DIAGONAL SUPERCOMPACT RADIN FORCING

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Abstract. Motivated by the goal of constructing a model in which there are no κ-Aronszajn trees for any regular $\kappa > \aleph_1$, we produce a model with many singular cardinals where both the singular cardinals hypothesis and weak square fail.

1. Introduction

In this paper, we produce a model of ZFC with some global behavior of the continuum function on singular cardinals and the failure of weak square. Our method is as an extension of Sinapova's work [18]. We define a diagonal supercompact Radin forcing which adds a club subset to a cardinal κ while forcing the failure of the Singular Cardinals Hypothesis (SCH) everywhere on the club and preserving the inaccessibility of κ. In the forcing extension, weak square will necessarily hold at some successors of singular cardinals below κ, but the set of these singular cardinals will be sufficiently sparse that it can be made non-stationary by κ-distributive forcing. We will thus obtain the following result.

Theorem 1.1. *If there are a supercompact cardinal κ and a weakly inaccessible cardinal $\theta > \kappa$, then there is a forcing extension in which κ is inaccessible and there is a club $E \subseteq \kappa$ of singular cardinals ν at which SCH and $\square^*_\nu$ both fail.*

We are motivated by the question of whether in ZFC one can construct a κ-Aronszajn tree for some $\kappa > \omega_1$. The question is also open if we ask for a *special* κ-Aronszajn tree. Forcing provides a possible path to a negative solution by showing that it is consistent with ZFC that there are no κ-Aronsajn trees on any regular $\kappa > \omega_1$. By a theorem of Jensen [11], $\square^*_\mu$ is equivalent to the existence of a special μ^+-Aronszajn tree. So our theorem is partial progress towards a model with no special Aronszajn trees.

The non-existence of κ-Aronszajn trees (the tree property at κ) and the non-existence of special κ-Aronszajn trees (failure of $\square^*$) are reflection principles which are closely connected with large cardinals. For example, theorems of Erdős and Tarski [6], and Monk and Scott [15], show that an inaccessible cardinal is weakly compact if and only if it has the tree property. Further, Mitchell and Silver [14]

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showed that the tree property at $\aleph_2$ is consistent with ZFC if and only if the existence of a weakly compact cardinal is.

Specker [22] showed that, if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. This theorem places an important restriction on models where there are no special Aronszajn trees. From Specker's theorem, a model with no special κ-Aronszajn trees for any $\kappa > \aleph_1$ must be one in which GCH fails everywhere. In particular GCH must fail at every singular strong limit cardinal, a failure of SCH. The consistency of the failure of SCH requires large cardinals [8]; a model in which GCH fails everywhere was first obtained by Foreman and Woodin [7].

There are many partial results towards constructing a model in which every regular cardinal greater than $\aleph_1$ has the tree property. There is a bottom up approach where one attempts to force longer and longer initial segments of the regular cardinals to have the tree property; see, for example [1, 3, 17, 25]. We refer the reader to [23] for some analogous results on successive failures of weak square. Another aspect of the problem comes from the interaction between cardinal arithmetic at singular strong limit cardinals μ and the tree property at μ^+. In the 1980's Woodin asked whether the failure of SCH at $\aleph_\omega$ is consistent with the tree property at $\aleph_{\omega+1}$. More generally, one can consider whether this situation is consistent at some larger singular cardinal. An important result in this direction is due to Gitik and Sharon [10], who showed that, relative to the existence of a supercompact cardinal, it is consistent that there is a singular cardinal κ of cofinality ω such that SCH fails at κ and there are no special κ^+-Aronszajn trees. In fact they show a stronger assertion ($\kappa^+ \notin I[\kappa^+]$), which we will define later. In the same paper, they show that it is possible to make κ into $\aleph_{\omega^2}$. Cummings and Foreman [4] showed that there is a PCF theoretic object called a bad scale in the models of Gitik and Sharon, which implies that $\kappa^+ \notin I[\kappa^+]$.

The key ingredient in Gitik and Sharon's argument was a new diagonal supercompact Prikry forcing. The basic idea is to start with supercompactness measures U_n on $\mathcal{P}_\kappa(\kappa^{+n})$ for $n < \omega$ and use them to define a Prikry forcing. This forcing adds a sequence $\langle x_n \mid n < \omega\rangle$, where each x_n is a typical point for U_n and $\bigcup_{n<\omega} x_n = \kappa^{+\omega}$. The result is that $\kappa^{+\omega}$ is collapsed to have to have size κ and $\kappa^{+\omega+1}$ becomes the new successor of κ. The fact that $\kappa^{+\omega+1} \notin I[\kappa^{+\omega+1}]$ in the ground model persists to provide $\kappa^+ \notin I[\kappa^+]$ in the extension. Moreover, if we start with $2^\kappa = \kappa^{+\omega+2}$ in the ground model, then we get the failure of SCH at κ in the extension.

Variations of Gitik and Sharon's poset have been used to construct many related models. We list a few such results:

(1) (Neeman [16]) From ω-many supercompact cardinals, there is a forcing extension in which there is a singular cardinal κ of cofinality ω such that SCH fails at κ and κ^+ has the tree property.
(2) (Sinapova [18]) From a supercompact cardinal κ, for any regular $\lambda < \kappa$, there is a forcing extension in which κ is a singular cardinal of cofinality λ, SCH fails at κ and κ carries a bad scale (in particular $\kappa^+ \notin I[\kappa^+]$ and there are no special κ^+-Aronszajn trees).
(3) (Sinapova [19]) From λ-many supercompact cardinals $\langle\kappa_\alpha \mid \alpha < \lambda\rangle$ with $\lambda < \kappa_0$ regular, there is a forcing extension in which κ_0 is a singular cardinal of cofinality λ, SCH fails at κ_0 and κ_0^+ has the tree property.

(4) (Sinapova [20]) From ω-many supercompact cardinals, it is consistent that Neeman's result above holds with $\kappa = \aleph_{\omega^2}$.

Woodin's original question remains open; see [21] for the best known partial result. A theme in the above results is that questions about the tree property are answered by first constructing a model where there are no special κ^+-Aronszajn trees (or even $\kappa^+ \notin I[\kappa^+]$). To obtain the tree property, one needs to increase the large cardinal assumption and to give a version of an argument of Magidor and Shelah [13], who showed that the tree property holds at μ^+ when μ is a singular limit of supercompact cardinals. The results of our paper are based on the ideas from Sinapova's [18], but we expect that they will generalize to give the tree property in the presence of stronger large cardinal assumptions.

The paper is organized as follows. In Section 2 we give some definitions and background material required for the main result. In Section 3 we describe the main forcing for Theorem 1.1 and prove some of its basic properties. In Section 4, we show that the main forcing gives a model with a club C of cardinals where SCH fails. In Section 5 we characterize which cardinals in this club C have weak square sequences and show that this set can be made non-stationary by κ-distributive forcing, thus completing the proof of Theorem 1.1. In Section 6 we make some concluding remarks and ask some open questions.

2. Background

In this section we will make the notions from the introduction precise and give some further definitions that are relevant to the rest of the paper.

Definition 2.1. We say that ν *has a weak square sequence* ($\square^*_\nu$) if there is a sequence $\langle C_\gamma \mid \gamma < \nu^+\rangle$ such that

(1) for all $\gamma < \nu^+$ limit, $C_\gamma \subset \mathcal{P}(\gamma)$ is nonempty of size at most ν such that, for every $c \in C_\gamma$, $c \subset \gamma$ is club in γ with $\mathrm{otp}(c) \leq \nu$, and
(2) for all $\beta < \gamma < \nu$, if β is a limit point of some $c \in C_\gamma$, then $c \cap \beta \in C_\beta$.

Definition 2.2. Let $\vec{z} = \langle z_\alpha \mid \alpha < \nu^+\rangle$ be a sequence of bounded subsets of ν^+. We say that a limit ordinal γ is *$\vec{z}$-approachable* if there is an unbounded set $A \subset \gamma$ with $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ such that, for every $\beta < \gamma$, $A \cap \beta = z_\alpha$ for some $\alpha < \gamma$. The *approachability ideal* $I[\nu^+]$ consists of all subsets $S \subset \nu^+$ for which there are $\vec{z}$ as above and a club $C \subset \nu^+$ so that every $\gamma \in C \cap S$ is $\vec{z}$-approachable.

By arranging that $z_{\alpha+1}$ is the closure of z_α for each $\alpha < \nu^+$, we may assume that for every $\vec{z}$-approachable point γ, there is a witness $A \subset \gamma$ which is closed.

2.1. Forcing preliminaries.

In this subsection, we describe the preparation of the ground model over which we will force with our diagonal supercompact Radin forcing. Begin with a model V_0 in which GCH holds and $\kappa < \theta$ are cardinals, with κ supercompact. Force over V_0 with Laver's forcing [12] to make the supercompactness of κ indestructible under κ-directed closed forcing, and then force over the resulting model to add θ-many Cohen subsets to κ. Call this final model V; it will be our ground model for the remainder of the paper.

The following lemma holds as in [18].

Lemma 2.3. *For all $\alpha < \theta$, for all $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\alpha}))$, there are a normal, fine ultrafilter U on $\mathcal{P}_\kappa(\kappa^{+\alpha})$ and functions $\langle f_\eta \mid \eta < \theta\rangle$ from κ to κ such that, letting $j : V \to M \cong \mathrm{Ult}(V, U)$, we have*

- *$\mathcal{X} \in M$;*
- *for all $\eta < \theta$, $j(f_\eta)(\kappa) = \eta$.*

Now, again as in [18], by recursion on $\alpha < \theta$, we can construct a sequence of ultrafilters $\vec{U} = \langle U_\alpha \mid \alpha < \theta \rangle$ and, for all $\alpha < \theta$, a sequence $\langle f^\alpha_\eta \mid \eta < \theta \rangle$ such that the following hold.

- For all $\alpha < \theta$, U_α is a normal, fine ultrafilter on $\mathcal{P}_\kappa(\kappa^{+\alpha})$. Let $j_\alpha : V \to M_\alpha \cong \mathrm{Ult}(V, U_\alpha)$ be the collapsed ultrapower map.
- For all $\alpha < \beta < \theta$, $U_\alpha \in M_\beta$.
- For all $\alpha < \theta$, κ is $\kappa^{+\alpha}$-supercompact in M_α.
- For all $\alpha, \eta < \theta$, we have $f^\alpha_\eta : \kappa \to \kappa$ and $j_\alpha(f^\alpha_\eta)(\kappa) = \eta$.

When we write that something happens for most (or for almost all) $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, we mean it happens for a U_α-measure one set. For $\alpha < \theta$, for most $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \cap \kappa$ is an inaccessible cardinal. We will always work with such x and will write κ_x for $x \cap \kappa$. For $x, y \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \prec y$ denotes the statement that $x \subseteq y$ and $\mathrm{otp}(x) < \kappa_y$.

For $\alpha < \beta < \theta$, let $\bar{u}^\beta_\alpha$ be a function on $\mathcal{P}_\kappa(\kappa^{+\beta})$ representing U_α in the ultrapower by U_β. For most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\bar{u}^\beta_\alpha(x)$ is a measure on $\mathcal{P}_{\kappa_x}(\kappa_x^{+f^\beta_\alpha(\kappa_x)})$. Also, for most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\mathrm{otp}(x \cap \kappa^{+\alpha}) = \kappa_x^{+f^\beta_\alpha(\kappa_x)}$. For such x, $\bar{u}^\beta_\alpha(x)$ is isomorphic to a measure $u^\beta_\alpha(x)$ on $\mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})$ via the order-isomorphism between $\kappa_x^{+f^\beta_\alpha(\kappa_x)}$ and $x \cap \kappa^{+\alpha}$.

For $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, let $Z^\beta_y = \{\alpha < \beta \mid \kappa^{+\alpha} \in y\}$. Note that, for most $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, we have $Z^\beta_y = y \cap \beta$, so the following results also hold with $y \cap \beta$ in place of Z^β_y. We feel that Z^β_y is the more natural set to consider in the context of the forcing defined in Section 3, so we will use it instead.

Lemma 2.4. *For most $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, the following hold.*

1. *Z^β_y is $<\kappa_y$-closed.*
2. *If $\mathrm{cf}(\beta) < \kappa$, then $\mathrm{cf}(\beta) < \kappa_y$ and Z^β_y is unbounded in β.*
3. *$\mathrm{otp}(Z^\beta_y) = f^\beta_\beta(\kappa_y)$ and, if β is a limit ordinal, then so is $f^\beta_\beta(\kappa_y)$. Also, if $cf(\beta) \geq \kappa$ then $\mathrm{cf}(f^\beta_\beta(\kappa_y)) \geq \kappa_y$.*
4. *For all $\alpha \in Z^\beta_y$, $\mathrm{otp}(y \cap \kappa^{+\alpha}) = \kappa_y^{+f^\beta_\alpha(\kappa_y)} = \kappa_y^{+\mathrm{otp}(\alpha \cap Z^\beta_y)}$.*
5. *κ_y is $\kappa_y^{+f^\beta_\beta(\kappa_y)}$-supercompact.*
6. *For all $\alpha \in Z^\beta_y$, $\bar{u}^\beta_\alpha(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+f^\beta_\alpha(\kappa_y)})$.*
7. *For all $\alpha_0 < \alpha_1$, both in Z^β_y, the function $x \mapsto \bar{u}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{u}^\beta_{\alpha_0}(y)$ in the ultrapower by $u^\beta_{\alpha_1}(y)$.*

Proof. Let $j = j_\beta$. Recall that a set A is in U_β iff $j``\kappa^{+\beta} \in j(A)$. Note first that, defining $g : \mathcal{P}_\kappa(\kappa^{+\beta}) \to V$ by $g(y) = Z^\beta_y$, we have $j(g)(j``\kappa^{+\beta}) = j``\beta$. Items (1)–(5) then follow easily.

To show (6), let $j(\langle \bar{u}^\beta_\alpha \mid \alpha < \beta \rangle) = \langle \bar{v}^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$ and $j(\langle f^\beta_\alpha \mid \alpha < \beta \rangle) = \langle g^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$. It suffices to show that, in M_β, for all $\alpha \in j``\beta$, $\bar{v}^{j(\beta)}_\alpha(j``\kappa^{+\beta})$ is a measure on $\mathcal{P}_\kappa(\kappa^{+g^{j(\beta)}_\alpha(\kappa)})$. Let $\alpha \in j``\beta$, with, say, $\alpha = j(\xi)$. Then $\bar{v}^{j(\beta)}_\alpha(j``\kappa^{+\beta}) = j(\bar{u}^\beta_\xi)(j``\kappa^{+\beta}) = U_\xi$, which is a measure on $\mathcal{P}_\kappa(\kappa^{+\xi}) = \mathcal{P}_\kappa(\kappa^{+g^{j(\beta)}_\alpha(\kappa)})$.

We finally show (7). Let $j(\langle \bar{u}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq \beta \rangle) = \langle \bar{v}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq j(\beta) \rangle$ and $j(\langle u^\beta_\alpha \mid \alpha < \beta \rangle) = \langle v^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$. It suffices to show that, in M_β, for all $\alpha_0 < \alpha_1$, both in $j``\beta$, the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta})$ in the ultrapower by $v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$. Fix $\alpha_0 < \alpha_1$ in $j``\beta$, with $\alpha_0 = j(\xi_0)$ and $\alpha_1 = j(\xi_1)$. Note that $\hat{U}_{\xi_1} := v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$ is a measure on $\mathcal{P}_\kappa(j``\kappa^{+\xi_1})$ that collapses to U_{ξ_1}. Also note that $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta}) = U_{\xi_0}$. Thus, we must show that the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents U_{ξ_0} in the ultrapower by $\hat{U}_{\xi_1}$.

Fix $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$. There is $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\beta})$ such that $x = j(\bar{x})$. Then $\bar{v}^{\alpha_1}_{\alpha_0}(x) = j(\bar{u}^{\xi_1}_{\xi_0}(\bar{x}))$. For most $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\xi_1})$, $\bar{u}^{\xi_1}_{\xi_0}(\bar{x})$ is a measure on $\mathcal{P}_{\kappa_{\bar{x}}}(\kappa_{\bar{x}}^{+f^{\xi_1}_{\xi_0}(\kappa_{\bar{x}})})$, and this is fixed by j. Thus, for most $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$, $\bar{v}^{\alpha_1}_{\alpha_0}(x) = \bar{u}^{\xi_1}_{\xi_0}(\bar{x})$. Therefore, since $\hat{U}_{\xi_1}$ collapses to U_{ξ_1}, $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents the same thing in the ultrapower by $\hat{U}_{\xi_1}$ as $\bar{x} \mapsto \bar{u}^{\xi_1}_{\xi_0}(x)$ represents in the ultrapower by U_{ξ_1}, which is U_{ξ_0}. This is true in V and, since M_β is sufficiently closed, it is true in M_β as well. □

Lemma 2.5. *Suppose $\beta < \theta$ and, for all $\alpha \le \beta$, $A_\alpha \in U_\alpha$. Let A^* be the set of all $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in U_\beta$.*

Proof. Let $j = j_\beta$. It suffices to show that $j``\kappa^{+\beta} \in j(A^*)$, i.e. for all $j(\alpha) \in j``\beta$, $\{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\} \in \hat{U}_\alpha$, where $\hat{U}_\alpha$ is the isomorphic copy of $U_\alpha$ living on $\mathcal{P}_\kappa(j``\kappa^{+\alpha})$. Fix such a $j(\alpha)$. Let $X = \{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\}$, and note that $X = j``A_\alpha \in \hat{U}_\alpha$. □

Lemma 2.6. *Suppose $\gamma < \theta$, $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$, and z satisfies all of the statements in Lemma 2.4. Suppose that, for all $\alpha \in Z^\gamma_z$, $A_\alpha \in u^\gamma_\alpha(z)$. Fix $\beta \in Z^\gamma_z$, and let A^* be the set of $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in u^\gamma_\beta(z)$.*

Proof. For each $\alpha \in Z^\gamma_z$, let $\bar{A}_\alpha$ be the collapsed version of A_α, so $\bar{A}_\alpha \in \bar{u}^\gamma_\alpha(z)$. Recall that $u^\gamma_\beta(z)$ is a measure on $\mathcal{P}_{\kappa_z}(z \cap \kappa^{+\beta})$. Let $k : V \to N \cong \mathrm{Ult}(V, u^\gamma_\beta(z))$ be the ultrapower map. By (6) of Lemma 2.4, for all $\alpha \in Z^\gamma_z \cap \beta$, the map $y \mapsto \bar{u}^\beta_\alpha(y)$ represents $\bar{u}^\gamma_\alpha(z)$ in the ultrapower. Note also that the map $y \mapsto Z^\beta_y$ represents $\{\eta < k(\beta) \mid k(\kappa)^{+\eta} \in k``(z \cap \kappa^{+\beta})\} = k``(Z^\gamma_z \cap \beta)$. To prove the lemma, it suffices to show that $k``(z \cap \kappa^{+\beta}) \in k(A^*)$, i.e. for all $\alpha \in Z^\gamma_z \cap \beta$, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\}$ is in the measure represented by the map $y \mapsto u^\beta_\alpha(y)$. Call this measure w and note that it is isomorphic to $\bar{u}^\gamma_\alpha(z)$. Also note that $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} = k``(A_\alpha)$, which collapses to $\bar{A}_\alpha \in \bar{u}^\gamma_\alpha(z)$. Thus, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} \in w$, completing the proof of the lemma. □

3. The main forcing

For $\beta < \theta$, let X_β be the set of $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$ satisfying all of the statements in Lemma 2.4. Fix $\eta < \theta$. We define a forcing notion, $\mathbb{P}_{\vec{U},\eta}$. Conditions of $\mathbb{P}_{\vec{U},\eta}$ are pairs (a, A) satisfying the following requirements.

1. (1) a and A are functions, $\mathrm{dom}(a)$ is a finite subset of $\theta \setminus \eta$, and $\mathrm{dom}(A) = \theta \setminus (\mathrm{dom}(a) \cup \eta)$.
2. (2) For all $\beta \in \mathrm{dom}(a)$, $a(\beta) \in X_\beta$.
3. (3) For all $\alpha < \beta$, both in $\mathrm{dom}(a)$, $a(\alpha) \prec a(\beta)$ and $\alpha \in Z^\beta_{a(\beta)}$.
4. (4) For all $\alpha \in \theta \setminus (\max(\mathrm{dom}(a)) + 1)$ (or, if $\mathrm{dom}(a) = \emptyset$, for all $\alpha \in \mathrm{dom}(A)$), $A(\alpha) \in U_\alpha$.

(5) For all $\alpha \in \mathrm{dom}(A) \cap \max(\mathrm{dom}(a))$, if $\beta = \min(\mathrm{dom}(a) \setminus \alpha)$, then $A(\alpha) \in u^\beta_\alpha(a(\beta))$ if $\alpha \in Z^\beta_{a(\beta)}$ and $A(\alpha) = \emptyset$ if $\alpha \notin Z^\beta_{a(\beta)}$.
(6) For all $\beta \in \mathrm{dom}(A)$ such that $A(\beta) \neq \emptyset$ and $\mathrm{dom}(a) \cap \beta \neq \emptyset$, if $\alpha = \max(\mathrm{dom}(a) \cap \beta)$, then for all $y \in A(\beta)$, $a(\alpha) \prec y$ and $\alpha \in Z^\beta_y$.

If $(a, A), (b, B) \in \mathbb{P}_{\vec{U},\eta}$, then $(b, B) \leq (a, A)$ iff the following requirements hold.

(1) $b \supseteq a$.
(2) For all $\alpha \in \mathrm{dom}(b) \setminus \mathrm{dom}(a)$, $b(\alpha) \in A(\alpha)$.
(3) For all $\alpha \in \mathrm{dom}(B)$, $B(\alpha) \subseteq A(\alpha)$.

$(b, B) \leq^* (a, A)$ if $(b, B) \leq (a, A)$ and $b = a$. In this case, (b, B) is called a *direct extension* of (a, A).

Remark 3.1. In our arguments, for notational simplicity we will typically assume that $\eta = 0$ and then denote $\mathbb{P}_{\vec{U},\eta}$ as $\mathbb{P}_{\vec{U}}$. Everything proved about $\mathbb{P}_{\vec{U}}$ can be proved for a general $\mathbb{P}_{\vec{U},\eta}$ in the same way by making the obvious changes. The reason we introduce the more general forcing is to be able to properly state the Factorization Lemma (3.6).

In what follows, let $\mathbb{P}$ denote $\mathbb{P}_{\vec{U}}$. For any condition $p = (a, A) \in \mathbb{P}$, we often denote (a, A) as (a^p, A^p) and let $\gamma^p = \max(\mathrm{dom}(a^p))$. We refer to a^p as the *stem* of p. Note that, if $p, q \in \mathbb{P}$ and $a^p = a^q$, then p and q are compatible. If a is a non-empty stem, then let γ^a denote $\max(\mathrm{dom}(a))$, and let $a^- = a \restriction \gamma^a$. Suppose a is a stem, $\alpha < \theta$, and $x \in X_\alpha$. Suppose moreover that either a is empty or $\gamma^a < \alpha$, $a(\gamma^a) \prec x$, and $\gamma^a \in Z^\alpha_x$. Then $a^\frown(\alpha, x)$ is a stem and $(a^\frown(\alpha, x))^- = a$. If $p \in \mathbb{P}$ and b is a stem, then b is *possible* for p if there is $q \leq p$ with $a^q = b$. If $p \in \mathbb{P}$ and b is possible for p, then $p \downarrow b$ denotes the maximal q such that $q \leq p$ and $a^q = b$. Such a q always exists.

Lemma 3.2. *Suppose that* $(a, A) \in \mathbb{P}$, $\beta \in \mathrm{dom}(A)$, *and* $A(\beta) \neq \emptyset$. *Then there is* $(b, B) \leq (a, A)$ *such that* $\beta \in \mathrm{dom}(b)$.

Proof. Let $\gamma := \min(\mathrm{dom}(a) \setminus \beta)$ if $\beta < \max(\mathrm{dom}(a))$, and let $\gamma := \theta$ otherwise. Let $\alpha_0 := \max(\mathrm{dom}(a) \cap \beta)$ if $\mathrm{dom}(a) \cap \beta \neq \emptyset$ and $\alpha_0 = -1$ otherwise. By Lemma 2.5 (if $\gamma = \theta$) or Lemma 2.6 (if $\gamma < \theta$), we can find $y \in A(\beta)$ such that

- if $\mathrm{dom}(a) \cap \beta \neq \emptyset$, then $\alpha_0 \in Z^\beta_y$ and $a(\alpha_0) \prec y$;
- for all $\alpha \in Z^\beta_y \setminus (\alpha_0 + 1)$, we have $\{x \in A(\alpha) \mid x \prec y\} \in u^\beta_\alpha(y)$.

Now define a condition (b, B) as follows. Let $\mathrm{dom}(b) = \mathrm{dom}(a) \cup \{\beta\}$, $b \restriction \mathrm{dom}(a) = a$, and $b(\beta) = y$. Let $\mathrm{dom}(B) = \theta \setminus \mathrm{dom}(b)$. For all $\alpha \in (\alpha_0, \beta)$, let $B(\alpha) = \{x \in A(\alpha) \mid x \prec y\}$. For $\alpha \in (\beta, \gamma)$, let $B(\alpha) = \{x \in A(\alpha) \mid y \prec x\}$. For all other $\alpha \in \mathrm{dom}(B)$, let $B(\alpha) = A(\alpha)$. It is easily verified that (b, B) is a condition in $\mathbb{P}$ extending (a, A). □

Definition 3.3. Suppose that G is a $\mathbb{P}$-generic over V. Let C^{sc}_G (sc for supercompact) be the set of all points $x = a(\beta)$ where $\beta \in \mathrm{dom}(a)$ for some $p = (a, A)$ in the generic filter G, and let $C_G = \{\kappa_x \mid x \in C^{\mathrm{sc}}_G\}$ be the generic Radin club.

Lemma 3.4. C_G *is club in* κ *and the assignment* $x \mapsto \kappa_x = x \cap \kappa$ *is an increasing bijection from* C^{sc}_G *to* C_G.

Proof. Straightforward by Lemma 3.2 and genericity. □

Lemma 3.5. *(Diagonal Intersection Lemma) Suppose that*

- *$\beta < \theta$;*
- *S is a set of stems with $\gamma^a < \beta$ for all $a \in S$;*
- *for each $a \in S$, we are given a set $Y_a \in U_\beta$.*

Let Z be the set of $y \in X_\beta$ such that, for all $a \in S$, if $\gamma^a \in Z^\beta_y$ and $a(\gamma^a) \prec y$, then $y \in Y_a$. Then $Z \in U_\beta$.

Proof. Let $j = j_\beta$. It suffices to show that $j``\kappa^{+\beta} \in j(Z)$. Notice that, if $a \in j(S)$ is such that $\max(\mathrm{dom}(a)) \in j``\beta$ and $a(\max(\mathrm{dom}(a))) \prec j``\beta$, then there is $\bar{a} \in S$ such that $j(\bar{a}) = a$. But then $Y_{\bar{a}} \in U_\beta$, so $j``\kappa^{+\beta} \in j(Y_{\bar{a}})$. It follows that $j``\kappa^{+\beta} \in j(Z)$, as desired. □

Suppose that $\beta < \theta$ and $y \in X_\beta$. Let $\vec{U}_y = \langle \bar{u}^\beta_\alpha(y) \mid \alpha \in Z^\beta_y \rangle$. For $\xi < f^\beta_\beta(\kappa_y)$, let $\alpha_\xi \in Z^\beta_y$ be such that $\mathrm{otp}(\alpha \cap Z^\beta_y) = \xi$. Then $\bar{u}^\beta_{\alpha_\xi}(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+\xi})$. Let $V_\xi = \bar{u}^\beta_{\alpha_\xi}(y)$. Then $\vec{U}_y = \langle V_\xi \mid \xi < f^\beta_\beta(y) \rangle$, and we can define $\mathbb{P}_{\vec{U}_y}$ as above.

If $p \in \mathbb{P}$, then $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$.

Lemma 3.6. *(Factorization Lemma) Let $p = (a, A) \in \mathbb{P}$. Suppose that $a \neq \emptyset$, $\gamma = \gamma^a$, and $y = a(\gamma)$. Then there is $p' \in \mathbb{P}_{\vec{U}_y}$ such that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\gamma+1}/(\emptyset, A \restriction (\gamma, \theta))$.*

Proof. Let $\pi : y \to \mathrm{otp}(y)$ be the unique order-preserving bijection. Define $p' = (a', A') \in \mathbb{P}_{\vec{U}_y}$ as follows. For $\xi < f^\gamma_\gamma(y)$, let $\alpha_\xi \in Z^\gamma_y$ be such that $\mathrm{otp}(\alpha_\xi \cap Z^\gamma_y) = \xi$. Let $\mathrm{dom}(a') = \{\xi < f^\gamma_\gamma(y) \mid \alpha_\xi \in \mathrm{dom}(a)\}$ and, for $\xi \in \mathrm{dom}(a')$, let $a'(\xi) = \pi``a(\alpha_\xi)$. Then $\mathrm{dom}(A') = f^\gamma_\gamma(y) \setminus \mathrm{dom}(a')$. If $\xi \in \mathrm{dom}(A')$, let $A'(\xi) = \{\pi``x \mid x \in A(\alpha_\xi)\}$. It is straightforward to verify that p' thus defined is in $\mathbb{P}_{\vec{U}_y}$ and that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\gamma+1}/(\emptyset, A \restriction (\gamma, \theta))$. □

By repeatedly applying the Factorization Lemma, standard arguments (see, e.g. [9]) allow us to assume we are working below a condition of the form $(\emptyset, A)$ when proving the following lemmas about $\mathbb{P}$.

Lemma 3.7. $(\mathbb{P}, \leq, \leq^*)$ *satisfies the Prikry property, i.e., if φ is a statement in the forcing language and $p \in \mathbb{P}$, then there is $q \leq^* p$ such that $q \parallel \varphi$.*

Proof. The proofs of this and the next few lemmas are similar to those for the classical Radin forcing, which can be found in [9]. Fix φ in the forcing language and $p \in \mathbb{P}$. By the Factorization Lemma (3.6), we may assume that $p = (\emptyset, A)$ for some A. Let a be a stem possible for p, and let $\alpha \in \theta \setminus (\gamma^a + 1)$. Let $Y_{a,\alpha} = \{x \in A(\alpha) \mid a(\gamma^a) \prec x \text{ and } \gamma^a \in Z^\alpha_x\}$. Note that $Y_{a,\alpha} \in U_\alpha$. Let $Y^0_{a,\alpha} = \{x \in Y_{a,\alpha} \mid \text{for some } B, (a^\frown(\alpha, x), B) \Vdash \varphi\}$, $Y^1_{a,\alpha} = \{x \in Y_{a,\alpha} \mid \text{for some } B, (a^\frown(\alpha, x), B) \Vdash \neg\varphi\}$, and $Y^2_{a,\alpha} = Y_{a,\alpha} \setminus (Y^0_{a,\alpha} \cup Y^1_{a,\alpha})$. Fix $i(a, \alpha) < 3$ such that $Y^{i(a,\alpha)}_{a,\alpha} \in U_\alpha$, and let $Y^*_{a,\alpha} = Y^{i(a,\alpha)}_{a,\alpha}$.

For $\alpha < \theta$, let $B(\alpha)$ be the set of $x \in A(\alpha)$ such that, for every stem a possible for p such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{a,\alpha}$. By the Diagonal Intersection Lemma (3.5), we have $B(\alpha) \in U_\alpha$. Thus, $(\emptyset, B) \in \mathbb{P}$ and $(\emptyset, B) \leq^* p$.

Suppose for sake of contradiction that no direct extension of $(\emptyset, B)$ decides φ. Find $(a, B^*) \leq (\emptyset, B)$ deciding φ with $|a|$ minimal. Without loss of generality, suppose that $(a, B^*) \Vdash \varphi$. Because of our assumption that no direct extension of $(\emptyset, B)$ decides φ, a is non-empty. Let $b = a^-$ and $\gamma = \gamma^a$. By our construction of B, we have $a(\gamma) \in Y^0_{b,\gamma}$, and, for any $x \in B(\gamma)$ such that $b^\frown(\gamma, x)$ is a stem, there is $\hat{B}_x$ such that $(b^\frown(\gamma, x), \hat{B}_x) \Vdash \varphi$. Let $p^* = (\emptyset, B) \downarrow b = (b, B^{**})$. We will find a direct extension (b, F) of p^* forcing φ, thus contradicting the minimality of $|a|$.

We first define $F \restriction \gamma^b$ (if $b = \emptyset$, then there is nothing to do here). Since there are fewer than κ-many possibilities for $F \restriction \gamma^b$ and U_γ is κ-complete, we may fix a function F^* on $\gamma^b \setminus \mathrm{dom}(b)$ such that $B_0(\gamma) := \{x \in B(\gamma) \mid b^\frown(\gamma, x)$ is a stem and $\hat{B}_x \restriction \gamma^b = F^*\} \in U_\gamma$. Then, for all $\alpha \in \gamma^b \setminus \mathrm{dom}(b)$, let $F(\alpha) = F^*(\alpha) \cap B^{**}(\alpha)$. We next define F on the interval (γ^b, γ) (or on all of γ, if $b = \emptyset$). If $\alpha \in (\gamma^b, \gamma)$, $x \in B_0(\gamma)$, and $\alpha \in Z^\gamma_x$, note that $\hat{B}_x(\alpha) \in u^\gamma_\alpha(x)$. Let $\bar{B}_x(\alpha)$ be the collapsed version of $\hat{B}_x(\alpha)$. Then $\bar{B}_x(\alpha) \in \bar{u}^\gamma_\alpha(x)$. Let $F^*(\alpha)$ be the set in U_α represented by the function $x \mapsto \bar{B}_x(\alpha)$ in the ultrapower by U_γ, and let $F(\alpha) = F^*(\alpha) \cap B^{**}(\alpha)$. Let $F(\gamma)$ be the set of $x \in B_0(\gamma) \cap B^{**}(\gamma)$ such that, for all $\alpha \in Z^\gamma_x \setminus (\gamma^b + 1)$, $\{y \in F^*(\alpha) \mid y \prec x\} = \hat{B}_x(\alpha)$. We claim that $F(\gamma) \in U_\gamma$. To see this, let $j = j_\gamma$. Note that the function $x \mapsto \hat{B}_x(\alpha)$ represents $\{j``y \mid y \in F^*(\alpha)\}$, which is equal to $\{z \in j(F^*(\alpha)) \mid z \prec j``\kappa^{+\gamma}\}$. Thus, $j``\kappa^{+\gamma} \in j(F(\gamma))$, so $F(\gamma) \in U_\gamma$. We finally define F on (γ, θ). If $\alpha \in (\gamma, \theta)$, let $F(\alpha)$ be the set of $y \in B^{**}(\alpha)$ such that $\gamma \in Z^\alpha_y$ and, for all $x \in F(\gamma)$ such that $x \prec y$, $y \in \hat{B}_x(\alpha)$. Then $F(\alpha) \in U_\alpha$. Notice that, by our construction, if $(c, H) \leq (b, F)$ and $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)})$.

Now suppose for sake of contradiction that $(b, F) \nVdash \varphi$. Find $(c, H) \leq (b, F)$ such that $(c, H) \Vdash \neg\varphi$. If $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)}) \Vdash \varphi$, which is a contradiction. Thus, suppose $\gamma \notin \mathrm{dom}(c)$. By our choice of $F(\alpha)$ for $\alpha \in (\gamma, \theta)$ (namely, our requirement that $\gamma \in Z^\alpha_y$ for all $y \in F(\alpha)$), it must be the case that $H(\gamma) \neq \emptyset$. But then (c, H) can be extended further to a condition (c', H') such that $\gamma \in \mathrm{dom}(c')$, and this again gives a contradiction. $\square$

The following definitions will play a crucial role in the proof that, if θ is weakly inaccessible, then κ remains strongly inaccessible in the extension by $\mathbb{P}$.

Definition 3.8. Let $n < \omega$. A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times X_\alpha)]^{\leq n}$ is *$\vec{U}$-fat* if the following conditions hold.

1. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} \in Z^{\alpha_{i_1}}_{x_{i_1}}$ and $x_{i_0} \prec x_{i_1}$.
2. For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t < \theta$ such that:
   - (a) for all (β, y) such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   - (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in U_{\alpha_t}$.

If T is as in the previous definition, then n is said to be the *height* of T.

Definition 3.9. Suppose that T is a fat tree, $\alpha < \theta$, and $x \in X_\alpha$. T is *$\vec{U}$-fat above* (α, x) if, for all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i \leq k$, we have $\alpha \in Z^{\alpha_i}_{x_i}$, and $x \prec x_i$.

Definition 3.10. Suppose that $\gamma < \theta$ and $z \in X_\gamma$. A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times X_\alpha)]^{\leq n}$ is *$\vec{U}$-fat below* (γ, z) if the following conditions hold.

1. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i \leq k$, we have $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$.
2. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} \in Z^{\alpha_{i_1}}_{x_{i_1}}$ and $x_{i_0} \prec x_{i_1}$.
3. For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t \in Z^\gamma_z$ such that:
   - (a) for all (β, y) such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   - (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in u^\gamma_{\alpha_t}(z)$.

Notice that, if T is $\vec{U}$-fat below (γ, z), then it is isomorphic to a $\vec{U}_z$-fat tree via the order-isomorphism from z to $\mathrm{otp}(z)$.

Definition 3.11. Suppose that $a = \langle(\beta_\ell, y_\ell) \mid \ell < m\rangle$ is a stem for $\mathbb{P}$, with $\beta_0 < \ldots < \beta_{\ell-1}$. A $(\vec{U}, a)$-*fat sequence of trees* is a sequence $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell \leq m\rangle$ such that

(1) $\mathcal{T}_0 = \{T_\emptyset\}$, where $T_\emptyset$ is a $\vec{U}$-fat tree below (β_0, y_0) (if $m = 0$, then $T_\emptyset$ is simply a $\vec{U}$-fat tree);
(2) $\mathcal{B}_0 = \{\langle b\rangle \mid b \text{ is a maximal element of } T_\emptyset\}$;
(3) for each $0 < \ell \leq m$, $\mathcal{T}_\ell = \{T_{\vec{b}} \mid \vec{b} \in \mathcal{B}_{\ell-1}\}$, where
   (a) if $\ell < m$, then each $T_{\vec{b}}$ is a $\vec{U}$-fat tree above $(\beta_{\ell-1}, y_{\ell-1})$ and below (β_ℓ, y_ℓ);
   (b) if $\ell = m$, then each $T_{\vec{b}}$ is a $\vec{U}$-fat tree above $(\beta_{\ell-1}, y_{\ell-1})$;
(4) for each $0 < \ell \leq m$, $\mathcal{B}_\ell$ is the set of all sequences $\vec{b} = \langle b_i \mid i \leq \ell\rangle$ such that
   (a) $\vec{b}^- := \langle b_i \mid i < \ell\rangle$ is an element of $\mathcal{B}_{\ell-1}$;
   (b) b_ℓ is a maximal element of $T_{\vec{b}^-}$.

Notice that, if $a = \langle(\beta_\ell, y_\ell) \mid \ell < m\rangle$ is a stem for $\mathbb{P}$ and $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell \leq m\rangle$ is a $(\vec{U}, a)$-fat sequence of trees, then every $\vec{b} = \langle b_i \mid i \leq m\rangle$ in $\mathcal{B}_m$ determines a stem $a_{\vec{b}}$ for $\mathbb{P}$ defined by letting

$$a_{\vec{b}} = b_0{}^\frown\langle(\beta_0, y_0)\rangle^\frown b_1{}^\frown\langle(\beta_1, y_1)\rangle^\frown \ldots {}^\frown \langle(\beta_{m-1}, y_{m-1})\rangle^\frown b_m.$$

Note also that since, for each $\ell < m$ and each $T \in \mathcal{T}_\ell$, we have that T is fat below (β_ℓ, y_ℓ), and since κ is strongly inaccessible, it follows that $|\mathcal{T}_m| < \kappa$.

The following fact is easily verified; the element $\vec{b}$ is constructed by recursion, taking advantage of the fact that branching in fat trees occurs on measure-one sets.

Fact 3.12. *Suppose that $(a, A) \in \mathbb{P}$ and $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell \leq m\rangle$ is a $(\vec{U}, a)$-fat sequence of trees. Then there is $\vec{b} \in \mathcal{B}_m$ such that the stem $a_{\vec{b}}$ is possible for (a, A).*

Lemma 3.13. *Suppose that $p = (a, A) \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ is a dense open set below p. Suppose that $a = \langle(\beta_\ell, y_\ell) \mid \ell < m\rangle$ is such that, for all $\ell_0 < \ell_1 < m$, $\beta_{i_0} < \beta_{i_1}$. Then there is a $(\vec{U}, a)$-fat sequence of trees $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell \leq m\rangle$ such that, for all $\vec{b} \in \mathcal{B}_m$, if $a_{\vec{b}}$ is possible for p, then there is B such that $(a_{\vec{b}}, B) \leq p$ and $(a_{\vec{b}}, B) \in D$.*

Proof. Let us first argue that it suffices to prove the lemma in the case $m = 0$, i.e., for conditions with an empty stem. To this end, suppose that $m > 0$, and let $(\beta, y) = (\beta_{m-1}, y_{m-1})$. By the Factorization Lemma (3.6), we have $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p_0 \times \mathbb{P}_{\vec{U},\beta+1}/p_1$, where $p_1 = (\emptyset, A \restriction [\beta + 1, \theta))$ and p_0 is of the form $(\bar{a}, \bar{B})$, where $\bar{a} = \langle(\bar{\beta}_\ell, \bar{y}_\ell) \mid \ell < m-1\rangle$ is such that, for all $\ell < m-1$, $\bar{y}_\ell$ is the image of y_ℓ under the order-preserving isomorphism between y and $\mathrm{otp}(y)$. We may assume by induction that we have established the full lemma for the forcing $\mathbb{P}_{\vec{U}_y}$. Let us also assume that we have established the lemma for $\mathbb{P}_{\vec{U},\beta+1}$ for conditions with empty stems. Let us regard D as a dense subset of $\mathbb{P}_{\vec{U}_y} \times \mathbb{P}_{\vec{U},\beta+1}$ in the natural way.

Let $\dot{D}_0$ be a $\mathbb{P}_{\vec{U},\beta+1}$-name for the set of $q_0 \in \mathbb{P}_{\vec{U}_y}$ such that, for some $q_1 \in \dot{G}_{\mathbb{P}_{\vec{U},\beta+1}}$, $(q_0, q_1) \in D$. Then $\dot{D}_0$ is forced to be a dense open subset of $\mathbb{P}_{\vec{U}_y}$ below p_0. By repeated applications of Lemma 3.7, we can find a condition $p_1^* = (\emptyset, A^*) \leq^* p_1$ in $\mathbb{P}_{\vec{U},\beta+1}$ and a dense open subset D_0 of $\mathbb{P}_{\vec{U}_y}/p_0$ such that $p_1^* \Vdash$ "$\dot{D}_0 = \check{D}_0$".

Apply the inductive hypothesis to $\mathbb{P}_{\vec{U}_y}$, p_0, and D_0 to find a $(\vec{U}_y, \bar{a})$-fat sequence of trees $\langle(\bar{\mathcal{T}}_\ell, \bar{\mathcal{B}}_\ell) \mid \ell < m\rangle$ such that, for all $\vec{b} \in \bar{\mathcal{B}}_{m-1}$, if $a_{\vec{b}}$ is possible for p_0, then there is $\bar{B}_{\vec{b}}$ such that $(a_{\vec{b}}, \bar{B}_{\vec{b}}) \leq p_0$ and $(a_{\vec{b}}, \bar{B}_{\vec{b}}) \in D_0$.

Now, for each such $\vec{b} \in \bar{\mathcal{B}}_{m-1}$, the set of $p \in \mathbb{P}_{\vec{U},\beta+1}$ such that $((a_{\vec{b}}, B_{\vec{b}}), p) \in D$ is dense below p_1^*. Denote this set by $D_{1,\vec{b}}$. We can now apply the lemma for $\mathbb{P}_{\vec{U},\beta+1}$ for conditions with empty stems to the condition p_1^* and the set $D_{1,\vec{b}}$ to find a $\vec{U}$-fat tree $T_{\vec{b}}$ such that

- for all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T_{\vec{b}}$, we have $\alpha_0 > \beta$;
- for all maximal $c \in T_{\vec{b}}$, if c is possible for p_1^*, then there is $C_{\vec{b}}$ such that $(c, C_{\vec{b}}) \leq p_1^*$ and $(c, C_{\vec{b}}) \in D_{1,\vec{b}}$.

By thinning out $T_{\vec{b}}$ if necessary, we may assume that it is $\vec{U}$-fat above (β, y).

Let $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell < m\rangle$ be the sequence obtained in the natural way by applying the order-isomorphism between $\operatorname{otp}(y)$ and y to the $(\vec{U}_y, \bar{a})$-fat sequence $\langle(\bar{\mathcal{T}}_\ell, \bar{\mathcal{B}}_\ell) \mid \ell < m\rangle$. Let $\mathcal{T}_m = \{T_{\vec{b}} \mid \vec{b} \in \bar{\mathcal{B}}_{m-1}\}$, and let $\mathcal{B}_m$ be the set of sequences of the form $\hat{b}^\frown\langle c\rangle$, where $\hat{b} \in \mathcal{B}_{m-1}$ and, letting $\vec{b}$ be the isomorphic copy of $\hat{b}$ in $\bar{\mathcal{B}}_{m-1}$, we have that c is a maximal element of $T_{\vec{b}}$. Then it is easily verified that $\langle(\mathcal{T}_\ell, \mathcal{B}_\ell) \mid \ell \leq m\rangle$ is a $(\vec{U}, a)$-fat sequence of trees as in the statement of the lemma.

It therefore suffices to consider p of the form $(\emptyset, A)$. We thus need to find a single $\vec{U}$-fat tree T such that, for every maximal element b of T, if b is possible for p, then there is B such that $(b, B) \leq p$ and $(b, B) \in D$.

We accomplish this by inductively constructing a decreasing sequence of conditions $\langle(\emptyset, A_n) \mid n < \omega\rangle$. Intuitively, A_n will take care of extensions $(b, B) \leq (\emptyset, A)$ such that $|b| = n$. We explicitly go through the first few steps of the construction.

Let $A_0 = A$. If there is a direct extension of $(\emptyset, A)$ in D, then we are done by setting $T = \{\emptyset\}$. Thus, suppose there is no such direct extension. For every stem a possible for $(\emptyset, A_0)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{0,a,\alpha} = \{x \in A_0(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z_x^\alpha\}$. Let $Y_{0,a,\alpha}^0 = \{x \in Y_{0,a,\alpha} \mid$ for some B, $(a^\frown(\alpha, x), B) \in D\}$, and let $Y_{0,a,\alpha}^1 = Y_{0,a,\alpha} \setminus Y_{0,a,\alpha}^0$. Find $i(0, a, \alpha) < 2$ such that $Y_{0,a,\alpha}^{i(0,a,\alpha)} \in U_\alpha$, and let $Y_{0,a,\alpha}^* = Y_{0,a,\alpha}^{i(0,a,\alpha)}$. For $\alpha < \theta$, let $A_1(\alpha)$ be the set of $x \in A_0(\alpha)$ such that, for all stems a possible for $(\emptyset, A_0)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z_x^\alpha$, $x \in Y_{0,a,\alpha}^*$. By the Diagonal Intersection Lemma (3.5), $A_1(\alpha) \in U_\alpha$ for all $\alpha < \theta$, so $(\emptyset, A_1) \leq^* (\emptyset, A_0)$. Note that $(\emptyset, A_1)$ satisfies the following property, which we denote $(*)_1$:

> Suppose that $q = (a^\frown(\alpha, x), B) \leq (\emptyset, A_1)$ and $q \in D$. Then, for every $y \in A_1(\alpha)$ such that $a(\gamma^a) \prec y$ and $\gamma^a \in Z_y^\alpha$, there is B_y such that $(a^\frown(\alpha, y), B_y) \in D$.

Now suppose that there is a stem $a = \{(\alpha, x)\}$ possible for $(\emptyset, A_1)$ and a B such that $(a, B) \in D$. We can then define a $\vec{U}$-fat tree T of height 1 whose maximal elements are all $\langle(\alpha, x)\rangle$ such that $x \in A_1(\alpha)$. We are then done, as T easily satisfies the requirements of the lemma. Thus, suppose there is no such a and proceed to define $(\emptyset, A_2)$ as follows.

For every stem a possible for $(\emptyset, A_1)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{1,a,\alpha} = \{x \in A_1(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z_x^\alpha\}$. Let $Y_{1,a,\alpha}^0$ be the set of all $x \in Y_{1,a,\alpha}$ such that there are $\beta_x^\alpha \in (\alpha, \theta)$ and $W_x^\alpha \in U_{\beta_x^\alpha}$ such that, for all $y \in W_x^\alpha$:

- $x \prec y$ and $\alpha \in Z_y^{\beta_x^\alpha}$;

- there is B such that $(a^\frown(\alpha, x)^\frown(\beta^\alpha_x, y), B) \in D$.

Let $Y^1_{1,a,\alpha} = Y_{1,a,\alpha} \setminus Y^0_{1,a,\alpha}$. Find $i(1,a,\alpha) < 2$ such that $Y^{i(1,a,\alpha)}_{1,a,\alpha} \in U_\alpha$, and let $Y^*_{1,a,\alpha} = Y^{i(1,a,\alpha)}_{1,a,\alpha}$. For $\alpha < \theta$, let $A_2(\alpha)$ be the set of $x \in A_1(\alpha)$ such that, for all stems a possible for $(\emptyset, A_1)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{1,a,\alpha}$. By the Diagonal Intersection Lemma (3.5), we have $A_2(\alpha) \in U_\alpha$ for all $\alpha < \theta$. Then $(\emptyset, A_2) \leq^* (\emptyset, A_1)$, and $(\emptyset, A_2)$ satisfies the following property, which we denote $(*)_2$:

> Suppose that $q = (a^\frown(\alpha, x)^\frown(\beta, y), B) \leq (\emptyset, A_2)$ and $q \in D$. Then, for every $x' \in A_2(\alpha)$ such that $a(\gamma^a) \prec x'$ and $\gamma^a \in Z^\alpha_{x'}$, there is $\beta^\alpha_{x'} \in (\alpha, \theta)$ and $W^\alpha_{x'} \in U_{\beta^\alpha_{x'}}$ such that, for all $y' \in W^\alpha_{x'}$, there is B' such that $(a^\frown(\alpha, x')^\frown(\beta^\alpha_{x'}, y'), B') \in D$.

Suppose that there is a stem $a = \{(\alpha, x), (\beta, y)\}$ possible for $(\emptyset, A_2)$ with $\alpha < \beta$ and a B such that $(a, B) \in D$. Using $(*)_2$, we can define a $\vec{U}$-fat tree T of height 2 whose maximal elements are all $\langle(\alpha, x'), (\beta^\alpha_{x'}, y')\rangle$ such that $x' \in A_2(\alpha)$ and $y' \in W^\alpha_{x'}$. We are then done, as T satisfies the requirements of the lemma. If there is no such stem a, then continue in the same manner.

In this way, we can construct A_n such that, if there is a stem a possible for $(\emptyset, A_n)$ with $|a| = n$ and a B such that $(a, B) \leq (\emptyset, A_n)$ and $(a, B) \in D$, then there is a $\vec{U}$-fat tree of height n as desired. For $\alpha < \theta$, let $A_\infty(\alpha) = \bigcap_{n<\omega} A_n(\alpha)$. For all $n < \omega$, $(\emptyset, A_\infty) \leq^* (\emptyset, A_n)$. Find $(a, B) \leq (\emptyset, A_\infty)$ such that $(a, B) \in D$. Let $n^* = |a|$. Then a is possible for $(\emptyset, A_{n^*})$, so there is a fat tree of height n^* as required by the lemma. □

Suppose that T is a fat tree, $\gamma < \theta$, and $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$. $T \restriction (\gamma, z)$ is the subtree of T consisting of all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ such that, for all $i \leq k$, $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$. Let $\gamma_T := \sup(\{\alpha \mid \text{for some } \langle(\alpha_i, x_i) \mid i \leq k\rangle \in T \text{ and } i \leq k,\ \alpha = \alpha_i\})$. Note that, if θ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, then $\gamma_T < \theta$.

Theorem 3.14. *If θ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, then κ remains regular in $V^{\mathbb{P}}$.*

Proof. Let $p = (a, A) \in \mathbb{P}$, let $\delta < \kappa$, and suppose that $\dot{f}$ is a $\mathbb{P}$-name forced by p to be a function from δ to κ. We will find $q \leq p$ forcing the range of $\dot{f}$ to be bounded below κ.

For all $\xi < \delta$, let D_ξ be the set of $(b, B) \in \mathbb{P}$ such that $(b, B) \Vdash$ "$\dot{f}(\xi) < \kappa_{b(\gamma^b)}$." Each D_ξ is a dense, open subset of $\mathbb{P}$ below p. For $\xi < \delta$, let S_ξ be the set of stems b such that, for some B, $(b, B) \leq p$ and $(b, B) \in D_\xi$. For all $b \in S_\xi$, fix a B^ξ_b witnessing this. For each $\beta \in (\gamma^a, \theta)$, let $A^*(\beta)$ be the set of $y \in A(\beta)$ such that, for all $\xi < \delta$ and all $b \in S_\xi$ such that $b(\gamma^b) \prec y$ and $\gamma^b \in Z^\beta_y$, $y \in B^\xi_b(\beta)$. By the Diagonal Intersection Lemma (3.5), $A^*(\beta) \in U_\beta$. For $\beta \in \mathrm{dom}(A) \cap \gamma^a$, let $A^*(\beta) = A(\beta)$. Then $(a, A^*) \leq (a, A)$. Let R be the set of stems possible for (a, A^*). For $\gamma < \theta$, let $R_{<\gamma} = \{c \in R \mid \gamma^c < \gamma\}$. For all $c \in R$, let $p_c = (a, A^*) \downarrow c$. For all $c \in R$ and $\xi < \delta$, apply Lemma 3.13 to p_c and D_ξ to obtain a $(\vec{U}, c)$-fat sequence of trees $\langle(\mathcal{T}_{c,\xi,\ell}, \mathcal{B}_{c,\xi,\ell}) \mid \ell \leq |c|\rangle$. Let E be the set of limit ordinals $\gamma < \theta$ such that, for all $c \in R_{<\gamma}$, all $\xi < \delta$, and all $T \in \mathcal{T}_{c,\xi,|c|}$, we have $\gamma_T < \gamma$. Then E is club in θ. Fix $\gamma \in E \setminus (\gamma^a + 1)$.

Claim 3.15. *Let Y_γ be the set of $z \in A^*(\gamma)$ such that, for all $c \in R_{<\gamma}$ such that $c(\gamma^c) \prec z$ and $\gamma^c \in Z^\gamma_z$, for all $\xi < \delta$, and for all $T \in \mathcal{T}_{c,\xi,|c|}$, we have that $T \restriction (\gamma, z)$ is $\vec{U}$-fat below (γ, z) and has the same height as T. Then $Y_\gamma \in U_\gamma$.*

Proof. Let $j = j_\gamma$. We show that $j``\kappa^{+\gamma} \in j(Y_\gamma)$. Note that, for all $c \in j(R_{<\gamma})$, if $c(\gamma^c) \prec j``\kappa^{+\gamma}$ and $\gamma^c \in j``\gamma$, then there is a stem $\bar{c} \in R_{<\gamma}$ such that $c = j(\bar{c})$. Fix such a c and an ordinal $\xi < \delta$. Since $|\mathcal{T}_{\bar{c},\xi,|\bar{c}|}| < \kappa$, we have $j(\mathcal{T}_{\bar{c},\xi,|\bar{c}|}) = \{j(T) \mid T \in \mathcal{T}_{\bar{c},\xi,|\bar{c}|}\}$. If $T \in \mathcal{T}_{\bar{c},\xi,|\bar{c}|}$, then $j(T) \restriction (j(\gamma), j``\kappa^{+\gamma}) = j``T$, which is fat below $(j(\gamma), j``\kappa^{+\gamma})$ of the same height as $j(T)$. Hence, $j``\kappa^{+\gamma} \in j(Y_\gamma)$. □

Choose $z \in Y_\gamma$. Then $q = (a^\frown(\gamma, z), A^{**}) \leq (a, A^*)$, where $A^{**}(\alpha) = A^*(\alpha)$ for all $\alpha \in \mathrm{dom}(A^*) \cap \gamma^a$, $A^{**}(\alpha) = \{x \in A^*(\alpha) \mid x \prec z\}$ for all $\alpha \in (\gamma^a, \gamma)$, and $A^{**}(\alpha) = \{x \in A^*(\alpha) \mid z \prec x\}$ for all $\alpha \in (\gamma, \theta)$.

We claim that q forces the range of $\dot{f}$ to be bounded below κ_z. Suppose for sake of contradiction that there is $\xi < \delta$ and $r \leq q$ such that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." Let $r = (d, F)$, and let $c = \{(\alpha, x) \in d \mid \alpha < \gamma\}$. Then $c \in R_{<\gamma}$, $c(\gamma^c) \prec z$, and $\gamma^c \in Z^\gamma_z$, so $T \restriction (\gamma, z)$ is fat below (γ, z), of the same height as T, for every $T \in \mathcal{T}_{c,\xi,|c|}$. Applying Fact 3.12 inside $\mathbb{P}_{\vec{U}_z}$, it follows that we can find $\vec{b} \in \mathcal{B}_{c,\xi,|c|}$ such that $a_{\vec{b}} \cup d$ is possible for r. By the definition of $\langle (\mathcal{T}_{c,\xi,\ell}, \mathcal{B}_{c,\xi,\ell}) \mid \ell \leq |c| \rangle$, it follows that there is B' such that $(a_{\vec{b}}, B') \in D_\xi$. Moreover, by our construction of A^*, we may assume that, for all $\alpha \in \mathrm{dom}(B') \cap (\gamma^{a_{\vec{b}}})$, we have $B'(\alpha) = A^*(\alpha)$. All of this together means that $(a_{\vec{b}}, B')$ and r are compatible. However, as $(a_{\vec{b}}, B') \in D_\xi$, we have $(a_{\vec{b}}, B') \Vdash$ "$\dot{f}(\xi) < \kappa_{a_{\vec{b}}(\gamma^{a_{\vec{b}}})} < \kappa_z$," contradicting the assumption that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." □

4. Cardinal arithmetic

For the rest of the paper, we will let θ be the least weakly inaccessible cardinal above κ. In this section, we show that, with this assumption, if G is $\mathbb{P}$-generic over V, then, in $V[G]$, every limit point of C_G below κ is a singular strong limit cardinal at which SCH fails. We begin by making the following definition.

Definition 4.1. For $\beta < \theta$ and $y \in X_\beta$, we write $o(y)$ for $f^\beta_\beta(\kappa_y)(= \mathrm{otp}(Z^\beta_y))$.

We note that $o(y)$ formally depends on β as well as y, but, as the value of β will always be clear from context, we supress its mention. For $\nu < \kappa$, we let $\theta(\nu)$ be the least weakly inaccessible cardinal greater than ν. Using this notation we have $o(y) < \theta(\kappa_y)$. Also, by the preparation of our ground model, we have $2^\kappa \geq \theta$ and, for all limit ordinals $\beta < \kappa$, we have $|\bigcup_{\alpha<\beta} \mathcal{P}_\kappa(\kappa^{+\alpha})| = \kappa^{+\beta}$. As a result, for all limit ordinals $\beta < \theta$, the following statements hold for almost all $y \in X_\beta$:

- $2^{\kappa_y} \geq \theta(\kappa_y)$;
- $|\bigcup_{\alpha<o(y)} \mathcal{P}_{\kappa_y}(\kappa_y^{+\alpha})| = \kappa_y^{+o(y)}$.

We henceforth assume that in fact all $y \in X_\beta$ satisfy these two conditions.

Lemma 4.2. *Suppose that $\beta < \theta$ is a limit ordinal and $y \in X_\beta$. Then $\mathbb{P}_{\vec{U}_y}$, as defined before Lemma 3.6, has the $\kappa_y^{+o(y)+1}$-Knaster property.*

Proof. It is not hard to see that there are just $\kappa_y^{+o(y)}$ many stems in this poset and that conditions with the same stem are compatible. □

Suppose now that G is $\mathbb{P}$-generic over V, and fix $\nu \in \lim(C_G)$. Also fix $p \in G$, $\beta \in \mathrm{dom}(a^p)$, and $y \in X_\beta$ such that $a^p(\beta) = y$ and $\kappa_y = \nu$. Since $\nu \in \lim(C_G)$, we know that β is a limit ordinal.

Lemma 4.3. *ν is a strong limit cardinal in $V[G]$.*

Proof. Fix $\mu < \nu$, and let $\beta_0 < \beta$ be the largest limit ordinal such that there exists $q \in G$ with $\beta_0 \in \mathrm{dom}(a^q)$ and $\kappa_{a^q(\beta_0)} \leq \mu$, if such a q exists. Let $\beta_0 = 0$ otherwise. Let $\beta_1 < \beta$ be the least ordinal such that there exists $q \in G$ such that $\beta_1 \in \mathrm{dom}(a^q)$ and $\mu < \kappa_{a^q(\beta_1)}$. Note that β_1 is a successor ordinal and there are only finitely many ordinals γ between β_0 and β_1 for which there exists $q \in G$ such that $\gamma \in \mathrm{dom}(a^q)$. We can now find $q \in G$ such that

- $\beta_1 \in \mathrm{dom}(a^q)$ and, if $\beta_0 > 0$, then $\beta_0 \in \mathrm{dom}(a^q)$ as well;
- for all γ in the interval $[\beta_0, \beta_1]$, either $\gamma \in a^q$ or $A^q(\gamma) = \emptyset$.

If $\beta_0 = 0$, then, in V the direct ordering $\leq^*$ is μ^+-closed in $\mathbb{P}/q$, so, by Lemma 3.7, forcing with $\mathbb{P}$ below q does not add any new subsets to μ. Since ν was strongly inaccessible and hence strong limit in V, it follows that $2^\mu < \nu$ continues to hold in $V[G]$.

If $\beta_0 > 0$, then let $y_0 = a^q(\beta_0)$. By the Factorization Lemma (3.6), $\mathbb{P}/q \cong \mathbb{P}_{\vec{U}_{y_0}}/q_0 \times \mathbb{P}_{\vec{U},\beta_0+1}/q_1$ for some conditions q_0 and q_1. As in the case in which $\beta_0 = 0$, forcing with $\mathbb{P}_{\vec{U},\beta_0+1}/q_1$ does not add any new subsets to μ. Moreover, in V, we have $|\mathbb{P}_{\vec{U}_{y_0}}| < \nu$ and ν is strongly inaccessible, so forcing with $\mathbb{P}_{\vec{U}_{y_0}}$ cannot add ν-many distinct subsets to μ. Again, it follows that $2^\mu < \nu$ continues to hold in $V[G]$. □

The argument of the above proof easily adapts to yield, together with Theorem 3.14, the following corollary.

Corollary 4.4. *κ is strongly inaccessible in $V[G]$.*

We next argue that ν is singular in $V[G]$. The proof breaks into two cases, depending on whether or not $\mathrm{cf}^V(\beta) < \kappa$.

Suppose first that $\mathrm{cf}^V(\beta) < \kappa$. Then, by Lemma 2.4, we have that $\mathrm{cf}^V(\beta) < \kappa_y = \nu$ and Z^β_y is unbounded in β. It follows by genericity that the set $A := \{\alpha \in Z^\beta_y \mid \exists p \in G[\alpha \in a^p]\}$ is unbounded in β and that

$$\nu = \sup\{\kappa_x \mid \exists \alpha \in A \exists p \in G[a^p(\alpha) = x]\}.$$

Therefore, in $V[G]$, we have $\mathrm{cf}(\nu) = \mathrm{cf}(\beta) < \nu$.

Next, suppose that $\mathrm{cf}^V(\beta) \geq \kappa$. The following lemma shows that $\mathrm{cf}^{V[G]}(\nu) = \omega$.

Lemma 4.5. *If $\mathrm{cf}^V(\beta) \geq \kappa$, then ν and $\nu^{+o(y)}$ change their cofinality to ω in $V[G]$.*

Proof. Work in V. Using the Factorization Lemma (3.6), $\mathbb{P}_{\vec{U}}/p \cong \mathbb{P}_{\vec{U}_y}/p_0 \times \mathbb{P}_{\vec{U},\beta+1}/p_1$ for some p_0 and p_1. As $\mathbb{P}_{\vec{U},\beta+1}/p_1$ does not add new bounded subsets to $\theta(\nu)$, it is sufficient to focus on the forcing $\mathbb{P}_{\vec{U}_y}$ which adds a Radin club to ν. For notational simplicity, let $\delta = f^\beta_\beta(y) = o(y)$, and let $\vec{U}_y = \langle V_\xi \mid \xi < \delta\rangle$. By Lemma 2.4, we have $\rho := \mathrm{cf}(\delta) \geq \nu$ and $\delta < \theta(\nu)$. We will show that ν and $\nu^{+\delta}$ change their cofinalities to ω after forcing with $\mathbb{P}_{\vec{U}_y}$.

Choose an increasing continuous sequence $\vec{\delta} = \langle \delta_\alpha \mid \alpha < \rho\rangle$ cofinal in δ. Let G_y be $\mathbb{P}_{\vec{U}_y}$-generic over V. For every $\beta' < \delta$, let $\alpha(\beta') < \rho$ be the minimal $\alpha < \rho$ so that $\beta' \leq \delta_\alpha$.

Since $\delta < \theta(\nu)$, we have $\nu^{+\delta} > \rho$. Let $\alpha_0 < \rho$ be the least ordinal so that $\rho < \nu^{+\delta_{\alpha_0}}$. By reindexing, we can assume that $\alpha_0 = 0$. Note that, for every β' with $\delta_0 \leq \beta' < \delta$, $Y_{\beta'} := \{x \in \mathcal{P}_\nu(\nu^{+\beta'}) \mid \alpha(\beta') \in x\}$ belongs to $V_{\beta'}$. For $x \in \mathcal{P}_\nu(\nu^{+\beta'})$, let $\nu_x := x \cap \nu$.

Move now to $V[G_y]$. Given $x \in C^{\mathrm{sc}}_{G_y}$, let $\beta(x)$ be the unique $\beta' < \delta$ such that there exists $q \in G_y$ for which $a^q(\beta') = x$, and let $\alpha(x) = \alpha(\beta(x))$. By the previous paragraph, there is some $\nu_0 \in C_{G_y}$ such that, for every $x \in C^{\mathrm{sc}}_{G_y}$, if $\nu_x > \nu_0$, then $x \in Y_{\beta(x)}$, and hence $\alpha(x) \in x$. Let x_0 be the minimal $x \in C^{\mathrm{sc}}_{G_y}$ satisfying the above. Starting from x_0, we define a sequence $\vec{x} = \langle x_n \mid n < \omega\rangle \subset C^{\mathrm{sc}}_{G_y}$. For each $n < \omega$, let x_{n+1} be the minimal x above x_n in $C^{\mathrm{sc}}_{G_y}$ so that $\sup(x_n \cap \rho) < \alpha(x) < \rho$. Let $\nu_\omega = \bigcup_{n<\omega} \nu_{x_n}$.

We claim that $\nu_\omega = \nu$. Suppose otherwise. Then $\nu_\omega = \nu_x$ for some $x \in C^{\mathrm{sc}}_{G_y}$. Let $\alpha = \alpha(x)$, and note that $\alpha \in x$. Since $\langle \nu_{x_n} \mid n < \omega\rangle$ is cofinal in ν_ω, we have $x \cap \rho = \bigcup_{n<\omega}(x_n \cap \rho)$. There is thus some $m < \omega$ such that $\alpha \in x_m$. But $\alpha \geq \alpha(x_{m+1}) > \sup(x_m \cap \rho)$, which is a contradiction.

It follows that $\nu = \nu_\omega$, so ν changes its cofinality to ω. The set $C^{\mathrm{sc}}_{G_y} \subset \mathcal{P}_\nu(\nu^{+\delta})$ is $\subseteq$-cofinal in $\mathcal{P}_\nu(\nu^{+\delta})$. Since $\langle \nu_{x_n} \mid n < \omega\rangle$ is cofinal in ν, $\langle x_n \mid n < \omega\rangle$ is $\subset$-cofinal in $C^{\mathrm{sc}}_{G_y}$. Thus, $\nu^{+\delta} = \bigcup_{n<\omega} x_n$. It follows that $\mathrm{cf}(\nu^\delta) = \omega$, as each x_n is bounded in $\nu^{+\delta}$. □

Remark 4.6. It follows easily from the above proof that, if $\mathrm{cf}^V(\beta) \geq \kappa$, then all V-regular cardinals between ν and $\nu^{+o(y)}$ change their cofinality to ω in $V[G]$, as well.

In either case, we have shown that ν is a singular strong limit cardinal in $V[G]$. To show that SCH fails at ν, we first make the following observation.

Lemma 4.7. $(\nu^+)^{V[G]} = (\nu^{+o(y)+1})^V$.

Proof. Exactly as in the start of the proof of Lemma 4.5, since $\mathbb{P}_{\vec{U},\beta+1}/p_1$ does not add new subsets to $\theta(\nu)$, it will suffice to show that forcing with $\mathbb{P}_{\vec{U}_y}$ collapses all V-cardinals in the interval $(\nu, (\nu^{+o(y)})^V]$ and preserves all cardinals greater than or equal to $\nu^{+o(y)+1}$. Observe first that, in $V[G_y]$, we have $\nu^{+o(y)} = \bigcup C^{\mathrm{sc}}_{G_y}$. Since $C^{\mathrm{sc}}_{G_y}$ is a $\prec$-increasing sequence of cofinality $\mathrm{cf}^{V[G]}(\nu) < \nu$, consisting sets of cardinality less than ν, it follows that $|(\nu^{+o(y)})^V| = \nu$ in $V[G]$. Next note that, in V, Lemma 4.2 implies that $\mathbb{P}_{\vec{U}_y}$ has the $\nu^{+o(y)+1}$-Knaster property and hence preserves all cardinals greater than or equal to $\nu^{+o(y)+1}$. □

To conclude that SCH fails at ν in $V[G]$, it is now enough to observe that, in V, we have $2^\nu \geq \theta(\nu)$ and $\theta(\nu)$ is a weakly inaccessible cardinal greater than $\kappa^{+o(y)+1}$. It follows that, in $V[G]$, we still have $2^\nu \geq \theta(\nu) > \kappa^{+o(y)+1}$, and $\theta(\nu)$ remains a cardinal. This completes the argument that, in $V[G]$, every limit point of C_G below κ is a singular strong limit cardinal at which SCH fails.

5. Approachability

In this section we characterize precisely which successors ν^+ for $\nu \in \lim(C_G)$ have reflection properties and then construct the final model in which the conclusion of Theorem 1.1 will hold. We begin with the following lemma. We will later need to apply the lemma to posets $\mathbb{P}_{\vec{U}_y}$ for $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, so note that its proof does not rely on the weak inaccessibility of θ.

Lemma 5.1. *Suppose that, in V,*

- *δ is an ordinal of cofinality $\mu < \kappa$;*
- *$p \in \mathbb{P}$*
- *$\nu_0 < \mu < \nu_1$ are such that one of the following four alternatives holds:*
  - *p forces ν_0 and ν_1 to be successive limit points of $\dot{C}_G$, there is $\alpha \in \mathrm{dom}(a^p)$ such that $a^p(\alpha) = y_0$, $\kappa_{y_0} = \nu_0$, and $\nu_0^{+o(y)} < \mu < \nu_1$;* ***or***
  - *$\nu_1 = \kappa$ and p forces that ν_0 is the largest limit point of $\dot{C}_G$ (so, in particular, p forces $\dot{C}_G$ to have a final segment of order type ω) and there are α and y_0 as in the previous alternative;* ***or***
  - *$\nu_0 = 0$ and p forces ν_1 to be the least limit point of $\dot{C}_G$;*
  - *$\nu_0 = 0$, $\nu_1 = \kappa$, and p forces that $\mathrm{otp}(\dot{C}_G) = \omega$.*
- *$\dot{C}$ is a $\mathbb{P}$-name forced by p to be a club in δ.*

Then there is a direct extension $p' \leq^* p$ and a club D in δ such that $p' \Vdash$ "$D \subseteq \dot{C}$".

Proof. First we show that it is enough to consider $\delta = \mu$. Assume for the moment that $\mu < \delta$. Let $\pi : \mu \to \delta$ be an increasing, continuous, and cofinal function. By passing to a name for a subset of $\dot{C}$ we can assume that it is forced that $\dot{C}$ is a subset of the range of π. Now a condition will force that there is a ground model club contained in $\dot{C}$ if and only if there is a ground model club contained in $\pi^{-1}(\dot{C})$.

So we may assume that $\delta = \mu$. If p forces either that ν_1 is the least limit point of $\dot{C}_G$ or that $\mathrm{otp}(\dot{C}_G) = \omega$, then, by applying Lemma 3.7 to $\mathbb{P}/p$, we see that forcing with $\mathbb{P}$ below p does not add any bounded subsets to ν_1, so there is in fact a direct extension p' of p deciding the value of $\dot{C}$.

Thus, assume we are in one of the first two alternatives, so p forces that ν_0 is a limit point of C_G and there is $\alpha \in \mathrm{dom}(a^p)$ such that $a^p(\alpha) = y_0$, $\kappa_{y_0} = \nu_0$, and $\nu_0^{+o(y)} < \mu < \nu_1$. Then, by the Factorization Lemma (3.6), $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_{y_0}}/p_0 \times \mathbb{P}_{\vec{U},\alpha+1}/p_1$ for some p_0 and p_1. Again, forcing with $\mathbb{P}_{\vec{U},\alpha+1}$ below p_1 does not add any bounded subsets to ν_1, so there is a direct extension p_1' forcing $\dot{C}$ to be equal to the interpretation of some $\mathbb{P}_{\vec{U}_{y_0}}$-name, $\dot{C}_0$. But then, by the $\nu_0^{+o(y)+1}$-cc of $\mathbb{P}_{\vec{U}_{y_0}}$, there is a club D in μ such that $p_0 \Vdash$ "$D \subseteq \dot{C}$". $\square$

We use Lemma 5.1 to show that the approachability property fails at certain points along our Radin club. Recall that we are assuming that θ is the least weakly inaccessible cardinal above κ.

Lemma 5.2. *Suppose that β is a limit ordinal with $\mathrm{cf}^V(\beta) < \kappa$ and p is a condition such that $a^p(\beta) = y$. Then p forces that $\kappa_y^{+o(y)+1} \notin I[\kappa_y^{+o(y)+1}]$.*

Proof. Assume for a contradiction that (some extension of) p forces $\kappa_y^{+o(y)+1} \in I[\kappa_y^{+o(y)+1}]$. Let $\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle$ be a name for the approachability witness. Recall that κ_y is forced to be singular in the extension by $\mathbb{P}$ and $\kappa_y^{+o(y)+1}$ is forced to be its successor. Therefore, for all limit $\gamma < \kappa_y^{+o(y)+1}$, $p \Vdash$ "$\mathrm{cf}(\gamma) < \kappa_y$". As a result, we can assume that the order type of each $\dot{z}_\gamma$ is forced to be less than κ_y. By the Factorization Lemma (3.6), $\mathbb{P}_{\vec{U}}/p \cong \mathbb{P}_{\vec{U}_y}/p_0 \times \mathbb{P}_{\vec{U},\beta+1}/p_1$ for some p_0 and p_1. By the Prikry property, $\mathbb{P}_{\vec{U},\beta+1}/p_1$ does not add any new subsets to $\kappa_y^{+o(y)+1}$, so we may in fact assume that $\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle$ is a $\mathbb{P}_{\vec{U}_y}$ name forced by p_0 to be a witness to approachability. Set $\nu := \kappa_y$ and, for $x \in \mathcal{P}_\nu(\nu^{+o(y)})$, let $\nu_x := x \cap \nu$ and $\bar{o}(x) = \mathrm{otp}(\{\eta < o(y) \mid \nu^{+\eta} \in x\})$.

Since $y \in X_\beta$, it follows that ν is $\nu^{+o(y)}$-supercompact. Since $\mathrm{cf}(\beta) < \kappa$ and hence $\mathrm{cf}(o(y)) < \nu$, it follows that ν is in fact $\nu^{+o(y)+1}$-supercompact. Let $j : V \to M$ witness that ν is $\nu^{+o(y)+1}$-supercompact. Set $\delta := \sup(j``\nu^{+o(y)+1})$ and $\mu = \mathrm{cf}(\delta) = \nu^{+o(y)+1}$. In M, $j(p_0)$ forces that δ is approachable with respect to $j(\langle \dot{z}_\gamma \mid \gamma < \nu^{+o(y)+1} \rangle)$, so there is a $j(\mathbb{P}_{\vec{U}_y})$-name $\dot{C}$ for a club subset of δ such that, for all $\gamma < \delta$, $j(p_0)$ forces that $\dot{C} \cap \gamma$ is enumerated as $j(\dot{z})_{\gamma'}$ for some $\gamma' < \delta$.

Let $p_0 = (a_0, A_0)$, and let $\eta = \max(\mathrm{dom}(a_0))$ if $a_0 \neq \emptyset$, or let $\eta = -1$ otherwise. Consider the condition $j(p_0) = (j(a_0), j(A_0))$. Note that, for all $(\alpha, x) \in A_0$, j fixes ν_x, and since $\mathrm{otp}(x) < \nu$, we have $j(x) = j``x$ and $j(\bar{o}(x)) = \bar{o}(x)$. For $\alpha \in (j(\eta), j(o(y)))$, we have that, in M, $j(A_0)(\alpha)$ is a measure-one set for a measure on $\mathcal{P}_{j(\nu)}(j(\nu)^{+\alpha})$. Since $j(\nu) > \nu^{+o(y)+1}$, we know that the set $A^*(\alpha) := \{z \in j(A_0)(\alpha) \mid \nu^{+o(y)+1} < (z \cap j(\nu))\}$ is still a measure-one set. For $\alpha \in \mathrm{dom}(j(A_0)) \cap j(\eta)$, set $A^*(\alpha) = j(A_0)(\alpha)$. Then $\hat{p}_0 = (j(a_0), A^*)$ is a direct extension of $j(p_0)$ in $j(\mathbb{P}_{U_y})$.

If there is a limit ordinal in $\mathrm{dom}(j(a_0))$, then let α_0^* be the largest such limit ordinal, and let $\nu_0 = a_0(\alpha_0^*) \cap j(\nu)$. Note that, letting $x = j^{-1}(a_0(\alpha_0^*))$, we have $\nu_0 = \nu_x$ and $\nu_x^{+\bar{o}(x)} < \nu < \mu$. If there is no limit ordinal in $\mathrm{dom}(j(a_0))$, then let $\nu_0 = 0$. If there is a limit ordinal in the interval $(j(\eta), j(o(y)))$, then let α_1^* be the least such ordinal, and let $\hat{p}_1$ be an extension of $\hat{p}_0$ with $\alpha^* \in \mathrm{dom}(a^{\hat{p}_1})$. Notice that, in this case, letting $\nu_1 = a^{\hat{p}_1}(\alpha_1^*) \cap j(\nu)$, our construction of A^* implies that $\mu < \nu_1$. If there is no such limit ordinal, then let $\hat{p}_1 = \hat{p}_0$ and $\nu_1 = j(\nu)$.

We are now in the setting of Lemma 5.1 applied to $j(\mathbb{P}_{\vec{U}_y})$, δ, $\nu_0 < \mu < \nu_1$, and $\hat{p}_1$. We can therefore find a direct extension $\hat{p}_2$ of $\hat{p}_1$ in $j(\mathbb{P}_{\vec{U}})$ and a club $D \subseteq \mu$ in M such that $\hat{p}_2$ forces $D \subseteq \dot{C}$. Let $E = \{\gamma < \nu^{+o(y)+1} \mid j(\gamma) \in D\}$. It is straightforward to see that E is $<\nu$-club in $\nu^{+o(y)+1}$. Let γ^* be the $\nu^{+o(y)}$-th element in an increasing enumeration of E. We can assume that there is an index $\hat{\gamma} < \nu^{+o(y)+1}$ such that $\hat{p}_2$ forces that $\dot{C} \cap j(\gamma^*)$ is enumerated as $j(\dot{z})_{\gamma'}$ for some $\gamma' < j(\hat{\gamma})$.

Recall that $\mathrm{cf}(o(y)) < \nu$. Now if $x \subseteq E \cap \gamma^*$ has order type at most $\mathrm{cf}(o(y))$, then there is a condition $p_x \leq p_0$ in $\mathbb{P}_{\vec{U}_y}$ which forces that $x \subseteq \dot{z}_\gamma$ for some $\gamma < \hat{\gamma}$. To see this, notice that j of this statement is witnessed by $\hat{p}_2$. Also note that the number of such x is $|E \cap \gamma^*|^{\mathrm{cf}(o(y))} = (\nu^{+o(y)})^{\mathrm{cf}(o(y))} = \nu^{+o(y)+1}$.

By the $\nu^{+o(y)+1}$-chain condition of $\mathbb{P}_{\vec{U}_y}$, we can find a condition which forces that, for $\nu^{+o(y)+1}$ many x, p_x is in the generic filter. This is impossible, since each $\dot{z}_\gamma$ is forced to have order type less than ν and hence, in the extension, where ν remains a strong limit cardinal, we have $|\bigcup_{\gamma < \hat{\gamma}} \mathcal{P}(\dot{z}_\gamma)| \leq \nu \cdot \nu^{+o(y)} = \nu^{+o(y)}$. □

Next, we show that weak square holds at points taken from X_β where $\mathrm{cf}(\beta) \geq \kappa$. We first need the definition of a partial square sequence.

Definition 5.3. Let $\lambda < \delta$ be regular cardinals, and let $S \subseteq \delta \cap \mathrm{cof}(\lambda)$. We say that S *carries a partial square sequence* if there is a sequence $\langle C_\gamma \mid \gamma \in S \rangle$ such that:

(1) for all $\gamma \in S$, C_γ is club in γ and $\mathrm{otp}(C_\gamma) = \lambda$;

(2) for all $\gamma < \gamma^*$ from S, if β is a limit point of C_γ and C_{γ^*}, then $C_\gamma \cap \beta = C_{\gamma^*} \cap \beta$.

Next, we need a theorem of Džamonja and Shelah [5].

Theorem 5.4. *Suppose that λ is a regular cardinal and $\mu > \lambda$ is singular. If $\mathrm{cf}([\mu]^{\leq\lambda}, \subseteq) = \mu$, then $\mu^+ \cap \mathrm{cof}(\lambda)$ is the union of μ-many sets, each of which carries a partial square sequence.*

Lemma 5.5. *Suppose that $\beta \in \theta \cap \mathrm{cof}(\geq\kappa)$ and $p \in \mathbb{P}$ is a condition such that $a^p(\beta) = y$ for some y. Then $p \Vdash \square^*_{\kappa_y}$.*

Proof. For each regular $\lambda < \kappa_y$, we have that $(\kappa_y^{+o(y)})^{\leq\lambda} = \kappa_y^{+o(y)}$ using the supercompactness of κ_y, and hence $\mathrm{cf}([\kappa_y^{+o(y)}]^{\leq\lambda}, \subseteq) = \kappa_y^{+o(y)}$. Therefore, by Theorem 5.4, we can write $\kappa_y^{+o(y)+1} \cap \mathrm{cof}(\lambda)$ as the union of $\kappa_y^{+o(y)}$-many sets which have partial squares. We call these partial square sequences $\vec{C}^{\lambda,i}$ for $i < \kappa_y^{+o(y)}$.

Now let G be $\mathbb{P}$-generic over V with $p \in G$. By Lemma 4.5, in $V[G]$, we have that each V-regular cardinal in the interval $[\kappa_y, \kappa_y^{+o(y)}]$ changes its cofinality to ω and $\kappa_y^{+o(y)+1}$ becomes the successor of κ_y. So, in $V[G]$, we can write $\kappa_y^{+o(y)+1}$ as the disjoint union of $(\kappa_y^{+o(y)+1} \cap \mathrm{cof}(<\kappa_y))^V$, which we call T_0, and a set T_1 of ordinals of countable cofinality.

We define a weak square sequence as follows. For $\gamma \in T_0$, we let $\mathcal{C}_\gamma = \{C^{\lambda,i}_{\gamma'} \cap \gamma \mid \lambda < \kappa_y,\ i < \kappa_y^{+o(y)}$ and γ is a limit point of $C^{\lambda,i}_{\gamma'}\}$. For $\gamma \in T_1$, we let $\mathcal{C}_\gamma = \{C\}$ where C is some cofinal ω-sequence in γ.

The coherence is obvious, so we just have to check that each $\mathcal{C}_\gamma$ is not too large. Suppose that there is γ such that $|\mathcal{C}_\gamma| \geq \kappa_y^{+o(y)+1}$. Then, by the pigeonhole principle, we can find two elements C and C' on which the indices λ and i are the same. But then we have that $C = C'$ by the coherence of the partial square sequence with indices λ and i, which is a contradiction. $\square$

We are now ready to complete the proof of Theorem 1.1. Let G be $\mathbb{P}$-generic over V. Our final model will be a further forcing extension of $V[G]$. In $V[G]$, let S be the set of $\nu \in \lim(C_G)$ such that $\square^*_\nu$ holds. By Lemmas 4.5, 5.2, and 5.5, and the fact that $\square^*_\nu$ implies $\nu^+ \in I[\nu^+]$, we know that $S \subseteq \kappa \cap \mathrm{cof}(\omega)$ and is precisely the set of $\nu < \kappa$ such that, for some $p \in G$ and some limit ordinal $\alpha \in a^p$ with $\mathrm{cf}^V(\alpha) \geq \kappa$, we have $\kappa_{a^p(\alpha)} = \nu$. By genericity, S is stationary in κ. However, we claim that S can be made non-stationary in a cofinality-preserving forcing extension of $V[G]$.

Lemma 5.6. *In $V[G]$, suppose that $\delta \in \lim(C_G) \cap \mathrm{cof}(>\omega)$. Then $S \cap \delta$ is non-stationary in δ.*

Proof. Fix $p = (a, A) \in G$ such that, for some $\beta \in (\theta \cap \mathrm{cof}(<\kappa))^V$, $a(\beta) = y$, where $\kappa_y = \delta$. Work in V, letting $\dot{S}$ be a canonical $\mathbb{P}_{\vec{U}}$-name for S. We will find $q \leq p$ such that $q \Vdash$ "$\dot{S} \cap \delta$ is non-stationary."

Let $\mu = \mathrm{cf}(\beta)$. Since $\mu < \kappa$ and $y \in X_\beta$, we have that $\mu < \kappa_y$ and Z^β_y is $<\kappa_y$-closed and unbounded in β. Find $D \subseteq Z^\beta_y$ such that:

- D is club in β;
- $\mathrm{otp}(D) = \mu$;
- $\min(D) > \max(\mathrm{dom}(a) \cap \beta)$.

For each $\alpha \in Z^\beta_y \setminus \min(D)$, let $Y_\alpha = \{x \in \mathcal{P}_\delta(y \cap \kappa^{+\alpha}) \mid D \cap \alpha \subseteq Z^\alpha_x\}$, and note that $Y_\alpha \in u^\beta_\alpha(y)$. Define $q = (a, B) \leq p$ by letting $B(\alpha) = A(\alpha) \cap Y_\alpha$ for $\alpha \in Z^\beta_y \setminus \min(D)$ and $B(\alpha) = A(\alpha)$ for all other values of α. Now $q \Vdash$ "$D \subseteq \dot{C}_G$." Let $\dot{E}$ be a $\mathbb{P}_{\vec{U}}/q$-name for $\{\kappa_x \mid$ for some $r \in G$ and $\alpha \in D$, $r(\alpha) = x\}$. $q \Vdash$ "$\dot{E}$ is club in δ" and, since $\lim(D) \subseteq \mathrm{cof}(<\kappa)$, Lemma 5.2 implies that $q \Vdash$ "$\lim(\dot{E}) \cap \dot{S} = \emptyset$.". □

Recall that a stationary subset T of a regular, uncountable cardinal λ is *fat* if for every club $D \subseteq \lambda$ and every ordinal $\eta < \lambda$, $D \cap T$ contains a closed subset of order type η. We claim that $\kappa \setminus S$ is a fat stationary subset of κ in $V[G]$. To see this, fix a club $D \subseteq \kappa$ and an infinite ordinal $\eta < \kappa$. Since κ is strongly inaccessible, we can find $\delta \in \lim(C_G \cap D)$ with $\mathrm{cf}(\delta) > \eta$. By Lemma 5.6, $S \cap \delta$ is non-stationary in δ, so we can find a club $E \subseteq \delta$ that is disjoint from S. But then $D \cap E$ is a closed subset of $D \cap (\kappa \setminus S)$. Thus, $\kappa \setminus S$ is a fat stationary subset of κ.

Let $\mathbb{Q}$ be the forcing notion whose conditions are closed, bounded subsets of κ disjoint from S, ordered by end-extension. $\mathbb{Q}$ adds a club in κ disjoint from S and, by a result of Abraham and Shelah [2, Theorem 1] and the fact that $\kappa \setminus S$ is fat, $\mathbb{Q}$ is κ-distributive. Thus, if H is $\mathbb{Q}$-generic over $V[G]$, D is the generic club added by Q, and $E = D \cap C_G$, then E witnesses that $V[G * H]$ satisfies the conclusion of Theorem 1.1. Moreover, if we let $N = (V[G * H])_\kappa = V^{V[G*H]}_\kappa$, then N is a model of GB (Gödel-Bernays) with a class club E through its cardinals such that, for every $\nu \in E$, ν is a singular cardinal, SCH fails at ν, and $\square^*_\nu$ fails.

6. Conclusion

In a forthcoming paper of the third author [24], a model is constructed in which $\aleph_{\omega^2}$ is strong limit and weak square fails for all cardinals in the interval $[\aleph_1, \aleph_{\omega^2+2}]$. In particular, it is shown that one can put collapses between the Prikry points of the Gitik-Sharon [10] construction which will make κ into $\aleph_{\omega^2}$ and enforce the failure of weak square below $\aleph_{\omega^2}$.

It is reasonable to believe that this construction could be combined with the forcing from Theorem 1.1, but we are left with the unsatisfactory result that weak square will hold at some successors of singulars in the extension. To make this precise, we formulate a question which seems to capture the limit of a naive combination of the two techniques.

Question 6.1. *Suppose that κ is a singular cardinal of cofinality ω such that $\square^*_\lambda$ fails for all $\lambda \in [\aleph_1, \kappa)$ and $|\{\lambda < \kappa \mid \lambda$ is singular strong limit $\}| = \kappa$. Is there a $\square^*_\kappa$-sequence?*

We also ask two other natural questions.

Question 6.2. *Is there a version of Theorem 1.1 in which the failure of $\square^*_\nu$ is replaced with the tree property at ν^+?*

Question 6.3. *Let $C_G \subset \kappa$ be a generic Radin club added by the poset $\mathbb{P}$ defined in Section 3. Does $\square_{\nu,\omega}$ fail at every ordinal $\nu \in C_G$?*

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