HIGHER-DIMENSIONAL DELTA-SYSTEMS

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Abstract. We investigate higher-dimensional Δ-systems indexed by finite sets of ordinals, isolating a particular definition thereof and proving a higher-dimensional version of the classical Δ-system lemma. We focus in particular on systems that consist of sets of ordinals, in which case useful order-theoretic uniformities can be ensured. We then present three applications of these higher-dimensional Δ-systems to problems involving the interplay between forcing and partition relations on the reals.

1. Introduction

The starting point for this paper is one of the basic concepts of combinatorial set theory: the Δ-*system*.

Definition 1.1. A family $\mathcal{U}$ of sets is a Δ-*system* if there is a set r, known as the *root* of the Δ-system, such that $u \cap v = r$ for all distinct $u, v \in \mathcal{U}$.

The uniformity provided by Δ-systems can be quite useful, so it is no surprise that the Δ-system lemma, which isolates conditions that guarantee that a given family of sets can be thinned out to form a large Δ-system, is one of the foundational results of combinatorial set theory. The most commonly stated form of the lemma, introduced by Shanin [15], is the following.

Lemma 1.2. *Suppose that $\mathcal{U}$ is an uncountable family of finite sets. Then there is an uncountable subfamily $\mathcal{U}^* \subseteq \mathcal{U}$ such that $\mathcal{U}^*$ is a Δ-system.*

The following is a less pithy but more general formulation. For a proof, we direct the reader to [13, Ch. II, §1].

Lemma 1.3. *Suppose that $\kappa < \lambda$ are infinite cardinals such that λ is regular and, for all $\nu < \lambda$, we have $\nu^{<\kappa} < \lambda$. Suppose also that $\mathcal{U}$ is a family of sets such that $|\mathcal{U}| \geq \lambda$ and $|u| < \kappa$ for all $u \in \mathcal{U}$. Then there is $\mathcal{U}^* \subseteq \mathcal{U}$ such that $|\mathcal{U}^*| = \lambda$ and $\mathcal{U}^*$ is a Δ-system.*

Δ-systems are inherently one-dimensional objects, in practice often enumerated as sequences indexed by ordinals. When investigating higher-dimensional combinatorial objects, however, one frequently encounters families of sets indexed by n-element sets of ordinals for some $n > 1$ and desires to find large subfamilies exhibiting certain uniformity properties analogous to the uniformities exhibited by Δ-systems. In this context, higher-dimensional analogues of the Δ-system lemma

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come into play. Such analogues were first developed in work of Todorcevic [20] and Shelah [16], [17], and have appeared with increasing frequency of late in works such as [2], [3], [5], [6], [12], [21], and [22].

The higher-dimensional Δ-systems in the aforementioned works have taken a number of slightly different forms. In this paper, we isolate one particular definition, based most directly on the 2-dimensional Δ-systems of [20] and [2] and on the n-dimensional Δ-systems of [3]. This definition generalizes the familiar 1-dimensional definition and, in the case in which the higher-dimensional Δ-system consists of sets of ordinals, it can be strengthened to incorporate some additional order-theoretic uniformities. The definition is presented in Section 2, where we also prove some basic properties of our higher-dimensional Δ-systems. In Section 3, we prove our main result, Theorem 3.8, which is an n-dimensional analogue of the classical Δ-system lemma, isolating conditions under which an n-dimensional Δ-system of a particular size can be guaranteed to exist inside of an arbitrary collection of sets indexed by n-element sets of ordinals. Theorem 3.8 is naturally seen as an elaboration of the Erdős-Rado theorem and is closely connected to the work on canonical partition relations of Erdős and Rado [9] and of Baumgartner [1]. After proving Theorem 3.8, we turn to a discussion of its optimality, proving that one of its parameters, the size of the arbitrary collection of sets inside of which we are guaranteed to find a large n-dimensional Δ-system, cannot be improved and indicating precisely the extent to which another of its parameters, the upper bound on the size of the members of our arbitrary collection of sets, can consistently be improved. The results of this section are summarized in Corollaries 3.18 and 3.20, which incorporate Theorem 3.8, our discussion of its optimality, and its connections with the Erdős-Rado theorem.

The remaining sections of the paper presents applications of our main result. Section 4 is a short section presenting a higher-dimensional analogue of the familiar use of Δ-systems to prove that Cohen forcing satisfies the Knaster property. In Section 5, we present an application to a problem involving the interplay of forcing and polarized partition relations. In Section 6, we show that, in certain arguments, the Δ-system lemma presented here can successfully replace a different lemma (from [16]) that, at least under the currently best known results, requires stronger assumptions. We apply this to a recent result of Zhang [22] regarding additive partition relations on the reals, obtaining a slight local improvement of his result.

Notation and conventions. For a set or proper class X and a cardinal κ, $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$, and $[X]^{<\kappa} = \{Y \subseteq X \mid |Y| < \kappa\}$. For a set u of ordinals, $\mathrm{otp}(u)$ denotes the order type of u. The class of ordinals is denoted by On. If ρ is an ordinal and X is a set or proper class of ordinals, then $[X]^\rho = \{Y \subseteq X \mid \mathrm{otp}(Y) = \rho\}$. This is a slight abuse of notation given the previous definition of $[X]^\kappa$ and the customary identification of a cardinal with its initial ordinal, but in practice we will use the Greek letter ρ precisely when the order type definition of $[X]^\rho$ is intended, so no confusion will arise from this.

We will often think of sets of ordinals as increasing sequences of ordinals in the natural way. So, for instance, if u is a set of ordinals, $\rho = \mathrm{otp}(u)$, and $i < \rho$, then $u(i)$ denotes the unique element $\alpha \in u$ such that $\mathrm{otp}(u \cap \alpha) = i$. If $\mathbf{i} \subseteq \rho$, then $u[\mathbf{i}]$ denotes $\{u(i) \mid i \in \mathbf{i}\}$. If X is a set of ordinals and $n < \omega$, then we will use the notation $(\alpha_0, \ldots, \alpha_{n-1}) \in [X]^n$ to denote the conjunction of the statements

$\{\alpha_0, \dots, \alpha_{n-1}\} \in [X]^n$ and $\alpha_0 < \dots < \alpha_{n-1}$. If A and B are nonempty sets of ordinals, then we write $A < B$ to assert that $\alpha < \beta$ for all $(\alpha, \beta) \in A \times B$. For improved readability, we will also sometimes omit commas and brackets when using small sets as subscripts or superscripts. For example, we may write $u^0_{\alpha\beta}$ instead of $u^{\{0\}}_{\{\alpha,\beta\}}$.

If κ is an infinite cardinal, then $\beth_n(\kappa)$ is defined by recursion on $n < \omega$ by setting $\beth_0(\kappa) := \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$ for all $n < \omega$. As is customary, we will denote $\beth_n(\aleph_0)$ simply by $\beth_n$. Suppose that $\kappa < \lambda$ are cardinals. We say that λ is κ-inaccessible if $\nu^\kappa < \lambda$ for all $\nu < \lambda$. Similarly, λ is $<\kappa$-inaccessible if $\nu^{<\kappa} < \lambda$ for all $\nu < \lambda$.

If μ, λ, and ν are cardinals and n is a natural number, then the partition relation $\mu \to (\lambda)^n_\nu$ is the assertion that, for all $c : [\mu]^n \to \nu$, there is $H \in [\mu]^\lambda$ such that $c \restriction [H]^n$ is constant. The negation of this partition relation is denoted by $\mu \not\to (\lambda)^n_\nu$.

If $\mathbb{P}$ is a forcing notion and $p, q \in \mathbb{P}$, then $p \parallel q$ asserts that p and q are compatible, i.e., there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$, and $p \perp q$ asserts that p and q are incompatible.

2. Uniform n-dimensional Δ-systems

In this section, we present the basic definitions of the paper and prove some of their basic properties. We begin by working towards our definition of an n-dimensional Δ-system indexed by finite sets of ordinals. Most of the paper will focus on the case in which the elements of the Δ-system are themselves sets of ordinals, in which case we can arrange for significant order-theoretic uniformities, but we first present a more general definition.

Our n-dimensional Δ-systems will be indexed by sets of the form $[H]^n$, where H is a set of ordinals, and for $n > 1$ they will have not single root witnessing the fact that they are n-dimensional Δ-systems, but rather a family of roots. When first attempting to generalize Δ-systems to higher dimensions, one might optimistically hope to require that, in an n-dimensional Δ-system $\langle u_b \mid b \in [H]^n \rangle$, the intersection $u_a \cap u_b$ depends only on $a \cap b$ for all $a, b \in [H]^n$. In other words, one might hope to require the existence of a family of roots $\langle R_a \mid a \in [H]^{\leq n} \rangle$ such that, for all $b, b' \in [H]^n$, we have $u_b \cap u_{b'} = R_{b \cap b'}$. However, this would be an overly restrictive requirement, even in the case $n = 2$. To see this, let μ be any infinite cardinal, and define a family of sets $\langle u_b \mid b \in [\mu]^2 \rangle$ by letting $u_{\alpha\beta} = \{\alpha, \beta+1\}$ for all $(\alpha, \beta) \in [\mu]^2$. Now observe that, if $\alpha < \beta < \gamma < \delta < \mu$, then

- $\beta \in u_{\beta\gamma} \cap u_{\beta\delta}$;
- $\beta \notin u_{\alpha\beta} \cap u_{\beta\gamma}$.

Hence, $u_{\beta\gamma} \cap u_{\beta\delta} \neq u_{\alpha\beta} \cap u_{\beta\gamma}$, yet $\{\beta, \gamma\} \cap \{\beta, \delta\} = \{\beta\} = \{\alpha, \beta\} \cap \{\beta, \gamma\}$. Therefore, if one adopts the requirement that $u_a \cap u_b$ must depend only on $a \cap b$, then one could not even find a subset $H \subseteq \mu$ of size 4 for which $\langle u_b \mid b \in [H]^n \rangle$ is a 2-dimensional Δ-system.

The starting point for what will become our actual definition is Todorcevic's 2-dimensional *double Δ-system* from [20]. According to Todorcevic's definition, if H is a set of ordinals, then a family of sets $\langle u_b \mid b \in [H]^2 \rangle$ is a *double Δ-system* if

- for all $\alpha \in H$, the family $\langle u_{\alpha\beta} \mid \beta \in H \setminus (\alpha + 1) \rangle$ is a Δ-system with root r^0_α (for simplicity, assume that H has no maximal element);

- for all $\beta \in H \setminus \{\min(H)\}$, the family $\langle u_{\alpha\beta} \mid \alpha \in H \cap \beta \rangle$ is a Δ-system with root r^1_β;
- both $\langle r^0_\alpha \mid \alpha \in H \rangle$ and $\langle r^1_\beta \mid \beta \in H \setminus \min\{H\} \rangle$ are Δ-systems, with roots r^0 and r^1, respectively.

Note that, if $\langle u_b \mid b \in [H]^2 \rangle$ is a double Δ-system, as witnessed by sets $\langle r^0_\alpha \mid \alpha \in H \rangle$, $\langle r^1_\beta \mid \beta \in H \setminus \{\min(H)\} \rangle$, r^0, and r^1, then it is in fact the case that $r^0 = r^1 = \bigcap_{b \in [H]^2} u_b$.

In order to succinctly generalize this definition to higher dimensions, and to help facilitate the later incorporation of further order-theoretic uniformities, the following notion will be useful.

Definition 2.1. Suppose that a and b are sets of ordinals.

(1) We say that a and b are *aligned* if $\mathrm{otp}(a) = \mathrm{otp}(b)$ and, for all $\gamma \in a \cap b$, we have $\mathrm{otp}(a \cap \gamma) = \mathrm{otp}(b \cap \gamma)$. In other words, if γ is a common element of a and b, then it occupies the same relative position in both a and b.
(2) If a and b are aligned then we let $\mathbf{r}(a, b) := \{i < \mathrm{otp}(a) \mid a(i) = b(i)\}$. Notice that, in this case, $a \cap b = a[\mathbf{r}(a, b)] = b[\mathbf{r}(a, b)]$.

Note that, in our counterexample to our initial overly restrictive attempt at a definition of a higher-dimensional Δ-system at the beginning of this section, the problem came about when we considered the *non-aligned sets* $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$. As we will see shortly, it turns out that this was the only insurmountable problem with our definition, and if one requires the family of roots in an n-dimensional Δ-system to control only the intersections of elements of the Δ-system indexed by *aligned* sets, then one obtains a much more workable definition, which we adopt as our general definition of an n-dimensional Δ-system indexed by n-element sets of ordinals.

Definition 2.2. Suppose that H is a set of ordinals, $1 \leq n < \omega$, and, for each $b \in [H]^n$, u_b is a set. We call $\langle u_b \mid b \in [H]^n \rangle$ an *n-dimensional Δ-system* if there is a family of *roots* $\langle R^{\mathbf{m}}_a \mid a \in [H]^{\leq n},\ \mathbf{m} \in [n]^{|a|} \rangle$ such that, for all $b, b' \in [H]^n$, if b and b' are aligned and $\mathbf{r}(b, b') = \mathbf{m}$, then $u_b \cap u_{b'} = R^{\mathbf{m}}_{b \cap b'}$.

We observe that, if $n = 1$, then this is precisely the classical definition of a Δ-system as given in Definition 1.1 modulo an enumeration of the Δ-system via a set of ordinals; the root r in Definition 1.1 corresponds to the root $R^\emptyset_\emptyset$ in Definition 2.2. When $n = 2$, we obtain Todorcevic's double Δ-systems; the roots r^0_α, r^1_α, and $r^0 (= r^1)$ of Todorcevic's definition correspond to the roots R^0_α, R^1_α, and $R^\emptyset_\emptyset$, respectively.

We now turn to the special setting in which the elements of our Δ-systems are set of ordinals. In this setting, we can ask for our Δ-systems to satisfy certain additional order-theoretic uniformities, and we will call n-dimensional Δ-systems that satisfy these uniformities *uniform n-dimensional Δ-systems*. Since any family of sets $\langle u_b \mid b \in [H]^n \rangle$ can be transformed into a family of sets of ordinals via a bijection between $\bigcup_{b \in [H]^n} u_b$ and an ordinal, and since the proof of our higher dimensional analogue of the Δ-system lemma yields uniform n-dimensional Δ-systems, there will be no loss of generality for us in focusing on this setting.

Let us first look at the 1-dimensional case to help us motivate our definition. In the context of families of sets of ordinals, the classical Δ-system lemma can easily be strengthened to require that the root of the Δ-system "sits inside" each of its elements in the same way, in the following sense.

Definition 2.3. A family $\mathcal{U}$ of sets of ordinals is a *uniform Δ-system* if there is a set r such that, for all distinct $u, v \in \mathcal{U}$, u and v are aligned and $u \cap v = r$.

Proposition 2.4. *Suppose that $\kappa < \lambda$ are infinite cardinals such that λ is regular and $<\kappa$-inaccessible. Suppose also that $\mathcal{U}$ is a family of sets of ordinals such that $|\mathcal{U}| \geq \lambda$ and $|u| < \kappa$ for all $u \in \mathcal{U}$. Then there is $\mathcal{U}^* \subseteq \mathcal{U}$ such that $|\mathcal{U}^*| = \lambda$ and $\mathcal{U}^*$ is a uniform Δ-system.*

Proof. First apply Lemma 1.3 to obtain $\mathcal{U}_0^* \subseteq \mathcal{U}$ such that $|\mathcal{U}_0^*| = \lambda$ and $\mathcal{U}_0^*$ is a Δ-system, with root r. Since λ is regular and $|u| < \kappa < \lambda$ for all $u \in \mathcal{U}$, by thinning out further if necessary, we may assume that there is an ordinal $\rho < \kappa$ such that $\mathrm{otp}(u) = \rho$ for all $u \in \mathcal{U}_0^*$. Now define a function $g : \mathcal{U}_0^* \to \mathcal{P}(\rho)$ by letting $g(u) = \{\eta < \rho \mid u(\eta) \in r\}$. Since λ is $<\kappa$-inaccessible, we can find a fixed set $\mathbf{r}^* \subseteq \rho$ and a set $\mathcal{U}^* \subseteq \mathcal{U}_0^*$ such that $|\mathcal{U}^*| = \lambda$ and $g(u) = \mathbf{r}^*$ for all $u \in \mathcal{U}^*$. Then, for all distinct $u, v \in \mathcal{U}^*$, it follows that u and v are aligned, with $\mathbf{r}(u, v) = \mathbf{r}^*$ and $u \cap v = r$. □

We are now ready for our definition of a uniform n-dimensional Δ-system. In the case $n = 1$, this will coincide with Definition 2.3, and it will strengthen Definition 2.2 in the same way that Definition 2.3 strengthens Definition 1.1.

Definition 2.5. Suppose that H is a set of ordinals, $1 \leq n < \omega$, and, for all $b \in [H]^n$, u_b is a set of ordinals. We call $\langle u_b \mid b \in [H]^n \rangle$ a *uniform n-dimensional Δ-system* if there is an ordinal ρ and, for each $\mathbf{m} \subseteq n$, a set $\mathbf{r}_{\mathbf{m}} \subseteq \rho$ satisfying the following statements.

(1) $\mathrm{otp}(u_b) = \rho$ for all $b \in [H]^n$.
(2) For all $a, b \in [H]^n$ and $\mathbf{m} \subseteq n$, if a and b are aligned with $\mathbf{r}(a, b) = \mathbf{m}$, then u_a and u_b are aligned with $\mathbf{r}(u_a, u_b) = \mathbf{r}_{\mathbf{m}}$.
(3) For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_0 \cap \mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.

We now show that Definition 2.5 does indeed strengthen Definition 2.2; clause (1) of the following proposition will also be useful in a number of other situations.

Proposition 2.6. *Suppose that $1 \leq n < \omega$, H is a set of ordinals, and $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system as witnessed by an ordinal ρ and sets $\langle \mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \subseteq n \rangle$. Then the following statements hold.*

(1) For all $\mathbf{m} \subseteq n$ and all $a, b \in [H]^n$, if $a[\mathbf{m}] = b[\mathbf{m}]$, then $u_a[\mathbf{r}_{\mathbf{m}}] = u_b[\mathbf{r}_{\mathbf{m}}]$.
(2) The family $\langle u_b \mid b \in [H]^n \rangle$ is an n-dimensional Δ-system in the sense of Definition 2.2.

Proof. (1) For all $a, b \in [H]^n$, let $\partial(a, b) := |\{\beta \in a \cap b \mid |a \cap \beta| \neq |b \cap \beta|\}|$. Our proof will be by induction on $\partial(a, b)$.

Fix $\mathbf{m}$, a, and b as in the statement of clause (1) of the proposition. If $\partial(a, b) = 0$, then a and b are aligned and $\mathbf{m} \subseteq \mathbf{r}(a, b)$. It follows from clauses (2) and (3) of Definition 2.5 that u_a and u_b are aligned and $\mathbf{r}(u_a, u_b) \supseteq \mathbf{r}_{\mathbf{m}}$. In particular, $u_a[\mathbf{r}_{\mathbf{m}}] = u_b[\mathbf{r}_{\mathbf{m}}]$, as desired.

Now suppose that $\partial(a, b) > 0$ and we have established all instances of clause (1) of the proposition for $a', b' \in [H]^n$ for which $a'[\mathbf{m}] = b'[\mathbf{m}]$ and $\partial(a', b') < \partial(a, b)$. Let $\alpha \in a \cap b$ be least such that $|a \cap \alpha| \neq |b \cap \alpha|$. Let $k_a, k_b < n$ be such that $a(k_a) = \alpha = b(k_b)$. Without loss of generality, we may assume that $k_a < k_b$.

We now alter a to form a new set $a' \in [H]^n$. If $a \cap b \cap \alpha \neq \emptyset$, then let $\alpha^* := \max(a \cap b \cap \alpha)$. In this case, by our choice of α, there must be $k^* < k_a$ such that $a(k^*) = b(k^*) = \alpha^*$. If $a \cap b \cap \alpha = \emptyset$, then let $k^* := -1$. In either case, note that, for all $\ell \in (k^*, k_a]$, we have $b(\ell) \notin a$ and, if $k^* \geq 0$, then $a(k^*) < b(\ell)$. Moreover, $\mathbf{m} \cap (k^*, k_a] = \emptyset$. We define a' by specifying $a'(\ell)$ for all $\ell < n$. If $\ell \leq k^*$ or $\ell > k_a$, then let $a'(\ell) = a(\ell)$. If $\ell \in (k^*, k_a]$, then let $a'(\ell) = b(\ell)$. The following observations are immediate.

(i) a and a' are aligned, with $\mathbf{r}(a, a') = n \setminus (k^*, k_a]$. In particular, $\mathbf{m} \subseteq \mathbf{r}(a, a')$.
(ii) $\partial(a', b) = \partial(a, b) - 1$, since

$$\{\beta \in a' \cap b \mid |a' \cap \beta| \neq |b \cap \beta|\} = \{\beta \in a \cap b \mid |a \cap \beta| \neq |b \cap \beta|\} \setminus \{\alpha\}.$$

We can therefore invoke the inductive hypothesis together with (i) to conclude that $u_a[\mathbf{r_m}] = u_{a'}[\mathbf{r_m}]$ and together with (ii) to conclude that $u_{a'}[\mathbf{r_m}] = u_b[\mathbf{r_m}]$, so it follows that $u_a[\mathbf{r_m}] = u_b[\mathbf{r_m}]$, as desired.

(2) To prove that $\langle u_b \mid b \in [H]^n \rangle$ satisfies Definition 2.2, we must specify roots

$$\langle R_a^{\mathbf{m}} \mid a \in [H]^{\leq n},\ \mathbf{m} \in [n]^{|a|} \rangle.$$

To this end, fix $a \in [H]^{\leq n}$ and $\mathbf{m} \in [n]^{|a|}$. If there are no $b \in [H]^n$ for which $b[\mathbf{m}] = a$, then simply let $R_a^{\mathbf{m}} = \emptyset$. Otherwise, choose $b \in [H]^n$ for which $b[\mathbf{m}] = a$ and set $R_a^{\mathbf{m}} = u_b[\mathbf{r_m}]$. By clause (1) of this proposition, the value of $R_a^{\mathbf{m}}$ is independent of our choice of b.

Now suppose that $b, b' \in [H]^n$ are aligned and $\mathbf{r}(b, b') = \mathbf{m}$, so, in particular, $b[\mathbf{m}] = b \cap b'$. Then u_b and $u_{b'}$ are aligned and $\mathbf{r}(u_b, u_{b'}) = \mathbf{r_m}$. Moreover, we defined $R_{b \cap b'}^{\mathbf{m}}$ so that $R_{b \cap b'}^{\mathbf{m}} = u_b[\mathbf{r_m}]$. It follows that $u_b \cap u_{b'} = R_{b \cap b'}^{\mathbf{m}}$, so $\langle R_a^{\mathbf{m}} \mid a \in [H]^{\leq n},\ \mathbf{m} \in [n]^{|a|} \rangle$ witnesses the fact that $\langle u_b \mid b \in [H]^n \rangle$ satisfies Definition 2.2. □

3. A higher-dimensional Δ-system lemma

In this section, we prove the main result of the paper (Theorem 3.8), a higher-dimensional analogue of the Δ-system lemma which asserts, roughly speaking, that inside every family of sets of ordinals indexed by n-element subsets of some sufficiently large cardinal μ, we can find a subset H of μ of some specified size such that $[H]^n$ indexes a uniform n-dimensional Δ-system. In the absence of weakly compact cardinals, this H will necessarily be smaller than μ. In the same way that the Δ-system lemma can fruitfully be seen as as an extension of the pigeonhole principle, this n-dimensional Δ-system lemma can fruitfully be seen as an elaboration of the Erdős-Rado theorem, and in fact a version of the Erdős-Rado theorem will be folded into our statement to carry along as an inductive hypothesis.

The result is also closely related to results on *canonical partition relations*, introduced by Erdős and Rado in [9], and in particular to work done by Baumgartner on canonical partition relations [1], which can also be seen as an elaboration of the Erdős-Rado theorem. Indeed, in the cases in which κ is a successor cardinal, much of our main result can be derived from the main result of [1]. When κ is a limit cardinal (and in particular in the important case $\kappa = \aleph_0$, $\lambda = \aleph_1$), this approach does not seem to work, so we provide a single proof that works for all cases. We first introduce the following notation, from [1], that allows us to indicate precisely the size of the family needed to ensure the existence of a large uniform n-dimensional Δ-system.

Definition 3.1. Given an infinite regular cardinal λ, recursively define $\sigma(\lambda, n)$ for $1 \leq n < \omega$ by letting $\sigma(\lambda, 1) = \lambda$ and, given $1 \leq n < \omega$, letting $\sigma(\lambda, n+1) = \left(2^{<\sigma(\lambda,n)}\right)^+$.

Remark 3.2. To connect Definition 3.1 with the already familiar $\beth$-notation and to help clarify the choice of cardinals in the statements of Corollary 3.15, Theorem 5.4, and Corollary 6.2, we make the following observations, which we leave the reader to verify.

(1) If $\lambda = \kappa^+$ and $1 \leq n < \omega$, then $\sigma(\lambda, n) = (\beth_{n-1}(\kappa))^+$. In particular, $\sigma(\aleph_1, n) = \beth_{n-1}^+$ and $\sigma(\beth_1^+, n) = \beth_n^+$.
(2) For every infinite regular λ, if $2 \leq n < \omega$, then $\sigma(\lambda, n) = (\beth_{n-2}(2^{<\lambda}))^+$.

Note in particular that $\sigma(\lambda, n)$ is regular for each regular infinite λ and each $1 \leq n < \omega$.

We also remark that $\sigma(\lambda, n)$ is precisely the cardinal resource needed to ensure a monochromatic set of size λ in the n-dimensional Erdős-Rado theorem, which can be formulated as follows: for every $1 \leq n < \omega$ and all infinite cardinals $\nu < \lambda$, with λ regular, the partition relation $\sigma(\lambda, n) \to (\lambda + (n-1))^n_\nu$ holds ([10, Theorem 39]; cf. also [1, Proposition 1]). See Corollary 3.18 for a more precise formulation of the connection between our main result and the Erdős-Rado theorem.

In the proof of Theorem 3.8, we will make use of the following notion of the *type* of a sequence of sets of ordinals, which describes the order-relations existing among the sets.

Definition 3.3. Suppose that I is a set and, for all $i \in I$, u_i is a set of ordinals. Then $\mathrm{tp}(\langle u_i \mid i \in I\rangle)$ (the *type of* $\langle u_i \mid i \in I\rangle$) is a function from $\mathrm{otp}(\bigcup_{i\in I} u_i)$ to $\mathcal{P}(I)$ defined as follows. First, let $\bigcup_{i \in I} u_i$ be enumerated in increasing order as $\langle \alpha_\eta \mid \eta < \mathrm{otp}(\bigcup_{i\in I} u_i)\rangle$. Then, for all $\eta < \mathrm{otp}(\bigcup_{i \in I} u_i)$, let $\mathrm{tp}(\langle u_i \mid i \in I\rangle)(\eta) := \{i \in I \mid \alpha_\eta \in u_i\}$.

We will often slightly abuse notation and write, for instance, $\mathrm{tp}(u_0, u_1, u_2)$ instead of $\mathrm{tp}(\langle u_0, u_1, u_2\rangle)$.

Remark 3.4. To connect Definition 3.3 with the earlier definition of *aligned sets*, we note that, if a and b are sets of ordinals, then a and b are aligned if and only if $\mathrm{tp}(a \cap b, a) = \mathrm{tp}(a \cap b, b)$. We also observe the following useful facts about the tp operator, which can easily be verified:

(1) Suppose that I is a set and, for all $i \in I$, u_i and u_i' are sets of ordinals. Suppose also that $\mathrm{tp}(\langle u_i \mid i \in I\rangle) = \mathrm{tp}(\langle u_i' \mid i \in I\rangle)$. Then the following statements hold.
   (a) For all $i \in I$, we have $\mathrm{otp}(u_i) = \mathrm{otp}(u_i')$.
   (b) For all $J \subseteq I$, we have $\mathrm{tp}(\langle u_i \mid i \in J\rangle) = \mathrm{tp}(\langle u_i' \mid i \in J\rangle)$.
(2) Suppose that u_0, u_1, u_0', and u_1' are sets of ordinals. If u_0 and u_1 are aligned and $\mathrm{tp}(u_0, u_1) = \mathrm{tp}(u_0', u_1')$, then u_0' and u_1' are also aligned nd $\mathbf{r}(u_0', u_1') = \mathbf{r}(u_0, u_1)$.

The higher-dimensional Δ-systems that we isolate in our main result will have an additional technical uniformity (the "moreover" clause of Theorem 3.8) that allows us to control the relationship between u_a and u_b for certain non-aligned pairs $a, b \in [H]^n$ and is useful in some applications. In order to properly state it, we need some further definitions. Readers can safely skip these technical considerations and the "moreover" clause of the theorem on first read, if desired, as they are not needed in our applications in Sections 4 and 5. They are used in the proof of Corollary 6.2, which is presented not in this paper but in [14].

Definition 3.5. Suppose that $i < \rho$ are ordinals and $a, b \in [\mathrm{On}]^\rho$. We say that a and b are *aligned above* i if $a[\rho \setminus i]$ and $b[\rho \setminus i]$ are aligned.

The following notion provides strictly less information than $\mathrm{tp}(a, b)$ but is sometimes easier to control.

Definition 3.6. Suppose that a and b are sets of ordinals. Then the *intersection type of* a *and* b, denoted $\mathrm{tp}_{\mathrm{int}}(a, b)$, is the set $\{(i, j) \in \mathrm{otp}(a) \times \mathrm{otp}(b) \mid a(i) = b(j)\}$.

Definition 3.7. Suppose that a is a nonempty set of ordinals and $i < \mathrm{otp}(a)$.

(1) We say that an ordinal α is i*-possible* for a if the following two statements hold:
   (a) if $i > 0$, then $\alpha > a(i - 1)$;
   (b) if $i + 1 < \mathrm{otp}(a)$, then $\alpha < a(i + 1)$.
   Intuitively, α is i-possible for a if $a(i)$ can be replaced by α without changing the relative positions of the other elements of a.
(2) If α is i-possible for a, then $a_{i \mapsto \alpha}$ is the set $(a \setminus \{a(i)\}) \cup \{\alpha\}$, i.e., the set obtained by replacing the i^{th} element of a with α.

We are now ready for our main result. As we will see at the end of this section, unless λ is a weakly compact cardinal, the theorem is optimal in the sense that μ cannot be lowered. We also note that clause (1) of the following theorem is essentially the Erdős-Rado theorem. In response to a query from the referee, we note that our proof does not yield an essentially new proof of the Erdős-Rado theorem; if one extracts the proof of just clause (1) from our proof, one obtains more or less a proof of the Erdős-Rado theorem originally given by Simpson in [18] (see also the proof of [4, Theorem 7.2.1]).

Theorem 3.8. *Suppose that*

- $1 \leq n < \omega$*;*
- $\kappa, \nu < \lambda$ *are infinite cardinals,* λ *is regular and* $<\kappa$*-inaccessible, and* $\mu = \sigma(\lambda, n)$*;*
- $g : [\mu]^n \to \nu$*;*
- *for all* $b \in [\mu]^n$*, we are given a set* $u_b \in [\mathrm{On}]^{<\kappa}$*.*

Then there are $H \in [\mu]^\lambda$ *and* $k < \nu$ *such that*

(1) $g(b) = k$ *for all* $b \in [H]^n$*;*
(2) $\langle u_b \mid b \in [H]^n \rangle$ *is a uniform* n*-dimensional* Δ*-system.*

Moreover, we can arrange our choice of H *so that, for all* $a, b \in [H]^n$ *and all* $m < n$*, if it is the case that* a *and* b *are aligned above* m *and* $a(m) = b(m)$*, then, for any ordinal* $\alpha \in H$ *that is* m*-possible for both* a *and* b*, we have* $\mathrm{tp}_{\mathrm{int}}(u_a, u_b) = \mathrm{tp}_{\mathrm{int}}(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}})$*.*

Proof. The proof is by induction on n. When $n = 1$, the result follows from Proposition 2.4 and the pigeonhole principle (note that the "moreover" clause of the theorem is trivial if $n = 1$). So suppose that $1 < n < \omega$ and we have established all instances of the theorem for $n - 1$.

Set $\mu^* := \sigma(\lambda, n - 1)$. Let θ be a sufficiently large regular cardinal, and let M be an elementary substructure of $(H(\theta), \in, g, \langle u_b \mid b \in [\mu]^n \rangle)$ such that M is closed under sequences of length less than μ^* and $\mu_M := M \cap \mu \in \mu$. This is possible, since μ^* is regular and $\mu = \sigma(\lambda, n) = \left(2^{<\mu^*}\right)^+$. Note that $\mathrm{cf}(\mu_M) \geq \mu^*$.

Temporarily fix $a \in [\mu_M]^{n-1}$, and consider $u_{a^\frown\langle\mu_M\rangle}$. Let $w_a := u_{a^\frown\langle\mu_M\rangle} \cap M$ and $\rho_a = \mathrm{otp}\left(u_{a^\frown\langle\mu_M\rangle}\right)$. Let $\mathbf{i}_a \subseteq \rho_a$ be such that $u_{a^\frown\langle\mu_M\rangle}[\mathbf{i}_a] = w_a$, and let $\mathbf{j}_a := \rho_a \setminus \mathbf{i}_a$. For each $j \in \mathbf{j}_a$, let $\gamma_{a,j}$ be the least ordinal γ in M such that $u_{a^\frown\langle\mu_M\rangle}(j) < \gamma$; to see that such an ordinal γ exists, note that $\sup(\bigcup_{b\in[H]^n} u_b)$ is definable in M and is therefore an element of M.

We now construct an increasing sequence $\langle \alpha_\eta \mid \eta < \mu^* \rangle$ of ordinals below μ_M as follows. Begin by letting $\alpha_\eta = \eta$ for all $\eta < n-1$. Now suppose that $n-1 \leq \eta < \mu^*$ and we have defined $\langle \alpha_\xi \mid \xi < \eta \rangle$. Let $A_\eta := \{\alpha_\xi \mid \xi < \eta\}$. By the closure of M and the fact that $[A_\eta]^{n-1}$ has size less than μ^*, we know that all of the following are elements of M:

- A_η;
- $\langle g(a^\frown\langle\mu_M\rangle) \mid a \in [A_\eta]^{n-1}\rangle$;
- $\langle (w_a, \rho_a, \mathbf{i}_a, \mathbf{j}_a) \mid a \in [A_\eta]^{n-1}\rangle$;
- $\langle \gamma_{a,j} \mid a \in [A_\eta]^{n-1},\ j \in \mathbf{j}_\alpha\rangle$.

Moreover, $\mathrm{tp}(\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [A_\eta]^{n-1}\rangle)$ is a function from an ordinal less than μ^* to $\mathcal{P}([A_\eta]^{n-1})$, so again the closure of M implies that $\mathrm{tp}(\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [A_\eta]^{n-1}\rangle)$ is in M.

For each $a \in [A_\eta]^{n-1}$ and $j \in \mathbf{j}_a$, let

$$\epsilon_{a,j} := \sup\{\sup(u_b \cap \gamma_{a,j}) \mid b \in [A_\eta]^n\}.$$

Note that $\mathrm{cf}(\gamma_{a,j}) \geq \mu^*$, since otherwise there would be a cofinal $x \subseteq \gamma_{a,j}$ such that $x \subseteq M$. Therefore, we have $\epsilon_{a,j} \in M \cap \gamma_{a,j}$ and, again by closure, $\langle \epsilon_{a,j} \mid a \in [A_\eta]^{n-1},\ j \in \mathbf{j}_a\rangle \in M$.

In $H(\theta)$, the ordinal μ_M witnesses the truth of the statement asserting the existence of an ordinal β such that:

- $\sup(A_\eta) < \beta < \mu$;
- $g(a^\frown\langle\beta\rangle) = g(a^\frown\langle\mu_M\rangle)$ for all $a \in [A_\eta]^{n-1}$;
- $\mathrm{tp}(\langle u_{a^\frown\langle\beta\rangle} \mid a \in [A_\eta]^{n-1}\rangle) = \mathrm{tp}(\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [A_\eta]^{n-1}\rangle)$;
- $u_{a^\frown\langle\beta\rangle}[\mathbf{i}_a] = w_a$ for all $a \in [A_\eta]^{n-1}$;
- $u_{a^\frown\langle\beta\rangle}(j)$ is in the interval $(\epsilon_{a,j}, \gamma_{a,j})$ for all $a \in [A_\eta]^{n-1}$ and all $j \in \mathbf{j}_a$.

All of the parameters in the above statement are in M (note, for instance, that, in the second item, $a^\frown\langle\mu_M\rangle$ is not in M, but $\langle g(a^\frown\langle\mu_M\rangle) \mid a \in [A_\eta]^{n-1}\rangle$ is). Therefore, by elementarity, we can choose $\alpha_\eta \in M$ satisfying the statement, and we continue to the next step of the construction. Notice that, as a consequence of clause (1a) of Remark 3.4 and the fact that α_η satisfies the third bullet point above, we have $\mathrm{otp}(u_{a^\frown\langle\beta\rangle}) = \mathrm{otp}(u_{a^\frown\langle\mu_M\rangle}) = \rho_a$ for all $a \in [A_\eta]^{n-1}$.

After completing the construction, let $A = \{\alpha_\eta \mid \eta < \mu^*\}$, and define a function g^* on $[A]^{n-1}$ by letting $g^*(a) = \langle g(a^\frown\langle\mu_M\rangle), \rho_a, \mathbf{i}_a, \mathbf{j}_a\rangle$ for all $a \in [A]^{n-1}$. Since we know that

- $g : [\mu]^n \to \nu$;
- $\rho_a < \kappa$; and
- $\mathbf{i}_a, \mathbf{j}_a \subseteq \rho_a$;

it follows that g^* can be coded as a function from $[A]^{n-1}$ to $\max\{\nu, 2^{<\kappa}\}$ which, by the hypothesis of the theorem, is less than λ. Recalling that $\mu^* = \sigma(\lambda, n-1) = |A|$, apply the induction hypothesis to g^* and $\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [A]^{n-1}\rangle$ to find $H_0 \subseteq A$, $k < \nu$, $\rho < \kappa$, and sets $\mathbf{i}, \mathbf{j} \subseteq \rho$ such that the following statements all hold:

- $\mathrm{otp}(H_0) = \lambda$;
- $g(a^\frown\langle\mu_M\rangle) = k$ for all $a \in [H_0]^{n-1}$;
- $\langle \rho_a, \mathbf{i}_a, \mathbf{j}_a\rangle = \langle \rho, \mathbf{i}, \mathbf{j}\rangle$ for all $a \in [H_0]^{n-1}$;
- $\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [H_0]^{n-1}\rangle$ is a uniform $(n-1)$-dimensional Δ-system, as witnessed by ρ and by sets $\mathbf{s_m} \subseteq \rho$ for each $\mathbf{m} \subseteq n-1$;
- $\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [H_0]^{n-1}\rangle$ satisfies the "moreover" clause in the statement of the theorem.

Let $E_0 := \{\eta < \mu^* \mid n-1 \leq \eta \text{ and } \alpha_\eta \in H_0\}$. We will thin out H_0 to a further unbounded subset $H \subseteq H_0$ before the end of the proof. For now, let us begin verifying clauses (1) and (2) in the statement of the theorem, noting that what we verify for H_0 will remain true after further thinning out.

We first take care of clause (1) of the theorem, simultaneously showing that $\mathrm{otp}(b) = \rho$ for all $b \in [H_0]^n$. To this end, fix $b \in [H_0]^n$. Then b is of the form $a^\frown\langle\alpha_\eta\rangle$ for some $\eta \in E_0$ and $a \in [A_\eta \cap H_0]^{n-1}$. By our choice of α_η, we have $g(a^\frown\langle\alpha_\eta\rangle) = g(a^\frown\langle\mu_M\rangle)$, and $\mathrm{otp}(u_{a^\frown\langle\alpha_\eta\rangle}) = \mathrm{otp}(u_{a^\frown\langle\mu_M\rangle}) = \rho_a$. Then, by our choice of H_0, k, and ρ, we have $g(a^\frown\langle\mu_M\rangle) = k$ and $\rho_a = \rho$. Therefore, $g(b) = k$ and $\mathrm{otp}(u_b) = \rho$, as desired.

We now turn our attention to clause (2). The value of ρ that we isolated above is the order type that will eventually witness that $\langle u_b \mid b \in [H]^n\rangle$ is a uniform n-dimensional Δ-system; indeed, by the previous paragraph we have $\mathrm{otp}(u_b) = \rho$ for all $b \in [H_0]^n$. We next specify the values for $\langle \mathbf{r_m} \mid \mathbf{m} \subseteq n\rangle$ that will witness that $\langle u_b \mid b \in [H]^n\rangle$ is a uniform n-dimensional Δ-system. For each $\mathbf{m} \subseteq n$, let $\mathbf{m}^- = \mathbf{m} \cap (n-1)$. If $n-1 \in \mathbf{m}$, then set $\mathbf{r_m} = \mathbf{s}_{\mathbf{m}^-}$. If $n-1 \notin \mathbf{m}$, then set $\mathbf{r_m} = \mathbf{s}_{\mathbf{m}^-} \cap \mathbf{i}$. Note that, in either case, we do indeed have $\mathbf{r_m} \subseteq \rho$.

Claim 3.9. *For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_0\cap\mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.*

Proof. This follows immediately from the fact that $\mathbf{s}_{\mathbf{m}_0^-\cap\mathbf{m}_1^-} = \mathbf{s}_{\mathbf{m}_0^-} \cap \mathbf{s}_{\mathbf{m}_1^-}$ for all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$. □

It remains to verify clause (2) of Definition 2.5, i.e., if $a, b \in [H]^n$ are aligned and $\mathbf{r}(a, b) = \mathbf{m}$, then u_a and u_b are aligned, and $\mathbf{r}(u_a, u_b) = \mathbf{r_m}$. We split this verification into two cases, depending on whether or not $n-1$ is in $\mathbf{m}$.

Claim 3.10. *Suppose that $b_0, b_1 \in [H_0]^n$ are aligned and $n-1 \in \mathbf{m} = \mathbf{r}(b_0, b_1)$. Then u_{b_0} and u_{b_1} are aligned and $\mathbf{r}(u_{b_0}, u_{b_1}) = \mathbf{r_m}$.*

Proof. Since $n-1 \in \mathbf{m}$, we have $\mathbf{r_m} = \mathbf{s}_{\mathbf{m}^-}$. It also follows from the fact that $n-1 \in \mathbf{m}$ that there is $\eta \in E_0$ such that b_0 and b_1 are of the form $a_0{}^\frown\langle\alpha_\eta\rangle$ and $a_1{}^\frown\langle\alpha_\eta\rangle$ respectively, where $a_0, a_1 \in [A_\eta \cap H_0]^{n-1}$ are aligned and $\mathbf{r}(a_0, a_1) = \mathbf{m}^-$. By our choice of H_0 and $\mathbf{s}_{\mathbf{m}^-}$, it follows that $u_{a_0{}^\frown\langle\mu_M\rangle}$ and $u_{a_1{}^\frown\langle\mu_M\rangle}$ are aligned and $\mathbf{r}(u_{a_0{}^\frown\langle\mu_M\rangle}, u_{a_1{}^\frown\langle\mu_M\rangle}) = \mathbf{s}_{\mathbf{m}^-}$. Now, recalling Remark 3.4, our choice of α_η implies that $\mathrm{tp}(u_{b_0}, u_{b_1}) = \mathrm{tp}(u_{a_0{}^\frown\langle\mu_M\rangle}, u_{a_1{}^\frown\langle\mu_M\rangle})$, and therefore u_{b_0} and u_{b_1} are aligned, with $\mathbf{r}(u_{b_0}, u_{b_1}) = \mathbf{s}_{\mathbf{m}^-} = \mathbf{r_m}$, as desired. □

We next deal with the case in which $\mathbf{m} \subseteq n-1$. This will take a bit more work. We first establish the following claim.

Claim 3.11. *Suppose that $b_0, b_1 \in [H_0]^n$, $\mathbf{m} \subseteq n-1$, and $b_0[\mathbf{m}] = b_1[\mathbf{m}]$. Then $u_{b_0}[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}]$.*

Proof. Since $\mathbf{m} \subseteq n-1$, we have $\mathbf{m}^- = \mathbf{m}$ and $\mathbf{r_m} = \mathbf{s_m} \cap \mathbf{i}$. We also know that b_0 and b_1 are of the form $a_0{}^\frown\langle\alpha_\eta\rangle$ and $a_1{}^\frown\langle\alpha_\varepsilon\rangle$, respectively, where $\eta, \varepsilon \in E_0$, $a_0, a_1 \in [H_0]^{n-1}$, and $a_0[\mathbf{m}] = a_1[\mathbf{m}]$. By Proposition 2.6(1) applied to $\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [H_0]^{n-1}\rangle$, $\mathbf{m}$, a_0, and a_1, we know that $u_{a_0{}^\frown\langle\mu_M\rangle}[\mathbf{s_m}] = u_{a_1{}^\frown\langle\mu_M\rangle}[\mathbf{s_m}]$. Now fix $i \in \mathbf{r_m}$. Since $i \in \mathbf{s_m}$, it follows that $u_{a_0{}^\frown\langle\mu_M\rangle}(i) = u_{a_1{}^\frown\langle\mu_M\rangle}(i)$. Since $i \in \mathbf{i}$, our choices of α_η and α_ε imply that $u_{b_0}(i) = u_{a_0{}^\frown\langle\mu_M\rangle}(i)$ and $u_{b_1}(i) = u_{a_1{}^\frown\langle\mu_M\rangle}(i)$. Together, this implies that $u_{b_0}(i) = u_{b_1}(i)$, and hence $u_{b_0}[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}]$. □

As an immediate consequence of Claim 3.11, if $b_0, b_1 \in [H_0]^n$ are aligned and $\mathbf{r}(b_0, b_1) = \mathbf{m} \subseteq n-1$, then $u_{b_0}[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}]$. Showing that u_{b_0} and u_{b_1} are disjoint outside of $u_{b_0}[\mathbf{r_m}]$ will take some more work and possibly a thinning out of H_0. For $m < n$ and $a \in [H_0]^m$, choose any $b \in [H_0]^n$ with $a = b[m]$ (i.e., b is an end-extension of a), and define $u_a := u_b[\mathbf{r}_m]$. By Claim 3.11, this definition is independent of our choice of b.

For the rest of the proof, we adopt the convention that $\max(\emptyset) = -1$.

Claim 3.12. *Suppose that $m < n$ and $a \in [H_0]^m$. Then*

$$\langle u_{a^\frown\langle\beta\rangle} \mid \beta \in H_0 \setminus (\max(a) + 1)\rangle$$

is a Δ-system with root u_a.

Proof. Suppose first that $m = n-1$, in which case $\mathbf{r}_m = \mathbf{i}$. Fix $(\eta, \varepsilon) \in [E_0]^2$ with $\alpha_\eta > \max(a)$, and consider $u_{a^\frown\langle\alpha_\eta\rangle} \cap u_{a^\frown\langle\alpha_\varepsilon\rangle}$. By Claim 3.11, we have $u_{a^\frown\langle\alpha_\eta\rangle}[\mathbf{i}] = u_{a^\frown\langle\alpha_\varepsilon\rangle}[\mathbf{i}]$. Furthermore, for all $j \in \mathbf{j}$, our choice of α_ε implies that

$$\sup(u_{a^\frown\langle\alpha_\eta\rangle} \cap \gamma_{a,j}) < u_{a^\frown\langle\alpha_\varepsilon\rangle}(j) < \gamma_{a,j},$$

and hence $u_{a^\frown\langle\alpha_\varepsilon\rangle}(j) \notin u_{a^\frown\langle\alpha_\eta\rangle}$. It follows that

$$u_{a^\frown\langle\alpha_\eta\rangle} \cap u_{a^\frown\langle\alpha_\varepsilon\rangle} = u_{a^\frown\langle\alpha_\varepsilon\rangle}[\mathbf{i}] = u_a,$$

as desired.

Next, suppose that $m < n-1$. Fix $(\beta_0, \beta_1) \in [H_0]^2$ with $\beta_0 > \max(a)$, and consider $u_{a^\frown\langle\beta_0\rangle} \cap u_{a^\frown\langle\beta_1\rangle}$. Fix $c \in [H_0]^{n-m-1}$ with $\min(c) > \beta_1$ and set $b_\ell := a^\frown\langle\beta_\ell\rangle^\frown c$ for $\ell < 2$. Note that $b_\ell \in [H_0]^n$, that $u_{a^\frown\langle\beta_\ell\rangle} = u_{b_\ell}[\mathbf{r}_{m+1}]$, and that $u_a = u_{b_\ell}[\mathbf{r}_m]$. Observe also that b_0 and b_1 are aligned and that $\mathbf{r}(b_0, b_1) = n \setminus \{m\}$, so, by Claim 3.10, we have $u_{b_0} \cap u_{b_1} = u_{b_0}[\mathbf{r}_{n\setminus\{m\}}] = u_{b_1}[\mathbf{r}_{n\setminus\{m\}}]$. Putting this together, we obtain

$$\begin{aligned}
u_{a^\frown\langle\beta_0\rangle} \cap u_{a^\frown\langle\beta_1\rangle} &= u_{b_0}[\mathbf{r}_{m+1}] \cap u_{b_1}[\mathbf{r}_{m+1}] \\
&= u_{b_0}[\mathbf{r}_{m+1}] \cap u_{b_1}[\mathbf{r}_{m+1}] \cap u_{b_0}[\mathbf{r}_{n\setminus\{m\}}] \cap u_{b_1}[\mathbf{r}_{n\setminus\{m\}}] \\
&= u_{b_0}[\mathbf{r}_m] \cap u_{b_1}[\mathbf{r}_m] \\
&= u_a,
\end{aligned}$$

where the passage from the second to the third line in the above sequence of equations follows from Claim 3.9 and the observation that $(m+1) \cap (n \setminus \{m\}) = m$. □

We are now ready to thin out H_0 to our final set H witnessing the conclusion of the theorem. We will recursively define an increasing sequence $\langle\beta_\xi \mid \xi < \lambda\rangle$ of ordinals from H_0 and then define $H := \{\beta_\xi \mid \xi < \lambda\}$.

Begin by letting $\beta_0 = \min(H_0)$. Next, suppose that $0 < \zeta < \lambda$ and $\langle\beta_\xi \mid \xi < \zeta\rangle$ has been defined. Let $B_\zeta := \{\beta_\xi \mid \xi < \zeta\}$. Suppose that $a_0 \in [B_\zeta]^{<n}$ and $a_1 \in [B_\zeta]^{\leq n}$. By Claim 3.12, the sequence $\langle u_{a_0{}^\frown\langle\beta\rangle} \setminus u_{a_0} \mid \beta \in H_0 \setminus (\sup(B_\zeta)+1)\rangle$ consists of pairwise disjoint sets. Since $|u_{a_1}| < \kappa$, it follows that, letting C_{a_0,a_1} be the set of $\beta \in H_0 \setminus (\sup(B_\zeta)+1)$ such that $u_{a_0 \frown \langle \beta \rangle} \setminus u_{a_0}$ has nonempty intersection with u_{a_1}, we have $|C_{a_0,a_1}| < \kappa$. Since the number of such pairs (a_0, a_1) is less than λ, we can find $\beta \in H_0 \setminus (\sup(B_\zeta)+1)$ such that, for all $a_0 \in [B_\zeta]^{<n}$ and all $a_1 \in [B_\zeta]^{\leq n}$, we have $\beta \notin C_{a_0,a_1}$. Let β_ζ be the least such β, and continue to the next step of the construction.

To verify that $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system as witnessed by ρ and $\langle \mathbf{r_m} \mid \mathbf{m} \subseteq n \rangle$, we must show that, for all $b_0, b_1 \in [H]^n$, if b_0 and b_1 are aligned and $\mathbf{m} = \mathbf{r}(b_0, b_1)$, then u_{b_0} and u_{b_1} are aligned with $\mathbf{r}(u_{b_0}, u_{b_1}) = \mathbf{r_m}$. To this end, fix $b_0, b_1 \in [H]^n$ such that b_0 and b_1 are aligned, and let $\mathbf{m} = \mathbf{r}(b_0, b_1)$. If $n-1 \in \mathbf{m}$, then the desired conclusion already follows from Claim 3.10, so assume that $n-1 \notin \mathbf{m}$.

Without loss of generality, assume that $\max(b_0) < \max(b_1)$. By Claim 3.11, we know that $u_{b_0}[\mathbf{r_m}] = u_{b_1}[\mathbf{r_m}]$. It will therefore suffice to show that, for all $i < \rho$, if $u_{b_1}(i) \in u_{b_0}$, then $i \in \mathbf{r_m}$.

To this end, fix $i < \rho$ such that $\gamma := u_{b_1}(i) \in u_{b_0}$. Let $m^* < n$ be least such that $b_1(m^*) > \max(b_0)$. Notice that this m^* exists, since $\max(b_1) > \max(b_0)$.

Claim 3.13. $\gamma \in u_{b_1[m^*]}$.

Proof. We will prove by induction on $\ell \leq n - m^*$ that $\gamma \in u_{b_1[n-\ell]}$. First, if $\ell = 0$, then $b_1[n-\ell] = b_1[n] = b_1$, and, by assumption, we have $\gamma \in u_{b_1}$. Next, suppose that $\ell < n - m^*$ and we have proven that $\gamma \in u_{b_1[n-\ell]}$. Then $b_1(n-\ell-1) > \max(b_0)$, so, by our thinning out of H_0 to H, we know that $u_{b_1[n-\ell]} \setminus u_{b_1[n-\ell-1]}$ is disjoint from u_{b_0}. Since $\gamma \in u_{b_0}$, it follows that $\gamma \in u_{b_1[n-\ell-1]}$. □

Claim 3.14. $\gamma \in u_{b_0[n-1]}$.

Proof. Since b_0 and b_1 are aligned and $\max(b_1) > \max(b_0)$, we know that $\max(b_0) \notin b_1$. Since m^* was least with $b_1(m^*) > \max(b_0)$, it follows that $\max(b_0) > \max(b_1[m^*])$. Therefore, by our thinning out of H_0 to H, we know that $u_{b_0} \setminus u_{b_0[n-1]}$ is disjoint from $u_{b_1[m^*]}$. Since $\gamma \in u_{b_1[m^*]}$ by the previous claim, it follows that $\gamma \in u_{b_0[n-1]}$. □

For $\ell < 2$, let $a_\ell = b_\ell[n-1]$ and $\beta_\ell = b_\ell(n-1)$. By the two previous claims and our choice of β_0 and β_1, we know that

$$\begin{aligned} \gamma \in u_{b_0[n-1]} \cap u_{b_1[m^*]} &= u_{b_0}[\mathbf{r}_{n-1}] \cap u_{b_1}[\mathbf{r}_{m^*}] \\ &\subseteq u_{b_0}[\mathbf{i}] \cap u_{b_1}[\mathbf{i}] \\ &= u_{a_0 \frown \langle \mu_M \rangle}[\mathbf{i}] \cap u_{a_1 \frown \langle \mu_M \rangle}[\mathbf{i}]. \end{aligned}$$

In particular, we have $i \in \mathbf{i}$ and, since $u_{b_1}[\mathbf{i}] = u_{a_1 \frown \langle \mu_M \rangle}[\mathbf{i}]$, we also know that $u_{a_1 \frown \langle \mu_M \rangle}(i) = \gamma$.

Since b_0 and b_1 are aligned, we know that a_0 and a_1 are aligned, and, since $n-1 \notin \mathbf{m}$, we also have $\mathbf{r}(a_0, a_1) = \mathbf{m}$. Therefore, by our choice of H_0, it follows that $u_{a_0 \frown \langle \mu_M \rangle}$ and $u_{a_1 \frown \langle \mu_M \rangle}$ are aligned and $\mathbf{r}(u_{a_0 \frown \langle \mu_M \rangle}, u_{a_1 \frown \langle \mu_M \rangle}) = \mathbf{s_m}$. Since $\gamma \in u_{a_0 \frown \langle \mu_M \rangle} \cap u_{a_1 \frown \langle \mu_M \rangle}$, it follows that $i \in \mathbf{s_m}$. But since $i \in \mathbf{i}$ and $\mathbf{r_m} = \mathbf{s_m} \cap \mathbf{i}$, it follows that $i \in \mathbf{r_m}$, which finishes the proof of clause (2).

We finally turn our attention to the "moreover" clause. To this end, fix $m < n$ and $a, b \in [H]^n$ such that a and b are aligned above m and $a(m) = b(m)$. Fix $\alpha \in H$ such that α is m-possible for both a and b. We must show that $\mathrm{tp_{int}}(u_a, u_b) = \mathrm{tp_{int}}(u_{a_{m \mapsto \alpha}}, u_{b_{m \mapsto \alpha}})$. Let $a^- = a[n-1]$ and $b^- = b[n-1]$, and let $a^+ = a^- \frown \langle \mu_M \rangle$ and $b^+ = b^- \frown \langle \mu_M \rangle$.

Suppose first that $m = n-1$, so $a(n-1) = b(n-1)$. By our construction of $\langle \alpha_\eta \mid \eta < \mu^* \rangle$, we know that $\mathrm{tp}(u_a, u_b) = \mathrm{tp}(u_{a^+}, u_{b^+}) = \mathrm{tp}(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}})$, and hence $\mathrm{tp}_{\mathrm{int}}(u_a, u_b) = \mathrm{tp}_{\mathrm{int}}(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}})$.

Suppose next that $m < n-1$. By the fact that $\langle u_{a^\frown\langle\mu_M\rangle} \mid a \in [H]^{n-1} \rangle$ satisfies the "moreover" clause in the statement of the theorem, we know that

$$(*) \qquad \mathrm{tp}_{\mathrm{int}}(u_{a^+}, u_{b^+}) = \mathrm{tp}_{\mathrm{int}}\left(u_{a^+_{m\mapsto\alpha}}, u_{b^+_{m\mapsto\alpha}}\right).$$

Suppose in addition that $a(n-1) = b(n-1)$. Then, by our construction of $\langle \alpha_\eta \mid \eta < \mu^* \rangle$, we know that

$$\mathrm{tp}(u_a, u_b) = \mathrm{tp}(u_{a^+}, u_{b^+}) \text{ and } \mathrm{tp}\left(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}}\right) = \mathrm{tp}\left(u_{a^+_{m\mapsto\alpha}}, u_{b^+_{m\mapsto\alpha}}\right).$$

Putting this together yields $\mathrm{tp}_{\mathrm{int}}(u_a, u_b) = \mathrm{tp}_{\mathrm{int}}(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}})$, as desired.

The remaining case is that in which $a(n-1) \neq b(n-1)$. We show that $\mathrm{tp}_{\mathrm{int}}(u_a, u_b) \subseteq \mathrm{tp}_{\mathrm{int}}(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}})$. A symmetric argument will yield the reverse inclusion. To this end, fix $(i,j) \in \mathrm{tp}_{\mathrm{int}}(u_a, u_b)$. Thus, we have $u_a(i) = u_b(j) = \gamma$ for some ordinal γ. Since a and b are aligned above m, $a(m) = b(m)$, and $a(n-1) \neq b(n-1)$, it follows that $b(n-1) \notin a$ and $a(n-1) \notin b$. An argument exactly as in the proofs of Claims 3.13 and 3.14 then shows that $\gamma \in u_{a^-} \cap u_{b^-} = u_a[\mathbf{i}] \cap u_b[\mathbf{i}]$, and hence we have $i, j \in \mathbf{i}$.

Since $i, j \in \mathbf{i}$, our construction of $\langle \alpha_\eta \mid \eta < \mu^* \rangle$ implies that $u_a(i) = u_{a^+}(i)$ and $u_b(j) = u_{b^+}(j)$, and hence $(i,j) \in \mathrm{tp}_{\mathrm{int}}(u_{a^+}, u_{b^+})$. By equation $(*)$ above, we have $(i,j) \in \mathrm{tp}_{\mathrm{int}}\left(u_{a^+_{m\mapsto\alpha}}, u_{b^+_{m\mapsto\alpha}}\right)$. Again by our construction of $\langle \alpha_\eta \mid \eta < \mu^* \rangle$ and the facts that $i, j \in \mathbf{i}$, we have $u_{a_{m\mapsto\alpha}}(i) = u_{a^+_{m\mapsto\alpha}}(i)$ and $u_{b_{m\mapsto\alpha}}(j) = u_{b^+_{m\mapsto\alpha}}(j)$, so $(i,j) \in \mathrm{tp}_{\mathrm{int}}(u_{a_{m\mapsto\alpha}}, u_{b_{m\mapsto\alpha}})$, thus finishing the proof. □

The following corollary gives an important special case, obtained from setting $\kappa = \aleph_0$ and $\lambda = \aleph_1$ in Theorem 3.8.

Corollary 3.15. *Suppose that $1 \leq n < \omega$, and let $\mu = \beth_{n-1}^+$. If $\langle u_b \mid b \in [\mu]^n \rangle$ is a sequence of finite sets of ordinals and $g : [\mu]^n \to \omega$ is a function, then there is $H \in [\mu]^{\aleph_1}$ such that $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system and $g \restriction [H]^n$ is constant.* □

We end this section with a discussion of the optimality of Theorem 3.8. It can be argued that, if $\kappa < \lambda \leq \mu$ are infinite cardinals, $1 \leq n < \omega$, and $\mu \to (\lambda)^{2n}_{2^{<\kappa}}$, then any sequence $\langle u_a \mid a \in [\mu]^n \rangle$ consisting of elements of $[\mathrm{On}]^{<\kappa}$ can be thinned out to a uniform n-dimensional Δ-system of size λ (see [3] for such an argument).

In general, $\mu \to (\lambda)^{2n}_{2^{<\kappa}}$ is a stronger assertion than $\mu \geq \sigma(\lambda, n)$, which is our assumption in Theorem 3.8, so this argument yields weaker results than those of Theorem 3.8. However, if λ is weakly compact, then we have $\lambda \to (\lambda)^{2n}_{2^{<\kappa}}$ for all $1 \leq n < \omega$ and all $\kappa < \lambda$, so we obtain the following corollary.

Corollary 3.16. *Suppose that $1 \leq n < \omega$ and that $\kappa < \lambda$ are infinite cardinals, with λ being weakly compact. Suppose also that $\langle u_a \mid a \in [\lambda]^n \rangle$ is a sequence consisting of elements of $[\mathrm{On}]^{<\kappa}$. Then there is $H \in [\lambda]^\lambda$ such that $\langle u_a \mid a \in [H]^n \rangle$ is a uniform n-dimensional Δ-system.* □

If λ is *not* weakly compact, though, then our result is optimal in the sense that the value of μ cannot be decreased. This is true even disregarding clause (1) or the "moreover clause" of Theorem 3.8 and focusing only on the higher-dimensional

Δ-systems (and not even requiring that the Δ-systems be uniform), for essentially the same reason that the Erdős-Rado theorem is optimal.

Proposition 3.17. *Suppose that $1 \leq n < \omega$ and λ is a regular uncountable cardinal that is not weakly compact, and suppose that $\mu < \sigma(\lambda, n)$. Then there is a sequence $\langle u_a \mid a \in [\mu]^n \rangle$ consisting of finite sets of ordinals such that there is no $H \in [\mu]^\lambda$ for which $\langle u_a \mid a \in [H]^n \rangle$ is an n-dimensional Δ-system.*

Proof. If $n = 1$, then we have $\mu < \lambda$, so the result is trivial. So assume that $n > 1$. Since λ is uncountable, regular, and not weakly compact, [8, Corollary 21.5] implies that $2^{<\lambda} \not\to (\lambda)^2_2$. Therefore, by successive applications of [7, Lemma 5A], which is the lemma establishing the optimality of the Erdős-Rado theorem, we have, for all $m < \omega$, $\beth_m(2^{<\lambda}) \not\to (\lambda)^{2+m}_2$. By Remark 3.2(2), $\sigma(\lambda, n) = \left(\beth_{n-2}(2^{<\lambda})\right)^+$ for all $2 \leq n < \omega$. Therefore, we have $\mu \leq \beth_{n-2}(2^{<\lambda})$, so there is a function $c : [\mu]^n \to 2$ that is not constant on $[H]^n$ for any $H \in [\mu]^\lambda$. For each $a \in [\mu]^n$, simply let $u_a = c(a)$. Now suppose that $H \in [\mu]^\lambda$, and suppose for sake of contradiction that $\langle u_a \mid a \in [H]^n \rangle$ is an n-dimensional Δ-system, as witnessed by roots $\langle R^{\mathbf{m}}_a \mid a \in [H]^{\leq n}, \ \mathbf{m} \in [n]^{a} \rangle$. Using the fact that c is not constant on $[H']^n$ for any unbounded $H' \subseteq H$, we can fix three sets $a_0 < a_1 < a_2$ in $[H]^n$ such that $c(a_0) = 0$ and $c(a_1) = c(a_2) = 1$. By the definition of an n-dimensional Δ-system, we should have $u_{a_0} \cap u_{a_1} = R^{\emptyset}_{\emptyset} = u_{a_1} \cap u_{a_2}$. However, we actually have $u_{a_0} \cap u_{a_1} = \emptyset$ and $u_{a_1} \cap u_{a_2} = 1$, which is our desired contradiction. $\square$

Before turning to the optimality of the value of κ in Theorem 3.8, we pause to summarize the results of this section thus far in a corollary connecting Theorem 3.8 and Proposition 3.17 with the Erdős-Rado theorem.

Corollary 3.18. *Suppose that $1 \leq n < \omega$ and that λ and μ are infinite regular cardinals such that λ is uncountable but not weakly compact. Then the following are equivalent:*

(1) $\mu \geq \sigma(\lambda, n)$;
(2) $\mu \to (\lambda)^n_2$;
(3) $\mu \to (\lambda + (n-1))^n_\nu$ for every $\nu < \lambda$;
(4) for every sequence $\langle u_b \mid b \in [\mu]^n \rangle$ such that each u_b is a finite set, there is $H \in [\mu]^\lambda$ such that $\langle u_b \mid b \in [H]^n \rangle$ is an n-dimensional Δ-system;
(5) the conclusion of Theorem 3.8 holds for n, λ, and μ, and for any choice of κ, ν, $g : [\mu]^n \to \nu$, and $\langle u_b \mid b \in [\mu]^n \rangle$ such that
  (a) $\nu < \lambda$;
  (b) λ is $<\kappa$-inaccessible; and
  (c) $u_b \in [\mathrm{On}]^{<\kappa}$ for every $b \in [\mu]^n$.

Proof. (1) $\Rightarrow$ (3) is the Erdős-Rado theorem, or the pigeonhole principle if $n = 1$ (it can also be extracted from our proof of Theorem 3.8), and (3) $\Rightarrow$ (2) is immediate. (1) $\Rightarrow$ (5) is Theorem 3.8, and (5) $\Rightarrow$ (4) follows by setting $\kappa = \aleph_0$ in Theorem 3.8 and invoking Proposition 2.6(2). (4) $\Rightarrow$ (2) is precisely the second half of the proof of Proposition 3.17. Finally, (2) $\Rightarrow$ (1) follows from the optimality of the Erdős-Rado theorem (the argument in the first half of the proof of Proposition 3.17). $\square$

We now turn to the optimality of κ in Theorem 3.8; in other words, we investigate the necessity of the requirement that λ be $<\kappa$-inaccessible in the statement of the theorem. It turns out that the optimality of κ is slightly more complicated than the optimality of μ, since even if λ is not $<\kappa$-inaccessible, it could be the case that $\sigma(\lambda, n) = \sigma(\lambda^*, n)$ for some $\lambda^* > \lambda$ such that λ^* *is* $<\kappa$-inaccessible. For example, suppose that $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$. Then $\sigma(\aleph_2, n) = \sigma(\aleph_3, n)$ for all $n \geq 2$. Also, $\aleph_3$ is $<\aleph_1$-inaccessible, so Theorem 3.8 holds for $\lambda = \aleph_3$ and $\kappa = \nu = \aleph_1$ (and any value of n). This immediately implies that the conclusion of Theorem 3.8 holds for $\lambda = \aleph_2$, $\kappa = \nu = \aleph_1$, and $2 \leq n < \omega$, despite the fact that $\aleph_2$ is not $<\aleph_1$-inaccessible. We can show however, that this is essentially the only way in which the value of κ in Theorem 3.8 can fail to be optimal.

Proposition 3.19. *Suppose that $1 \leq n < \omega$ and $\kappa < \lambda$ are infinite cardinals such that λ is regular and not $<\kappa$-inaccessible. Let $\lambda^* = (\lambda^{<\kappa})^+$, and suppose that $\mu < \sigma(\lambda^*, n)$. Then there is a sequence $\langle u_a \mid a \in [\mu]^n\rangle$ consisting of elements of $[\lambda]^{<\kappa}$ such that there is no $H \in [\mu]^\lambda$ for which $\langle u_a \mid a \in [H]^n\rangle$ is an n-dimensional Δ-system.*

Proof. Fix a cardinal $\nu < \lambda$ such that $\nu \geq \kappa$ and $\nu^{<\kappa} \geq \lambda$. Next, fix an injective sequence $\langle x_\eta \mid \eta < \lambda\rangle$ of elements of $[\nu]^{<\kappa}$ such that, for all distinct $\eta, \xi < \lambda$, neither of x_η nor x_ξ is a subset of the other. One way to see that this can be done is the following. Let $\langle \kappa_i \mid i < \theta\rangle$ be such that

- if κ is a successor cardinal, then $\theta = 1$ and κ_0 is its immediate predecessor (so $\nu^{<\kappa} = \nu^{\kappa_0}$); or
- if κ is a limit cardinal, then $\theta = \mathrm{cf}(\kappa)$ and $\langle \kappa_i \mid i < \theta\rangle$ is a strictly increasing sequence of cardinals converging to κ.

Now let $\langle f_\eta \mid \eta < \lambda\rangle$ be an injective sequence of elements of $\bigcup_{i<\theta} {}^{\kappa_i}\nu$. Partition ν into pairwise disjoint pieces $\langle A_i \mid i < \theta\rangle$, each of size ν and, for each $i < \theta$, let $\pi_i : \kappa_i \times \nu \to A_i$ be a bijection. Now, viewing elements of ${}^{\kappa_i}\nu$ as subsets of $\kappa_i \times \nu$, for each $\eta < \lambda$, let i_η be the unique $i < \theta$ such that $f_\eta \in {}^{\kappa_i}\nu$, and let $x_\eta = \pi_i``f_\eta$. Then $\langle x_\eta \mid \eta < \lambda\rangle$ is as desired. Also, fix an injective sequence $\langle y_\alpha \mid \alpha < \lambda^{<\kappa}\rangle$ of elements of $[\lambda]^{<\kappa}$ such that

- for all $\alpha < \lambda^{<\kappa}$, we have $y_\alpha \cap \nu = \emptyset$;
- for all $\alpha < \beta < \lambda^{<\kappa}$, neither of y_α nor y_β is a subset of the other.

For all $\alpha < \lambda^{<\kappa}$, let $\eta_\alpha = \sup(y_\alpha)$. Since λ is regular, we have $\eta_\alpha < \lambda$.

Suppose first that $n = 1$. Then $\sigma(\lambda^*, 1) = \lambda^* = (\lambda^{<\kappa})^+$, so we can assume that $\mu = \lambda^{<\kappa}$. For all $\alpha < \mu$, let $u_\alpha = y_\alpha \cup x_{\eta_\alpha}$. Fix $H \in [\mu]^\lambda$, and suppose for sake of contradiction that $\langle u_\alpha \mid \alpha \in H\rangle$ is a Δ-system, with root r. Let $r^- := r \cap \nu$ and $r^+ := r \setminus \nu$. Note that, for all distinct $\alpha, \beta \in H$, we have $y_\alpha \cap y_\beta = r^+$ and $x_{\eta_\alpha} \cap x_{\eta_\beta} = r^-$.

There are now two cases to consider, depending on whether or not $\{\eta_\alpha \mid \alpha \in H\}$ is unbounded in λ. Suppose first that $\eta^* := \sup\{\eta_\alpha \mid \alpha \in H\}$ is less than λ. Then $\langle y_\alpha \setminus r^+ \mid \alpha \in H\rangle$ is an injective sequence of pairwise disjoint subsets of $\eta^* + 1$, contradicting the fact that $|H| = \lambda > \eta^* + 1$. Suppose next that $\eta^* = \lambda$. Then $\{x_{\eta_\alpha} \setminus r^- \mid \alpha \in H\}$ is a set of size λ consisting of pairwise disjoint subsets of ν, contradicting the fact that $|H| = \lambda > \nu$.

Now suppose that $n > 1$. Then, by Remark 3.2(1), we know that $\sigma(\lambda^*, n) = (\beth_{n-1}(\lambda^{<\kappa}))^+$, so we can assume that $\mu = \beth_{n-1}(\lambda^{<\kappa})$. For any infinite cardinal χ, the coloring $d : [{}^\chi 2]^2 \to \chi$ defined by letting $d(f, g)$ be the least $\xi < \chi$ for which $f(\xi) \neq g(\xi)$ for all distinct $f, g \in {}^\chi 2$ witnesses the negative partition relation

$2^\chi \not\to (3)^2_\chi$. Therefore, setting $\chi = \lambda^{<\kappa}$ and repeatedly applying [7, Lemma 5A], we have $\mu \not\to (\aleph_0)^n_{\lambda^{<\kappa}}$. Let $c : [\mu]^n \to \lambda^{<\kappa}$ witness this negative partition relation.

Now define $\langle u_a \mid a \in [\mu]^n \rangle$ by letting $u_a = y_{c(a)} \cup x_{\eta_{c(a)}}$ for all $a \in [\mu]^n$. Fix $H \in [\mu]^\lambda$, and suppose for sake of contradiction that $\langle u_a \mid a \in [H]^n \rangle$ is an n-dimensional Δ-system, as witnessed by roots $\langle R^{\mathbf{m}}_a \mid a \in [H]^{\leq n},\ \mathbf{m} \in [n]^a \rangle$. Let $r := R^0_0$, $r^- := r \cap \nu$, and $r^+ := r \setminus \nu$. Since c witnesses $\mu \not\to (\aleph_0)^n_{\lambda^{<\kappa}}$, we can find disjoint sets $a_0, a_1 \in [H]^n$ such that $c(a_0) \neq c(a_1)$. Now arbitrarily fix a set $a_\gamma \in [H]^n$ for each $2 \leq \gamma < \lambda$ in such a way that $\langle a_\gamma \mid \gamma < \lambda \rangle$ is an injective sequence of pairwise disjoint sets. Since $\langle u_a \mid a \in [H]^n \rangle$ is an n-dimensional Δ-system, we know that, for all $\gamma < \delta < \lambda$, we have $u_{a_\gamma} \cap u_{a_\delta} = r$, and hence $y_{c(a_\gamma)} \cap y_{c(a_\delta)} = r^+$ and $x_{\eta_{c(a_\gamma)}} \cap x_{\eta_{c(a_\delta)}} = r^-$.

There are now two possibilities. First, suppose that there are $\gamma < \delta < \lambda$ for which $u_{a_\gamma} = u_{a_\delta}$. Then we can find $\ell < 2$ for which $u_{a_\ell} \neq u_{a_\gamma}$ (and hence $u_{a_\gamma} \not\subseteq u_{a_\ell}$). But now we are in the same situation as in the proof of Proposition 3.17: we must have $u_{a_\gamma} \cap u_{a_\delta} = r = u_{a_\ell} \cap u_{a_\gamma}$, but $u_{a_\gamma} \cap u_{a_\delta} = u_{a_\gamma}$, and since $u_{a_\gamma} \not\subseteq u_{a_\ell}$, we have $u_{a_\ell} \cap u_{a_\gamma} \neq u_{a_\gamma}$, which is a contradiction.

The other possibility is that the sets $\langle u_{a_\gamma} \mid \gamma < \lambda \rangle$ are all pairwise disjoint. There are now two subcases, depending on whether or not $\eta^* := \sup\{\eta_{c(a_\gamma)} \mid \gamma < \lambda\}$ is equal to λ. If $\eta^* < \lambda$, then $\langle u_{a_\gamma} \setminus r \mid \gamma < \lambda \rangle$ is an injective sequence of pairwise disjoint subsets of $\max\{\eta^* + 1, \nu\}$. If $\eta^* = \lambda$, then $\{x_{\eta_{c(a_\gamma)}} \setminus r^- \mid \gamma < \lambda\}$ is a set of size λ consisting of pairwise disjoint subsets of ν. In either case, we contradict the fact that $\lambda > \max\{\eta^* + 1, \nu\}$. □

If $\kappa < \lambda$ are both regular infinite cardinals and λ is not $<\kappa$-inaccessible, then $(\lambda^{<\kappa})^+$ is the least $<\kappa$-inaccessible cardinal greater than or equal to λ (it can fail to be $<\kappa$-inaccessible if κ is singular). We can therefore combine the results of this section in the following equivalence.

Corollary 3.20. *Suppose that $1 \leq n < \omega$ and $\kappa < \lambda$ are infinite regular cardinals. Let λ^* be the least $<\kappa$-inaccessible cardinal greater than or equal to λ, and suppose that μ is an infinite cardinal. Then the following are equivalent.*

1. *$\mu \geq \sigma(\lambda^*, n)$;*
2. *the conclusion of Theorem 3.8 holds for n, κ, λ, and μ with any choice of $\nu < \lambda$, $g : [\mu]^n \to \nu$, and $\langle u_b \mid b \in [\mu]^n \rangle$ with each u_b in $[\mathrm{On}]^{<\kappa}$;*
3. *for every sequence $\langle u_a \mid a \in [\mu]^n \rangle$ such that each u_a is a set of cardinality less than κ, there is $H \in [\mu]^\lambda$ such that $\langle u_a \mid a \in [H]^n \rangle$ is an n-dimensional Δ-system.*

Proof. (1) ⇒ (2) follows from Theorem 3.8, and (2) ⇒ (3) is immediate. If λ is $<\kappa$-inaccessible, then $\lambda^* = \lambda$, in which case (3) ⇒ (1) follows from Proposition 3.17. If λ is not $<\kappa$-inaccessible, then (3) ⇒ (1) follows from Proposition 3.19 and the observation that $\lambda^* = \lambda^{<\kappa}$ in this case. □

4. Chain conditions

One of the primary uses of the classical Δ-system lemma is in proving that certain forcing notions satisfy chain conditions. For example, one of the first applications that many people learn is in the proof that the forcing notion to add any number of Cohen reals is κ-Knaster for every regular uncountable κ:

Lemma 4.1. *Let χ be any infinite cardinal, and let $\mathbb{P} = \mathrm{Add}(\omega, \chi)$ be the forcing to add χ-many Cohen reals. Suppose that κ is a regular uncountable cardinal and $\langle p_\alpha \mid \alpha < \kappa \rangle$ is a sequence of conditions from $\mathbb{P}$. Then there is an unbounded $A \subseteq \kappa$ such that $\langle p_\alpha \mid \alpha \in A \rangle$ consists of pairwise compatible conditions.*

During forcing constructions involving higher-dimensional combinatorial statements, one frequently encounters sequences of conditions indexed not by single ordinals but by n-element sets of ordinals for some $n > 1$. One would then like to find a large set such that the restriction of the sequence to that set satisfies certain uniformities analogous to the uniformities exhibited by $\langle p_\alpha \mid \alpha \in A \rangle$ in Lemma 4.1. A first, naïve attempt at formulating a statement to this effect, similar to our overly optimistic first attempt to define higher-dimensional Δ-systems at the start of Section 2, might look vaguely as follows:

> Let χ be an infinite cardinal and $n < \omega$, and let $\mathbb{P}$ be the forcing to add χ-many Cohen reals. Then there is a sufficiently large regular cardinal $\mu \leq \chi$ such that, for every sequence $\langle p_a \mid a \in [\mu]^n \rangle$ of conditions in $\mathbb{P}$, there is a "large" set $H \subseteq \mu$ such that $\langle p_a \mid a \in [H]^n \rangle$ consists of pairwise compatible conditions.

It is easily seen that such a statement cannot possibly hold if $n > 1$, however. Indeed suppose that $n = 2$ and, for all $(\alpha, \beta) \in [\mu]^2$, define a condition $p_{\alpha\beta} \in \mathbb{P}$ by letting $\mathrm{dom}(p_{\alpha\beta}) = \{\alpha, \beta\}$, $p_{\alpha\beta}(\alpha) = 0$, and $p_{\alpha\beta}(\beta) = 1$ (we are thinking of conditions in $\mathbb{P}$ as being finite partial functions from χ to 2). Then $p_{\alpha\beta} \perp p_{\beta\gamma}$ for all $(\alpha, \beta, \gamma) \in [\mu]^3$, so we could not even find a set H of size 3 as in the above statement. The obvious problem here is that the sets $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are not aligned, and it turns out that this is the only obstacle. By requiring the compatibility of p_a and p_b only when a and b are aligned, we obtain a consistent statement. For example:

Lemma 4.2. *Suppose that λ is a regular uncountable cardinal, $1 \leq n < \omega$, and $\mu = \sigma(\lambda, n)$, and suppose that $\mathbb{P}$ is the forcing notion to add χ-many Cohen reals for some infinite cardinal χ. Then, for every sequence $\langle p_a \mid a \in [\mu]^n \rangle$ of conditions in $\mathbb{P}$, there is a set $H \in [\mu]^\lambda$ such that, for all $a, b \in [H]^n$, if a and b are aligned, then $p_a \parallel p_b$.*

Proof. Fix a sequence $\langle p_a \mid a \in [\mu]^n \rangle$ consisting of conditions in $\mathbb{P}$. For each $a \in [\mu]^n$, let $u_a = \mathrm{dom}(p_a)$ and $k_a = \mathrm{otp}(u_a)$, and let $\bar{p}_a : k_a \to 2$ denote the condition isomorphic to p_a, i.e., $\bar{p}_a(i) = p_a(u_a(i))$ for all $i < k_a$. Now apply Theorem 3.8 to $\langle u_a \mid a \in [\mu]^n \rangle$ and the function $a \mapsto \bar{p}_a$ to find an $H \in [\mu]^\lambda$, a $k < \omega$, and a function $\bar{p} : k \to 2$ such that $\langle u_a \mid a \in [H]^n \rangle$ is a uniform n-dimensional Δ-system and $\bar{p}_a = \bar{p}$ for all $a \in [H]^n$.

We claim that $p_a \parallel p_b$ for all aligned $a, b \in [H]^n$. To this end, fix $a, b \in [H]^n$ such that a and b are aligned. The only way we could have $p_a \perp p_b$ is if there is $\alpha \in u_a \cap u_b$ such that $p_a(\alpha) \neq p_b(\alpha)$. Since $\langle u_a \mid a \in [H]^n \rangle$ is a uniform n-dimensional Δ-system, we know that u_a and u_b are aligned. Moreover, we know that $\bar{p}_a = \bar{p}_b = \bar{p}$. Therefore, if $\alpha \in u_a \cap u_b$, then there is $i < k$ such that $\alpha = u_a(i) = u_b(i)$. But then $p_a(\alpha) = \bar{p}(i) = p_b(\alpha)$. Therefore, we have $p_a \parallel p_b$. $\square$

Remark 4.3. If λ is weakly compact, then, by Corollary 3.16, Lemma 4.2 still holds with $\mu = \lambda$ rather than $\mu = \sigma(\lambda, n)$.

5. An application to polarized partition relations

In this section, we give a relatively simple application illustrating a typical use of Theorem 3.8 in a forcing argument. The following definition was introduced by Todorcevic.

Definition 5.1 ([19, Remark 9.3.3])**.** Let $1 \leq n < \omega$. Then Θ_n is the least cardinal θ such that, for every function $f : \theta^n \to \omega$, there is a sequence $\langle A_i \mid i < n \rangle$ of infinite subsets of θ such that $f \restriction \prod_{i<n} A_i$ is constant.

We clearly have $\Theta_1 = \aleph_1$. The next proposition establishes lower bounds for Θ_n for $n > 1$.

Proposition 5.2. *Suppose that $1 \leq n < \omega$, κ is a cardinal, and $\Theta_n > \kappa$. Then $\Theta_{n+1} > \kappa^+$.*

Proof. Since $\Theta_n > \kappa$, we can fix a function $g : \kappa^n \to \omega$ such that g is not constant on any product of n infinite subsets of κ. For each $\beta < \kappa^+$, fix an injective function $e_\beta : \beta \to \kappa$. Then the function $g_\beta : \beta^n \to \omega$ defined by letting

$$g_\beta(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle) = g(\langle e_\beta(\alpha_0), \ldots, e_\beta(\alpha_{n-1}) \rangle)$$

for all $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in \beta^n$ has the property that g_β is not constant on any product of n infinite subsets of β.

We now define a function $f : (\kappa^+)^{n+1} \to (n+2) \times \omega$ that will not be constant on any product of $(n+1)$ infinite subsets of κ^+. This can easily be coded as a function into ω, so this suffices to prove the proposition.

Given $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_n \rangle \in (\kappa^+)^{n+1}$ and $i \leq n$, let $\vec{\alpha}^i$ denote the sequence formed by removing α_i from $\vec{\alpha}$, i.e., $\vec{\alpha}^i = \langle \alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \rangle$. Let us now define $f(\vec{\alpha})$. If there are $i < j \leq n$ such that $\alpha_i = \alpha_j$, then let $f(\vec{\alpha}) = (n+1, 0)$. Otherwise, let $i \leq n$ be such that $\alpha_j < \alpha_i$ for all $j \in (n+1) \setminus \{i\}$, and let $f(\vec{\alpha}) = (i, g_{\alpha_i}(\vec{\alpha}^i))$.

Suppose for sake of contradiction that $\langle A_i \mid i \leq n \rangle$ is a sequence of infinite subsets of κ^+ such that $f \restriction \prod_{i \leq n} A_i$ is constant, taking value (m, k). First note that we can always find a sequence $\vec{\alpha} \in \prod_{i \leq n} A_i$ whose coordinates are all distinct, so it cannot be the case that $m = n+1$. Thus, $m \leq n$, so, by our definition of f, it follows that $A_j < A_m$ for all $j \in (n+1) \setminus \{m\}$. Fix $\beta \in A_m$, and define $\langle A^*_j \mid j < n \rangle$ by letting $A^*_j = A_j$ for $j < m$ and $A^*_j = A_{j+1}$ for $m \leq j < n$. Then each A^*_j is an infinite subset of β and, by our definition of f, it follows that $g_\beta \restriction \prod_{j<n} A^*_j$ is constant, taking value k, contradicting our assumptions about g_β. □

In particular, we immediately obtain the following corollary, answering a part of Question 9.3.4 from [19].

Corollary 5.3. $\Theta_n \geq \aleph_n$ *for all* $1 \leq n < \omega$. □

It follows easily from the Erdős-Rado theorem that $\Theta_n \leq \beth^+_{n-1}$ for all $1 \leq n < \omega$. In particular, if GCH holds, then $\Theta_n = \aleph_n$ for all $1 \leq n < \omega$. We now apply Theorem 3.8 to prove that adding any number of Cohen reals preserves the inequality $\Theta_n \leq (\beth^+_{n-1})^V$. (In fact, we will prove that a slightly stronger partition relation, which easily implies $\Theta_n \leq (\beth^+_{n-1})^V$, holds after forcing to add the Cohen reals.) Thus, if we start with a model of GCH, then we can force to make the continuum arbitrarily large while keeping $\Theta_n = \aleph_n$ for all $1 \leq n < \omega$.

Theorem 5.4. *Suppose that $1 \leq n < \omega$ and χ is an infinite cardinal. Let $\mu = \beth_{n-1}^+$, and let $\mathbb{P}$ be the forcing to add χ-many Cohen reals. Then the following statement holds in $V^{\mathbb{P}}$:*

> *For every function $c : [\mu]^n \to \omega$, there is a sequence $\langle A_m \mid m < n \rangle$ such that*
> - *for all $m < n$, A_m is a subset of μ of order type $\omega + 1$;*
> - *for all $m < m' < n$, we have $A_m < A_{m'}$;*
> - *$c \restriction \prod_{m<n} A_m$ is constant.*

Proof. We think of conditions in $\mathbb{P}$ as being finite partial functions from χ to 2, ordered by reverse inclusion. Given a condition $p \in \mathbb{P}$, let $\bar{p}$ denote the function from $|\mathrm{dom}(p)|$ to 2 defined by letting $\bar{p}(i) = p(\mathrm{dom}(p)(i))$ for all $i < |\mathrm{dom}(p)|$.

Since the conclusion of the theorem is trivial if $n = 1$, we may assume that $n > 1$. Fix a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{c}$ forced by p to be a function from $[\mu]^n$ to ω. For each $b \in [\mu]^n$, find a condition $q_b \leq p$ and a color $k_b < \omega$ such that $q_b \Vdash$ "$\dot{c}(b) = k_b$". Let $u_b = \mathrm{dom}(q_b)$, and define a function $g : [\mu]^n \to {}^{<\omega}2 \times \omega$ by letting $g(b) = \langle \bar{q}_b, k_b \rangle$ for all $b \in [\mu]^n$. Apply Theorem 3.8 to find $H \in [\mu]^{\aleph_1}$ such that $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system and $g \restriction [H]^n$ is constant, taking value $\langle \bar{q}, k \rangle$. By taking an initial segment if necessary, assume that we in fact have $\mathrm{otp}(H) = \omega_1$. Note that, if b and b' are aligned elements of $[H]^n$, then q_b and $q_{b'}$ are compatible in $\mathbb{P}$.

Let $\rho = |\bar{q}|$, and let $\langle \mathbf{r_m} \subseteq \rho \mid \mathbf{m} \subseteq n \rangle$ witness the fact that $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system. For each $m < n$ and each $a \in [H]^m$, define u_a by letting b be any element of $[H]^n$ such that $b[m] = a$ and then letting $u_a := u_b[\mathbf{r}_m]$ (we are thinking of m as an initial subset of n here). Then set $q_a := q_b \restriction u_a$. By Proposition 2.6 and the fact that $\bar{q}_b = \bar{q}$ for all $b \in [H]^n$, it follows that our definition of u_a and q_a is independent of our choice of b.

By the arguments of Claim 3.12, we know that, for every $m < n$ and every $a \in [H]^m$, the sequence $\langle u_{a ^\frown \langle \beta \rangle} \mid \beta \in H \setminus (\max(a) + 1) \rangle$ is a 1-dimensional Δ-system, with root u_a. Since $q_b \leq p$ for all $b \in [H]^n$, it follows that $\mathrm{dom}(p) \subseteq u_\emptyset$ and $q_\emptyset \leq p$. We will show that $q_\emptyset$ forces the existence of a sequence $\langle A_m \mid m < n \rangle$ in $V^{\mathbb{P}}$ such that

- each A_m is a subset of μ of order type $\omega + 1$;
- $A_m < A_{m'}$ for all $m < m' < n$;
- the realization of $\dot{c}$ is constant when restricted to $\prod_{m<n} A_m$, with value k.

Since p was arbitrary, this suffices to prove the theorem. We first need the following claim.

Claim 5.5. *Suppose that $m < n$, $a \in [H]^m$, and $\gamma \in H \setminus (\max(a) + 1)$. Then the set $D_{a,\gamma} = \{q_{a ^\frown \langle \beta \rangle} \mid \beta \in H \setminus \gamma\}$ is predense below q_a in $\mathbb{P}$.*

Proof. Fix a condition $r \leq q_a$. We will find an element of $D_{a,\gamma}$ compatible with r. Since $\langle u_{a ^\frown \langle \beta \rangle} \mid \beta \in H \setminus \gamma \rangle$ is an infinite 1-dimensional Δ-system with root u_a, and since $\mathrm{dom}(r)$ is finite, we can find $\beta \in H \setminus \gamma$ such that $u_{a ^\frown \langle \beta \rangle} \setminus u_a$ is disjoint from $\mathrm{dom}(r)$. But then $q_{a ^\frown \langle \beta \rangle} \restriction \mathrm{dom}(r) = q_a$, so, since $r \leq q_a$, it follows that $r \cup q_{a ^\frown \langle \beta \rangle}$ is a condition in $\mathbb{P}$, so $q_{a ^\frown \langle \beta \rangle}$ is an element of $D_{a,\gamma}$ compatible with r. $\square$

Now suppose that G is $\mathbb{P}$-generic over V with $q_\emptyset \in G$, and let c be the realization of $\dot{c}$ in $V[G]$. By applying Claim 5.5 n times, working in $V[G]$, we can recursively choose an increasing sequence $\langle \delta_m \mid m < n\rangle$ of elements of H such that, letting $d = \{\delta_m \mid m < n\}$, we have

- $q_d \in G$;
- $H \cap \delta_0$ is infinite;
- for all $m < n-1$, $H \cap (\delta_{m+1} \setminus (\delta_m + 1))$ is infinite.

Let A^*_0 denote the set of the first ω-many elements of $H \cap \delta_0$ and, for all $m < n-1$, let A^*_{m+1} denote the set of the first ω-many elements of $H \cap (\delta_{m+1} \setminus (\delta_m + 1))$.

We now construct an $n \times \omega$ matrix $\langle \alpha_{m,\ell} \mid m < n,\ \ell < \omega\rangle$ such that

- for all $m < n$, $\langle \alpha_{m,\ell} \mid \ell < \omega\rangle$ is an increasing sequence of elements of A^*_m;
- letting $A_m = \{\alpha_{m,\ell} \mid \ell < \omega\} \cup \{\delta_m\}$ for each $m < n$, we have $q_b \in G$ for all $b \in \prod_{m<n} A_m$.

The construction is by recursion on the anti-lexicographical order on $n \times \omega$, i.e., we set $(m,\ell) < (m',\ell')$ if $\ell < \ell'$ or ($\ell = \ell'$ and $m < m'$). During the construction, at stage (m,ℓ), for all $m' < n$, we will let $A_{m'} \restriction (m,\ell)$ denote the set $\{\alpha_{m',\ell'} \mid \ell' \leq \ell\} \cup \{\delta_{m'}\}$ if $m' < m$ and $\{\alpha_{m',\ell'} \mid \ell' < \ell\} \cup \{\delta_{m'}\}$ if $m \leq m'$. In other words, $A_{m'} \restriction (m,\ell)$ is simply the portion of $A_{m'}$ that we have specified before stage (m,ℓ) of the construction. Our recursion hypothesis will be the assumption that, when we reach stage (m,ℓ), for all $b \in \prod_{m'<n} A_{m'} \restriction (m,\ell)$, we have $q_b \in G$. It will then follow that $q^*_{m,\ell} := \bigcup\{q_b \mid b \in \prod_{m'<n} A_{m'} \restriction (m,\ell)\}$ is also an element of G.

To begin the construction, note that, for all $m' < n$, we have $A_{m'} \restriction (0,0) = \{\delta_{m'}\}$, so $q^*_{0,0} = q_d \in G$. Thus, our recursion hypothesis is initially satisfied. Now suppose that $(m,\ell) \in n \times \omega$ and we have defined $\langle \alpha_{m',\ell'} \mid (m',\ell') < (m,\ell)\rangle$ so that the resulting condition $q^*_{m,\ell}$ is in G. Temporarily move back to V, noting that each $A_{m'} \restriction (m,\ell)$ is finite and hence in V, and A^*_m is also in V, as it is definable from H (and δ_{m-1}, if $m > 0$).

Let $B_0 = \prod_{m'<m}(A_{m'} \restriction (m,\ell))$ and $B_1 = \prod_{m<m'<n}(A_{m'} \restriction (m,\ell))$. If $\ell > 0$, then let $\gamma = \alpha_{m,\ell-1} + 1$; if $\ell = 0$, then let $\gamma = 0$. For each $\alpha \in A^*_m \setminus \gamma$, let $q^*_\alpha = \bigcup\{q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \mid b_0 \in B_0,\ b_1 \in B_1\}$. Notice that, if $b_0, b'_0 \in B_0$ and $b_1, b'_1 \in B_1$, then $b_0{}^\frown\langle\alpha\rangle^\frown b_1$ and $b'_0{}^\frown\langle\alpha\rangle^\frown b'_1$ are aligned, and hence $q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}$ and $q_{b'_0{}^\frown\langle\alpha\rangle^\frown b'_1}$ are compatible. It follows that q^*_α is a condition in $\mathbb{P}$.

Claim 5.6. *The set $E = \{q^*_\alpha \mid \alpha \in A^*_m \setminus \gamma\}$ is predense below $q^*_{m,\ell}$ in $\mathbb{P}$.*

Proof. Fix $r \leq q^*_{m,\ell}$. We will find an element of E compatible with r. Let $\mathbf{m} = n \setminus \{m\}$. For each $(b_0, b_1) \in B_0 \times B_1$, the sequence $\langle u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \mid \alpha \in A^*_m \setminus \gamma\rangle$ forms a 1-dimensional Δ-system whose root is equal to $u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}[\mathbf{r_m}]$ for all $\alpha \in A^*_m \setminus \gamma$. Since $A^*_m \setminus \gamma$ is infinite and since $\mathrm{dom}(r)$, B_0, and B_1 are all finite, we can find $\alpha \in A^*_m \setminus \gamma$ such that, for all $(b_0, b_1) \in B_0 \times B_1$, the set $u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \setminus (u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}[\mathbf{r_m}])$ is disjoint from $\mathrm{dom}(r)$.

We claim that q^*_α and r are compatible. To see this, it suffices to show that $q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}$ and r are compatible for every $(b_0, b_1) \in B_0 \times B_1$. Thus, fix $(b_0, b_1) \in B_0 \times B_1$. We know that $u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \cap \mathrm{dom}(r) \subseteq u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}[\mathbf{r_m}]$. Since $b_0{}^\frown\langle\alpha\rangle^\frown b_1$ and $b_0{}^\frown\langle\delta_m\rangle^\frown b_1$ are aligned with $\mathbf{r}(b_0{}^\frown\langle\alpha\rangle^\frown b_1, b_0{}^\frown\langle\delta_m\rangle^\frown b_1) = \mathbf{m}$, we also know that $q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \parallel q_{b_0{}^\frown\langle\delta_m\rangle^\frown b_1}$ and $u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}[\mathbf{r_m}] = u_{b_0{}^\frown\langle\delta_m\rangle^\frown b_1}[\mathbf{r_m}]$. Then

$$q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \restriction (u_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}[\mathbf{r_m}]) = q_{b_0{}^\frown\langle\delta_m\rangle^\frown b_1} \restriction (u_{b_0{}^\frown\langle\delta_m\rangle^\frown b_1}[\mathbf{r_m}]).$$

But we have $q^*_{m,\ell} \leq q_{b_0{}^\frown\langle\delta_m\rangle^\frown b_1}$, since $b_0{}^\frown\langle\delta_m\rangle^\frown b_1 \in \prod_{m'<n} A_{m'} \restriction (m,\ell)$. It follows that $r \leq q^*_{m,\ell} \leq q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1} \restriction \mathrm{dom}(r)$. Therefore, r and $q_{b_0{}^\frown\langle\alpha\rangle^\frown b_1}$ are compatible. $\square$

Returning to $V[G]$, we can find $\alpha \in A^*_m \setminus \gamma$ such that $q^*_\alpha \in G$. But notice that, if we were to set $\alpha_{m,\ell} = \alpha$, then, letting $(m,\ell)^+$ denote the anti-lexicographic successor of (m,ℓ), we would have $q^*_{(m,\ell)^+} = q^*_{m,\ell} \cup q^*_\alpha \in G$. We can therefore set $\alpha_{m,\ell} = \alpha$ while maintaining the recursion hypothesis, and continue to the next step of the construction.

At the end of the construction, we have built sets $\langle A_m \mid m < n \rangle$ such that

- for each $m < n$, A_m is a subset of H and $\mathrm{otp}(A_m) = \omega + 1$;
- for each $m < m' < n$, $A_m < A_{m'}$;
- for each $b \in \prod_{m<n} A_m$, we have $q_b \in G$, and hence $c(b) = k$.

Therefore, $\langle A_m \mid m < n \rangle$ witnesses this instance of the theorem. □

Remark 5.7. With some appropriate bookkeeping, the order type $\omega + 1$ in the statement of Theorem 5.4 can be replaced by any countable ordinal α.

6. A variation, and monochromatic sumsets of reals

In this section, we discuss an alternative form of higher-dimensional Δ-system that has appeared in the literature. The following theorem is due to Shelah and follows from the proof of [16, Lemma 4.1] (cf. also [6, Claim 7.2.a] and [21, Lemma 3.6] for more complete proofs of similar statements).

Theorem 6.1. *Suppose that $\nu \le \lambda \le \mu$ are infinite cardinals, $1 \le n < \omega$, and $\mu \to (\lambda)^{2n}_{2^\nu}$. Suppose moreover that $\langle u_a \mid a \in [\mu]^n \rangle$ is a sequence of elements from $[\mathrm{On}]^{\le \nu}$. Then there is $H \in [\mu]^\lambda$ and a sequence $\langle u^*_a \mid a \in [H]^{\le n} \rangle$ of elements from $[\mathrm{On}]^{\le \nu}$ such that*

1. *$u^*_a \supseteq u_a$ for all $a \in [H]^n$;*
2. *for all $a, b \in [H]^n$, we have $\mathrm{tp}(u^*_a, u_a) = \mathrm{tp}(u^*_b, u_b)$;*
3. *for all $a, b \in [H]^{\le n}$, we have $u^*_a \cap u^*_b = u^*_{a \cap b}$;*
4. *for all $a_0 \subseteq a_1$ and $b_0 \subseteq b_1$, where $a_1, b_1 \in [H]^{\le n}$, if $\mathrm{tp}(a_1, a_0) = \mathrm{tp}(b_1, b_0)$, then $\mathrm{tp}(u^*_{a_1}, u^*_{a_0}) = \mathrm{tp}(u^*_{b_1}, u^*_{b_0})$.*

It is currently unclear whether arguments similar to those in the proof of Theorem 3.8 can be used to obtain the conclusion of Theorem 6.1 from a weaker assumption on μ, such as $\mu \ge \sigma(\lambda, n)$. It is the case, however, that certain results that have been proven using Theorem 6.1 can be proven by instead using Theorem 3.8. This can yield some improvements, since Theorem 3.8 places weaker assumptions on the cardinal μ. We give one example of such a result here.

In [22], Zhang uses Theorem 6.1 to prove that, in the forcing extension obtained by adding $\beth_\omega$-many Cohen reals, we have $\mathbb{R} \to^+ (\aleph_0)_r$ for every $r < \omega$, i.e., for every $r < \omega$ and every function $f : \mathbb{R} \to r$, there is an infinite set $X \subseteq \mathbb{R}$ such that $f \restriction (X + X)$ is constant. We remark that, by a result of Hindman, Leader, and Strauss [11], if $2^{\aleph_0} < \aleph_\omega$, then there is $r < \omega$ such that $\mathbb{R} \not\to^+ (\aleph_0)_r$, so, over a model of GCH, it is necessary to add at least $\beth_\omega$-many reals to obtain $\mathbb{R} \to^+ (\aleph_0)_r$ for every $r < \omega$.

Let us examine, though, the number of reals that must be added to obtain $\mathbb{R} \to^+ (\aleph_0)_r$ for some fixed $r < \omega$. Zhang in fact proves that $\mathbb{R} \to^+ (\aleph_0)_2$ holds in ZFC and, for a fixed $r > 2$, in proving that $\mathbb{R} \to^+ (\aleph_0)_r$ holds in the forcing extension, Theorem 6.1 is employed with $\kappa = \aleph_0$, $\lambda = \aleph_1$, and $n = 2r$. Hence, μ can be taken to be least such that $\mu \to (\aleph_1)^{4r}_{2^{\aleph_0}}$. By the Erdős-Rado theorem, then, we can take $\mu = \beth^+_{4r}$. Zhang's proof uses the fact that we have added at least μ-many

Cohen reals and therefore shows that, for this fixed value of $r > 2$, the statement $\mathbb{R} \to^+ (\aleph_0)_r$ holds in the forcing extension obtained by adding $\beth_{4r}^+$-many Cohen reals.

Inspection of Zhang's proof reveals that Theorem 3.8, with $\kappa = \aleph_1$, $\lambda = \beth_1^+$, and $n = 2r$, can be used in place of Theorem 6.1. We can therefore take $\mu = \sigma(\beth_1^+, 2r) = \beth_{2r}^+$, obtaining the following corollary:

Corollary 6.2. *Suppose that $2 < r < \omega$ and $\mathbb{P}$ is the forcing to add at least $\beth_{2r}^+$-many Cohen reals. Then, in $V^{\mathbb{P}}$, we have $\mathbb{R} \to^+ (\aleph_0)_r$.*

This is an improvement on the bound of $\beth_{4r}^+$ given by Zhang's proof, though of course it does not improve on Zhang's bound for obtaining $\mathbb{R} \to^+ (\aleph_0)_r$ simultaneously for all $r < \omega$. We omit the adaptation of Zhang's proof using Theorem 3.8 instead of Theorem 6.1 here, as it would entail introducing a considerable number of definitions and only involves very minor changes to Zhang's proof. Instead, we direct the reader to [22] and [14], in which Zhang's original proof and the adaptation using Theorem 3.8 are spelled out in detail.

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