ON THE GROWTH RATE OF CHROMATIC NUMBERS OF FINITE SUBGRAPHS

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Abstract. We prove that, for every function $f : \mathbb{N} \to \mathbb{N}$, there is a graph G with uncountable chromatic number such that, for every $k \geq 3$, every subgraph of G with fewer than $f(k)$ vertices has chromatic number less than k. This answers a question of Erdős, Hajnal, and Szemerédi. We also show that, if $\diamondsuit$ holds, then we can take this graph G to be a Hajnal-Máté graph on ω_1.

1. Introduction

The De Bruijn-Erdős compactness theorem [2] states that, for every natural number d and every graph G, if every finite subgraph of G has chromatic number at most d, then G also has chromatic number at most d. It follows that, for every graph G with infinite chromatic number, we can define a function $f_G : \mathbb{N} \to \mathbb{N}$ by letting $f_G(k)$ be the least natural number m for which there exists a subgraph of G with m vertices and chromatic number at least k. This function is clearly increasing, and the question naturally arises: how quickly can f_G grow?

By a result of Erdős [3], for any function $f : \mathbb{N} \to \mathbb{N}$, there is a graph G with chromatic number $\aleph_0$ such that f_G grows faster than f. For this reason, investigation of this question has been focused on graphs with uncountable chromatic number. Indeed, there are relevant ways in which graphs with uncountable chromatic number behave fundamentally differently from graphs with finite or countable chromatic number. For example, for every natural number k, there are graphs of girth k and arbitrarily large finite chromatic number. By taking disjoint unions of such graphs, it follows that there are graphs of girth k and chromatic number $\aleph_0$. On the other hand, Erdős and Hajnal prove in [4] that every uncountably chromatic graph must contain a copy of every finite bipartite graph; in particular, it contains a cycle of every even length.

In [4], Erdős and Hajnal introduce the shift graphs $G_0(\alpha, n, s)$ for ordinals α, and natural numbers n and s with $1 \leq s < n$. The following facts about shift graphs are proven in [4] and [5]. In what follows, the expression $\log^{(n)}$ denotes the n-times iterated base 2 logarithm and $\exp_n$ denotes the n-times iterated exponential function.

Theorem 1.1. *Suppose that α is an ordinal and n and s are natural numbers with $1 \leq s < n$.*

(1) *If κ is an infinite cardinal and $\alpha \geq (\exp_{n-1}(\kappa))^+$, then $\chi(G_0(\alpha, n, s)) > \kappa$.*

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(2) *There is a constant* $c_n > 0$ *such that, for every natural number* k *and every subgraph* H *of* $G_0(\alpha, n, 1)$ *with* k *vertices, we have* $\chi(H) \leq c_n \log^{(n-1)}(k)$.

As a result, it follows that, for every $n < \omega$, there is an uncountably chromatic graph G such that f_G grows more quickly than $\exp_n$. Erdős, Hajnal, and Szemerédi then stated the following general question, asking if f_G can grow arbitrarily quickly for graphs G with uncountable chromatic number. (The question is mentioned in many places in slightly different forms; see, e.g., [5] and [6]).

Question 1.2. *Is it true that, for every function* $f : \mathbb{N} \to \mathbb{N}$*, there is an uncountably chromatic graph* G *such that* $\lim_{k\to\infty} f(k)/f_G(k) = 0$*?*

In [12], Komjáth and Shelah prove that Question 1.2 consistently has a positive answer. In particular, given a model of ZFC, they construct a forcing extension of that model in which, for every function $f : \mathbb{N} \to \mathbb{N}$, there is a graph G such that $|G| = \chi(G) = \aleph_1$ and, for every natural number $k \geq 3$, $f_G(k) \geq f(k)$.

In this paper, we prove outright that Question 1.2 has a positive answer in ZFC.

Theorem A. *For every function* $f : \mathbb{N} \to \mathbb{N}$*, there is a graph* G *such that* $|G| = 2^{\aleph_1}$*,* $\chi(G) = \aleph_1$ *and, for every natural number* $k \geq 3$*,* $f_G(k) \geq f(k)$*.*

Notice that the cardinality of the graph given by the theorem is strictly greater than $\aleph_1$. It is unclear whether this is necessary in general. However, we also prove that, under the additional assumption of $\diamondsuit$, we can obtain graphs of size $\aleph_1$ and can even require them to be particular types of graphs known as *Hajnal-Máté graphs*.

Theorem B. *Suppose that* $\diamondsuit$ *holds. Then, for every function* $f : \mathbb{N} \to \mathbb{N}$*, there is a Hajnal-Máté graph* G *such that* $|G| = \chi(G) = \aleph_1$ *and, for every natural number* $k \geq 3$*,* $f_G(k) \geq f(k)$*.*

The structure of the paper is as follows. In Section 2, we introduce some of the basic graph-theoretic notions we will be using and prove some basic propositions. In Section 3, we review the set-theoretic technology of club guessing and relate it to some of the notions introduced in Section 2. In Section 4, we prove Theorem A. In Section 5, we prove Theorem B. We conclude by noting some questions that remain open.

1.1. **Notation and conventions.**

The notation and definitions used here are mostly standard. We refer the reader to [8] for any undefined set-theoretic notions. We include 0 in the set $\mathbb{N}$ of natural numbers. Ord denotes the class of ordinals. Any ordinal is thought of as a set whose elements are all strictly smaller ordinals. If n is a natural number, then $[\mathrm{Ord}]^n$ denotes the class of n-element sets of ordinals. Members of $[\mathrm{Ord}]^n$ will typically be presented as $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1}$.

If A is a set of ordinals, then $\mathrm{otp}(A)$ denotes the order type of A. If $i < \mathrm{otp}(A)$, then $A(i)$ denotes the unique element α of A such that $\mathrm{otp}(A \cap \alpha) = i$. If $I \subseteq \mathrm{otp}(A)$, then $A[I]$ denotes $\{A(i) \mid i \in I\}$. The *strong supremum of* A, denoted $\mathrm{ssup}(A)$, is defined to be $\sup\{\alpha + 1 \mid \alpha \in A\}$. It is the least ordinal that is strictly greater than every ordinal in A. If A and B are two sets of ordinals, then we say that B *end-extends* A, written $A \sqsubseteq B$, if $B \cap \mathrm{ssup}(A) = A$. If σ and τ are functions whose domains are sets of ordinals, then we say that τ *end-extends* σ, written $\sigma \sqsubseteq \tau$, if $\mathrm{dom}(\sigma) \sqsubseteq \mathrm{dom}(\tau)$ and $\tau \restriction \mathrm{dom}(\sigma) = \sigma$.

The *cofinality* of an ordinal α is denoted by $\mathrm{cf}(\alpha)$. If β is an ordinal and μ is an infinite regular cardinal, then $S^\beta_\mu = \{\alpha < \beta \mid \mathrm{cf}(\alpha) = \mu\}$. If θ is an infinite cardinal, then $H(\theta)$ denotes the collection of sets hereditarily of cardinality less than θ.

All of our graphs are simple undirected graphs (in particular, they have no loops). If $G = (V, E)$ is a graph and $\vec{v} = \langle v_0, \ldots, v_n\rangle$ is a finite sequence of elements of V, then $\vec{v}$ is a *walk* in G if $\{v_i, v_{i+1}\} \in E$ for every $i < n$. $\vec{v}$ is a *closed walk* if $\vec{v}$ is a walk and, moreover, $v_0 = v_n$. Finally, $\vec{v}$ is a *cycle* if it is a closed walk, $n \geq 3$, and $v_i \neq v_j$ for all $i < j < n$. In such a case, n is the *length* of the cycle. The cycle is an *odd cycle* if n is odd.

If $G = (V, E)$ is a graph, then we will sometimes write $|G|$ to mean $|V|$. The chromatic number of G is denoted by $\chi(G)$.

2. Types and Specker graphs

In this section, we introduce some of the basic notions that we will be using.

Definition 2.1. Suppose that n is a natural number.

(1) A *disjoint type of width* n is a function $t : 2n \to 2$ such that
$$|\{i < 2n \mid t(i) = 0\}| = |\{i < 2n \mid t(i) = 1\}| = n.$$

(2) If a and b are disjoint elements of $[\mathrm{Ord}]^n$, then $\mathrm{tp}(a, b)$ is the unique disjoint type $t : 2n \to 2$ such that, letting $a \cup b = \{\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}\}$, enumerated in increasing order, we have $a = \{\alpha_i \mid i < 2n \text{ and } t(n) = 0\}$ and $b = \{\alpha_i \mid i < 2n \text{ and } t(n) = 1\}$.

A disjoint type of width n will often be represented as a sequence of 0s and 1s of length $2n$. For example, if $a = \{0, 1, 3\}$ and $b = \{2, 4, 5\}$, then $\mathrm{tp}(a, b) = 001011$. If t_0 and t_1 are two disjoint types of widths n_0 and n_1, respectively, then $t_0{}^\frown t_1$ denotes the disjoint type of width $n_0 + n_1$ represented by the concatenation of the sequences of 0s and 1s representing t_0 and t_1. Formally, $t_0{}^\frown t_1$ is the function $t : 2n_0 + 2n_1 \to 2$ defined by letting
$$t(i) = \begin{cases} t_0(i) & \text{if } i < 2n_0 \\ t_1(i - 2n_0) & \text{if } 2n_0 \leq i < 2n_0 + 2n_1. \end{cases}$$
We will be particularly interested in the following family of types.

Definition 2.2. Suppose that s and n are natural numbers with $1 \leq s < n$. Then t^n_s is the disjoint type of width n defined by letting, for all $i < 2n$,
$$t^n_s(i) = \begin{cases} 0 & \text{if } i < s \\ 0 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is even} \\ 1 & \text{if } s \leq i < 2n - s \text{ and } i - s \text{ is odd} \\ 1 & \text{if } i \geq 2n - s. \end{cases}$$

This definition might initially be difficult to parse. Essentially, t^n_s is the type consisting of s copies of 0, followed by $n - s$ copies of 01, followed by s copies of 1. For example, $t^5_2 = 0001010111$.

Definition 2.3. Suppose that n is a natural number, t is a disjoint type of width n, and α is an ordinal. Then $G(\alpha, t) = ([\alpha]^n, E(\alpha, t))$ is the graph with vertex set $[\alpha]^n$ and edge set $E(\alpha, t)$ defined by putting $\{a, b\} \in E(\alpha, t)$ if and only if a and b are disjoint elements of $[\alpha]^n$ and either $\mathrm{tp}(a, b) = t$ or $\mathrm{tp}(b, a) = t$.

Graphs of the form $G(\alpha, t^n_s)$, where s and n are natural numbers with $1 \leq s < n-1$, are sometimes known as *Specker graphs* (see [5]). Erdős and Hajnal proved the following facts about Specker graphs.

Theorem 2.4 ([4, Theorem 7.4])**.** *Suppose that s and n are natural numbers with $1 \leq s \leq n-1$, and suppose that α is an ordinal.*

(1) *If α is an infinite cardinal, then $\chi(G(\alpha, t^n_s)) = |G(\alpha, t^n_s)| = \alpha$.*
(2) *If $n \geq 2s^2 + 1$, then $G(\alpha, t^n_s)$ contains no odd cycles of length $2s + 1$ or shorter.*

Remark 2.5. The proof of the above theorem is not given in [4]. For a proof of item (1), we direct the reader to [1], where, in addition, the authors investigated the functions f_G for G of the form $G(\alpha, t^n_1)$. For a proof of item (2), we direct the reader to [13]. We note that the proof of item (2) in [13] requires $n > 2s^2 + 3s + 1$, which is a slightly worse lower bound than that obtained by Erdős and Hajnal. As we will see in our application, the precise lower bound for n is irrelevant for us; what matters is that such a lower bound exists.

We end this section with some basic propositions about graphs that will be useful for us in the proof of Theorem A.

Proposition 2.6. *Suppose that $G = (V, E)$ is a graph, $k \in \mathbb{N}$, and $E = \bigcup_{j<k} E_j$. For $j < k$, let $G_j = (V, E_j)$. Then $\chi(G) \leq \prod_{j<k} \chi(G_j)$.*

Proof. For each $j < k$, let $c_j : V \to \chi(G_j)$ be a proper coloring of G_j. Let $\mathcal{H}$ be the set of functions $h : k \to \mathrm{Ord}$ such that, for all $j < k$, $h(k) < \chi(G_j)$. Clearly, $|\mathcal{H}| = \prod_{j<k} \chi(G_j)$. Define a coloring $c : V \to \mathcal{H}$ by letting $c(v)(j) = c_j(v)$ for all $v \in V$ and $j < k$. It is easily verified that c is a proper coloring of G. □

Proposition 2.7. *Suppose that $j \geq 1$ is a natural number, G is a graph, and G has a closed walk of length $2j + 1$. Then G has an odd cycle of length $2j + 1$ or shorter.*

Proof. The proof is by induction on j. To take care of the case $j = 1$, simply note that a closed walk of length 3 must be a cycle (since our graphs have no loops). Suppose now that $j > 1$ and $\vec{v} = \{v_0, v_1, \ldots, v_{2j+1}\}$ is a closed walk in G. If $\vec{v}$ is a cycle, then we are done. Otherwise, by rotating the walk if necessary, we may assume that there is i with $0 < i < 2j$ such that $v_0 = v_i$. Then $\{v_0, \ldots, v_i\}$ and $\{v_i, \ldots, v_{2j+1}\}$ are both closed walks in G. One of them must have an odd length (and neither can have a length of 1, since our graph has no loops), so we can appeal to the inductive hypothesis to obtain our desired conclusion. □

Proposition 2.8. *Suppose that $m \in \mathbb{N}$, $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are graphs, H has no odd cycles of length m or shorter, and there is a graph homomorphism from G to H. Then G has no odd cycles of length m or shorter.*

Proof. Let $\varphi : V_G \to V_H$ induce a graph homomorphism from G to H. If $j \geq 1$ and $\{v_0, \ldots, v_{2j+1}\}$ is an odd cycle in G, then $\{\varphi(v_0), \ldots, \varphi(v_{2j+1})\}$ is a closed path in H. By Proposition 2.7, H must then contain an odd cycle of length $2j + 1$ or shorter. The result follows. □

3. Club guessing

In this section, we review the machinery of club guessing, which will be used in the proof of Theorem A, and then prove a key lemma regarding the interaction between club guessing and disjoint types.

Definition 3.1. Suppose that $\kappa < \lambda$ are regular cardinals and $S \subseteq S^\lambda_\kappa$ is stationary. A *club-guessing sequence* on S is a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$ such that

- for every $\alpha \in S$, C_α is a club in α of order type κ;
- for every club D in λ, there are stationarily many $\alpha \in S$ such that $C_\alpha \subseteq D$.

Shelah proved that, if there is at least a one-cardinal gap between κ and λ, then club-guessing sequences always exist.

Theorem 3.2 (Shelah [14])**.** *Suppose that $\kappa < \lambda$ are regular cardinals, $\kappa^+ < \lambda$, and $S \subseteq S^\lambda_\kappa$ is stationary. Then there is a club-guessing sequence on S.*

Proposition 3.3. *Suppose that $\kappa < \lambda$ are regular cardinals, $S \subseteq S^\lambda_\kappa$ is stationary, and $\langle C_\alpha \mid \alpha \in S \rangle$ is a club-guessing sequence. Suppose moreover that $\mu < \lambda$ and $S = \bigcup_{\eta < \mu} S_\eta$. Then there is $\eta < \mu$ such that $\langle C_\alpha \mid \alpha \in S_\eta \rangle$ is a club-guessing sequence.*

Proof. Suppose not. This means that, for every $\eta < \mu$, there is a club $D_\eta \subseteq \lambda$ such that, for all $\alpha \in S_\eta$, $C_\alpha \not\subseteq D_\eta$. Let $D = \bigcap_{\eta < \mu} D_\eta$. Since λ is regular and $\mu < \lambda$, it follows that D is a club in λ. Also, $D \subseteq D_\eta$ for every $\eta < \mu$, so it follows that, for every $\eta < \mu$ and every $\alpha \in S_\eta$, $C_\alpha \not\subseteq D$. But $S = \bigcup_{\eta < \mu} S_\eta$, so, for all $\alpha \in S$, $C_\alpha \not\subseteq D$, contradicting the fact that $\langle C_\alpha \mid \alpha \in S \rangle$ is a club-guessing sequence. □

We now prove that the initial segments of the elements of a club-guessing sequence indexed by a subset of S^λ_ω realize every disjoint type.

Lemma 3.4. *Suppose that λ is an uncountable regular cardinal, $S \subseteq S^\lambda_\omega$ is stationary, and $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ is a club-guessing sequence. Suppose moreover that $n < \omega$ and $t : 2n \to 2$ is a disjoint type of width n. Then there are $\gamma < \delta$, both in S, such that $C_\gamma[n]$ and $C_\delta[n]$ are disjoint and $\mathrm{tp}(C_\gamma[n], C_\delta[n]) = t$.*

Proof. Let θ be a sufficiently large regular cardinal, let $\lhd$ be a fixed well-ordering of $H(\theta)$, and let $\langle M_\xi \mid \xi < \lambda \rangle$ be a continuous $\in$-increasing chain of elementary submodels of $(H(\theta), \in, \lhd)$ such that

- $\vec{C}, S \in M_0$;
- for all $\xi < \lambda$, $|M_\xi| < \lambda$;
- for all $\xi < \lambda$, $\beta_\xi := M_\xi \cap \lambda$ is an ordinal.

Let $D = \{\beta_\xi \mid \xi < \lambda\}$. Then D is a club in λ, so we can fix $\delta \in S$ such that $\beta_\delta = \delta$ and $C_\delta \subseteq D$. For each $i < \omega$, let ξ_i be the unique ordinal ξ such that $\beta_\xi = C_\delta(i)$, and let $N_i = M_{\xi_i}$. Let $N_\omega = M_\delta$. Then $\langle N_i \mid i \leq \omega \rangle$ is an $\in$-increasing chain of elementary submodels, $\delta = N_\omega \cap \lambda$ and, for $i < \omega$, $C_\delta(i) = N_i \cap \lambda$.

Let $\exists^\infty \alpha$ stand for the quantifier "there are unboundedly many $\alpha < \lambda$ such that…". (This is the same as $\forall \eta < \lambda \exists \alpha < \lambda (\eta < \alpha \text{ and} \ldots)$.) Notice that, for all $\eta < C_\delta(n-1)$, there is $\beta > \eta$ for which there is $\gamma \in S$ such that $C_\gamma[n] = C_\gamma[n-1] \cup \{\beta\}$ (namely, $\beta = C_\delta(n-1)$ and $\gamma = \delta$ witness this statement). By elementarity, observing that $C_\delta[n-1] \in N_{n-1}$ it follows that

$$N_{n-1} \models \exists^\infty \beta_{n-1} \, \exists \gamma \in S \, (C_\gamma[n] = C_\delta[n-1] \cup \{\beta_{n-1}\}) \, .$$

By another application of elementarity, $H(\theta)$ satisfies the same statement, so, for all $\eta < C_\delta(n-2)$, there is $\beta > \eta$ for which the following statement holds:

$$\exists^\infty \beta_{n-1}\ \exists \gamma \in S\,(C_\gamma[n] = C_\delta[n-2] \cup \{\beta, \beta_{n-1}\})\,.$$

Namely, $\beta = C_\delta(n-2)$ witnesses this statement. As above, we obtain

$$N_{n-2} \models \exists^\infty \beta_{n-2}\ \exists^\infty \beta_{n-1}\ \exists \gamma \in S\,(C_\gamma[n] = C_\delta[n-2] \cup \{\beta_{n-2}, \beta_{n-1}\})\,.$$

Again, it follows that $H(\theta)$ satisfies the same statement. Continuing in this way, the following statement holds in $H(\theta)$ and hence in every N_i:

$$\exists^\infty \beta_0 \ldots \exists^\infty \beta_{n-1}\ \exists \gamma \in S\,(C_\gamma[n] = \{\beta_0, \ldots, \beta_{n-1}\})\,.$$

Define an auxiliary function $s : n \to n+1$ as follows. Given $i < n$, let $s(i)$ be the number of 1s appearing before the i^{th} (starting at zero) 0 in the sequence representation of t. For example, if $t = 001011$, then $s(0) = s(1) = 0$ and $s(2) = 1$, or, if $t = 100110$, then $s(0) = s(1) = 1$ and $s(2) = 3$. We now recursively choose $\beta_0^* < \beta_1^* < \ldots < \beta_{n-1}^*$, ensuring that, for all $i < n$, $\beta_i^* < C_\delta(s(i))$ and

$$\exists^\infty \beta_{i+1} \ldots \exists^\infty \beta_{n-1}\ \exists \gamma \in S\,(C_\gamma[n] = \{\beta_0^*, \ldots, \beta_i^*\} \cup \{\beta_{i+1}, \ldots, \beta_{n-1}\})\,.$$

We will also arrange so that, if $s(i) > 0$, then $\beta_i^* > C_\delta(s(i)-1)$.

The construction is straightforward. If $i < n$ and we have already chosen $\{\beta_j^* \mid j < i\}$, then, by our recursion hypotheses, we have $\{\beta_j^* \mid j < i\} \in N_{s(i)}$ and

$$N_{s(i)} \models \exists^\infty \beta_i \ldots \exists^\infty \beta_{n-1}\ \exists \gamma \in S\left(C_\gamma[n] = \{\beta_j^* \mid j < i\} \cup \{\beta_i, \ldots, \beta_{n-1}\}\right),$$

so we can choose β_i^* witnessing this statement such that $\max\{\beta_j^* \mid j < i\} < \beta_i^* < C_\gamma(s(i))$ and such that, if $s(i) > 0$, then $\beta_i^* > C_\gamma(s(i)-1)$.

At the end of the construction, we have

$$N_\omega \models \exists \gamma \in S\left(C_\gamma[n] = \{\beta_0^*, \ldots, \beta_{n-1}^*\}\right),$$

so we can choose $\gamma \in S \cap \delta$ witnessing this statement. We constructed $\{\beta_i^* \mid i < n\}$ precisely so that $\text{tp}(\{\beta_i^* \mid i < n\}, C_\delta[n]) = t$, so γ and δ are as desired. □

4. Proof of Theorem A

We are now ready to prove Theorem A; we restate it here for convenience.

Theorem A. *For every function $f : \mathbb{N} \to \mathbb{N}$, there is a graph G such that $|G| = 2^{\aleph_1}$, $\chi(G) = \aleph_1$ and, for every natural number $k \geq 3$, $f_G(k) \geq f(k)$.*

Proof. Fix a function $f : \mathbb{N} \to \mathbb{N}$. We will construct a graph G such that $|G| = 2^{\aleph_1}$, $\chi(G) = \aleph_1$ and, for every $k \in \mathbb{N}$, if H is a subgraph of G with at most $f(k)$ vertices, then $\chi(H) \leq 2^{k+1}$. This clearly suffices for the theorem.

For each $k \in \mathbb{N}$, let s_k be the smallest natural number $s \geq 1$ such that $2s+1 \geq f(k)$, and let $n_k = 2s_k^2 + 1$. Note that, by Clause (2) of Theorem 2.4, the graph $G(\omega_2, t_{s_k}^{n_k})$ has no odd cycles of length $f(k)$ or shorter. Partition $\mathbb{N}$ into adjacent intervals $\{I_0, I_1, \ldots\}$, with $|I_k| = n_k$. More precisely, set $I_0 = \{0, 1, \ldots, n_0 - 1\}$ and, if $k \in \mathbb{N}$ and I_k has been specified, then let $m_{k+1} = \max(I_k) + 1$ and set $I_{k+1} = \{m_{k+1}, m_{k+1}+1, \ldots, m_{k+1} + n_{k+1} - 1\}$.

Let $S = S^{\omega_2}_\omega$. By Theorem 3.2, we can fix a club-guessing sequence $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$. For $\beta \in S$, let Σ_β be the set of all functions $\sigma : S \cap \beta \to \mathbb{N}$, let $\Sigma_{<\beta} = \bigcup_{\alpha \in S \cap \beta} \Sigma_\alpha$, and let $\Sigma = \bigcup_{\alpha \in S} \Sigma_\alpha$. Notice, in particular, that $\Sigma_{\min(S)} = \{\emptyset\}$. For $\sigma \in \Sigma$, let α_σ be the unique $\alpha \in S$ such that $\sigma \in \Sigma_\alpha$. Clearly, $|\Sigma| = \aleph_2 \cdot \aleph_0^{\aleph_1} = 2^{\aleph_1}$.

We will define a graph $G = (\Sigma, E)$ that will be as desired. We will simultaneously be defining an auxiliary function $h : E \to \mathbb{N}$. These objects will satisfy the following requirements.

(1) If $\sigma, \tau \in \Sigma$ and $\{\sigma, \tau\} \in E$, then either $\sigma \sqsubseteq \tau$ or $\tau \sqsubseteq \sigma$.
(2) For all $\sigma \sqsubseteq \tau$ in Σ and all $k \in \mathbb{N}$, if $\{\sigma, \tau\} \in E$ and $h(\{\sigma, \tau\}) = k$, then
  (a) $\tau(\alpha_\sigma) = k$;
  (b) for all $j \leq k$, we have that $C_{\alpha_\sigma}[I_j]$ and $C_{\alpha_\tau}[I_j]$ are disjoint and $\mathrm{tp}(C_{\alpha_\sigma}[I_j], C_{\alpha_\tau}[I_j]) = t^{n_j}_{s_j}$.
(3) For all $\tau \in \Sigma$ and all $k \in \mathbb{N}$, there is at most one $\sigma \sqsubseteq \tau$ such that $\{\sigma, \tau\} \in E$ and $h(\{\sigma, \tau\}) = k$.

To define G, it suffices to specify, for each $\tau \in \Sigma$, the set of $\sigma \sqsubseteq \tau$ such that $\{\sigma, \tau\} \in E$. The value of $h(\{\sigma, \tau\})$ for such σ will then be determined by requirement (2)(a) above as $h(\{\sigma, \tau\}) = \tau(\alpha_\sigma)$. To this end, fix $\beta \in S$ and $\tau \in \Sigma_\beta$. For each $k \in \mathbb{N}$, ask the following question:

> Is there $\alpha \in S \cap \beta$ such that $\tau(\alpha) = k$ and, for all $j \leq k$, $C_\alpha[I_j]$ and $C_\beta[I_j]$ are disjoint and $\mathrm{tp}(C_\alpha[I_j], C_\beta[I_j]) = t^{n_j}_{s_j}$?

If the answer is "yes", then let α^τ_k be the least such α and place $\{\tau \restriction (S \cap \alpha^\tau_k), \tau\}$ in E. If the answer is "no", then there will be no $\sigma \sqsubseteq \tau$ such that $\{\sigma, \tau\} \in E$ and $h(\sigma, \tau) = k$. This completes the construction of G; it is clear that we have satisfied the requirements listed above.

For each $k \in \mathbb{N}$, set $E_k = \{\{\sigma, \tau\} \in E \mid h(\{\sigma, \tau\}) = k\}$ and $E_{\geq k} = \{\{\sigma, \tau\} \in E \mid h(\{\sigma, \tau\}) \geq k\}$.

Claim 4.1. *For all $k \in \mathbb{N}$,*

(1) *(Σ, E_k) is cycle-free;*
(2) *$(\Sigma, E_{\geq k})$ has no odd cycles of length $f(k)$ or shorter.*

Proof. (1) Suppose that $k, \ell \in \mathbb{N}$ and $\langle \sigma_0, \ldots, \sigma_\ell \rangle$ enumerates a cycle in (Σ, E_k) (in particular, we have $\ell \geq 3$). By rotating the elements if necessary, we may assume that $\alpha_{\sigma_0} \geq \alpha_{\sigma_j}$ for all $j \leq \ell$. In particular, we have $\sigma_1, \sigma_{\ell-1} \sqsubseteq \sigma_0$ and $h(\{\sigma_1, \sigma_0\}) = k = h(\{\sigma_{\ell-1}, \sigma_0\})$. But this contradicts requirement (3) in the construction of G, stating that there can be at most one $\sigma \sqsubseteq \sigma_0$ such that $\{\sigma, \sigma_0\} \in E$ and $h(\{\sigma, \sigma_0\}) = k$..

(2) Fix $k \in \mathbb{N}$. By requirement (2)(b) in our construction of G, the function $\varphi : \Sigma \to [\omega_2]^{n_k}$ defined by $\varphi(\sigma) = C_{\alpha_\sigma}[I_k]$ induces a graph homomorphism from $(\Sigma, E_{\geq k})$ to $G(\omega_2, t^{n_k}_{s_k})$. Since $G(\omega_2, t^{n_k}_{s_k})$ contains no odd cycles of length $f(k)$ or shorter, Proposition 2.8 implies that $(\Sigma, E_{\geq k})$ also contains no such odd cycles. □

We can now show that the finite subgraphs of G behave as desired.

Claim 4.2. *Suppose that $k \in \mathbb{N}$ and $H = (V_H, E_H)$ is a subgraph of G with $|V_H| \leq f(k)$. Then $\chi(H) \leq 2^{k+1}$.*

Proof. Note that $E_H = (E_H \cap E_{\geq k}) \cup \bigcup_{j<k} (E_H \cap E_j)$. By Clause (1) of Claim 4.1, for all $j < k$, $H_j := (V_H, E_H \cap E_j)$ is cycle-free and hence has chromatic number at most 2. By Clause (2) of Claim 4.1 and the fact that $|V_H| \leq f(k)$, it follows that $H_{\geq k} := (V_H, E_H \cap E_{\geq k})$ has no odd cycles and thus also has chromatic number at most 2. Proposition 2.6 then implies that

$$\chi(H) \leq \chi(H_{\geq k}) \cdot \prod_{j<k} \chi(H_j) \leq 2 \cdot 2^k = 2^{k+1}.$$

□

To finish the proof of the theorem, it remains to show that $\chi(G) = \aleph_1$. First note that, by construction, for each $\tau \in \Sigma$, there are only countably many $\sigma \sqsubseteq \tau$ with $\{\sigma, \tau\} \in E$. It is thus straightforward to define a proper coloring $c : \Sigma \to \omega_1$ of G by recursion on α_σ. Therefore, $\chi(G) \leq \aleph_1$.

To see that $\chi(G) \geq \aleph_1$, suppose for sake of contradiction that $c : \Sigma \to \mathbb{N}$ is a proper coloring of G. Recursively define a function $\rho : S \to \mathbb{N}$ by letting

$$\rho(\alpha) = c(\rho \restriction (S \cap \alpha))$$

for all $\alpha \in S$. For $k \in \mathbb{N}$, let $S_k = \{\alpha \in S \mid \rho(\alpha) = k\}$. By Proposition 3.3, we can fix $k \in \mathbb{N}$ such that $\{C_\alpha \mid \alpha \in S_k\}$ is a club-guessing sequence. By Lemma 3.4, we can find $\alpha^* < \beta^*$ in S_k such that, for all $j \leq k$, $C_{\alpha^*}[I_j]$ and $C_{\beta^*}[I_j]$ are disjoint and $\mathrm{tp}(C_{\alpha^*}[I_j], C_{\beta^*}[I_j]) = t^{n_j}_{s_j}$. (The disjoint type to which Lemma 3.4 is applied here is the concatenation $t^{n_0}_{s_0} {}^\frown t^{n_1}_{s_1} {}^\frown \ldots {}^\frown t^{n_k}_{s_k}$.)

It follows that, when considering $\rho \restriction (S \cap \beta^*)$ in the construction of G, the answer to the question about k was "yes", since α^* is a witness. There is therefore an $\alpha \in S \cap \beta^*$ such that $\rho(\alpha) = k$ and $\{\rho \restriction (S \cap \alpha), \rho \restriction (S \cap \beta^*)\} \in E$. But, by our definition of ρ, we have

$$c(\rho \restriction (S \cap \alpha)) = \rho(\alpha) = k = \rho(\beta^*) = c(\rho \restriction (S \cap \beta^*)),$$

contradicting the assumption that c is a proper coloring of G. Thus, $\chi(G) = \aleph_1$, so we have completed the proof. □

5. Diamond and Hajnal-Máté graphs

We begin this section by recalling the combinatorial principle $\diamondsuit$.

Definition 5.1. $\diamondsuit$ is the assertion that there is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that

(1) for every $\alpha < \omega_1$, $A_\alpha \subseteq \alpha$;
(2) for every $A \subseteq \omega_1$, there are stationarily many $\alpha < \omega_1$ for which $A \cap \alpha = A_\alpha$.

$\diamondsuit$ is easily seen to be a strengthening of the Continuum Hypothesis, and it holds in many canonical inner models of set theory, such as Gödel's constructible universe L.

Recalling the previous section, notice that, if we had a club-guessing sequence on $S^{\omega_1}_\omega$, then we could modify the proof of Theorem A to obtain graphs of cardinality $2^{\aleph_0}$ witnessing its conclusion (the vertex set of the graphs would be $\bigcup_{\alpha \in S^{\omega_1}_\omega} \Delta_\alpha$, where Δ_α is the set of all functions from S^α_ω to $\mathbb{N}$). Since $\diamondsuit$ implies both the existence of such a club-guessing sequence and the Continuum Hypothesis, $\diamondsuit$ implies the existence of such graphs of cardinality $\aleph_1$. We can do slightly better than this, though, and ensure that these graphs have a particular structure.

Definition 5.2. A graph $G = (\omega_1, E)$ is a *Hajnal-Máté graph* if, for all $\beta < \omega_1$, the set $N^<_G(\beta) := \{\alpha < \beta \mid \{\alpha, \beta\} \in E\}$ is either finite or a set of order type ω converging to β.

In [7], Hajnal and Máté prove that $\diamondsuit^+$, which is a strengthening of $\diamondsuit$ that also holds in L, implies the existence of Hajnal-Máté graphs with uncountable chromatic number. On the other hand, they prove in the same paper that Martin's Axiom implies that every Hajnal-Máté graph has countable chromatic number. Komjáth, in [9] proves that $\diamondsuit$ is sufficient to obtain uncountably chromatic Hajnal-Máté graphs that are, moreover, triangle-free. In [11], Komjáth and Shelah improve this result and show that, if $\diamondsuit$ holds, then for every natural number k, there is an uncountably chromatic Hajnal-Máté graph with no odd cycles of length $2k+1$ or shorter. In [10], Komjáth shows that, if the principle $\diamondsuit^*$, which is a strengthening of $\diamondsuit$, holds, then there is a Hajnal-Máté graph G such that $\chi(G) = \aleph_1$ and, for all $k \geq 1$, we have $f_G(k) \geq 2^{k-1}$.

We now show how to use $\diamondsuit$ to adjust the proof of Theorem A to obtain Hajnal-Máté graphs. For convenience, we restate Theorem B here.

Theorem B. *Suppose that $\diamondsuit$ holds. Then, for every function $f : \mathbb{N} \to \mathbb{N}$, there is a Hajnal-Máté graph G such that $|G| = \chi(G) = \aleph_1$ and, for every natural number $k \geq 3$, $f_G(k) \geq f(k)$.*

Proof. Fix a function $f : \mathbb{N} \to \mathbb{N}$. We will construct a Hajnal-Máté graph G such that $\chi(G) = \aleph_1$ and, for every $k \in \mathbb{N}$, if H is a subgraph of G with at most $f(k)$ vertices, then $\chi(H) \leq 2^{k+1}$. As in the proof of Theorem A, this will suffice to prove the theorem. For notational convenience, the vertex set of our graph will actually be $S := S^{\omega_1}_{\omega}$ rather than ω_1. Our graph can easily be transferred to a Hajnal-Máté graph on ω_1 by, for instance, using the unique order preserving map from $S^{\omega_1}_{\omega}$ to ω_1.

For $k \in \mathbb{N}$, define the natural numbers s_k and n_k and the interval I_k exactly as in the proof of Theorem A. By a straightforward coding argument, $\diamondsuit$ is easily seen to be equivalent to the existence of a sequence $\langle (C_\alpha, f_\alpha) \mid \alpha \in S \rangle$ such that

- for all $\alpha \in S$, C_α is a cofinal subset of α of order type ω and $f_\alpha : S \cap \alpha \to \mathbb{N}$;
- for every club $D \subseteq \omega_1$ and every function $f : S \to \mathbb{N}$, there are stationarily many $\alpha \in S$ such that
  - $C_\alpha \subseteq D$;
  - $f \restriction (S \cap \alpha) = f_\alpha$.

Fix such a sequence. As in the proof of Theorem A, we will define our graph $G = (S, E)$ together with an auxiliary function $h : E \to \mathbb{N}$. These will satisfy the following requirements.

(1) For all $\alpha < \beta$ in S and all $k \in \mathbb{N}$, if $\{\alpha, \beta\} \in E$ and $h(\{\alpha, \beta\}) = k$, then
   (a) $f_\beta(\alpha) = k$;
   (b) for all $j \leq k$, $C_\alpha[I_j]$ and $C_\beta[I_j]$ are disjoint and $\mathrm{tp}(C_\alpha[I_j], C_\beta[I_j]) = t^{n_j}_{s_j}$.

(2) For all $\beta \in S$ and all $k \in \mathbb{N}$, there is at most one $\alpha < \beta$ such that $\{\alpha, \beta\} \in E$ and $h(\{\alpha, \beta\}) = k$.

To define G, it suffices to specify $N^<_G(\beta) := \{\alpha < \beta \mid \{\alpha, \beta\} \in E\}$ for each $\beta \in S$. The value of $h(\{\alpha, \beta\})$ for $\alpha \in N^<_G(\beta)$ will then be forced to be $f_\beta(\alpha)$. To this end, fix $\beta \in S$. For each $k \in \mathbb{N}$, ask the following question:

> Is there $\alpha \in S \cap \beta$ such that $f_\beta(\alpha) = k$ and, for all $j \leq k$, $C_\alpha[I_j]$ and $C_\beta[I_j]$ are disjoint and $\mathrm{tp}(C_\alpha[I_j], C_\beta[I_j]) = t^{n_j}_{s_j}$?

If the answer is "yes", then let α^β_k be the least such α and place $\{\alpha^\beta_k, \beta\}$ in E. If the answer is "no", then there will be no $\alpha < \beta$ such that $\{\alpha, \beta\} \in E$ and $h(\{\alpha, \beta\}) = k$. This completes the construction of G.

Notice that, for all $\beta \in S$ and all natural numbers $k \geq 1$, if α^β_k is defined, then $\alpha^\beta_k > C_\beta(\max[I_{k-1}])$. It follows that, if $N^<_G(\beta)$ is infinite, then it is a set of order type ω converging to β. Therefore, G is indeed a Hajnal-Máté graph. The verification that the finite subgraphs of G behave as desired is exactly as in the proof of Theorem A, so we omit it.

It remains to show that $\chi(G) = \aleph_1$. Suppose for sake of contradiction that $c : S \to \mathbb{N}$ is a proper coloring of G. Let $S^* = \{\alpha \in S \mid c \restriction (S \cap \alpha) = f_\alpha\}$. By the properties of our $\diamondsuit$ sequence, S^* is stationary and, moreover, $\langle C_\alpha \mid \alpha \in S^* \rangle$ is a club-guessing sequence. By Proposition 3.3, we can fix $k \in \mathbb{N}$ such that, letting $S^*_k := \{\alpha \in S^* \mid c(\alpha) = k\}$, $\langle C_\alpha \mid \alpha \in S^*_k \rangle$ is a club-guessing sequence.

By Lemma 3.4, we can find $\alpha^* < \beta^*$ in S^*_k such that, for all $j \leq k$, $C_{\alpha^*}[I_j]$ and $C_{\beta^*}[I_j]$ are disjoint and $\mathrm{tp}(C_{\alpha^*}[I_j], C_{\beta^*}[I_j]) = t^{n_j}_{s_j}$. It follows that, when considering β^* in the construction of G, the answer to the question about k was "yes", since α^* is a witness. There is therefore an $\alpha \in S \cap \beta^*$ such that $f_{\beta^*}(\alpha) = k$ and $\{\alpha, \beta^*\} \in E$. But then we have

$$c(\alpha) = f_{\beta^*}(\alpha) = k = c(\beta^*),$$

contradicting the assumption that c is a proper coloring of G. Thus, $\chi(G) = \aleph_1$, and we have completed the proof. $\square$

6. Questions

We end with a couple of questions that remain open. First, as mentioned above, it is unknown whether the existence of graphs of size $\aleph_1$ such that f_G grows arbitrarily quickly follows simply from ZFC.

Question 6.1. *Is it true in* ZFC *that, for every function* $f : \mathbb{N} \to \mathbb{N}$*, there is a graph* G *such that* $|G| = \chi(G) = \aleph_1$ *and, for all sufficiently large* $k \in \mathbb{N}$*,* $f_G(k) \geq f(k)$*?*

Next, our method only produces graphs of chromatic number precisely $\aleph_1$. It is unclear whether we can get such graphs of arbitrarily large chromatic number.

Question 6.2. *Is it true that, for every function* $f : \mathbb{N} \to \mathbb{N}$ *and every cardinal* κ*, there is a graph* G *such that* $\chi(G) \geq \kappa$ *and, for all sufficiently large* $k \in \mathbb{N}$*,* $f_G(k) \geq f(k)$*?*

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