{cases} \sigma(n) & \text{if } n \in A \\ d(n^*) & \text{if } n = n^* \\ 0 & \text{otherwise.} \end{cases}$$

Then, in W, we have $y \in O \setminus \hat{U}_{\mathcal{K}}$, so $\hat{U}_{\mathcal{K}} \not\subseteq O$. □

Since $\{\hat{U}_{\mathcal{K}} \mid U \in \tau\}$ is a base for $\hat{\tau}_{\mathcal{K}}$, this completes the proof. □

We now show that $\hat{X}_{\mathcal{K}}$ is homeomorphic to a natural quotient of $C(S,X)^V$ in the extension by $\mathbb{B}$. Recall that we endow $C(S,X)^V$ with the compact-open topology. Now suppose that G is a generic ultrafilter on $\mathbb{B}$, and, in $V[G]$, define an equivalence relation $\sim_G$ on $C(S,X)^V$ by letting $g \sim_G h$ if and only if there is $b \in G$ such that $g \restriction N_b = h \restriction N_b$. Let $C(S,X)^V/G$ denote the quotient space with respect to this equivalence relation. For each $U \in \tau$, define a set $N_U \subseteq C(S,X)^V/G$ as follows: for all $g \in C(S,X)^V$, put $[g] \in N_U$ if and only if there is $b \in G$ such that $g[N_b] \subseteq U$; note that this is well-defined.

Proposition 8.8. *$\{N_U \mid U \in \tau\}$ is a base for $C(S,X)^V/G$.*

Proof. It is immediate from the definition that each N_U is open. Now let $\hat{O} \subseteq C(S,X)^V/G$ be open, and fix $g \in C(S,X)^V$ such that $[g] \in \hat{O}$. Let $O = \{h \in C(S,X)^V \mid [h] \in \hat{O}\}$. We can then find a compact $K \subseteq S$ and an open set $U \in \tau$ such that $g[K] \subseteq U$ and $\{h \in C(S,X)^V \mid h[K] \subseteq U\} \subseteq O$. Since K is compact and g is continuous, we can find a finite collection $b_0, \ldots, b_{n-1}$ from $\mathbb{B}$ such that $K \subseteq \bigcup\{N_{b_i} \mid i < n\}$ and, for all $i < n$, we have $g[N_{b_i}] \subseteq U$. Letting $b := \bigvee\{b_i \mid i < n\}$, we get $K \subseteq N_b$ and $g[N_b] \subseteq U$.

Suppose first that $b \in G$. In this case, we claim that $N_U \subseteq \hat{O}$. To see this, fix $h \in C(S,X)^V$ such that $[h] \in N_U$. We can therefore find $c \in G$ such that $h[N_c] \subseteq U$. Define $h' \in C(S,X)^V$ by letting $h' \restriction N_c = h \restriction N_c$ and letting $h' \restriction (S \setminus N_c)$ be constant, taking an arbitrary value in U. Then $h'[K] \subseteq U$, so $[h'] \in \hat{O}$. Moreover, since $c \in G$, we have $h' \sim_G h$, so $[h'] = [h]$. Thus, $N_U \subseteq \hat{O}$.

Suppose next that $b \notin G$. In this case, we claim that $\hat{O}$ is actually all of $C(S,X)^V/G$. To see this, let $h \in C(S,X)^V$ be arbitrary. Define $h' \in C(S,X)^V$ by letting $h' \restriction N_b = g \restriction N_b$ and $h' \restriction (S \setminus N_b) = h \restriction (S \setminus N_b)$. Then $h'[N_b] = g[N_b] \subseteq U$, so $[h'] \in \hat{O}$. Since $b \notin G$, we have $[h'] = [h]$, so $[h] \in \hat{O}$. □

Now, in $V[G]$, define a map $k : C(S,X)^V/G \to \hat{X}_{\mathcal{K}}$ as follows. Given $g \in C(S,X)^V$, let $K := g[S]$. Then $K \in \mathcal{K}(X)$, so g corresponds in V to a $\mathbb{B}$-name $\dot{F}_g$ for an element of $\hat{K}$. Let F_g be the evaluation of this name in $V[G]$, and set $k([g]) = [(F_g, K)]$.

Proposition 8.9. *k is a bijection.*

Proof. To show that k is injective, fix two functions $g, h \in C(S,X)^V$. Familiar arguments yield the fact that, in V, the set

$$\{b \in \mathbb{B} \mid g \restriction N_b = h \restriction N_b \textbf{ or } g[N_b] \cap h[N_b] = \emptyset\}$$

is dense in $\mathbb{B}$. The genericity of G then yields the injectivity of k. To show that k is surjective, fix an element $[(F,K)] \in \hat{X}_{\mathcal{K}}$, and let $\dot{F}$ be a $\mathbb{B}$-name for F that is forced by $1_{\mathbb{B}}$ to be a maximal filter on $(\tau_K)_c$. Then $f_{\dot{F}}$, as defined at the beginning of this section, is in $C(S,X)^V$, and it is routine to verify that $k([f_{\dot{F}}]) = [(F,K)]$. □

The continuity of k and k^{-1} will now follow immediately from the following proposition.

Proposition 8.10. *For all $U \in \tau$, we have $k[N_U] = \hat{U}_K$.*

Proof. Suppose that $g \in C(S,X)^V$ and $[g] \in N_U$. Let $K := g[S]$, and fix $b \in G$ such that $g[N_b] \subseteq U$. Then, by construction, we have $(K \setminus U) \notin F_g$, so $k([g]) = [(F_g, K)] \in \hat{U}_{\mathcal{K}}$. Conversely, suppose that $g \in C(S,X)^V$ and $[g] \notin N_U$. Again, let $K := g[S]$. By familiar arguments, the following set is dense in $\mathbb{B}$ (in V):

$$\{b \in \mathbb{B} \mid g[N_b] \subseteq U \textbf{ or } g[N_b] \subseteq K \setminus U\}.$$

Since $[g] \notin N_U$, the genericity of G implies that $K \setminus U \in F_g$, so $k([g]) = [(F_g, K))] \notin \hat{U}_{\mathcal{K}}$. □

Altogether, we have established the following theorem.

Theorem 8.11. *Suppose that X is a Hausdorff space, $\mathbb{B}$ is a complete Boolean algebra, and $G \subseteq \mathbb{B}$ is a generic ultrafilter. Then, in $V[G]$, we have*

$$C(S(\mathbb{B}), X)^V / G \cong \hat{X}_{\mathcal{K}}.$$

We end this section with a few remarks. First, as was mentioned above, if X is a locally compact Hausdorff space, then $\hat{X}_{\mathcal{K}}$ is the canonical interpretation of X in $V^{\mathbb{B}}$ in the sense of [29]. Recall also that the category ED is precisely the category of Stone spaces of complete Boolean algebras; recall as well that $\underline{X}(S)$ equals, by definition, $C(S,X)$ for any $S \in \mathsf{ED}$. Therefore, the results of this section show that one way to think of the embedding of the category of locally compact Hausdorff spaces into Cond is to recognize that, for all such spaces $\underline{X}$, the condensed set $\underline{X}$ is precisely an organized presentation of all forcing names for points in the canonical interpretations of X in all possible set forcing extensions.

Second, we note that the operation $X \mapsto \hat{X}_{\mathcal{K}}$ commutes with products. Namely, if X_i is a Hausdorff space for all $i \in I$, then $(\widehat{\prod_{i \in I} X_i})_{\mathcal{K}} \cong \prod_{i \in I} (\hat{X}_i)_{\mathcal{K}}$. This is because the collection

$$\{\prod_{i \in I} K_i \mid \forall i \in I \ K_i \in \mathcal{K}(X_i)\}$$

is $\subseteq$-cofinal in $\mathcal{K}(\prod_{i \in I} X_i)$.

Third, even though Theorem 8.11 holds for *all* Hausdorff spaces X, there is an important sense in which the correspondence is meaningfully stronger in case X is compact. In particular, let $\dot{k}$ be the $\mathbb{B}$-name for the homeomorphism we constructed to witness Theorem 8.11. If X is compact, then, for every $\mathbb{B}$-name $\dot{x}$ for an element of $\hat{X}_{\mathcal{K}} (= \hat{X})$, there is a $g \in C(S,X)$ such that $1_{\mathbb{B}} \Vdash \dot{k}([g]) = \dot{x}$. In particular, in this case we can take $\dot{x}$ to be a $\mathbb{B}$-name for a maximal filter on τ_c, then then let $g = f_{\dot{x}}$, as defined at the outset of this section. However, in general this may not be the case, and there may be $\mathbb{B}$-names $\dot{x}$ for elements of $\hat{X}_{\mathcal{K}}$ for which there is *no* $g \in C(S,X)$ such that $1_{\mathbb{B}} \Vdash \dot{k}([g]) = \dot{x}$. For such $\dot{x}$, one may need to work below some condition $b \in \mathbb{B}$ that forces $\dot{x}$ to lie inside the interpretation of some particular $K \in \mathcal{K}(X)$; once one does so, one may then find $g \in C(S,X)$ such that $b \Vdash \dot{k}([g]) = \dot{x}$. This issue will lead to some slight complications and asymmetries in the next section.

We end this section by noting that these observations are of wider significance than might initially be suspected, by reason of the variety of expressions within condensed mathematics in which sets of the form $C(S, X)$, sometimes covertly, partake. The reader is referred to Lemma 11.1 below, and its argument, for a representative instance.

9. Compact interpretations and continuous functions

Fix for now a complete Boolean algebra $\mathbb{B}$, and let $S = S(\mathbb{B})$. In this section, we establish an equivalence between the settings of $C(S, X)$ on the one hand and $\mathbb{B}$-names for elements of $\hat{X}_{\mathcal{K}}$ on the other, with respect to interpretations of certain sentences. Fix for now a positive integer n and Hausdorff spaces $\langle X_i \mid i < n\rangle = \vec{X}$. We will specify a collection $\mathcal{L}(\vec{X})$ of formulas. We will do this by recursively defining $\mathcal{L}_j(\vec{X})$ for $j \leq n$ and then letting $\mathcal{L}(\vec{X}) := \bigcup_{j\leq n} \mathcal{L}_j(\vec{X})$. For each $j \leq n$ and $\varphi \in \mathcal{L}_j(\vec{X})$, the free variables in φ will be precisely $\{v_i \mid j \leq i < n\}$.

First, let $\mathcal{L}_0(\vec{X})$ consist of all formulas of the form $(v_0, v_1, \ldots, v_{n-1}) \in U$, where U is an open subset of $\prod_{i<n} X_i$. Now suppose that $j < n$ and we have specified $\mathcal{L}_j(\vec{X})$. Then $\mathcal{L}_{j+1}(\vec{X})$ consists precisely of the formulas that can be formed in one of the following ways. First, fix a formula $\varphi(v_j, v_{j+1}, \ldots, v_{n-1}) \in \mathcal{L}_j(\vec{X})$. Then

(1) the formula $\forall x\varphi(x, v_{j+1}, \ldots, v_{n-1})$ is in $\mathcal{L}_{j+1}(\vec{X})$;
(2) the formula $\exists x\varphi(x, v_{j+1}, \ldots, v_{n-1})$ is in $\mathcal{L}_{j+1}(\vec{X})$;
(3) for every compact $K \subseteq X_j$, the formula $\exists x \in K\varphi(x, v_{j+1}, \ldots, v_{n-1})$ is in $\mathcal{L}_{j+1}(\vec{X})$.

We call a formula $\varphi \in \mathcal{L}(\vec{X})$ *bounded* if all of its existential quantifiers are bounded by compact sets (i.e., they arise through option (3) above rather than option (2)). In their most basic setting, that of $\prod_{i<n} X_i$, the free variables v_i occurring in formulas in $\mathcal{L}(\vec{X})$ should be understood as standing for points in X_i, and the truth value of sentences arising from the substitution of these points for the free variables is evaluated in the natural way. But we also want to interpret these formulas in other contexts, namely in the context of $C(S, \prod_{i<n} X_i)$ and of $V^{\mathbb{B}}$.

Let us first deal with $C(S, \prod_{i<n} X_i)$, where the free variables v_i stand for elements of $C(S, X_i)$. We describe now what it means for $C(S, \prod_{i<n} X_i)$ to satisfy a sentence formed by substituting appropriate continuous functions for free variables in formulas in $\mathcal{L}(\vec{X})$. By induction on $j \leq n$, we will deal with $\mathcal{L}_j(\vec{X})$, in fact specifying what it means for a sentence to be satisfied *below a condition* $b \in \mathbb{B}$.

Suppose that $\varphi \in \mathcal{L}_0(\vec{X})$, $b \in \mathbb{B}$, and, for all $i < n$, $f_i \in C(S, X_i)$. Then φ is of the form $(v_0, \ldots, v_{n-1}) \in U$ for some open set $U \subseteq \prod_{i<n} X_i$. We say that $C(S, \prod_{i<n} X_i)$ *satisfies* $\varphi(f_0, \ldots, f_{n-1})$ *below* b, written

$$C(S, \prod_{i<n} X_i) \models_b \varphi(f_0, \ldots, f_{n-1}),$$

if and only if the set $\{s \in N_b \mid (f_0(s), \ldots, f_{n-1}(s)) \in U\}$ is dense in N_b. If $b = 1_{\mathbb{B}}$, then it is omitted from the terminology and notation.

Now suppose that $j < n$, $\varphi \in \mathcal{L}_{j+1}$, $b \in \mathbb{B}$, and, for all $j < i < n$, $f_i \in C(S, X_i)$.

- If φ is of the form $\forall x\psi(x, v_{j+1}, \ldots, v_{n-1})$, then

$$C(S, \prod_{i<n} X_i) \models_b \varphi(f_{j+1}, \ldots, f_{n-1})$$

if and only if, for all $g \in C(N_b, X_j)$, we have

$$C(S, \prod_{i<n} X_i) \models_b \psi(g, f_{j+1}, \ldots, f_{n-1}).$$

- If φ is of the form $\exists x \psi(x, v_{j+1}, \ldots, v_{n-1})$, then

$$C(S, \prod_{i<n} X_i) \models_b \varphi(f_{j+1}, \ldots, f_{n-1})$$

  if and only if there exists $g \in C(N_b, X_j)$ such that

$$C(S, \prod_{i<n} X_i) \models_b \psi(g, f_{j+1}, \ldots, f_{n-1}).$$

- If $K \subseteq X_j$ is compact and φ is of the form $\exists x \in K \psi(x, v_{j+1}, \ldots, v_{n-1})$, then

$$C(S, \prod_{i<n} X_i) \models_b \varphi(f_{j+1}, \ldots, f_{n-1})$$

  if and only if there exists $g \in C(N_b, K)$ such that

$$C(S, \prod_{i<n} X_i) \models_b \psi(g, f_{j+1}, \ldots, f_{n-1}).$$

In $V^{\mathbb{B}}$, the free variables v_i stand for $\mathbb{B}$-names for elements of $(\hat{X}_i)_{\mathcal{K}}$. We now define satisfaction in this context. Suppose that $\varphi \in \mathcal{L}_0(\vec{X})$ is of the form $(v_0, \ldots, v_{n-1}) \in U$, $b \in \mathbb{B}$, and, for all $i < n$, $\dot{x}_i$ is a $\mathbb{B}$-name for an element of $(\hat{X}_i)_{\mathcal{K}}$. Then $V^{\mathbb{B}} \models_b \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ if and only if $b \Vdash_{\mathbb{B}} (\dot{x}_0, \ldots, \dot{x}_{n-1}) \in \hat{U}_{\mathcal{K}}$.

Now suppose that $j < n$, $\varphi \in \mathcal{L}_{j+1}$, $b \in \mathbb{B}$, and, for all $j < i < n$, $\dot{x}_i$ is a $\mathbb{B}$-name for an element of $(\hat{X}_i)_{\mathcal{K}}$.

- If φ is of the form $\forall x \psi(x, v_{j+1}, \ldots, v_{n-1})$, then $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$ if and only if, for every $\mathbb{B}$-name $\dot{y}$ for an element of $(\hat{X}_j)_{\mathcal{K}}$, we have $V^{\mathbb{B}} \models_b \psi(\dot{y}, \dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$.
- If φ is of the form $\exists x \psi(x, v_{j+1}, \ldots, v_{n-1})$, then $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$ if and only if there exists a $\mathbb{B}$-name $\dot{y}$ for an element of $(\hat{X}_j)_{\mathcal{K}}$ such that $V^{\mathbb{B}} \models_b \psi(\dot{y}, \dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$.
- If $K \subseteq X_j$ is compact and φ is of the form $\exists x \in K \psi(x, v_{j+1}, \ldots, v_{n-1})$, then $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$ if and only if there exists a $\mathbb{B}$-name $\dot{y}$ such that $b \Vdash_{\mathbb{B}} \dot{y} \in \hat{K}$ and $V^{\mathbb{B}} \models_b \psi(\dot{y}, \dot{x}_{j+1}, \ldots, \dot{x}_{n-1})$.

Note that there is some monotonicity in these definitions. Namely, if $c \leq_{\mathbb{B}} b$ and $C(S, \prod_{i<n} X_i) \models_b \varphi$, then $C(S, \prod_{i<n} X_i) \models_c \varphi$, and similarly for $V^{\mathbb{B}}$.

Given $i < n$ and $f \in C(S, X_i)$, let $\dot{x}_f$ be a $\mathbb{B}$-name for $\dot{k}([f])$, where $\dot{k}$ is a name for the homeomorphism constructed to witness Theorem 8.11. Concretely, if $K = f[S]$, then $\dot{x}_f$ is a $\mathbb{B}$-name for $[(\dot{F}_f, K)] \in \hat{X}_{\mathcal{K}}$.

Theorem 9.1. *Suppose that $j \leq n$, $\varphi \in \mathcal{L}_j(\vec{X})$, $b \in \mathbb{B}$ and, for all $j \leq i < n$, we have $f_i \in C(S, X_i)$.*

(1) If $C(S, \prod_{i<n} X_i) \models_b \varphi(f_j, \ldots, f_{n-1})$, then $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{f_j}, \ldots, \dot{x}_{f_{n-1}})$.
(2) If φ is bounded and $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{f_j}, \ldots, \dot{x}_{f_{n-1}})$, then $C(S, \prod_{i<n} X_i) \models_b \varphi(f_j, \ldots, f_{n-1})$.

Proof. The proof is by induction on j. Suppose first that $j = 0$. For notational convenience, we can assume here that $n = 1$ by considering the single space $X = \prod_{i<n} X_i$ (recalling that $(\widehat{\prod_{i<n} X_i})_{\mathcal{K}} = \prod_{i<n}(\hat{X}_i)_{\mathcal{K}}$). Then there is an open set $U \subseteq X$ such that φ is the formula $v_1 \in U$, and we are given a function $f \in C(S, X)$. Let $K := \mathrm{im}(f)$, so K is a compact subset of X, and $\dot{x}_f$ is forced to be of the form $[(\dot{F}_f, K)]$.

To verify (1), suppose that $C(S, X) \models_b \varphi(f)$, and hence $\{s \in N_b \mid f(s) \in U\}$ is dense in N_b. Suppose for sake of contradiction that $b \not\Vdash_{\mathbb{B}} \dot{x}_f \in \hat{U}$. Then, by the definition of $\dot{F}_f$, there is $c \leq b$ such that $(\check{K} \setminus \check{U}, c) \in \dot{F}_f$, and therefore, again by the definition of $\dot{F}_f$, we have $N_c \subseteq f^{-1}[K \setminus U]$, which is a contradiction.

For (2), suppose that $b \Vdash_{\mathbb{B}} \dot{x}_f \in \hat{U}$, i.e., $b \Vdash_{\mathbb{B}} (K \setminus U) \notin \dot{F}_f$. Suppose for sake of contradiction that $\{s \in N_b \mid f(s) \in U\}$ is not dense in N_b. Then there is $c \leq b$ such that $N_c \subseteq f^{-1}[K \setminus U]$. But then $c \Vdash_{\mathbb{B}} (K \setminus U) \in \dot{F}_f$, which is a contradiction.

Now suppose that $j = j' + 1$ for some $j' < n$. To verify (1), suppose that $C(S, X) \models_b \varphi(f_j, \dots f_{n-1})$ but, for sake of contradiction, assume that $V^{\mathbb{B}} \not\models_b \varphi(\dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$. Suppose first that φ is of the form $\forall x \psi$ for some $\psi \in \mathcal{L}_{j'}(\vec{X})$. By assumption, there is a $\mathbb{B}$-name $\dot{y}$ for an element of $(\hat{X}_{j'})_{\mathcal{K}}$ such that $V^{\mathbb{B}} \not\models_b \psi(\dot{y}, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$. We can then find a condition $c \leq_{\mathbb{B}} b$ and a compact set $K \subseteq X_j$ such that $V^{\mathbb{B}} \not\models_c \varphi(\dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$ and $c \Vdash_{\mathbb{B}} \dot{y} \in \hat{K}$. We can therefore find a function $g \in C(S, K)$ such that $c \Vdash_{\mathbb{B}} \dot{y} = [(\dot{F}_g, \hat{K})] = \dot{x}_g$. Then $V^{\mathbb{B}} \not\models_c \psi(\dot{x}_g, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$, so by the induction hypothesis we have $C(S, X) \not\models_c \psi(g, f_j, \dots, f_{n-1})$, contradicting the assumption that $C(S, X) \models_b \varphi(f_j, \dots, f_{n-1})$.

Suppose next that φ is of the form $\exists x \psi$ for some $\psi \in \mathcal{L}_{j'}(\vec{X})$. Then we can fix $g \in C(S, X_{j'})$ such that $C(S, \vec{X}) \models_b \psi(g, f_j, \dots, f_{n-1})$. By the induction hypothesis, it follows that $V^{\mathbb{B}} \models \psi(\dot{x}_g, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$, so $V^{\mathbb{B}} \models \varphi(\dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$. The case in which φ is of the form $\exists x \in K \psi$ is the same.

To verify (2), suppose that φ is bounded and $V^{\mathbb{B}} \models_b \varphi(\dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$. Assume first that φ is of the form $\forall x \psi$ for some (bounded) $\psi \in \mathcal{L}_{j'}(\vec{X})$. Then, for all $g \in C(S, X_{j'})$, we have $V^{\mathbb{B}} \models_b \psi(\dot{x}_g, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$, so, by the induction hypothesis, we have $C(S, \vec{X}) \models_b \psi(g, f_j, \dots, f_{n-1})$. Since g was arbitrary, we have $C(S, \vec{X}) \models_b \varphi(f_j, \dots, f_{n-1})$.

Finally, assume that φ is of the form $\exists x \in K \psi$ for some compact $K \subseteq X_{j'}$ and some (bounded) $\psi \in \mathcal{L}_{j'}(\vec{X})$. Then we can find a $\mathbb{B}$-name $\dot{y}$ for an element of $\hat{K}$ such that $V^{\mathbb{B}} \models_b \psi(\dot{y}, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$. We can then find a function $g \in C(S, K)$ such that $b \Vdash_{\mathbb{B}} \dot{y} = \dot{x}_g$. Then $V^{\mathbb{B}} \models_b \psi(\dot{x}_g, \dot{x}_{f_j}, \dots, \dot{x}_{f_{n-1}})$, so, by the induction hypothesis, we have $C(S, \vec{X}) \models_b \psi(g, f_j, \dots, f_{n-1})$. Since $g \in C(S, K)$, it follows that $C(S, \vec{X}) \models_b \varphi(f_j, \dots, f_{n-1})$, as desired. □

We now finally return to Whitehead's problem, where the work of this and the previous section allows us to establish the following general result.

Theorem 9.2. *Suppose that A is an abelian group and $0 \to K \xrightarrow{\subseteq} F \to A \to 0$ is a free resolution of A, and let B_K be a basis for K, and let $e : B_K \to \omega$ be a function. Suppose that $\mathbb{B}$ is a complete Boolean algebra that forces the existence of a homomorphism $\tau : K \to \mathbb{Z}$ such that*

- *in $V^{\mathbb{B}}$, τ does not lift to a homomorphism $\nu : F \to \mathbb{Z}$;*
- *for all $t \in B_K$, we have $|\tau(t)| \leq e(t)$.*

Then, in V, we have $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$.

Proof. Let B_F be a basis for F, and note that elements of $\mathrm{Hom}(K, \mathbb{Z})$ and $\mathrm{Hom}(F, \mathbb{Z})$ can be identified with elements of ${}^{B_K}\mathbb{Z}$ and ${}^{B_F}\mathbb{Z}$, respectively, and the restriction map from $\mathrm{Hom}(F, \mathbb{Z})$ to $\mathrm{Hom}(K, \mathbb{Z})$ induces a continuous map $\pi : {}^{B_F}\mathbb{Z} \to {}^{B_K}\mathbb{Z}$. Note that the assertion that A is not Whitehead is precisely the assertion that π is not surjective, i.e., it corresponds to the sentence $\exists x \in {}^{B_K}\mathbb{Z}\ \forall y \in {}^{B_F}\mathbb{Z}\ \pi(y) \neq x$. Noting that the set $U := \{(y, x) \in {}^{B_F}\mathbb{Z} \times {}^{B_K}\mathbb{Z} \mid \pi(y) \neq x\}$ is an open subset of ${}^{B_F}\mathbb{Z} \times {}^{B_K}\mathbb{Z}$, we see that this sentence is an element of $\mathcal{L}(({}^{B_F}\mathbb{Z}, {}^{B_K}\mathbb{Z}))$.

By assumption, A is not Whitehead in $V^{\mathbb{B}}$, and in fact more is true. Let $\hat{\pi}$ be the extension of π to $V^{\mathbb{B}}$ induced by the restriction map from $\mathrm{Hom}(F, \mathbb{Z})$ to $\mathrm{Hom}(K, \mathbb{Z})$. Then $\mathbb{B}$ forces that there is an element of $\prod_{t \in B_K}[-e(t), e(t)]$ that is not in the range of π. Let X_K and X_F denote ${}^{B_K}\mathbb{Z}$ and ${}^{B_F}\mathbb{Z}$, respectively, and note that $K^* := \prod_{t \in B_K}[-e(t), e(t)]$ is a compact subset of X_K. Let φ be the (bounded) sentence $\exists x \in K^* \forall y \in X_F\ \pi(y) \neq x$. It is readily verified that the following facts hold in $V^{\mathbb{B}}$.

- $(\hat{X}_K)_{\mathcal{K}}$ is (homeomorphic to) the set of all $x \in ({}^{B_K}\mathbb{Z})^{V^{\mathbb{B}}}$ such that there exists $e : B_K \to \omega$ in V such that $|x(t)| \leq e(t)$ for all $t \in B_K$. A similar characterization holds for $(\hat{X}_F)_{\mathcal{K}}$.
- $\hat{U}_{\mathcal{K}} = \{(y, x) \in (\hat{X}_K)_{\mathcal{K}} \times (\hat{X}_F)_{\mathcal{K}} \mid \hat{\pi}(y) \neq x\}$.

By assumption, we then have $V^{\mathbb{B}} \models \varphi$. By Theorem 9.1, it follows that $C(S(\mathbb{B}), X_F \times X_K) \models \varphi$, and in fact $C(S(\mathbb{B}), X_F \times K^*) \models \varphi$, i.e., there is a continuous function $f : S(\mathbb{B}) \to K^*$ such that, for all continuous $g : S(\mathbb{B}) \to X_F$, the set of $s \in S(\mathbb{B})$ such that $\pi(g(s)) \neq f(s)$ is dense in $S(\mathbb{B})$. But this immediately implies that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$, completing the proof of the theorem. □

We now immediately obtain the following corollary, establishing clause (3) of Theorem 1.3.

Corollary 9.3. *Suppose that A is a nonfree abelian group and κ is the least cardinality of a nonfree subgroup of A. Let S be the Stone space of the Boolean completion of the forcing to add κ-many Cohen reals. Then* $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) \neq \emptyset$.

Proof. Let $\mathbb{B}$ denote the Boolean completion of the forcing to add κ-many Cohen reals, let $0 \to K \xrightarrow{\subseteq} F \to A \to 0$ be a free resolution of A, and let B_K be a basis for K. By Theorem 6.3, $\mathbb{B}$ forces the existence of a homomorphism $\tau : K \to \mathbb{Z}$ such that

- in $V^{\mathbb{B}}$, τ does not lift to a homomorphism $\nu : F \to \mathbb{Z}$; and
- for all $t \in B_K$, $\tau(t) \in \{0, 1\}$.

Therefore, Theorem 9.2 implies that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) \neq 0$, as desired. □

We end this section with some observations regarding forcing extensions in which Whitehead's problem has a negative answer. We first note that the converse of Theorem 9.2 is not true in general; namely, if A is an abelian group and $\mathbb{B}$ is a complete Boolean algebra, then it does not follow from $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$ that A is not Whitehead in $V^{\mathbb{B}}$. To see this, first recall that an abelian group A is $\aleph_1$-free if all of its countable subgroups are free, and a subgroup B of an $\aleph_1$-free abelian group A is said to be $\aleph_1$-*pure* if A/B is $\aleph_1$-free. An abelian group A satisfies *Chase's condition* if A is $\aleph_1$-free and every countable subgroup of A is contained in a countable $\aleph_1$-pure subgroup of A. In [9], Griffith showed that one can construct in ZFC an abelian group of size $\aleph_1$ that is not free but satisfies Chase's condition. One half of Shelah's independence result can then be stated as follows.

Theorem 9.4 (Shelah, [21], cf. [5])**.** *Suppose that MA_{ω_1} holds and A is an abelian group of size $\aleph_1$ that satisfies Chase's condition. Then A is Whitehead.*

Now suppose that $V = L$, and let A be an abelian group of size $\aleph_1$ that satisfies Chase's condition but is not free. For concreteness, suppose it is constructed as in [5, Theorem 7.3]. It is readily verified that this construction is absolute between V and outer models with the same $\aleph_1$; in particular, in any ccc forcing extension, it remains true that A satisfies Chase's condition and is not free. Let $\mathbb{B}$ be the Boolean completion of the standard ccc forcing iteration to force MA_{ω_1}, and let $0 \to K \to F \to A \to 0$ be a free resolution of A. Since $V = L$, A is not Whitehead in V, so there is a map $\tau \in \mathrm{Hom}(K, \mathbb{Z})$ that does not lift to a map $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$. Then the constant map $\varphi : S(\mathbb{B}) \to \mathrm{Hom}(K, \mathbb{Z})$ taking value τ witnesses that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$. However, since A still satisfies Chase's condition in $V^{\mathbb{B}}$ and MA_{ω_1} holds there, it follows that $\mathbb{B}$ forces that A is Whitehead.

However, we can recover an informative partial converse to Theorem 9.2, in the following sense. Note that, in the situation described in the paragraph above, forcing with $\mathbb{B}$ necessarily adds a new element $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$ that extends τ. However, this new homomorphism σ must be quite far from any ground model homomorphism; in particular, it cannot be bounded by any ground model function when restricted to any basis for F that lies in V. More generally, we have the following result.

Theorem 9.5. *Suppose that A is a non-Whitehead abelian group, $0 \to K \to F \to A \to 0$ is a free resolution of A, and $\tau \in \mathrm{Hom}(K, \mathbb{Z})$ is a homomorphism that does not extend to a homomorphism $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$. Suppose moreover that B_F is a basis for F, $e : B_F \to \omega$ is a function, and $\mathbb{B}$ is a complete Boolean algebra. Then, in $V^{\mathbb{B}}$, there is no homomorphism $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$ such that*

- *$\sigma \restriction K = \tau$; and*
- *for all $y \in B_F$, $|\sigma(y)| \leq e(y)$.*

Proof. Let B_K be a basis for K, let $X_K := {}^{B_K}\mathbb{Z}$ and $X_F := {}^{B_F}\mathbb{Z}$, let $\pi : X_F \to X_K$ be the continuous map from the proof of Theorem 9.2, and again let $U := \{(y, x) \in X_F \times X_K \mid \pi(y) \neq x\}$. Let $\varphi(x)$ be the formula $\forall y \in X_F (y, x) \in U$, where x is a free variable. Let $f : S(\mathbb{B}) \to X_K$ be the constant function taking value $\tau \restriction B_K$. Then, by choice of τ, we have $C(S(\mathbb{B}), X_F \times X_K) \models \varphi(f)$. By Theorem 9.1, it follows that $V^{\mathbb{B}} \models \varphi(\dot{x}_f)$. Since f is a constant function, it is readily verified that $\dot{x}_f$ is forced to be equal to the value of f, i.e., to $\tau \restriction B_K$. Therefore, it follows that, in $V^{\mathbb{B}}$, for all $\sigma \in \mathrm{Hom}(F, \mathbb{Z})$, if $\sigma \restriction B_F \in (\hat{X}_F)_{\mathcal{K}}$, then $\sigma \restriction K \neq \tau$. But $\sigma \restriction B_F \in (\hat{X}_F)_{\mathcal{K}}$ is equivalent to the existence of an $e : B_F \to \omega$ in V such that $|\sigma(y)| \leq e(y)$ for all $y \in B_F$, so the theorem follows. $\square$

10. Martin's Axiom and $\mathsf{CondAb}_{\omega_1}$

In this section, we show that the independence of Whitehead's problem in the classical setting can persist if we restrict ourselves to CondAb_κ for certain cardinal κ. Both for concreteness and because the category $\mathsf{CondAb}_{\omega_1}$ of light condensed sets provides the setting for [4], we focus on the case $\kappa = \omega_1$. The arguments in this section will be almost entirely set theoretic and combinatorial. We present a more algebraic path to related results in Section 11.

For a given abelian group A, if $\underline{A}$ is Whitehead in $\mathsf{CondAb}_{\aleph_1}$, then A is Whitehead in the classical sense, since

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_{\omega_1}}(\underline{A}, \underline{\mathbb{Z}})(*) = \mathrm{Ext}^1_{\mathsf{Ab}}(A, \mathbb{Z}).$$

Therefore, one side of the independence of Whitehead's problem in $\mathsf{CondAb}_{\omega_1}$ is clear: if it is the case that, classically, every Whitehead abelian group is free, then it follows that, for every abelian group A, if $\underline{A}$ is Whitehead in $\mathsf{CondAb}_{\omega_1}$, then A is free. In this section, we establish the nontrivial side of this independence. Namely, we prove the consistency of the existence of a nonfree abelian group A such that $\underline{A}$ is Whitehead in $\mathsf{CondAb}_{\omega_1}$. In fact, we shall prove that this follows from MA_{ω_1}, the same hypothesis used by Shelah in his proof of Theorem 1.1.

In what follows, when we write, e.g., that $S = \lim_{i \in \Lambda} S_i \in \mathsf{Prof}$, it should be implicitly understood that Λ is a directed partial order and each S_i is a finite (discrete) space. For each $i \in \Lambda$, we get a projection map $\pi_i : S \to S_i$. The basic open subsets of S are of the form $\pi_i^{-1}\{\bar{s}\}$, where $i \in \Lambda$ and $\bar{s} \in S_i$. If we write $S = \lim_{i \in \Lambda} S_i \in \mathsf{Prof}_{\omega_1}$, then this means, moreover, that Λ is countable.

Proposition 10.1. *Suppose that $S = \lim_{i \in \Lambda} S_i \in \mathsf{Prof}$ and $\{O_k \mid k < n\}$ is a finite partition of S into nonempty clopen sets. Then there is an $i \in \Lambda$ and a function $f : S_i \to n$ such that, for all $k < n$, we have $O_k = \{s \in S \mid f(\pi_i(s)) = k\}$.*

Proof. This follows immediately from the fact that each O_k is compact and open and can thus be written as a finite union of basic open subsets of S, together with the directedness of Λ. $\square$

Proposition 10.2. *Suppose that $S = \lim_{i \in \Lambda} S_i \in \mathsf{Prof}$, A and B are (discrete) abelian groups, and $\varphi : S \to \mathrm{Hom}(A, B)$ is a continuous map. For every finitely generated subgroup A_0 of A, there is an $i \in \Lambda$ and a map $\varphi_0 : S_i \to \mathrm{Hom}(A_0, B)$ such that, for all $s \in S$, we have*

$$\varphi(s) \restriction A_0 = \varphi_0(\pi_i(s)).$$

Proof. Let X_0 be a finite subset of A, and let A_0 be the subgroup of A generated by X_0. For each $\psi : X_0 \to B$, the set $S_\psi := \{s \in S \mid \varphi(s) \restriction X_0 = \psi\}$ is a clopen subset of S, by the continuity of φ. By the compactness of S, there are only finitely many ψ for which S_ψ is nonempty. Therefore, by Proposition 10.1, there is $i \in \Lambda$ and $f : S_i \to {}^{X_0}B$ such that, for all $s \in S$, we have $\varphi(s) \restriction X_0 = f(\pi_i(s))$.

Since A_0 is generated by X_0, every element of $\mathrm{Hom}(A_0, B)$ is determined by its restriction to X_0. Therefore, for each $\bar{s} \in S_i$, there is a unique extension $\varphi_0(\bar{s})$ of $f(\bar{s})$ to an element of $\mathrm{Hom}(A_0, B)$.[4] It then follows that, for all $s \in S$, we have $\varphi(s) \restriction A_0 = \varphi_0(\pi_i(s))$, as desired. $\square$

Recall the definition of Chase's condition from the end of Section 9. We now show that Theorem 9.4 transfers to the richer setting of $\mathsf{CondAb}_{\omega_1}$. By clause (1) of Theorem 1.3, this is sharp, since for any nonfree abelian group of size $\aleph_1$, the profinite set 2^{ω_1} witnesses that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_{\omega_2}}(\underline{A}, \underline{\mathbb{Z}}) \neq 0$.

[4]Such an extension must exist because, if $s \in S$ is such that $\pi_i(s) = \bar{s}$, then $\varphi(s) \restriction A_0$ is such an extension; the uniqueness then follows from the previous sentence.

Theorem 10.3. *Suppose that MA_{ω_1} holds and A is an abelian group of cardinality ω_1 satisfying Chase's condition. Then*

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}_{\omega_1}}(\underline{A}, \underline{\mathbb{Z}}) = 0.$$

Proof. Let $0 \to K \to F \to A \to 0$ be the canonical free resolution of A, let $S = \lim_{i \in \Lambda} S_i \in \mathsf{Prof}_{\omega_1}$, and let $\varphi : S \to \mathrm{Hom}(K, \mathbb{Z})$ be continuous. We must produce a continuous map $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ such that $\psi(s) \restriction K = \varphi(s)$ for all $s \in S$. We fix a basis $\{\bar{a} \mid a \in A\}$ for F; the surjection from F to A in the free resolution above is then generated by the map that sends $\bar{a}$ to a for all $a \in A$. For each subgroup $A' \subseteq A$, let $F_{A'}$ be the free group generated by $\{\bar{a} \mid a \in A'\}$, and let $K_{A'}$ be the kernel of the canonical surjection from $F_{A'}$ to A'.

Claim 10.4. *Suppose that A' is a free subgroup of A, with a basis B', and let $f : S \to {}^{B'}\mathbb{Z}$ be a continuous function (where $\mathbb{Z}$ is discrete and ${}^{B'}\mathbb{Z}$ is given the product topology). Then there is a unique continuous map $\psi' : S \to \mathrm{Hom}(F_{A'}, \mathbb{Z})$ such that*

1. *(1) for all $s \in S$ and $b \in B'$, $\psi'(s)(\bar{b}) = f(s)(b)$;*
2. *(2) for all $s \in S$, $\psi'(s) \restriction K_{A'} = \varphi(s) \restriction K_{A'}$.*

Proof. Fix $s \in S$. To fulfill requirement (1), we are obliged to let $\psi'(s)(\bar{b}) := f(s)(b)$ for all $b \in B'$. Now fix $a \in A' \setminus B'$. Then there is a unique way to express a in the form $\sum_{b \in B^*} c_b b$, where $B^* \subseteq B'$ is finite and, for all $b \in B^*$, c_b is a nonzero integer. To satisfy requirement (2), we are obliged to let

$$\psi'(s)(\bar{a}) := \varphi(s)\left(\bar{a} - \sum_{b \in B^*} c_b \bar{b}\right) + \sum_{b \in B^*} c_b f(s)(b).$$

We must then extend $\psi'(s)$ linearly to the rest of $F_{A'}$. It is routine to verify that $\psi'(s)$ thusly defined is in $\mathrm{Hom}(F_{A'}, \mathbb{Z})$ satisfying (1) and (2), and the continuity of ψ follows immediately from the continuity of φ and f. The uniqueness of ψ' is evident from the fact that, as the above construction makes clear, all values of $\psi'(s)$ were entirely determined by requirements (1) and (2) and the necessity for $\psi'(s)$ to be a homomorphism. □

We define a forcing notion $\mathbb{P}$ to which we will apply MA_{ω_1}. Conditions in $\mathbb{P}$ are all pairs of the form $p = (A_p, \psi_p)$ such that

- A_p is a finitely-generated pure subgroup of A;
- $\psi_p : S \to \mathrm{Hom}(F_{A_p}, \mathbb{Z})$ is continuous;
- for all $s \in S$, $\psi_p(s) \restriction K_{A_p} = \varphi(s) \restriction K_{A_p}$.

If $p, q \in \mathbb{P}$, then $q \leq p$ if and only if $A_q \supseteq A_p$ and, for all $s \in S$, $\psi_q(s) \restriction F_{A_p} = \psi_q(s)$.

Claim 10.5. *For all $a \in A$, the set $D_a := \{p \in \mathbb{P} \mid a \in A_p\}$ is dense in $\mathbb{P}$.*

Proof. Fix $a \in A$ and $p \in \mathbb{P}$. We will find $q \leq p$ with $q \in D_a$. Since A is $\aleph_1$-free, we can find a finitely generated pure subgroup A' of A such that $A_p \cup \{a\} \subseteq A'$. Then A'/A_p is free, so we can fix a basis for A' of the form $B_0 \cup B_1$, where B_0 is a basis for A_p. Define a function $f : S \to {}^{B_0 \cup B_1}\mathbb{Z}$ as follows: for all $s \in S$ and $b \in b_0 \cup B_1$, let

$$f(s)(b) := \begin{cases} \psi_p(s)(b) & \text{if } b \in B_0 \\ 0 & \text{if } b \in B_1. \end{cases}$$

Since ψ_p is continuous, it follows that f is continuous. Let $\psi' : S \to \mathrm{Hom}(F_{A'}, \mathbb{Z})$ be the unique continuous map given by Claim 10.4 applied to A', $B_0 \cup B_1$, and f. Then $q := (A', \psi') \in D_a$. Moreover, the uniqueness clause of Claim 10.4 together with the definition of f implies that $\psi'(s) \restriction F_{A_p} = \psi_p(s)$ for all $s \in S$, so we have $q \leq p$. □

Claim 10.6. *$\mathbb{P}$ has the ccc.*

Proof. Let $\vec{p} = \langle p_\eta \mid \eta < \omega_1 \rangle$ be a sequence of conditions in $\mathbb{P}$. By (the proof of) [5, Lemma 7.5], by thinning out $\vec{p}$ to some cofinal subsequence if necessary, we can assume that there is a free subgroup A' of A that is pure in A and such that $A_{p_\eta} \subseteq A'$ for all $\eta < \omega_1$ (this is the only use of the full power of Chase's condition in the proof). Let B' be a basis for A'. Using Claim 10.5 and extending each p_η if necessary, we can assume that, for all $\eta < \omega_1$, A_{p_η} is freely generated by some finite subset B'_η of B'. By thinning out $\vec{p}$ again if necessary, we can assume that the sets $\{B'_\eta \mid \eta < \omega_1\}$ form a Δ-system, with root R.

For all $\eta < \omega_1$, define a map $f_\eta : S \to {}^{B'_\eta}\mathbb{Z}$ by letting $f_\eta(s)(b) := \psi_{p_\eta}(s)(\bar{b})$ for all $s \in S$ and $b \in B'_\eta$. By Proposition 10.2, we can fix $i_\eta \in \Lambda$ and a map $g_\eta : S_{i_\eta} \to {}^{R}\mathbb{Z}$ such that, for all $s \in S$ and $b \in R$, we have $f_\eta(s)(b) = g_\eta(\pi_i(s))(b)$. Since there are only countably many choices for i_η and g_η, we can find a fixed i and g and ordinals $\eta < \xi < \omega_1$ such that $i_\eta = i_\xi = i$ and $g_\eta = g_\xi = g$. Note that, for all $s \in S$ and $b \in R$, this implies that $f_\eta(s)(b) = f_\xi(s)(b)$.

Let $B^* := B_\eta \cup B_\xi$, and let A^* be the free group generated by B^*. Then A^* is a pure subgroup of A'; since A' is pure in A, this implies that A^* is also pure in A. Define a function $f : S \to {}^{B^*}\mathbb{Z}$ by letting $f(s) = f_\eta(s) \cup f_\xi(s)$ for all $s \in S$. Since ψ_{p_η} and ψ_{p_ξ} are continuous (and hence f_η and f_ξ are continuous), it follows that f is continuous. Let $\psi^* : S \to \mathrm{Hom}(F_{A^*}, \mathbb{Z})$ be the unique continuous map obtained by applying Claim 10.4 to A^*, B^*, and f. Then $q = (A^*, \psi^*) \in \mathbb{P}$ and, by the uniqueness clause of Claim 10.4, it extends both p_η and p_ξ. Therefore, $\vec{p}$ does not enumerate an antichain, and hence $\mathbb{P}$ has the ccc. □

Now apply MA_{ω_1} to obtain a filter $G \subseteq \mathbb{P}$ such that, for all $a \in a$, we have $G \cap D_a \neq \emptyset$. Define $\psi : S \to \mathrm{Hom}(F, \mathbb{Z})$ by letting $\psi(s) = \bigcup\{\psi_p(s) \mid p \in G\}$. The fact that each ψ_p is a continuous map such that, for all $s \in S$, we have $\psi_p(s) \restriction K_{A_p} = \varphi(s) \restriction K_{A_p}$ implies that ψ is a continuous map such that, for all $s \in S$, we have $\psi_p(s) \restriction K = \varphi(s)$, as desired. □

We thus obtain the following corollary, establishing Theorem 1.4(1).

Corollary 10.7. *If MA_{ω_1} holds, then there is a nonfree abelian group A of size $\aleph_1$ such that $\underline{A}$ is Whitehead in $\mathsf{CondAb}_{\omega_1}$.*

11. A different vantage point

In this section, we describe and apply a lemma providing an alternative, more purely algebraic path to several of our results, as well as to a few new ones. Recall from Section 8 that if $\mathbb{B}$ is the complete Boolean algebra of regular open subsets of an extremally disconnected compact Hausdorff space S, then $C(S, \mathbb{Z})$ is precisely the group of $\mathbb{B}$-names for integers.

Lemma 11.1. *For all abelian groups A and extremally disconnected profinite sets S,*

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) = \mathrm{Ext}^1_{\mathsf{Ab}}(A, C(S, \mathbb{Z})).$$

Proof. The chain of equalities is as follows:

$$\begin{aligned}\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A},\underline{\mathbb{Z}})(S) &= \mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A}\otimes\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}})\\ &= \mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A},\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}}))\\ &= \mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A},\underline{\mathrm{Hom}}_{\mathsf{Cond}}(\underline{S},\underline{\mathbb{Z}}))\\ &= \mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A},\underline{C(S,\mathbb{Z})})\\ &= \mathrm{Ext}^1_{\mathsf{Ab}}(A,C(S,\mathbb{Z}))\end{aligned}$$

To see the fifth equality, apply Lemma 3.2; for the third and fourth, respectively, apply [13, Prop. 3.5 and Prop. 3.2]. The first equality follows from [20, p. 13 (iii) (cf. p. 25)]. This leaves us only with the second. Here the idea, evidently, is to apply the Hom-tensor adjunction at the level of $\mathrm{Ext}^1_{\mathsf{CondAb}}$, and we may do so by the naturality of the adjunction at the $\mathrm{Hom}_{\mathsf{CondAb}}$-level, the flatness of $\mathbb{Z}[\underline{S}]$ [20, p. 13], and the fact, noted already, that $\underline{\bigoplus_I \mathbb{Z}} = \bigoplus_I \underline{\mathbb{Z}}$ for any set I. More particularly, letting $\underline{\mathcal{P}}$, much as in the proof of Lemma 3.2, denote the condensation of a free resolution of $\underline{A}$, we have:

$$\begin{aligned}\mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A}\otimes\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}}) &= \mathrm{H}^1(\mathrm{Hom}_{\mathsf{CondAb}}(\underline{\mathcal{P}}\otimes\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}}))\\ &= \mathrm{H}^1(\mathrm{Hom}_{\mathsf{CondAb}}(\underline{\mathcal{P}},\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}}))\\ &= \mathrm{Ext}^1_{\mathsf{CondAb}}(\underline{A},\underline{\mathrm{Hom}}_{\mathsf{CondAb}}(\mathbb{Z}[\underline{S}],\underline{\mathbb{Z}})),\end{aligned}$$

and this chain of equalities, if sound, will complete the argument. Let us check the details. Observe first that $\underline{\mathcal{P}}$ is a projective resolution of $\underline{A}$, by [13, Prop. 2.18] and Proposition 12.3 below; this shows the third equality. The second is the aforementioned adjunction. For the first, we have noted already that $\mathbb{Z}[\underline{S}]$ is flat, so the question reduces to that of whether the terms of $\underline{\mathcal{P}}\otimes\mathbb{Z}[\underline{S}]$ are projective. For this observe simply that $\left(\underline{\bigoplus_I \mathbb{Z}}\right)\otimes\mathbb{Z}[\underline{S}] = \left(\bigoplus_I \underline{\mathbb{Z}}\right)\otimes\mathbb{Z}[\underline{S}] = \bigoplus_I \left(\underline{\mathbb{Z}}\otimes\mathbb{Z}[\underline{S}]\right) = \bigoplus_I \mathbb{Z}[\underline{S}]$ for any set I; that such sums are projective follows from Proposition 12.3 below. □

Note that the lemma allows us to recast this paper's essential background in uniformly classical terms: Shelah's result was that the implication

$$\left[\,\mathrm{Ext}^1_{\mathsf{Ab}}(A,C(S,\mathbb{Z}))=0 \text{ for } S=*\,\right] \Rightarrow \text{ the abelian group } A \text{ is free}$$

is independent of the ZFC axioms. Clausen and Scholze's result, in contrast, can now be seen to follow from the fact that the implication

$$\left[\,\mathrm{Ext}^1_{\mathsf{Ab}}(A,C(S,\mathbb{Z}))=0 \text{ for all } S\in\mathsf{ED}\,\right] \Rightarrow \text{ the abelian group } A \text{ is free}$$

is indeed a ZFC theorem. This implication is, moreover, straightforward to prove, since by a result of Nöbeling ([16], see also [6, Proposition XI.4.4]), generalizing work of Specker [23], the group $C(S,\mathbb{Z})$ is itself a free abelian group, one isomorphic to the free abelian group on the topological weight of S.

In fact, we may be more precise. Observe that, if A is a nonfree abelian group of cardinality κ, then the canonical free resolution of A yields $\mathrm{Ext}^1_{\mathsf{Ab}}(A,\mathbb{Z}^{(\kappa)})\neq 0$, where $\mathbb{Z}^{(\kappa)}$ denotes the free abelian group on κ-many generators. Moreover, if A is an abelian group and A_0 is a nonfree *subgroup* of A of cardinality κ, then an application of $\mathrm{Hom}(\cdot,\mathbb{Z}^{(\kappa)})$ to the short exact sequence $0\to A_0\to A\to A/A_0\to 0$ yields a long exact sequence, a portion of which is

$$\ldots\to\mathrm{Ext}^1_{\mathsf{Ab}}(A,\mathbb{Z}^{(\kappa)})\to\mathrm{Ext}^1_{\mathsf{Ab}}(A_0,\mathbb{Z}^{(\kappa)})\to 0.$$

It follows that $\mathrm{Ext}^1_{\mathsf{Ab}}(A, \mathbb{Z}^{(\kappa)}) \neq 0$. Therefore, Lemma 11.1 immediately yields the following corollary, providing an alternative proof of clauses (1) and (3) of Theorem 1.3.

Corollary 11.2. *Suppose that A is a nonfree abelian group and κ is the least cardinality of a nonfree subgroup of A. Then*

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) \neq 0$$

for every $S \in \mathsf{ED}$ of weight at least κ. □

Lemma 11.1 also sheds additional light on the results of Section 10, both providing an alternative proof of a variation on the main result of that section, as well as showing that the existence of large cardinals provides a bound on the extent to which this result can be generalized.

We first note that, by results in [7], it is consistent with the axioms of ZFC that, for every cardinal κ, there is a nonfree abelian group A such that $\mathrm{Ext}^1_{\mathsf{Ab}}(A, B) = 0$ for every abelian group B of cardinality $\leq\kappa$. Thus, Lemma 11.1 yields the following corollary, establishing clause (2) of Theorem 1.4 and indicating that, consistently, the positive answer to the interpretation of Whitehead's problem in CondAb established by Theorem 1.2 fails in CondAb_κ for *every* uncountable cardinal κ.

Corollary 11.3. *It is consistent with the axioms of ZFC that, for every cardinal κ, there is a nonfree abelian group A such that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) = 0$ for all $S \in \mathsf{ED}_\kappa$. In particular, it is consistent that, for every uncountable cardinal κ, there is a nonfree abelian group A such that $\underline{A}$ is Whitehead in CondAb_κ.*

On the other hand, the existence of sufficiently large cardinals provides a limit on the extent to which the phenomena of Corollary 11.3 can occur. In particular, if κ is a strongly compact cardinal, then CondAb_κ is sufficiently rich to guarantee a positive answer to the appropriate interpretation of Whitehead's problem. The following corollary yields clause (3) of Theorem 1.4.

Corollary 11.4. *Suppose that κ is a strongly compact cardinal and A is an abelian group. If $\underline{A}$ is Whitehead in CondAb_κ, then A is free.*

Proof. Suppose that $\underline{A}$ is Whitehead in CondAb_κ. In particular, for for every $S \in \mathsf{ED}_\kappa$, we have $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S) = 0$. By Lemma 11.1, it follows that, for every cardinal $\lambda < \kappa$, we have $\mathrm{Ext}^1_{\mathsf{Ab}}(A, \mathbb{Z}^{(\lambda)}) = 0$. By the observations in the paragraph immediately preceding Corollary 11.2, it follows that A is κ-free, i.e., every subgroup of A of cardinality $<\kappa$ is free. By [6, Corollary II.3.11], since κ is strongly compact, A is free, as desired.[5] □

Remark 11.5. In the proof of Corollary 11.4, the crucial property of the cardinal κ was the fact that every κ-free group is free. This is true if κ is a strongly compact cardinal, but we remark that, consistently, this can also be true for smaller values of κ. In particular, it is shown in [12, §4] that, assuming the consistency of the existence of infinitely many supercompact cardinals, there exists a model of ZFC in which this condition holds for the first cardinal fixed point of the $\aleph$ function, i.e., the least κ such that $\kappa = \aleph_\kappa$.

[5] As [6, Corollary II.3.11] makes clear, we could have merely assumed that κ is only $L_{\omega_1\omega}$-compact, a weaker but lesser-known hypothesis than the strong compactness of κ.

12. Locally compact abelian groups

In this section, we show that Theorem 1.2 extends from the category of (discrete) abelian groups to the category of locally compact abelian groups; here it is useful to recall that the projective objects of the former category are exactly the free ones. In particular, we prove the following theorem.

Theorem 12.1. *Let A be an object of the category* LCA *of locally compact abelian groups. Then*

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}}) = 0$$

if and only if A is projective in LCA.

Proof. By the structure theorem for locally compact abelian groups (see [20, Thm. 4.1(i)]), any locally compact abelian group A is of the form $\mathbb{R}^n \times B$ for some $n \in \mathbb{N}$, where B is an extension of a compact group by a discrete one, or where, in other words, there exists a short exact sequence

$$0 \to C \to B \to D \to 0 \tag{1}$$

with C compact and D discrete whose image in CondAb is exact as well.[6] Since $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) = 0$ [20, p. 25], we have

$$(\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}}) = 0) \Leftrightarrow (\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{B}, \underline{\mathbb{Z}}) = 0).$$

Consider the following portion of the long exact sequence deriving from equation (1):

$$\cdots \to 0 \to \underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{D}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{B}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{C}, \underline{\mathbb{Z}}) \to 0 \to \cdots$$

(The rightmost 0 follows from [20, p. 27] (e.g., from the comment following Cor. 4.9); the leftmost 0 follows from [20, Prop. 4.2].) Hence

$$\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}}) = \underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{B}, \underline{\mathbb{Z}}) = 0$$

if and only if

(1) $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{D}, \underline{\mathbb{Z}}) = 0$ and
(2) $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{C}, \underline{\mathbb{Z}}) = 0$.

By Theorem 1.2, item (1) holds if and only if $D = \bigoplus_I \mathbb{Z}$ for some I. Also, the argument of [13, Prop. 4.11] together with [20, Prop. 4.3(ii)] shows that $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{C}, \underline{\mathbb{R}}) = 0$; this together with the fact that $\underline{\mathrm{Hom}}(\underline{C}, \underline{\mathbb{R}}) = \underline{\mathrm{Hom}(C, \mathbb{R})} = 0$ ([20, Prop. 4.2]) implies that the map $\underline{\mathrm{Hom}}(\underline{C}, \underline{\mathbb{R}/\mathbb{Z}}) \to \underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{C}, \underline{\mathbb{Z}})$ appearing in the long exact sequence induced by applying $\underline{\mathrm{Hom}}(\underline{C}, -)$ to $\underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{R}/\mathbb{Z}}$ is an isomorphism. Again applying [20, Prop. 4.2], we have $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{C}, \underline{\mathbb{Z}}) = \underline{\mathrm{Hom}(C, \mathbb{R}/\mathbb{Z})} = \underline{\mathbb{D}(C)}$, the condensed image of the Pontryagin dual group of C; in particular, item (2) holds if and only if $C = 0$. Summing up, $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}}) = 0$ if and only if A is of the form $\mathbb{R}^n \times D$ for some $D = \bigoplus_I \mathbb{Z}$. As noted, by [15, Prop. 3.3], such A are precisely the projective objects of LCA. $\square$

This theorem carries as corollary a negative answer to one other condensed version of Whitehead's Problem:

Corollary 12.2. *As the condensed abelian group $\underline{\mathbb{R}}$ witnesses, $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(A, \underline{\mathbb{Z}}) = 0$ does not imply that A is projective in* CondAb.

---

[6] See [13, Props. 4.9, 2.18].

To see this, recall that, as noted, the compact projective condensed abelian groups of the form $\mathbb{Z}[\underline{S}]$ with $S \in \mathsf{ED}$ generate CondAb ([3, p. 16–17 and Proposition 3.5]); in particular, for any condensed abelian group X there exists a set I, a collection $\{S_i \mid i \in I\}$ of spaces in ED, and an epimorphism $r : \bigoplus_I \mathbb{Z}[\underline{S_i}] \to X$. If X is projective then r admits a right inverse s, hence:

Proposition 12.3. *The projective objects of* CondAb *are exactly the retracts of direct sums of groups of the form* $\mathbb{Z}[\underline{S}]$ *with* S *extremally disconnected.*

Hence it will suffice to show that $\mathbb{R}$ is not a retract of such a sum. This is immediate from the following lemma.

Lemma 12.4. *The underlying group of* $\mathbb{Z}[\underline{S}]$ *is the free abelian group* $\bigoplus_S \mathbb{Z}$.

By the lemma, any section $s : \underline{\mathbb{R}} \to \bigoplus_I \mathbb{Z}[\underline{S_i}]$ of an r as above would evaluate over a point $*$ as an injection $s(*) : \mathbb{R} \to \bigoplus_I \bigoplus_{S_i} \mathbb{Z}$, but any nonzero element in the image of $s(*)$ would need to be divisible by every positive n, a contradiction. The following, therefore, will complete the proof of Corollary 12.2.

Proof of Lemma 12.4. As noted, $\mathbb{Z}[\underline{S}]$ is the sheafification of the presheaf

$$T \mapsto \bigoplus_{\mathrm{Cont}(T,S)} \mathbb{Z} \tag{2}$$

Recall also that the sheafification $\mathcal{F}^\sharp$ of a presheaf $\mathcal{F}$ on a site is given by the twofold iteration of the $+$ operation, where

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{U} \in \mathrm{cov}(U)}\ \mathrm{H}^0(\mathcal{U}; \mathcal{F}).$$

Here

$$\mathrm{H}^0(\mathcal{U}; \mathcal{F}) = \{(s_i) \in \prod_{i \in I} \mathcal{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I\}$$

with respect to a given cover $\mathcal{U} = \{U_i \to U \mid i \in I\}$ (see [24, Tag 00W1]), and $\mathcal{U} \prec \mathcal{V} = \{V_j \to U \mid j \in J\}$ in $\mathrm{cov}(U)$ if there exists an $a : J \to I$ and maps $V_j \to U_{a(j)}$ factoring $V_j \to U$ through $U_{a(j)} \to U$ for all $j \in J$. Observe now that if $U = *$ and $\mathcal{F}$ is the presheaf (2) over the condensed site $*_{\text{proét}}$ then the cover $\mathcal{U}_{\mathrm{id}} := \{\mathrm{id} : * \to *\}$ is cofinal in $\mathrm{cov}(U)$; it follows that $\mathcal{F}^\sharp(*) = \mathcal{F}^+(*) = \bigoplus_S \mathbb{Z}$, as claimed. □

13. Conclusion

Though the project leading to this article began with a desire simply to understand the combinatorial phenomena underlying considerations of Whitehead's problem in the setting of condensed abelian groups, it quickly expanded into the beginnings of a more general investigation into connections between condensed mathematics and forcing, an investigation that we hope to see further developed in the future. These connections have manifested herein in a number of ways, notably in

(1) the recognition that condensed objects can often be viewed as organized presentations of forcing names; and
(2) the observation that interesting phenomena arise via the evaluation of condensed objects at the Stone spaces of complete Boolean algebras significant in set theoretic forcing (cf. clause (3) of Theorem 1.3).

In the context of Whitehead's problem, these investigations culminated in our Theorem 9.2 that, if A is an abelian group and $\mathbb{B}$ is a complete Boolean algebra which forces a certain technical strengthening of the failure of A to be Whitehead, then $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$. We conjecture that this technical strengthening can be foregone:

Conjecture 13.1. *Suppose that A is an abelian group, $\mathbb{B}$ is a complete Boolean algebra, and*

$$\Vdash_{\mathbb{B}} \text{“}A \textit{ is not Whitehead”}.$$

Then $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{A}, \underline{\mathbb{Z}})(S(\mathbb{B})) \neq 0$.

Note that the converse of this conjecture does not hold, as shown in the discussion following Corollary 9.3. We feel that a resolution of this conjecture will shed light not just on the relationship between forcing and condensed mathematics, but also on considerations of Whitehead's problem in the classical setting.

Finally, there remain questions about alternative interpretations of Whitehead's problem in the condensed setting. In Section 12, we initiated some further exploration of two main parameters in play in condensed interpretations of Whitehead's problem, namely the subscript s of the $\underline{\mathrm{Ext}}^1$ term, and the domain d of $\underline{\mathrm{Ext}}^1_s(\cdot, \underline{\mathbb{Z}})$. After all, in retrospect, it's not altogether surprising that ($s = \mathsf{CondAb}$)-valued invariants $\underline{\mathrm{Ext}}^1_s$ should be strong enough to settle the problem over the comparatively weaker domains of $d = \mathsf{Ab}$ or even $d = \mathsf{LCA}$ (we hope to have shown that it is altogether *interesting* nevertheless). A natural next question, then, is how much further this d may be pushed. By Corollary 12.2, of course, Whitehead's problem has a ZFC solution when $d = \mathsf{CondAb}$ as well, but it is an unsatisfying one, deriving as it does from the arguably spurious case of $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) = 0$. Against this background, particularly interesting is the problem when d equals the subcategory Solid of CondAb mentioned briefly in our introduction. Forming, as it does, the paradigmatic example of an analytic ring, this is one of CondAb's most critical subcategories (see [20, §5–7] and [19, §2]); moreover, as noted, it was this category's features which Clausen and Scholze leveraged in their original proof of Theorem 1.2. Within it, in addition, $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}$ and $\underline{\mathrm{Ext}}^1_{\mathsf{Solid}}$ coincide [20, Lemma 5.9], so that this instance of the problem recovers the $s = d$ symmetry of the classical one:

Question 13.2. *Let A be a solid abelian group. Does $\underline{\mathrm{Ext}}^1_{\mathsf{Solid}}(A, \underline{\mathbb{Z}}) = 0$ (or equivalently $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(A, \underline{\mathbb{Z}}) = 0$) imply that A is projective?*

Part of the question is whether the answer will follow from the ZFC axioms alone. The solid setting, being "very much a 'nonarchimedean' notion" [20, footnote 10 on p. 32], carries the additional virtue of circumventing the aforementioned issue of $\underline{\mathbb{R}}$. More precisely, the inclusion $\mathsf{Solid} \to \mathsf{CondAb}$ possesses a left adjoint *solidification* functor notated $M \mapsto M^{\blacksquare}$, and the solidification $\underline{\mathbb{R}}^{\blacksquare}$ of $\underline{\mathbb{R}}$ is 0 [20, Corollary 6.1(iii)]; in particular, $\underline{\mathbb{R}}$ is not solid. This leads to the following natural variation on Question 13.2: *Suppose that A is a condensed abelian group and $\underline{\mathrm{Ext}}^1_{\mathsf{CondAb}}(A, \underline{\mathbb{Z}}) = 0$. Must $A^{\blacksquare}$ be projective in the category of solid abelian groups?*

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