HIGHER-DIMENSIONAL HIGHLY CONNECTED RAMSEY THEORY

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Abstract. Bergfalk, Hrušak, and Shelah recently introduced a weakening of the classical partition relation for pairs in which the complete monochromatic subgraph of the classical relation is replaced by a highly connected monochromatic subgraph. In subsequent work, we proved that, assuming the consistency of the existence of a weakly compact cardinal, it is consistent that an optimal square-bracket version of this highly connected partition relation holds at the continuum. In this paper, we introduce a higher-dimensional generalization of the highly connected partition relation and prove an analogous consistency result indicating that, if the existence of a weakly compact cardinal is consistent, then it is consistent that an optimal square-bracket version of the higher-dimensional highly connected partition relation holds at the continuum.

1. Introduction

The infinite Ramsey theorem for pairs can be stated in terms of graph colorings as follows: for any edge-coloring of a countably infinite complete graph using finitely many colors, there is an infinite complete subgraph all of whose edges receive the same color. As was realized early on in the study of Ramsey theory, though, if one tries to generalize this statement to uncountable cardinals in the most natural way, one obtains a statement that is only consistent at large cardinals.

In [1], Bergfalk, Hrušák, and Shelah introduce a natural weakening of the partition relation from the classical Ramsey theorem that, unlike the classical version, can consistently hold at accessible cardinals. In this weakening, one requires in the conclusion of the statement only the existence of a monochromatic subgraph that exhibits a high degree of connectedness rather than the complete monochromatic subgraph of the classical relation. To state this more precisely, we recall the following definition. (See the end of this section for an explanation of notation.)

Definition 1.1. Given a graph $G = (X, E)$ and a cardinal κ, we say that G is *κ-connected* if $G \setminus Y$ is connected for every $Y \in [X]^{<\kappa}$. We say that G is *highly connected* if it is $|X|$-connected.

Definition 1.2. (Bergfalk-Hrušák-Shelah [1]) Suppose that μ, ν, and λ are cardinals. The partition relation $\nu \rightarrow_{hc} (\mu)^2_\lambda$ is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there is an $X \in [\nu]^\mu$ and a highly connected subgraph (X, E) of $(\nu, [\nu]^2)$ such that $c \restriction E$ is constant.

The negation of $\nu \rightarrow_{hc} (\mu)^2_\lambda$ is indicated by $\nu \not\rightarrow_{hc} (\mu)^2_\lambda$ (and we will use analogous notation to negate other partition relations without further comment). We

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remark that a complete graph is evidently highly connected, and, if G is a *finite* highly connected graph, then G is complete. On the other hand, it is straightforward to construct a highly connected non-complete graph of any prescribed infinite cardinality. The relation $\nu \rightarrow_{hc} (\mu)^2_\lambda$ is therefore a weakening of the classical relation $\nu \rightarrow (\mu)^2_\lambda$ that coincides with the classical relation for finite μ.

If κ is an uncountable cardinal, then the classical partition relation $\kappa \rightarrow (\kappa)^2_2$ holds if and only if κ is weakly compact. On the other hand, the highly-connected version of this partition relation is consistent at, for instance, $2^{\aleph_1}$.

Theorem 1.3 (Bergfalk-Hrušák-Shelah [1]). *It is consistent, relative to the consistency of a weakly compact cardinal, that*

$$2^{\aleph_1} \rightarrow_{hc} \left(2^{\aleph_1}\right)^2_\lambda$$

holds for all $\lambda < 2^{\aleph_1}$.

It is proven in [1] that $\aleph_1$ cannot be replaced by $\aleph_0$ in the previous theorem. In particular, $2^{\aleph_0} \not\rightarrow_{hc} \left(2^{\aleph_0}\right)^2_{\aleph_0}$ is a theorem of ZFC. However, as the author shows in [4], if one further weakens the highly connected partition relation to require the existence of a large highly connected subgraph on which only *two* colors appear, then one does obtain a nontrivial statement consistent at $2^{\aleph_0}$. More precisely:

Definition 1.4. Suppose that μ, ν, λ, and κ are cardinals. The partition relation $\nu \rightarrow_{hc} [\mu]^2_{\lambda,\kappa}$ (*resp.* $\nu \rightarrow_{hc} [\mu]^2_{\lambda,<\kappa}$) is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there is an $X \in [\nu]^\mu$, a highly connected subgraph (X, E) of $(\nu, [\nu]^2)$, and a set $\Lambda \in [\lambda]^{\leq\kappa}$ (*resp.* $\Lambda \in [\lambda]^{<\kappa}$) such that $c``E \subseteq \Lambda$.

Theorem 1.5 ([4]). *The following two statements are equiconsistent over* ZFC.

(1) *There exists a weakly compact cardinal.*
(2) $2^{\aleph_0} \rightarrow_{hc} [2^{\aleph_0}]^2_{\lambda,2}$ *holds for all* $\lambda < 2^{\aleph_0}$.

Note that all of the definitions and theorems presented thus far about highly connected Ramsey theory are two-dimensional. In this paper, we present a generalization of highly connected Ramsey theory to higher dimensions, prove the analogue of Theorem 1.5 in the context of higher dimensions, and prove that our generalization is sharp in the same sense that Theorem 1.5 is sharp: namely, that the maximum number of colors allowed to appear in the highly connected subgraph cannot be reduced.

Before we embark on this task, though, let us make a preliminary remark about the nature of our generalization to higher dimensions. In two dimensions, i.e., in the context of graphs, the notion of "connectedness" is unambiguous: a graph is connected if and only if every pair of distinct vertices in the graph is connected by a path. However, there are a number of natural and nonequivalent ways to generalize the notion of "connectedness" to the realm of k-uniform hypergraphs when $k > 2$. One approach, and the one we take here, is to generalize the notion of a "path" from the setting of graphs to the setting of k-uniform hypergraphs and assert that a hypergraph is connected if and only if every pair of vertices is connected by such a generalized path. Even here, there are multiple nonequivalent ways of defining the higher-dimensional analogue of "path". We choose the notion of "tight path" (see Section 2 for the definition), but, as the reader can verify, our proofs go through with any of the other common higher-dimensional analogues of "path" extant in the literature.

However, there are other meaningful approaches to connectivity of k-uniform hypergraphs, including some that come from homological algebra (cf. [5], for example). In the context of graphs, homological connectivity and path connectivity are equivalent, but they become distinct notions in higher dimensions. We feel that generalizations of highly connected Ramsey theory using homological connectivity in place of path connectivity may be quite interesting and fruitful, but this lies outside of the scope of this paper. We refrain from making any assertion that the precise generalization considered in this paper represents the "correct" generalization of highly connected Ramsey theory to higher dimensions; we encourage further investigation in other directions, and we plan to carry out some of these investigations in future work.

The structure of the paper is as follows. In Section 2, we introduce our higher-dimensional version of the highly connected partition relation and present some of its basic properties, including a negative ZFC result (Proposition 2.5) that will show that the main result of the paper is sharp. In Section 3, we review some definitions and results about higher-dimensional Δ-systems that will be important for the proof of our main theorem. Finally, in Section 4, we prove the main result of this paper (Theorem 4.1), indicating that, consistently, nontrivial higher-dimensional highly connected partition relations can hold at $2^{\aleph_0}$.

1.1. Notation and conventions.

Our notation is for the most part standard. For all set-theoretic notions that are used here without definition, we refer the reader to [2]. If A is a set or proper class and κ is a cardinal, then $[A]^\kappa = \{B \subseteq A \mid |B| = \kappa\}$. The sets $[A]^{\leq\kappa}$ and $[A]^{<\kappa}$ are then defined in the obvious way. For an integer $k \geq 2$, a *k-uniform hypergraph* is a pair $H = (X, E)$, where X is a set and $E \subseteq [X]^k$. The elements of X are called the *vertices* of H, and the elements of E are called the *edges* of H. A 2-uniform hypergraph is simply called a *graph*. A k-uniform hypergraph $H = (X, E)$ is *complete* if $E = [X]^k$.

If a is a set of ordinals, then we will often conflate a with the sequence enumerating a in increasing fashion. In particular, if $\eta < \mathrm{otp}(a)$, then $a(\eta)$ denotes the unique $\beta \in a$ such that $\mathrm{otp}(a \cap \beta) = \eta$. Similarly, if $\mathbf{r} \subseteq \mathrm{otp}(a)$, then $a[\mathbf{r}]$ denotes the set $\{a(\eta) \mid \eta \in \mathbf{r}\}$. We denote the class of ordinals by On. If β is an ordinal and X is a set, then ${}^\beta X$ denotes the set of functions from β to X, and ${}^{<\beta}X$ denotes $\bigcup_{\alpha<\beta} {}^\alpha X$.

2. Highly tight-path-connected hypergraphs

In this section, we introduce our higher-dimensional versions of the partition relations considered above. To do so, we will need to generalize the notion of *highly connected* from the realm of graphs to the realm of k-uniform hypergraphs for an arbitrary $2 \leq k < \omega$. As mentioned in the introduction, we choose to focus in this paper on notions of connectivity arising from generalized paths.

Definition 2.1. Suppose that $2 \leq t \leq k < \omega$, $0 < \ell < \omega$, and $H = (X, E)$ is a k-uniform hypergraph. A *t-tight Berge path of length ℓ* in H is a pair $P = (\vec{x}, \vec{e})$ consisting of an injective sequence $\vec{x} = \langle x_i \mid i < \ell + t - 1\rangle$ of elements of X and an injective sequence $\vec{e} = \langle e_i \mid i < \ell\rangle$ of elements of E such that, for all $i < \ell$, we have $\{x_{i+s} \mid s < t\} \subseteq e_i$. In such a situation, we say that P is a t-tight Berge path *from* x_0 *to* $x_{\ell+t-2}$ or that x_0 and $x_{\ell+t-2}$ are *connected* by P.

If $t = k$, then a t-tight Berge path in H is simply called a *tight path*, and if $t = 2$, then a t-tight Berge path is simply called a *Berge path*.

Note that, for graphs, tight paths and Berge paths are the same and coincide with what are typically simply called "paths". In this paper, for concreteness and ease of notation, we will be working only with tight paths. However, the reader may verify that all of our proofs, with only superficial modifications, can be made to prove the analogous results formulated in terms of t-tight Berge paths for any t with $2 \leq t \leq k$, where k is the uniformity of the hypergraph under consideration.

Also note that, if $P = (\vec{x}, \vec{e})$ is a tight path in a k-uniform hypergraph, then $\vec{e}$ is uniquely determined by $\vec{x}$: if $\vec{x} = \langle x_i \mid i < \ell + k - 1 \rangle$ and $\vec{e} = \langle e_i \mid i < \ell \rangle$, then $e_i = \{x_{i+s} \mid s < k\}$ for all $i < \ell$. Therefore, in practice we identify a tight path with its sequence of vertices and will slightly abuse notation by writing, for instance, $P = \vec{x}$.

With the higher-dimensional analogue of *path* fixed, our higher dimensional generalization of *highly connected* now follows naturally.

Definition 2.2. Suppose that $2 \leq k < \omega$ and $H = (X, E)$ is a k-uniform hypergraph.

(1) H is *tight-path-connected* if, for all distinct $x, y \in H$, there is a tight path in H from x to y.
(2) For a cardinal κ, we say that H is *κ-tight-path-connected* if, for all $Y \in [X]^{<\kappa}$, the k-uniform hypergraph
$$H \setminus Y := (X \setminus Y,\ E \cap [X \setminus Y]^k)$$
is tight-path-connected.
(3) H is *highly tight-path-connected* if it is $(|X| - k + 1)$-tight-path-connected.

We note that, if H is an *infinite* k-uniform hypergraph, then Definition 2.2(3) is equivalent to the more natural statement asserting that H is $|X|$-tight-path-connected, which is also more directly analogous to Definition 1.1. The reason for the slightly less natural formulation is the fact that, if $k > 2$, then a finite k-uniform hypergraph $H = (X, E)$ with at least two vertices cannot be $(|X| - k + 2)$-tight-path-connected, since if one removes $(|X| - k + 1)$-many vertices from X, then one is left with $(k - 1)$-many vertices and zero edges. (Definition 2.2 can easily be seen to be equivalent to Definition 1.1 in the case of $k = 2$.) We also note that, as was true in the 2-dimensional case, for every $k \geq 2$, a finite k-uniform hypergraph is highly tight-path-connected if and only if it is complete, but it is straightforward to construct highly tight-path-connected k-uniform hypergraphs of any prescribed infinite cardinality that are not complete.

With these definitions in place, we can generalize the highly connected partition relations of [1] and introduce natural square-bracket variants. In what follows, we use the subscript *htc* to stand for "highly tight-path-connected".

Definition 2.3. Suppose that μ, ν, λ, and κ are cardinals and $2 \leq k < \omega$.

(1) The partition relation $\nu \to_{htc} (\mu)^k_\lambda$ is the assertion that, for every coloring $c : [\nu]^k \to \lambda$, there is an $X \in [\nu]^\mu$ and a highly tight-path-connected k-uniform hypergraph (X, E) such that $c \restriction E$ is constant.

(2) The partition relation $\nu \to_{htc} [\mu]^k_{\lambda,\kappa}$ (*resp.* $\nu \to_{htc} [\mu]^k_{\lambda,<\kappa}$) is the assertion that, for every coloring $c : [\nu]^k \to \lambda$, there is an $X \in [\nu]^\mu$, a highly tight-path-connected k-uniform hypergraph (X, E), and a set $\Lambda \in [\lambda]^{\leq\kappa}$ (*resp.* $\Lambda \in [\lambda]^{<\kappa}$) such that $c``E \subseteq \Lambda$.

We remind the reader that the classical partition relation $\nu \to (\mu)^k_\lambda$ is the strengthening of Definition 2.3 in which (X, E) is required to be *complete* (and similarly for $\nu \to [\mu]^k_{\lambda,\kappa}$), and that the infinite Ramsey theorem asserts that $\aleph_0 \to (\aleph_0)^k_m$ for all finite k and m. In light of the following proposition (which was observed in [1] in the case $k = 2$) and the fact that finite k-uniform hypergraphs are highly tight-path-connected if and only if they are complete, the relation $\nu \to_{htc} (\mu)^k_\lambda$ can be seen as a natural generalization of the finite instances of the classical Ramsey partition relation to uncountable cardinalities. The proposition also stands in stark contrast to the fact that, if μ is an uncountable cardinal that is not weakly compact, then $\mu \not\to (\mu)^2_2$.

Proposition 2.4. *Suppose that μ is an infinite cardinal and $1 < k, m < \omega$. Then $\mu \to_{htc} (\mu)^k_m$.*

Proof. Fix a coloring $c : [\mu]^k \to m$ and a uniform ultrafilter U over μ. Recall that, for $1 \leq n < \omega$, we can define ultrafilters U_n over $[\mu]^n$ by induction on n as follows. First, if $X \subseteq [\mu]^1$, then $X \in U_1$ if and only if $\bigcup X \in U$. Now suppose that $1 \leq n < \omega$ and we have defined U_n. For every set $X \subseteq [\mu]^{n+1}$ and every ordinal $\alpha < \mu$, let

$$X_\alpha := \{b \in [\mu \setminus (\alpha + 1)]^n \mid \{\alpha\} \cup b \in X\},$$

and put X in U_{n+1} if and only if

$$\{\alpha \in X \mid X_\alpha \in U_n\} \in U.$$

It is easy to check that, for all $1 \leq n < \omega$, U_n is an ultrafilter over $[\mu]^n$ and, if $X \in U$, then $[X]^n \in U_n$.

Now, for each $\alpha < \mu$, since m is finite, we can fix a color $i_\alpha < m$ and a set $Z(\alpha) \in U_{k-1}$ such that, for all $b \in Z(\alpha)$, we have $\min(b) > \alpha$ and $c(\{\alpha\} \cup b) = i_\alpha$. We can then fix a set $X \in U$ and a color $i < m$ such that $i_\alpha = i$ for all $\alpha \in X$.

We claim that $(X, [X]^k \cap c^{-1}\{i\})$ is a highly tight-path-connected hypergraph. To this end, fix $Y \in [X]^{<\mu}$ and distinct $\alpha, \beta \in X \setminus Y$. We will find $b \in [X \setminus Y]^{k-1}$ such that $\min(b) > \max\{\alpha, \beta\}$ and $c(\{\alpha\} \cup b) = c(\{\beta\} \cup b) = i$. This will finish the proof since then the sequence $\langle\alpha\rangle^\frown b^\frown\langle\beta\rangle$ will be a tight path of length 2 from α to β in $(X \setminus Y, [X \setminus Y]^k \cap c^{-1}\{i\})$.

Notice that, since U is a uniform ultrafilter, we have $X \setminus Y \in U$, and hence $[X \setminus Y]^{k-1} \in U_{k-1}$. We can therefore fix

$$b \in [X \setminus Y]^{k-1} \cap Z(\alpha) \cap Z(\beta).$$

Then b is as desired. □

In light of this proposition, we will focus on colorings using infinitely many colors. We now prove a proposition that places a limit on the positive instances of the partition relation that can consistently hold.

Proposition 2.5. *Suppose that λ and μ are infinite cardinals such that $2^{<\lambda} = \lambda$ and $\mu \leq 2^\lambda$, and let $k \geq 2$ be a natural number. Then*

$$\mu \not\to_{htc} [\mu]^k_{\lambda,<k}.$$

Proof. For $F \in [{}^{\lambda}2]^k$, let $\Delta(F)$ denote the least $j < \lambda$ such that

$$|\{f \restriction (j+1) \mid f \in F\}| = k.$$

In other words, it is the least j such that all distinct $f, g \in F$ differ at or before place j.

We will define a coloring $c : [\mu]^k \to ({}^{<\lambda}2)^k$ witnessing $\mu \not\to_{htc} [\mu]^k_{\lambda,<k}$. Note that, by our assumption, we have $|({}^{<\lambda}2)^k| = \lambda$, so c takes the correct number of colors. For simplicity, when discussing sets $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\} \in [\mu]^k$, we will implicitly assume that $\alpha_0 < \alpha_1 < \ldots < \alpha_{k-1}$.

Let $\{f_\alpha \mid \alpha < \mu\}$ be such that f_α and f_β are distinct elements of ${}^{\lambda}2$ for all $\alpha < \beta < \mu$. Given $A = \{\alpha_i \mid i < k\} \in [\mu]^k$, let $\Delta(A) = \Delta(\{f_{\alpha_i} \mid i < k\})$, and let

$$c(A) = \langle f_{\alpha_i} \restriction (\Delta(A) + 1) \mid i < k \rangle .$$

To see that c witnesses $\mu \not\to_{htc} [\mu]^k_{\lambda,<k}$, fix a set $X \in [\mu]^\mu$ and a highly tight-path-connected k-uniform hypergraph (X, E), and let $\Lambda = c``E$. We will show that $|\Lambda| \geq k$.

Claim 2.6. *For every $\alpha \in X$, there is $e \in E$ such that $\alpha = \min(e)$.*

Proof. Fix $\alpha \in X$. Since (X, E) is highly connected, it follows that the hypergraph $(X \setminus \alpha,\ E \cap [X \setminus \alpha]^k)$ is connected. Therefore, there is an element e of $E \cap [X \setminus \alpha]^k$ containing α; since $e \subseteq (X \setminus \alpha)$, it follows that α must be the least element of e. □

Elements of Λ are of the form $\vec{\sigma} = \langle \sigma_i \mid i < k \rangle$ where there is a fixed $\delta(\vec{\sigma}) < \lambda$ such that $\sigma_i \in {}^{\delta(\vec{\sigma})+1}2$ for all $i < k$. Fix $\vec{\tau} = \langle \tau_i \mid i < k \rangle$ in Λ such that $\delta(\vec{\tau})$ is minimal, and fix an element $b = \{\beta_0, \ldots, \beta_{k-1}\}$ of E such that $c(b) = \vec{\tau}$. By Claim 2.6, for each $i < k$, we can fix $e_i \in E$ such that $\beta_i = \min(e_i)$.

Claim 2.7. *For all $i < i' < k$, we have $c(e_i) \neq c(e_{i'})$.*

Proof. Fix $i < i' < k$. Since $c(b) = \vec{\tau}$, we have

$$f_{\beta_i} \restriction (\delta(\vec{\tau}) + 1) = \tau_i \neq \tau_{i'} = f_{\beta_{i'}} \restriction (\delta(\vec{\tau}) + 1).$$

Let $c(e_i) = \vec{\sigma}^i = \langle \sigma^i_0, \sigma^i_1, \ldots, \sigma^i_{k-1} \rangle$ and $c(e_{i'}) = \vec{\sigma}^{i'} = \langle \sigma^{i'}_0, \sigma^{i'}_1, \ldots \sigma^{i'}_{k-1} \rangle$. Then $f_{\beta_i} \restriction (\delta(\vec{\sigma}^i) + 1) = \sigma^i_0$, and, by choice of $\vec{\tau}$, we have $\delta(\vec{\tau}) \leq \delta(\vec{\sigma}^i)$. Similarly, $f_{\beta_{i'}} \restriction (\delta(\vec{\sigma}^{i'}) + 1) = \sigma^{i'}_0$ and $\delta(\vec{\tau}) \leq \delta(\vec{\sigma}^{i'})$. In particular, τ_i is an initial segment of σ^i_0 and $\tau_{i'}$ is an initial segment of $\sigma^{i'}_0$. Since $\tau_i \neq \tau_{i'}$, it follows that $c(e_i) \neq c(e_{i'})$. □

By Claim 2.7, $\{c(e_i) \mid i < k\}$ is a subset of Λ of size k, and thus $|\Lambda| \geq k$, as desired. □

Corollary 2.8. *For all $2 \leq k < \omega$, $2^{\aleph_0} \not\to_{htc} [2^{\aleph_0}]^k_{\aleph_0,<k}$.* □

In Section 4, we will show that this corollary is sharp by proving the consistency of $2^{\aleph_0} \to_{htc} [2^{\aleph_0}]^k_{\aleph_0,k}$, modulo the consistency of the existence of a weakly compact cardinal.

3. Higher-dimensional Δ-systems

We now introduce one of the primary technical tools that we will use in the proof of our main theorem: higher-dimensional Δ-systems. The use of higher-dimensional analogues of Δ-systems dates back to work of Todorcevic [8] and Shelah [6], [7] in the 1980s; the precise definitions and results in this section come primarily from [3]. We first need the following crucial definitions.

Definition 3.1. (Aligned sets) Suppose that a and b are sets of ordinals.

(1) We say that a and b are *aligned* if $\mathrm{otp}(a) = \mathrm{otp}(b)$ and, for all $\gamma \in a \cap b$, we have $\mathrm{otp}(a \cap \gamma) = \mathrm{otp}(b \cap \gamma)$. In other words, if γ is a common element of a and b, then it occupies the same relative position in both a and b.

(2) If a and b are aligned then we let $\mathbf{r}(a, b) := \{i < \mathrm{otp}(a) \mid a(i) = b(i)\}$. Notice that, in this case, $a \cap b = a[\mathbf{r}(a, b)] = b[\mathbf{r}(a, b)]$.

Definition 3.2. (Uniform k-dimensional Δ-systems) Suppose that X is a set of ordinals, $1 \leq k < \omega$, and, for all $b \in [X]^k$, u_b is a set of ordinals. We call $\langle u_b \mid b \in [X]^k\rangle$ a *uniform k-dimensional Δ-system* if there is an ordinal ρ and, for each $\mathbf{m} \subseteq k$, a set $\mathbf{r}_{\mathbf{m}} \subseteq \rho$ satisfying the following statements.

(1) $\mathrm{otp}(u_b) = \rho$ for all $b \in [X]^k$.

(2) For all $a, b \in [X]^k$ and $\mathbf{m} \subseteq k$, if a and b are aligned with $\mathbf{r}(a, b) = \mathbf{m}$, then u_a and u_b are aligned with $\mathbf{r}(u_a, u_b) = \mathbf{r}_{\mathbf{m}}$.

(3) For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq k$, we have $\mathbf{r}_{\mathbf{m}_0 \cap \mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.

The result about existence of higher-dimensional Δ-systems that will be relevant for us is the following.

Theorem 3.3 ([3])**.** *Suppose that $1 \leq k < \omega$ and that $\kappa < \mu$ are infinite cardinals, with μ being weakly compact. Suppose also that $g : [\mu]^k \to \kappa$ is a function and $\langle u_a \mid a \in [\mu]^k\rangle$ is a sequence consisting of elements of $[\mathrm{On}]^{<\kappa}$. Then there is $X \in [\mu]^\mu$ such that $g \restriction [X]^k$ is constant and $\langle u_b \mid b \in [X]^k\rangle$ is a uniform k-dimensional Δ-system.*

These higher-dimensional Δ-systems are especially useful in forcing arguments involving higher-dimensional combinatorial statements. We end this section with some lemmas indicating their utility, followed by a general discussion of a process that will be put into use in the proof of our main theorem in Section 4.

For the rest of this paper, μ will denote a fixed weakly compact cardinal. The particular forcing notion we will be working with is $\mathbb{P} := \mathrm{Add}(\omega, \mu)$, the forcing to add μ-many Cohen reals. We think of conditions in $\mathbb{P}$ as being finite partial functions from μ to 2, ordered by reverse inclusion. For each $p \in \mathbb{P}$, let $u_p := \mathrm{dom}(p)$, and let $\bar{p}$ denote the "collapse" of p. More formally, $\bar{p} : \mathrm{otp}(u_p) \to 2$ is defined by letting $\bar{p}(j) = p(u_p(j))$ for every $j < \mathrm{otp}(u_p)$. Note that $\{\bar{p} \mid p \in \mathbb{P}\} = {}^{<\omega}2$.

Proposition 3.4. *Suppose that $p, q \in \mathbb{P}$ are such that*

- *u_p and u_q are aligned; and*
- *$\bar{p} = \bar{q}$.*

Then p and q are compatible.

Proof. It suffices to show that, for all $\alpha \in u_p \cap u_q$, we have $p(\alpha) = q(\alpha)$. To this end, fix such an α. Since u_p and u_q are aligned, there is a single $\eta < \mathrm{otp}(u_p)$ such that $\alpha = u_p(\eta) = u_q(\eta)$. But then $p(\alpha) = \bar{p}(\eta) = \bar{q}(\eta) = q(\alpha)$, as desired. □

Lemma 3.5. *Fix $1 \leq k < \omega$, and suppose that $\langle p_a \mid a \in [\mu]^k\rangle$ is a sequence of conditions in $\mathbb{P}$. Then there is $X \in [\mu]^\mu$ such that, for all $a, b \in [X]^k$, if a and b are aligned, then p_a and p_b are compatible.*

Proof. Define a function $g : [\mu]^k \to {}^{<\omega}2$ by letting $g(a) = \bar{p}_a$ for all $a \in [\mu]^k$. Also, for notational ease, let u_a denote u_{p_a} for all $a \in [\mu]^k$. Now apply Lemma 3.3 to g

and the sequence $\langle u_a \mid a \in [\mu]^k \rangle$ to obtain an $X \in [\mu]^\mu$ such that $g \restriction [X]^k$ is constant and $\langle u_a \mid a \in [X]^k \rangle$ is a uniform n-dimensional Δ-system. Now, if $a, b \in [X]^k$ are aligned, then u_a and u_b are aligned and $\bar{p}_a = \bar{p}_b$, so, by Proposition 3.4, p_a and p_b are compatible. Thus, X satisfies the conclusion of the lemma. $\square$

Discussion 3.6. Suppose that $1 \leq k < \omega$, X is a set of ordinals whose order type is a limit of limit ordinals, and $\langle u_b \mid b \in [X]^k \rangle$ is a uniform k-dimensional Δ-system, as witnessed by an ordinal ρ and subsets $\langle \mathbf{r_m} \mid \mathbf{m} \subseteq k \rangle$. Assume also that ρ is finite, i.e., each u_b is a finite set of ordinals. Let

$$X^* := \{\alpha \in X \mid \mathrm{otp}(X \cap \alpha) \text{ is a limit ordinal}\}.$$

Now, for any $a \in [X^*]^{<k}$ and any $\mathbf{m} \in [n]^{|a|}$, define a set $u_a^{\mathbf{m}}$ as follows. First, fix a set $b \in [X]^k$ such that $b[\mathbf{m}] = a$. Such a b exists because $a \subseteq X^*$ and hence there are infinitely many elements of X between any two elements of a. Then let $u_a^{\mathbf{m}} = u_b[\mathbf{r_m}]$.

We claim that this definition of $u_a^{\mathbf{m}}$ is independent of our choice of b. To see this, suppose that $b_0, b_1 \in [X]^k$ are such that $b_0[\mathbf{m}] = a = b_1[\mathbf{m}]$. We must show that $u_{b_0}[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}]$. Fix $c \in [X]^k$ such that

- $c[\mathbf{m}] = a$;
- $c \setminus a$ is disjoint from both b_0 and b_1.

Again, it is possible to find such a c because of the fact that there are infinitely many elements of X between any two elements of a. Note that, for each $i < 2$, b_i and c are aligned and $\mathbf{r}(b_i, c) = \mathbf{m}$. It follows that u_{b_i} and u_c are aligned and $\mathbf{r}(u_{b_i}, u_c) = \mathbf{r_m}$. In particular, we have

$$u_{b_0}[\mathbf{r_m}] = u_c[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}],$$

as desired.

We now argue that, for any $m < k$ and any $a \in [X^*]^m$, the sequence

$$\langle u^{m+1}_{a \cup \{\alpha\}} \mid \alpha \in X^* \setminus (\max(a) + 1) \rangle$$

is a (1-dimensional) Δ-system with root u_a^m. (Here, in expressions like u_a^m, the m in the superscript should be interpreted as the subset of k consisting of all natural numbers less than m.) To see this, fix $\alpha < \beta$, both in $X^* \setminus (\max(a) + 1)$. Let $b_\alpha, b_\beta \in [X^* \setminus (\beta + 1)]^{k-m-1}$ be disjoint sets, and let $c_\alpha := a \cup \{\alpha\} \cup b_\alpha$ and $c_\beta := a \cup \{\beta\} \cup b_\beta$. Then c_α and c_β are aligned and $\mathbf{r}(c_\alpha, c_\beta) = m$, and hence $u_{c_\alpha} \cap u_{c_\beta} = u_{c_\alpha}[\mathbf{r}_m] = u_{c_\beta}[\mathbf{r}_m] = u_a^m$. Moreover, we have $u^{m+1}_{a \cup \{\alpha\}} = u_{c_\alpha}[\mathbf{r}_{m+1}]$ and $u^{m+1}_{a \cup \{\beta\}} = u_{c_\beta}[\mathbf{r}_{m+1}]$. Putting this all together, we obtain $u^{m+1}_{a \cup \{\alpha\}} \cap u^{m+1}_{a \cup \{\beta\}} = u_a^m$, as desired.

It also follows that, if $m < k$, $a, a' \in [X^*]^m$ are aligned, and $\mathbf{m} \in [k]^m$, then $u_a^{\mathbf{m}}$ and $u_{a'}^{\mathbf{m}}$ are aligned. To see this, fix such m, a, a', and $\mathbf{m}$. Let $\mathbf{t} = \mathbf{r}(a, a') \subseteq m$. Now fix $b, b' \in [X]^k$ such that

- $b[\mathbf{m}] = a$ and $b'[\mathbf{m}] = a'$;
- $b \cap b' = a \cap a'$.

Then b and b' are aligned with $\mathbf{r}(b, b') = \mathbf{m}[\mathbf{t}] =: \mathbf{m}^*$, and hence u_b and $u_{b'}$ are aligned, with $\mathbf{r}(u_b, u_{b'}) = \mathbf{r}_{\mathbf{m}^*}$. Also, we have $u_a^{\mathbf{m}} = u_b[\mathbf{r_m}]$ and $u_{a'}^{\mathbf{m}} = u_{b'}[\mathbf{r_m}]$. Thus, if $\gamma \in u_a^{\mathbf{m}} \cap u_{a'}^{\mathbf{m}}$, then there must be $\eta, \eta' \in \mathbf{r_m}$ such that $\gamma = u_b(\eta) = u_{b'}(\eta')$. Since u_b and $u_{b'}$ are aligned, we must have $\eta = \eta'$. Therefore, we have

$$\mathrm{otp}(u_a^{\mathbf{m}} \cap \gamma) = \mathrm{otp}(\mathbf{r_m} \cap \eta) = \mathrm{otp}(u_{a'}^{\mathbf{m}} \cap \gamma).$$

It follows that $u^{\mathbf{m}}_a$ and $u^{\mathbf{m}}_{a'}$ are aligned.

Suppose in addition that for each $b \in [X]^k$ we have a condition $p_b \in \mathbb{P}$ such that $u_{p_b} = u_b$. Suppose also that there is a fixed $\bar{p}^*$ such that $\bar{p}_b = \bar{p}^*$ for all $b \in [X]^k$. Now, for any $a \in [X^*]^{<k}$ and any $\mathbf{m} \in [n]^{|a|}$, define a condition $p^{\mathbf{m}}_a \in \mathbb{P}$ as follows. First, fix any $b \in [X]^k$ such that $b[\mathbf{m}] = a$. Then, let $p^{\mathbf{m}}_a := p_b \restriction u^{\mathbf{m}}_a$. By the previous discussion, it follows that this definition of $p^{\mathbf{m}}_a$ is independent of our choice of b. Also, if $m < k$, $\mathbf{m} \in [k]^m$, and $a, a' \in [X^*]^m$ are aligned, then it follows from our previous discussion and Proposition 3.4 that

- $\bar{p}^{\mathbf{m}}_a = \bar{p}^{\mathbf{m}}_{a'} = \overline{\bar{p}^* \restriction \mathbf{r}_{\mathbf{m}}}$; and
- $p^{\mathbf{m}}_a$ and $p^{\mathbf{m}}_b$ are compatible.

4. A positive consistency result at $2^{\aleph_0}$

We are now ready to prove our main theorem, a positive consistency result indicating that Corollary 2.8 is sharp. For concreteness, we are stating and proving the theorem in a very specific form, but the same proof, *mutatis mutandis*, will yield the modified statement in which $\mathrm{Add}(\omega, \mu)$ is replaced by $\mathrm{Add}(\theta, \mu)$ and both instances of $2^{\aleph_0}$ in the partition relation are replaced by 2^θ, where θ is an arbitrary regular infinite cardinal less than μ.

Theorem 4.1. *Suppose that μ is a weakly compact cardinal, and let $\mathbb{P} = \mathrm{Add}(\omega, \mu)$ be the forcing to add μ-many Cohen reals. Then, in $V^{\mathbb{P}}$, we have*

$$2^{\aleph_0} \rightarrow_{htc} [2^{\aleph_0}]^k_{\lambda,k}$$

for every $k \geq 2$ and every $\lambda < \mu$.

Proof. In $V^{\mathbb{P}}$, we have $2^{\aleph_0} = \mu$. Fix an integer $k \geq 2$, a cardinal $\lambda < \mu$, a condition $p \in \mathbb{P}$, and a $\mathbb{P}$-name $\dot{c}$ forced by p to be a function from $[\mu]^k$ to λ. We will find a condition $q^* \leq p$, a set $\Lambda \in [\lambda]^{\leq k}$, and a $\mathbb{P}$-name $\dot{A}$ such that q^* forces both that $\dot{A}$ is an unbounded subset of μ and that $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$ is highly tight-path-connected.

We begin by recursively constructing a $\subseteq$-decreasing sequence $\langle X_j \mid j < k \rangle$ of unbounded subsets of μ exhibiting increasing amounts of uniformity with respect to $\dot{c}$. More precisely, we will construct sequences $\langle X_j \mid j < k \rangle$, $\langle q_{b,j} \mid j < k,\ b \in [X_j]^k \rangle$, $\langle \bar{q}_{*,j} \mid j < k \rangle$, and $\langle i_{*,j} \mid j < k \rangle$ that satisfy certain requirements, which we enumerate below. We first need a bit of notation. Throughout the rest of the paper, we will use the convention that, if X is an unbounded subset of μ, then X^* denotes the set

$$\{\alpha \in X \mid \mathrm{otp}(X \cap \alpha) \text{ is a limit ordinal}\}.$$

We are now ready to state the properties we will require of our sequences.

(1) For all $j < k$, $X_j \subseteq \mu$ is unbounded and, if $j < k - 1$, then $X_{j+1} \subseteq X^*_j$.
(2) For all $j < k$, we have $\bar{q}_{*,j} \in {}^{<\omega}2$ and $i_{*,j} < \lambda$.
(3) For all $j < k$ and $b \in [X_j]^k$, we have $q_{b,j} \in \mathbb{P}$, $q_{b,j} \leq p$, $q_{b,j} \Vdash$ "$\dot{c}(b) = i_{*,j}$", and $\bar{q}_{b,j} = \bar{q}_{*,j}$.
(4) For all $j < k$ and $b \in [X_j]^k$, let $u_{b,j} := \mathrm{dom}(q_{b,j})$. Then, for all $j < k$, $\langle u_{b,j} \mid b \in [X]^k \rangle$ is a uniform k-dimensional Δ-system, as witnessed by a natural number ρ_j and a sequence $\langle \mathbf{r}_{\mathbf{m},j} \mid \mathbf{m} \subseteq k \rangle$ of subsets of ρ_j.

There is one additional requirement, which we need some further notation to describe. For all $j < k$, $a \in [X^*_j]^{<k}$, and $\mathbf{m} \in [k]^{|a|}$, we will define $u^{\mathbf{m}}_{a,j}$ and $q^{\mathbf{m}}_{a,j}$ by choosing a set $b \in [X_j]^k$ such that $b[\mathbf{m}] = a$ and letting $u^{\mathbf{m}}_{a,j} = u_{b,j}[\mathbf{r}_{\mathbf{m},j}]$ and

$q^{\mathbf{m}}_{a,j} = q_{b,j} \restriction u^{\mathbf{m}}_{a,j}$. Then Discussion 3.6, together with the fact that X_j will satisfy requirements (1)–(4) above, will imply that this definition is independent of our choice of b.

We will be interested in particular values of $\mathbf{m} \in [k]^{<k}$, which we therefore give names. For $0 < k' < k$, let $\mathbf{m}^+_{k'} := k'$ and $\mathbf{m}^-_{k'} := k \setminus (k - k')$. Given $b \in [\mu]^k$, let $b[+k'] := b[\mathbf{m}^+_{k'}]$ and $b[-k'] := b[\mathbf{m}^-_{k'}]$. In other words, $b[+k']$ consists of the first k'-many elements of b and $b[-k']$ consists of the last k'-many elements of b. (Note that $b[+k']$ is the same as $b[k']$; we will often include the + for symmetry, though.) We can now state the final requirement of our construction.

(5) For all $j < k-1$ and all $b \in [X_{j+1}]^k$, the conditions $q^{\mathbf{m}^-_1}_{b[+1],j}$ and $q^{\mathbf{m}^+_{k-1}}_{b[-(k-1)],j}$ are compatible with one another, and $q_{b,j+1}$ extends both of them.

We begin with the first step of our construction. For all $b \in [\mu]^k$, fix a condition $q_{b,0} \leq p$ that decides the value of $\dot{c}(b)$, say as $i_{b,0} < \lambda$. Set $u_{b,0} := \mathrm{dom}(q_{b,0})$. Since μ is weakly compact, we can use Theorem 3.3 to find an unbounded set $X_0 \subseteq \mu$, a fixed collapsed condition $\bar{q}_{*,0} \in {}^{<\omega}2$, and a color $i_{*,0} < \lambda$ such that

- for all $b \in [X_0]^k$, we have $\bar{q}_{b,0} = \bar{q}_{*,0}$ and $i_{b,0} = i_{*,0}$;
- $\langle u_{b,0} \mid b \in [X_0]^k \rangle$ is a uniform k-dimensional Δ-system.

Let $\rho_0 := \mathrm{dom}(\bar{q})$ and $\langle \mathbf{r}_{\mathbf{m},0} \mid \mathbf{m} \subseteq k \rangle$ witness that $\langle u_{b,0} \mid b \in [X_0]^k \rangle$ is a uniform k-dimensional Δ-system. This evidently satisfies requirements (1)–(4) of our construction, and requirement (5) is irrelevant for the first step.

Now suppose that $j < k-1$ and we have constructed $\langle X_{j'} \mid j' \leq j \rangle$, $\langle q_{b,j'} \mid j' \leq j,\ b \in [X_{j'}]^k \rangle$, $\langle \bar{q}_{*,j'} \mid j' \leq j \rangle$, and $\langle i_{*,j'} \mid j' \leq j \rangle$ satisfying all relevant instances of requirements (1)–(5). We will construct X_{j+1}, $\langle q_{b,j+1} \mid b \in [X_{j+1}]^k \rangle$, $\bar{q}_{*,j+1}$, and $i_{*,j+1}$.

Recall that X^*_j is the set of $\alpha \in X_j$ such that $\mathrm{otp}(X_j \cap \alpha)$ is a limit ordinal. Recall also that, for each $a \in [X^*_j]^{<k}$ and $\mathbf{m} \in [k]^{|a|}$, we have defined a set $u^{\mathbf{m}}_{a,j}$ and a condition $q^{\mathbf{m}}_{a,j}$. Begin by defining a function $f_j : [X^*_j]^k \to 2$ by setting $f_j(b) = 0$ if $q^{\mathbf{m}^-_1}_{b[+1],j}$ and $q^{\mathbf{m}^+_{k-1}}_{b[-(k-1)],j}$ are compatible, and $f_j(b) = 1$ otherwise. Since μ is weakly compact and X^*_j is unbounded in μ, we can find an unbounded set $Y_j \subseteq X^*_j$ such that $f_j \restriction [Y_j]^k$ is constant.

Claim 4.2. *$f_j(b) = 0$ for all $b \in [Y_j]^k$.*

Proof. Since $f_j \restriction [Y_j]^k$ is constant, it suffices to find a single $b \in [Y_j]^k$ for which $f_j(b) = 0$. Begin by fixing an arbitrary $\alpha \in Y_j$. Let $\langle a_\ell \mid \ell < \omega \rangle$ be a sequence of pairwise disjoint elements of $[Y_j \setminus (\alpha+1)]^{k-1}$. Since $\langle u_{b,j} \mid b \in [X_j]^k \rangle$ is a uniform k-dimensional Δ-system, we know that $\langle u^{\mathbf{m}^+_{k-1}}_{a_\ell,j} \mid \ell < \omega \rangle$ is a (1-dimensional) Δ-system with root $u^{\emptyset}_{\emptyset,j}$. Therefore, we can fix $\ell < \omega$ such that $u^{\mathbf{m}^+_{k-1}}_{a_\ell,j} \cap u^{\mathbf{m}^-_1}_{\{\alpha\},j} \subseteq u^{\emptyset}_{\emptyset,j}$. Notice that $q^{\mathbf{m}^+_{k-1}}_{a_\ell,j} \restriction u^{\emptyset}_{\emptyset,j} = q^{\emptyset}_{\emptyset,j} = q^{\mathbf{m}^-_1}_{\{\alpha\},j} \restriction u^{\emptyset}_{\emptyset,j}$, so it follows that $q^{\mathbf{m}^-_1}_{\{\alpha\},j}$ and $q^{\mathbf{m}^+_{k-1}}_{a_\ell,j}$ are compatible. Let $b = \{\alpha\} \cup a_\ell$. Then $b \in [Y_j]^k$, $b[+1] = \{\alpha\}$, and $b[-(k-1)] = a_\ell$, so $f_j(b) = 0$, as desired. $\square$

We now proceed much as we did in the first step. For each $b \in [Y_j]^k$, fix a condition $q_{b,j+1} \leq q^{\mathbf{m}^-_1}_{b[+1],j} \cup q^{\mathbf{m}^+_{k-1}}_{b[-(k-1)],j}$ that decides the value of $\dot{c}(b)$, say as $i_{b,j+1}$.

Set $u_{b,j+1} := \mathrm{dom}(q_{b,j+1})$. Since μ is weakly compact, we can find an unbounded set $X_{j+1} \subseteq Y_j$, a collapsed condition $\bar{q}_{*,j+1} \in {}^{<\omega}2$, and a color $i_{*,j+1} < \lambda$ such that

- for all $b \in [X_{j+1}]^k$, we have $\bar{q}_{b,j+1} = \bar{q}_{*,j+1}$ and $i_{b,j+1} = i_{*,j+1}$;
- $\langle u_{b,j+1} \mid b \in [X_{j+1}]^k \rangle$ is a uniform k-dimensional Δ-system.

It is now easily verified that our construction continues to satisfy requirements (1)–(5). The only requirement that is not immediately evident from our construction is the fact that $q_{b,j+1} \leq p$ for all $b \in [X_{j+1}]^k$. But note that, by our assumption, $q_{b,j} \leq p$ for all such b. Since $q^{\emptyset}_{\emptyset,j} = \bigcap_{b \in [H_{j+1}]^k} q_{b,j}$, it follows that $q^{\emptyset}_{\emptyset,j} \leq p$. But we chose $q_{b,j+1}$ to extend $q^{\mathbf{m}^-_1}_{b[+1],j}$, which itself extends $q^{\emptyset}_{\emptyset,j}$, so $q_{b,j+1} \leq p$, as desired.

At the end of the construction, we thin out X^*_{k-1} one final time, using the following claim.

Claim 4.3. *There is an unbounded $X \subseteq X^*_{k-1}$ such that, for all $j < k-1$ and all $b \in [X]^k$, the conditions $q_{b,k-j-2}$ and $q^{\{j\}}_{b[-1],k-1}$ are compatible.*

Proof. For each $j < k-1$, define a function $g_j : [X^*_{k-1}]^k \to 2$ by letting $g_j(b) = 0$ if $q_{b,k-j-2}$ and $q^{\{j\}}_{b[-1],k-1}$ are compatible, and letting $g_j(b) = 1$ otherwise. Since μ is weakly compact and X^*_{k-1} is unbounded in μ, we can find an unbounded $X \subseteq X^*_{k-1}$ such that, for all $j < k-1$, $g_j \restriction [X]^k$ is constant. We claim that, for all $j < k-1$ and all $b \in [X]^k$, we have $g_j(b) = 0$. To show this, fix $j < k-1$. Since $g_j \restriction [X]^k$ is constant, it suffices to find a single $b \in [X]^k$ such that $g_j(b) = 0$.

Fix $a_0 = \{\alpha_{j'} \mid j' < k\} \in [X]^k$ such that $X \cap \alpha_0$ is infinite. We now recursively define an increasing sequence of ordinals $\langle \alpha_{k+j'} \mid j' < j \rangle$ in $X \setminus (\alpha_{k-1}+1)$. As we do so, for each $\epsilon \leq j$, we will set $a_\epsilon := \{\alpha_{\epsilon+j'} \mid j' < k\}$, and we will arrange so that the conditions $\{q_{a_\epsilon,k-\epsilon-1} \mid \epsilon \leq j\}$ are pairwise compatible.

Suppose that $j^* < j$ and we have defined $\{\alpha_{j'} \mid j' < k+j^*\}$, and therefore have also defined $\{a_\epsilon \mid \epsilon \leq j^*\}$. Let $s := \bigcup_{\epsilon \leq j^*} q_{a_\epsilon,k-\epsilon-1}$, which we have arranged to be a condition in $\mathbb{P}$. Notice that we have also already specified

$$a_{j^*+1}[+(k-1)] = \{\alpha_{j^*+1+j'} \mid j' < k-1\} = a_{j^*}[-(k-1)].$$

Therefore, by requirement (5) of the construction at the beginning of this proof, we know that

$$q_{a_{j^*},k-j^*-1} \leq q^{\mathbf{m}^+_{k-1}}_{a_{j^*}[-(k-1)],k-j^*-2} = q^{\mathbf{m}^+_{k-1}}_{a_{j^*+1}[+(k-1)],k-j^*-2}.$$

Let $\hat{a} := a_{j^*+1}[+(k-1)]$. Now, by Discussion 3.6 the sequence

$$\langle u_{\hat{a}\cup\{\alpha\},k-j^*-2} \mid \alpha \in X \setminus (\max(\hat{a})+1) \rangle$$

is a (1-dimensional) Δ-system with root $u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j^*-2}$. We can therefore find an $\alpha \in X \setminus (\max(\hat{a})+1)$ such that $u_{\hat{a}\cup\{\alpha\},k-j^*-2} \cap \mathrm{dom}(s) \subseteq u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j^*-2}$. Let α_{k+j^*} be such an α, which also completes the definition of a_{j^*+1}. Note that

$$q_{a_{j^*+1},k-j^*-2} \restriction u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j^*-2} = q^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j^*-2},$$

and recall that $s \leq q^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j^*-2}$. It follows that s and $q_{a_{j^*+1},k-j^*-2}$ are compatible, as desired, and we can continue with the construction.

After defining $\langle \alpha_{j'} \mid j' < k+j \rangle$, we are ready to construct b such that $g_j(b) = 0$. We first set $b(k-1) := a_j(0)$, noting that this is the same as $a_0(j)$. Let

$s = q_{a_0,k-1} \cup q_{a_j,k-j-1}$. By the previous paragraphs, s is indeed a condition in $\mathbb{P}$. Recall that we chose a_0 so that $X \cap a_0(0)$ is infinite. We can therefore fix a sequence $\langle d_\ell \mid \ell < \omega \rangle$ of pairwise disjoint elements of $[X \cap a_0(0)]^{k-1}$. For each $\ell < \omega$, let $b_\ell := d_\ell \cup \{b(k-1)\}$. Since $\langle u_{b',k-j-2} \mid b' \in [X]^k \rangle$ is a uniform k-dimensional Δ-system, the sequence $\langle u_{b_\ell,k-j-2} \mid \ell < \omega \rangle$ is a Δ-system with root $u^{\mathbf{m}_1^-}_{\{b(k-1)\},k-j-2}$, so we can fix $\ell < \omega$ such that $u_{b_\ell,k-j-2} \cap \mathrm{dom}(s) \subseteq u^{\mathbf{m}_1^-}_{\{b(k-1)\},k-j-2}$. Set $b := b_\ell$. Notice that $b[-1] = \{b(k-1)\} = a_j[+1]$. By requirement (5) from the construction at the start of the proof of this theorem, we know that

$$s \leq q_{a_j,k-j-1} \leq q^{\mathbf{m}_1^-}_{a_j[+1],k-j-2} = q^{\mathbf{m}_1^-}_{b[-1],k-j-2}.$$

Since $q_{b,k-j-2} \restriction u^{\mathbf{m}_1^-}_{b[-1],k-j-2} = q^{\mathbf{m}_1^-}_{b[-1],k-j-2}$, it follows that $q_{b,k-j-2}$ and s are compatible. Since $s \leq q_{a_0,k-1} \leq q^{\{j\}}_{\{a_0(j)\},k-1} = q^{\{j\}}_{b[-1],k-1}$, it follows that $q_{b,k-j-2}$ and $q^{\{j\}}_{b[-1],k-1}$ are compatible, so $g_j(b) = 0$, as desired. □

Fix an unbounded $X \subseteq X^*_{k-1}$ as given by Claim 4.3, and, as usual, let X^* be the set of $\alpha \in X$ such that $\mathrm{otp}(X \cap \alpha)$ is a limit ordinal. Let $q^* = q^{\emptyset}_{\emptyset,k-1}$. By the paragraph preceding Claim 4.3, we have $q^* \leq p$. Let $\Lambda = \{i_{*,j} \mid j < k\}$, and let $\dot{A}$ be a $\mathbb{P}$-name for the set of $\alpha \in X^*$ such that there exists $j < k$ for which $q^{\{j\}}_{\{\alpha\},k-1} \in \dot{G}$, where $\dot{G}$ is the canonical name for the $\mathbb{P}$-generic filter. We claim that q^* forces that $\dot{A}$ is unbounded in μ and that $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$ is highly tight-path-connected.

We first show that q^* forces that $\dot{A}$ is unbounded, in fact establishing the following stronger claim.

Claim 4.4. *For all $j < k$, q^* forces that $\dot{A}_j := \left\{\alpha \in X^* \;\middle|\; q^{\{j\}}_{\{\alpha\},k-1} \in \dot{G}\right\}$ is unbounded in μ.*

Proof. Fix $j < k$ and $\gamma < \mu$, and fix an arbitrary condition $r \leq q^*$. We will find an $\alpha \in X^* \setminus \gamma$ such that $q^{\{j\}}_{\{\alpha\},k-1}$ is compatible with r. This clearly suffices to prove the claim.

By Discussion 3.6, we know that $\langle u^{\{j\}}_{\{\alpha\},k-1} \mid \alpha \in X^* \setminus \gamma \rangle$ is a (1-dimensional) Δ-system with root $u^{\emptyset}_{\emptyset,k-1}$. Therefore, we can find $\alpha \in X^* \setminus \gamma$ such that $u^{\{j\}}_{\{\alpha\},k-1} \cap \mathrm{dom}(r) \subseteq u^{\emptyset}_{\emptyset,k-1}$. Moreover, we know that $q^{\{j\}}_{\{\alpha\},k-1} \restriction u^{\emptyset}_{\emptyset,k-1} = q^{\emptyset}_{\emptyset,k-1} = q^*$, and $r \leq q^*$. It follows that r and $q^{\{j\}}_{\{\alpha\},k-1}$ are compatible, as desired. □

We conclude by proving that q^* forces that $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$ is highly tight-path-connected. It suffices to prove that q^* forces the following statement: for all distinct $\alpha, \beta \in \dot{A}$ and all $\gamma < \mu$, there is a tight path $\vec{x}$ in $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$ from α to β such that every vertex in $\vec{x}$ except for α and β is greater than γ.

To this end, fix a condition $r \leq q^*$, an ordinal $\gamma < \mu$, and distinct ordinals $\alpha, \beta \in X^*$ such that r forces α and β to be in $\dot{A}$. We will find a condition $s \leq r$ forcing the existence of a tight path from α to β as in the previous paragraph.

Since r forces α and β to be in $\dot{A}$, we can assume without loss of generality that there are $j_\alpha, j_\beta < k$ such that $r \leq q^{\{j_\alpha\}}_{\{\alpha\},k-1} \cup q^{\{j_\beta\}}_{\{\beta\},k-1}$. By switching α and β if necessary, we can also assume that $j_\alpha \leq j_\beta$.

We begin by constructing $a, b \in [X]^k$ such that

- $\alpha = a(j_\alpha)$ and $\beta = b(j_\beta)$;
- $a[j_\beta + 1] \cap b[j_\beta + 1] = \emptyset$;
- for all $j \in (j_\alpha, k)$, we have $a(j) \in X^* \setminus (\gamma + 1)$;
- for all $j \in (j_\beta, k)$, we have $a(j) = b(j)$;
- $q_{a,k-1}$ and $q_{b,k-1}$ are both compatible with r.

Let us first construct $a[j_\alpha + 1]$. Let $\{d_\ell \mid \ell < \omega\}$ be a family of pairwise disjoint elements of $[X \cap \alpha]^{j_\alpha}$. Since $\langle u_{b',k-1} \mid b' \in [X_{k-1}]^k \rangle$ is a uniform k-dimensional Δ-system, we know that $\langle u^{j_\alpha+1}_{d_\ell \cup \{\alpha\},k-1} \mid \ell < \omega \rangle$ is a (1-dimensional) Δ-system, with root $u^{\{j_\alpha\}}_{\{\alpha\},k-1}$. We can therefore fix an $\ell < \omega$ such that $u^{j_\alpha+1}_{d_\ell \cup \{\alpha\},k-1} \cap \mathrm{dom}(r) \subseteq u^{\{j_\alpha\}}_{\{\alpha\},k-1}$. Let $a[j_\alpha] = d_\ell$ and $a(j_\alpha) = \alpha$. Note that $q^{j_\alpha+1}_{a[j_\alpha+1],k-1} \restriction u^{\{j_\alpha\}}_{\{\alpha\},k-1} = q^{\{j_\alpha\}}_{\{\alpha\},k-1}$ and $r \leq q^{\{j_\alpha\}}_{\{\alpha\},k-1}$; it follows that $q^{j_\alpha+1}_{a[j_\alpha+1],k-1}$ and r are compatible.

By exactly the same reasoning, we can define $b[j_\beta] \in [X \cap \beta]^{j_\beta}$ such that $b[j_\beta] \cap a[j_\alpha] = \emptyset$ and, letting $b(j_\beta) = \beta$, the conditions $q^{j_\beta+1}_{b[j_\beta+1],k-1}$ and r are compatible.

We next construct $\{a(j) \mid j_\alpha < j \leq j_\beta\}$. Let $\{d'_\ell \mid \ell < \omega\}$ be a family of pairwise disjoint elements of $[X^* \setminus (\max\{\alpha, \beta, \gamma\} + 1)]^{j_\beta - j_\alpha}$. For each $\ell < \omega$, let $e_\ell = a[j_\alpha + 1] \cup d'_\ell$. Since $\langle u_{b',k-1} \mid b' \in [X_{k-1}]^k \rangle$ is a uniform k-dimensional Δ-system, it follows that $\langle u^{j_\beta+1}_{e_\ell,k-1} \mid \ell < \omega \rangle$ is a (1-dimensional) Δ-system with root $u^{j_\alpha+1}_{a[j_\alpha+1],k-1}$. We can therefore fix an $\ell < \omega$ such that $u^{j_\beta+1}_{e_\ell,k-1} \cap \mathrm{dom}(r) \subseteq u^{j_\alpha+1}_{a[j_\alpha+1],k-1}$. Let $\{a(j) \mid j_\alpha < j \leq j_\beta\} = d'_\ell$. Note that $q^{j_\beta+1}_{a[j_\beta+1],k-1} \restriction u^{j_\alpha+1}_{a[j_\alpha+1],k-1} = q^{j_\alpha+1}_{a[j_\alpha+1],k-1}$, which we previously showed is compatible with r. It follows that $q^{j_\beta+1}_{a[j_\beta+1],k-1}$ is also compatible with r.

We finally define $\{a(j) \mid j_\beta < j < k\} = \{b(j) \mid j_\beta < j < k\}$. Let $\{d''_\ell \mid \ell < \omega\}$ be a family of pairwise disjoint elements of $[X^* \setminus (\max\{a(j_\beta), \beta, \gamma\} + 1)]^{k - j_\beta - 1}$. For each $\ell < \omega$, let $e^a_\ell = a[j_\beta + 1] \cup d''_\ell$ and $e^b_\ell = b[j_\beta + 1] \cup d''_\ell$. As above, $\langle u_{e^a_\ell,k-1} \mid \ell < \omega \rangle$ and $\langle u_{e^b_\ell,k-1} \mid \ell < \omega \rangle$ are both Δ-systems, with roots $u^{j_\beta+1}_{a[j_\beta+1],k-1}$ and $u^{j_\beta+1}_{b[j_\beta+1],k-1}$, respectively. We can therefore fix an $\ell < \omega$ such that $u_{e^a_\ell,k-1} \cap \mathrm{dom}(r) \subseteq u^{j_\beta+1}_{a[j_\beta+1],k-1}$ and $u_{e^b_\ell,k-1} \cap \mathrm{dom}(r) \subseteq u^{j_\beta+1}_{b[j_\beta+1],k-1}$. Let $\{a(j) \mid j_\beta < j < k\} = \{b(j) \mid j_\beta < j < k\} = d''_\ell$, which finishes the construction of a and b. Note that $q_{a,k-1} \restriction u^{j_\beta+1}_{a[j_\beta+1],k-1} = q^{j_\beta+1}_{a[j_\beta+1],k-1}$, which we previously showed is compatible with r. It follows that $q_{a,k-1}$ is compatible with r. The same argument shows that $q_{b,k-1}$ is compatible with r. Moreover, since a and b are aligned and $\bar{q}_{a,k-1} = \bar{q}_{b,k-1} = \bar{q}_{*,k-1}$, Proposition 3.4 implies that $q_{a,k-1}$ and $q_{b,k-1}$ are compatible.

Let $s_0 = r \cup q_{a,k-1} \cup q_{b,k-1}$. By the previous paragraphs, s_0 is a condition in $\mathbb{P}$. We next recursively construct an increasing sequence $\langle \delta_j \mid j < j_\beta \rangle$ of ordinals in X^* such that $\delta_0 > \max\{a(k-1), b(k-1), \gamma\}$. We will then define a_j and b_j for $j \leq j_\beta$ by letting $a_j := a[-(k-j)] \cup \{\delta_{j'} \mid j' < j\}$ and $b_j := b[-(k-j)] \cup \{\delta_{j'} \mid j' < j\}$. Note that $a_0 = a$ and $b_0 = b$. We will construct $\langle \delta_j \mid j < j_\beta \rangle$ in such a way that all of the following conditions are pairwise compatible with one another:

- s_0;
- $q_{a_j,k-j-1}$ for all $j \leq j_\beta$;
- $q_{b_j,k-j-1}$ for all $j \leq j_\beta$;
- $q^{\{j\}}_{\{\delta_j\},k-1}$ for all $j < j_\beta$.

Suppose that $j < j_\beta$ is fixed and we have constructed $\{\delta_{j'} \mid j' < j\}$. Notice that this determines the value of $a_{j'}$ and $b_{j'}$ for all $j' \leq j$. Suppose we have carried out the construction in such a way that

$$s_j := r \cup \bigcup_{j' \leq j} (q_{a_{j'},k-j'-1} \cup q_{b_{j'},k-j'-1}) \cup \bigcup_{j'<j} q^{\{j'\}}_{\{\delta_{j'}\},k-1}$$

is a condition in $\mathbb{P}$. We now show how to find an ordinal δ_j as desired.

By our construction at the beginning of this proof, we know that $q_{a_j,k-j-1} \leq q^{\mathbf{m}^+_{k-1}}_{a_j[-(k-1)],k-j-2}$ and $q_{b_j,k-j-1} \leq q^{\mathbf{m}^+_{k-1}}_{b_j[-(k-1)],k-j-2}$. For ease of notation, let $\hat{a} := a_j[-(k-1)]$ and $\hat{b} := b_j[-(k-1)]$. If $j > 0$, let $\delta^* = \delta_{j-1}$, and if $j = 0$, let $\delta^* = \max\{a(k-1), b(k-1), \gamma\}$. Since $\langle u_{b',k-j-2} \mid b' \in [X^*]^k\rangle$ is a uniform k-dimensional Δ-system, we know that $\langle u_{\hat{a}\cup\{\delta\},k-j-2} \mid \delta \in X^* \setminus (\delta^*+1)\rangle$ and $\langle u_{\hat{b}\cup\{\delta\},k-j-2} \mid \delta \in X^* \setminus (\delta^*+1)\rangle$ are Δ-systems with roots $u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j-2}$ and $u^{\mathbf{m}^+_{k-1}}_{\hat{b},k-j-2}$, respectively. Also, since $\langle u_{b',k-1} \mid b' \in [X^*]^k\rangle$ is a uniform k-dimensional Δ-system, we know from Discussion 3.6 that $\langle u^{\{j\}}_{\{\delta\},k-1} \mid \delta \in X^* \setminus (\delta^*+1)\rangle$ is a Δ-system with root $u^{\emptyset}_{\emptyset,k-1}$. We can therefore fix $\delta_j \in X^* \setminus (\delta^*+1)$ such that $u_{\hat{a}\cup\{\delta_j\},k-j-2} \cap \mathrm{dom}(s_j) \subseteq u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j-2}$, $u_{\hat{b}\cup\{\delta_j\},k-j-2} \cap \mathrm{dom}(s_j) \subseteq u^{\mathbf{m}^+_{k-1}}_{\hat{b},k-j-2}$, and $u^{\{j\}}_{\{\delta_j\},k-1} \cap \mathrm{dom}(s_j) \subseteq u^{\emptyset}_{\emptyset,k-1}$.

Note that we have $a_{j+1} = \hat{a} \cup \{\delta_j\}$ and $b_{j+1} = \hat{b} \cup \{\delta_j\}$. Note also that $q_{a_{j+1},k-j-2} \restriction u^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j-2} = q^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j-2}$, and by the beginning of the previous paragraph we have $s_j \leq q_{a_j,k-j-1} \leq q^{\mathbf{m}^+_{k-1}}_{\hat{a},k-j-2}$. Therefore, $q_{a_{j+1},k-j-2}$ is compatible with s_j. By the same argument, $q_{b_{j+1},k-j-2}$ is compatible with s_j. Also, we have $q^{\{j\}}_{\{\delta_j\},k-1} \restriction u^{\emptyset}_{\emptyset,k-1} = q^{\emptyset}_{\emptyset,k-1}$, and $s_j \leq q^{\emptyset}_{\emptyset,k-1}$, so $q^{\{j\}}_{\{\delta_j\},k-1}$ and s_j are compatible. Moreover, since a_{j+1} and b_{j+1} are aligned and $\bar{q}_{a_{j+1},k-j-2} = \bar{q}_{b_{j+1},k-j-2} = \bar{q}_{*,k-j-2}$, we know that $q_{a_{j+1},k-j-2}$ and $q_{b_{j+1},k-j-2}$ are compatible. Finally, since we chose X to satisfy Claim 4.3, and since $\{\delta_j\} = a_{j+1}[-1] = b_{j+1}[-1]$, we know that $q^{\{j\}}_{\{\delta_j\},k-1}$ is compatible with both $q_{a_{j+1},k-j-2}$ and $q_{b_{j+1},k-j-2}$. Therefore, our choice of δ_j satisfies all of our requirements, and we can continue with the construction.

At the end of the construction, let

$$s = r \cup \bigcup_{j \leq j_\beta} (q_{a_j,k-j-1} \cup q_{b_j,k-j-1}) \cup \bigcup_{j<j_\beta} q^{\{j\}}_{\{\delta_j\},k-1}.$$

Our construction ensures that s is a condition in $\mathbb{P}$ that extends r. Now let $\vec{x} := \langle a(j) \mid j_\alpha \leq j < k\rangle^\frown\langle\delta_j \mid j < j_\beta\rangle^\frown\langle\beta\rangle$. We claim that s forces $\vec{x}$ to enumerate the vertices in a tight path from α to β in $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$. Since every vertex in $\vec{x}$ except for α and β is greater than γ, this will finish the proof of the theorem.

First recall that s extends both $q_{a,k-1}$ and $q_{b,k-1}$, so, for each $j_\alpha \leq j < k$, we have $s \leq q^{\{j\}}_{\{a(j)\},k-1}$, and also $s \leq q^{\{j_\beta\}}_{\{\beta\},k-1}$. Finally, for all $j < j_\beta$, we have $s \leq q^{\{j\}}_{\{\delta_j\},k-1}$. Therefore, s forces every vertex in $\vec{x}$ to be in $\dot{A}$. Finally, note that the sequence of edges of the tight path enumerated by $\vec{x}$ is $\langle a_j \mid j_\alpha \leq j \leq j_\beta\rangle^\frown\langle b_{j_\beta}\rangle$. By our construction, for each $j_\alpha \leq j \leq j_\beta$, we have $s \leq q_{a_j,k-j-1}$, and hence $s \Vdash$ "$\dot{c}(a_j) = i_{*,k-j-1} \in \Lambda$". Also, $s \leq q_{b_{j_\beta},k-j_\beta-1}$, so $s \Vdash$ "$\dot{c}(b_{j_\beta}) = i_{*,k-j_\beta-1} \in \Lambda$".

Therefore, s does indeed force $\vec{x}$ to be a tight path from α to β in $(\dot{A}, [\dot{A}]^k \cap \dot{c}^{-1}[\Lambda])$, thus completing the proof of the theorem. $\square$

We end the paper with a couple of related open questions. In [4], we show that $2^{\aleph_0} \to_{hc} [2^{\aleph_0}]^2_{\aleph_0,2}$ is in fact *equiconsistent* with the existence of a weakly compact cardinal. The main tool for showing this is a proof showing that, if $\lambda < \mu$ are infinite regular cardinals and $\square(\mu)$ holds, then, $\mu \not\to_{hc} [\mu]^2_{\lambda,<\lambda}$. We ask if something similar holds for our higher-dimensional generalizations.

Question 4.5. *Is it the case that, for every $2 < k < \omega$, the positive partition relation $2^{\aleph_0} \to_{htc} [2^{\aleph_0}]^k_{\aleph_0,k}$ is equiconsistent over ZFC with the existence of a weakly compact cardinal?*

Question 4.6. *Suppose that $2 < k < \omega$, $\lambda < \mu$ are infinite regular cardinals, and $\square(\mu)$ holds. Must it be the case that $\mu \not\to_{htc} [\mu]^k_{\lambda,<\lambda}$?*

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