KNASTER AND FRIENDS III: SUBADDITIVE COLORINGS

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ABSTRACT. We continue our study of strongly unbounded colorings, this time focusing on subadditive maps. In Part I of this series, we showed that, for many pairs of infinite cardinals $\theta < \kappa$, the existence of a strongly unbounded coloring $c : [\kappa]^2 \to \theta$ is a theorem of ZFC. Adding the requirement of subadditivity to a strongly unbounded coloring is a significant strengthening, though, and here we see that in many cases the existence of a *subadditive* strongly unbounded coloring $c : [\kappa]^2 \to \theta$ is independent of ZFC. We connect the existence of subadditive strongly unbounded colorings with a number of other infinitary combinatorial principles, including the narrow system property, the existence of κ-Aronszajn trees with ascent paths, and square principles. In particular, we show that the existence of a closed, subadditive, strongly unbounded coloring $c : [\kappa]^2 \to \theta$ is equivalent to a certain weak indexed square principle $\boxminus^{\mathrm{ind}}(\kappa, \theta)$. We conclude the paper with an application to the failure of the infinite productivity of κ-stationarily layered posets, answering a question of Cox.

1. INTRODUCTION

For any pair $\theta < \kappa$ of infinite regular cardinals, the positive partition relation $\kappa \to (\kappa)_\theta^2$, which asserts that every coloring $c : [\kappa]^2 \to \theta$ has a homogeneous set of cardinality κ, is equivalent to κ being weakly compact. For non-weakly-compact cardinals κ, though, one can seek to measure the incompactness of κ by asking whether certain strengthenings of the negative relation $\kappa \nrightarrow (\kappa)_\theta^2$ hold. One natural such strengthening is to require that there exist colorings $c : [\kappa]^2 \to \theta$ witnessing certain strong unboundedness properties. In [LHR18], which forms Part I of this series of papers, the authors introduce the following coloring principle, which asserts the existence of such strongly unbounded colorings, and use it to answer questions about the infinite productivity of the κ-Knaster condition for uncountable κ.

Definition 1.1. $\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, and every $i < \theta$, there exists $\mathcal{B} \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) > i$ for all $a, b \in \mathcal{B}$ with $\sup(a) < \min(b)$.

Much of [LHR18] is devoted to analyzing situations in which $\mathrm{U}(\ldots)$ necessarily holds and, moreover, is witnessed by *closed* or *somewhere-closed* colorings (see Definition 2.1 below). In Part II of this series [LHR21], we studied $\mathrm{Cspec}(\kappa)$, the *C-sequence spectrum of* κ (see Definition 5.5 below), which is another measure of the incompactness of κ, and found some unexpected connections between $\mathrm{Cspec}(\kappa)$ and the validity of instances of $\mathrm{U}(\kappa, \ldots)$.

In this paper, which can be read largely independently of [LHR18, LHR21], we investigate *subadditive* witnesses to $\mathrm{U}(\ldots)$.

Definition 1.2. A coloring $c : [\kappa]^2 \to \theta$ is *subadditive* if, for all $\alpha < \beta < \gamma < \kappa$, the following inequalities hold:

(1) $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$;
(2) $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$.

Adding the requirement of subadditivity significantly strengthens the coloring principle, and we prove that the existence of closed, subadditive witnesses to $\mathrm{U}(\ldots)$ is equivalent to a certain indexed square principle. Our first main result improves Clause (1) of [LHR18, Theorem A].[1]

Theorem A. *Let $\theta < \kappa$ be a pair of infinite regular cardinals. The following are equivalent:*

(1) $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ *holds;*
(2) *There is a closed, subadditive witness to $\mathrm{U}(\kappa, 2, \theta, 2)$;*
(3) *There is a closed, subadditive witness to $\mathrm{U}(\kappa, \kappa, \theta, \sup(\mathrm{Reg}(\kappa)))$.*

In addition, $\square(\kappa, \sqsubseteq_\theta)$ implies (1)–(3).

We also prove that a version of square with built-in diamond for a singular cardinal λ gives rise to somewhere-closed subadditive witnesses to $\mathrm{U}(\lambda^+, \ldots)$, which in turn imply that the C-sequence spectrum of λ^+ is rich:

Theorem B. *Suppose that λ is a singular cardinal, $\vec{f}$ is a scale for λ of length λ^+ in some product $\prod \vec{\lambda}$, and $\boxtimes(\vec{\lambda})$ holds. Let Σ denote the set of good points for $\vec{f}$.*

Then, for every $\theta \in \mathrm{Reg}(\lambda) \setminus (\mathrm{cf}(\lambda) + 1)$, there exists a Σ-closed, subadditive witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$. In particular, $\mathrm{Reg}(\lambda) \subseteq \mathrm{Cspec}(\lambda^+)$.

For a pair of infinite regular cardinals $\theta < \kappa$ and a coloring $c : [\kappa]^2 \to \theta$, an interesting facet of the study of the unboundedness properties of c is the set $\partial(c)$ of its levels of divergence (see Definition 3.21 below). Any coloring c for which $\partial(c)$ is stationary is automatically a somewhere-closed witness to $\mathrm{U}(\kappa, \kappa, \theta, \theta)$. We prove that the existence of a (fully) closed witness c to $\mathrm{U}(\kappa, \kappa, \theta, \theta)$ for which $\partial(c)$ is stationary is equivalent to the existence of a nonreflecting stationary subset of E_θ^κ, and that the existence of a nonreflecting stationary subset of E_θ^κ does not suffice to yield a subadditive witness to $\mathrm{U}(\kappa, 2, \theta, 2)$. We have three main consistency results concerning the characteristic $\partial(c)$:

Theorem C.   (1) *For any pair of infinite regular cardinals $\theta < \kappa$, there is a κ-strategically closed, θ^+-directed closed forcing notion that adds a subadditive witness c to $\mathrm{U}(\kappa, \kappa, \theta, \theta)$ for which $\partial(c)$ is stationary;*

(2) *For any pair of infinite regular cardinals $\theta < \kappa$, there is a κ-strategically closed, θ-directed closed forcing notion that adds a closed subadditive witness c to $\mathrm{U}(\kappa, \kappa, \theta, \theta)$ for which $\partial(c)$ is stationary;*

(3) *For regular uncountable cardinals $\theta < \lambda < \kappa$ such that λ is supercompact and κ is weakly compact, there is a forcing extension in which $\square(\kappa, \theta)$ fails, yet, there is a closed, subadditive witness c to $\mathrm{U}(\kappa, \kappa, \theta, \theta)$ for which $\partial(c)$ is stationary.*

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[1]Subsection 1.1 below contains information about precisely where in the paper to find the proof of each of these theorems and definitions of the terms therein.

On the Ramsey-theoretic side, we prove that in the presence of large cardinals, for many pairs of infinite regular cardinals $\theta < \kappa$, there are no subadditive witnesses to $U(\kappa, 2, \theta, 2)$, i.e., $\kappa \to [\kappa]^2_{\theta, <\theta}$ holds restricted to the class of subadditive colorings. We also show that similar results hold at small cardinals in forcing extensions or in the presence of forcing axioms.

On the anti-Ramsey-theoretic side, we have a result reminiscent of the motivating result of [LHR18] concerning the infinite productivity of strong forms of the κ-chain condition. When combined with Theorem A, the next theorem shows that $\Box(\kappa)$ yields a gallery of counterexamples to productivity of κ-stationarily layered posets, answering a question of Cox [Cox17].

Theorem D. *Suppose that $\theta \leq \chi < \kappa$ are infinite, regular cardinals, κ is $<\chi$-inaccessible, and there is a closed and subadditive witness c to $U(\kappa, 2, \theta, 2)$. Then there is a sequence of posets $\langle \mathbb{P}_i \mid i < \theta \rangle$ such that:*

(1) *for all $i < \theta$, $\mathbb{P}_i$ is well-met and χ-directed closed with greatest lower bounds;*
(2) *for all $j < \theta$, $\prod_{i<j} \mathbb{P}_i$ is κ-stationarily layered;*
(3) *$\prod_{i<\theta} \mathbb{P}_i$ is not κ-cc.*

If, in addition, $\partial(c) \cap E^\kappa_\chi$ is stationary, then the sequence $\langle \mathbb{P}_i \mid i < \theta \rangle$ can be made constant.

As a corollary, we get that Magidor's forcing for changing the cofinality of a measurable cardinal λ to a regular cardinal $\theta < \lambda$ adds a poset $\mathbb{P}$ whose θ^{th} power is not λ^+-cc, but all of whose lower powers are λ^+-stationarily layered.

1.1. **Organization of this paper.** In Section 2, we present some useful definitions and facts about $U(\kappa, \mu, \theta, \chi)$, largely derived from Part I of this series. We also present a pseudo-inverse to the fact, observed in Part I, that Shelah's principle $\text{Pr}_1(\kappa, \kappa, \theta, \chi)$ implies $U(\kappa, 2, \theta, \chi)$.

In Section 3, we review the notion of *subadditivity* and some of its variations and prove that any subadditive witness to $U(\kappa, 2, \theta, 2)$ is in fact a witness to $U(\kappa, \mu, \theta, \chi)$ for all $\mu < \kappa$ and all $\chi \leq \text{cf}(\theta)$ (and, under certain closure assumptions, even stronger principles). Subsection 3.1 contains results connecting subadditive strongly unbounded colorings to narrow systems and trees with ascent paths. In Subsection 3.2, we discuss *locally small* colorings of the form $c : [\lambda^+]^2 \to \text{cf}(\lambda)$, focusing in particular on the case in which λ is a singular cardinal. Locally small colorings are necessarily witnesses to $U(\lambda^+, 2, \text{cf}(\lambda), \text{cf}(\lambda))$, and retain this property in any outer model with the same cardinals. In Subsection 3.3, we introduce a subset $\partial(c) \subseteq \kappa$ associated with a coloring $c : [\kappa]^2 \to \theta$ that is useful in the analysis of $U(\kappa, \mu, \theta, \chi)$, particularly in the context of subadditive colorings. We then introduce a forcing notion that establishes Clause (1) of Theorem C. Subsection 3.4 contains a number of results indicating the extent to which various compactness principles place limits on the existence of certain subadditive witnesses to $U(\kappa, \mu, \theta, \chi)$. In particular, it is shown that simultaneous stationary reflection, the existence of highly complete or indecomposable ultrafilters, and the P-ideal dichotomy all have such an effect.

In Section 4, we introduce an indexed square principle $\boxminus^{\text{ind}}(\kappa, \theta)$ and prove that it is equivalent to the existence of a closed, subadditive witness to $U(\kappa, 2, \theta, 2)$, thereby establishing the first part of Theorem A. We also prove a consistency result indicating that $\boxminus^{\text{ind}}(\kappa, \theta)$ is a proper weakening of $\Box^{\text{ind}}(\kappa, \theta)$ and does not even imply $\Box(\kappa, \theta)$, in the process proving Clause (3) of Theorem C. Section 4 also

contains the proof of the second part of Theorem A and the proof of Clause (2) of Theorem C.

Section 5 is concerned with successors of singular cardinals. We begin by proving Theorem B, showing that a certain square with built-in diamond sequence on a singular cardinal λ entails the existence of a subadditive witness to $U(\lambda^+, \lambda^+, \theta, \lambda)$ for all $\theta \in \text{Reg}(\lambda) \setminus (\text{cf}(\lambda)+1)$. We then present an improvement upon a result from Part I of this series proving the existence of closed witnesses to $U(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda))$ for all singular λ whose cofinality is not greatly Mahlo and all $\theta \leq \text{cf}(\lambda)$.

Section 6 deals with the infinity productivity of κ-stationarily layered posets and contains our proof of Theorem D.

1.2. **Notation and conventions.** Throughout the paper, κ denotes a regular uncountable cardinal, and χ, θ, and μ denote cardinals $\leq \kappa$. λ will always denote an infinite cardinal. We say that κ is χ-*inaccessible* iff, for all $\nu < \kappa$, $\nu^\chi < \kappa$, and say that κ is $<\chi$-*inaccessible* iff, for all $\nu < \kappa$ and $\mu < \chi$, $\nu^\mu < \kappa$. For a regular cardinal Υ, we denote by H_Υ the collection of all sets of hereditary cardinality less than Υ.

Reg denotes the class of infinite regular cardinals, and $\text{Reg}(\kappa)$ denotes $\text{Reg} \cap \kappa$. E^κ_χ denotes the set $\{\alpha < \kappa \mid \text{cf}(\alpha) = \chi\}$, and $E^\kappa_{\geq \chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\neq \chi}$, etc. are defined analogously.

For a set of ordinals a, we write $\text{ssup}(a) := \sup\{\alpha + 1 \mid \alpha \in a\}$, $\text{acc}^+(a) := \{\alpha < \text{ssup}(a) \mid \sup(a \cap \alpha) = \alpha > 0\}$, $\text{acc}(a) := a \cap \text{acc}^+(a)$, $\text{nacc}(a) := a \setminus \text{acc}(a)$, and $\text{cl}(a) := a \cup \text{acc}^+(a)$. For sets of ordinals a and b, we write $a < b$ if, for all $\alpha \in a$ and all $\beta \in b$, we have $\alpha < \beta$. For a set of ordinals a and an ordinal β, we write $a < \beta$ instead of $a < \{\beta\}$ and $\beta < a$ instead of $\{\beta\} < a$.

For any set $\mathcal{A}$, we write $[\mathcal{A}]^\chi := \{\mathcal{B} \subseteq \mathcal{A} \mid |\mathcal{B}| = \chi\}$ and $[\mathcal{A}]^{<\chi} := \{\mathcal{B} \subseteq \mathcal{A} \mid |\mathcal{B}| < \chi\}$. In particular, $[\mathcal{A}]^2$ consists of all unordered pairs from $\mathcal{A}$. In some scenarios, we will also be interested in ordered pairs from $\mathcal{A}$. In particular, if $\mathcal{A}$ is either a set of ordinals or a collection of sets of ordinals, then we will abuse notation and write $(a, b) \in [\mathcal{A}]^2$ to mean $\{a, b\} \in [\mathcal{A}]^2$ and $a < b$.

2. Preliminaries

In this brief section, we recall a key definition and present a few useful facts about $U(\ldots)$. We start by recalling the following definition from [LHR18] concerning *closed* colorings.

Definition 2.1. Suppose that $c : [\kappa]^2 \to \theta$ is a coloring.

  (1) For all $\beta < \kappa$ and $i \leq \theta$, we let $D^c_{\leq i}(\beta)$ denote the set $\{\alpha < \beta \mid c(\alpha, \beta) \leq i\}$.
  (2) For all $\Sigma \subseteq \kappa$, c is Σ-*closed* if, for all $\beta < \kappa$ and $i \leq \theta$,

$$\text{acc}^+(D^c_{\leq i}(\beta)) \cap \Sigma \subseteq D^c_{\leq i}(\beta).$$

  (3) c is *somewhere-closed* if it is Σ-closed for some stationary $\Sigma \subseteq \kappa$.
  (4) c is *closed* if it is κ-closed.

The following fact is a useful tool for proving that certain colorings satisfy strong instances of $U(\ldots)$.

Fact 2.2 ([LHR18]). *Suppose that $c : [\kappa]^2 \to \theta$ is a coloring and $\omega \leq \chi < \kappa$. Then* (1) $\implies$ (2) $\implies$ (3):

  (1) *For some stationary $\Sigma \subseteq E^\kappa_{\geq \chi}$, c is a Σ-closed witness to $U(\kappa, 2, \theta, \chi)$.*

(2) *For every family $\mathcal{A} \subseteq [\kappa]^{<\chi}$ consisting of κ-many pairwise disjoint sets, for every club $D \subseteq \kappa$, and for every $i < \theta$, there exist $\gamma \in D$, $a \in \mathcal{A}$, and $\epsilon < \gamma$ such that:*
  - $\gamma < a$;
  - *for all $\alpha \in (\epsilon, \gamma)$ and all $\beta \in a$, we have $c(\alpha, \beta) > i$.*

(3) *c witnesses $\mathrm{U}(\kappa, \kappa, \theta, \chi)$.*

It is sometimes useful to consider the following unbalanced form of $\mathrm{U}(\cdots)$.

Definition 2.3. $\mathrm{U}(\kappa, \mu \circledast \nu, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, and every $i < \theta$, there exist $\mathcal{A}' \in [\mathcal{A}]^\mu$ and $\mathcal{B}' \in [\mathcal{A}]^\nu$ such that, for every $(a, b) \in \mathcal{A}' \times \mathcal{B}'$, $a < b$ and $\min(c[a \times b]) > i$.

Lemma 2.4. *Suppose that $c : [\kappa]^2 \to \theta$ witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, with $\theta < \kappa$.*

(1) *For every cofinal $A \subseteq \kappa$, there exists $\epsilon < \kappa$ such that*
$$\sup\{\beta \in A \setminus \epsilon \mid \sup\{c(\alpha, \beta) \mid \alpha \in A \cap \epsilon\} = \theta\} = \kappa;$$

(2) *For every cofinal $A \subseteq \kappa$, there exists $\beta \in A$ such that $\{c(\alpha, \beta) \mid \alpha \in A \cap \beta\}$ is unbounded in θ. In particular, c witnesses $\mathrm{U}(\kappa, \mathrm{cf}(\theta) \circledast 1, \theta, 2)$;*

(3) *For every stationary $S \subseteq \kappa$, there exists $\epsilon < \kappa$ such that, for every $i < \theta$, $\{\beta \in S \mid \epsilon < \beta,\ c(\epsilon, \beta) > i\}$ is stationary;*

(4) *For every stationary $S \subseteq \kappa$ and a family of functions $\mathcal{H} \in [{}^S\theta]^{<\kappa}$, there exists $\epsilon \in S$ such that, for every $h \in \mathcal{H}$, the following set is stationary:*
$$\{\beta \in S \mid \epsilon < \beta,\ c(\epsilon, \beta) > \max\{h(\epsilon), h(\beta)\}\}.$$

Proof. Clause (2) follows from Clause (1), and Clause (3) follows from Clause (4).

(1) Towards a contradiction, suppose that A is a counterexample. Then, for every $\epsilon < \kappa$, there exists a large enough $\eta_\epsilon \in [\epsilon, \kappa)$ such that, for every $\beta \in A \setminus \eta_\epsilon$, for some $i < \theta$, $A \cap \epsilon \subseteq D^c_{\leq i}(\beta)$. Fix a sparse enough cofinal subset $A' \subseteq A$ such that, for every $\epsilon \in A'$, $\min(A' \setminus (\epsilon + 1)) \geq \eta_{\epsilon+1}$. It follows that for every $\beta \in A'$, for some $i_\beta < \theta$, $A \cap \beta \subseteq D^c_{\leq i_\beta}(\beta)$. Fix $i^* < \theta$ and an unbounded $A^* \subseteq A'$ such that $i_\beta = i^*$ for all $\beta \in A^*$. Since c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, we can find $(\alpha, \beta) \in [A^*]^2$ such that $c(\alpha, \beta) > i^*$, contradicting the fact that $\alpha \in A \cap \beta \subseteq D^c_{\leq i^*}(\beta)$.

(4) Towards a contradiction, suppose that S and $\mathcal{H}$ form a counterexample. For every $\epsilon \in S$, fix a function $h_\epsilon \in \mathcal{H}$ and a club $D_\epsilon \subseteq \kappa$ disjoint from $\{\beta \in S \mid \epsilon < \beta,\ c(\epsilon, \beta) > \max\{h_\epsilon(\epsilon), h_\epsilon(\beta)\}\}$. Let $D := \bigwedge_{\epsilon < \kappa} D_\epsilon$. As $|\mathcal{H}| < \kappa$, find $h \in \mathcal{H}$ for which $A := \{\epsilon \in D \cap S \mid h_\epsilon = h\}$ is cofinal in κ. As $\theta < \kappa$, find $i < \theta$ for which $B := \{\beta \in A \mid h(\beta) = i\}$ is cofinal in κ. Now, as c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, we may pick $(\epsilon, \beta) \in [B]^2$ such that $c(\epsilon, \beta) > i$. But $i = \max\{h_\epsilon(\epsilon), h_\epsilon(\beta)\}$, contradicting the fact that $\beta \in D_\epsilon \cap S$. $\square$

The following is a corollary to a result from [LHR18] which we never took the time to derive.

Proposition 2.5. *If κ is a Mahlo cardinal admitting a nonreflecting stationary subset of $\mathrm{Reg}(\kappa)$, then $\mathrm{U}(\kappa, \kappa, \theta, \kappa)$ holds for every $\theta \leq \kappa$.*

Proof. By [LHR18, Theorem 4.11]. $\square$

Recall that Shelah's principle $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of

size κ, and every $i < \theta$, there exist $(a, b) \in [\mathcal{A}]^2$ such that $c[a \times b] = \{i\}$. By [LHR18, Lemma 2.3], $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ implies $\mathrm{U}(\kappa, 2, \theta, \chi)$. Here we deal with a pseudo-inverse:

Proposition 2.6. *Suppose that* $\mathrm{U}(\kappa, 2, \theta, \chi)$ *holds with* $\chi \leq \mathrm{cf}(\theta) = \theta \leq \theta^{<\theta} < \kappa$. *Then* $V^{\mathrm{Add}(\theta,1)} \models \mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$.

Proof. In V, let $c : [\kappa]^2 \to \theta$ be a witness to $\mathrm{U}(\kappa, 2, \theta, \chi)$. Let G be $\mathrm{Add}(\theta, 1)$-generic over V, and work for now in $V^{\mathrm{Add}(\theta,1)}$. Let $g : \theta \to \theta$ be the Cohen-generic function, and set $d := g \circ c$. To see that d witnesses $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$, let $\mathcal{A}$ be a κ-sized pairwise disjoint subfamily of $[\kappa]^{<\chi}$, and let $\tau < \theta$; we need to find $(a, b) \in [\mathcal{A}]^2$ such that $d[a \times b] = \{\tau\}$. Since $\chi \leq \theta$, every element of $\mathcal{A}$ is in V. Also, letting $\dot{\mathcal{A}} \in V$ be an $\mathrm{Add}(\theta, 1)$-name for $\mathcal{A}$, we can fix for each $a \in \mathcal{A}$ a condition $p_a \in G$ such that $p_a \Vdash$ "$a \in \dot{\mathcal{A}}$". As $\chi \leq \theta^{<\theta} < \kappa$, and hence $|\mathrm{Add}(\theta, 1)| < \kappa$, by passing to a subfamily if necessary, we may assume that there is a fixed $p^* \in G$ such that $p_a = p^*$ for all $a \in \mathcal{A}$, and therefore $\mathcal{A} = \{a \in [\kappa]^{<\chi} \mid p \Vdash$ "$a \in \dot{\mathcal{A}}$"$\}$ is in V. We therefore move back to V and run a density argument. Let $p : i \to \theta$ be an arbitrary condition in $\mathrm{Add}(\theta, 1)$ below p^*. By the hypothesis on c, pick $(a, b) \in [\mathcal{A}]^2$ such that $x := c[a \times b]$ is disjont from i. Let $j := \mathrm{ssup}(x)$, and let $q : j \to \theta$ be some condition extending p and satisfying $q(\xi) = \tau$ for all $\xi \in x$. Clearly, $q(c(\alpha, \beta)) = \tau$ for all $(\alpha, \beta) \in a \times b$, so q forces that $d[a \times b] = \{\tau\}$, as sought. $\square$

We conclude this short section with another simple fact worth recording.

Proposition 2.7. *If κ is weakly compact, then, in $V^{\mathrm{Add}(\omega,\kappa)}$:*

(1) $\mathrm{U}(\kappa, \kappa, \omega, 2)$ *holds;*

(2) $\mathrm{U}(\kappa, 2, \omega, \omega)$ *holds;*

(3) $\mathrm{U}(\kappa, 2, \theta, 2)$ *fails for every regular uncountable $\theta < \kappa$.*

Proof. (1) By [LHR18, Lemma 2.7].

(2) It is not hard to see that, in $V^{\mathrm{Add}(\omega,\kappa)}$, $\mathrm{Pr}_1(\kappa, \omega_1, \omega, \omega)$ holds. In particular, by [LHR18, Lemma 2.3], $\mathrm{U}(\kappa, 2, \omega, \omega)$ holds.

(3) Evidently, for every *ccc* forcing $\mathbb{P}$ and every infinite cardinal θ, $\kappa \to [\kappa]^2_\theta$ implies $V^{\mathbb{P}} \models \kappa \to [\kappa]^2_{\theta,\omega}$. In particular, in $V^{\mathrm{Add}(\omega,\kappa)}$, $\mathrm{U}(\kappa, 2, \theta, 2)$ fails for every regular uncountable $\theta < \kappa$. $\square$

3. SUBADDITIVE COLORINGS

We now turn to *subadditive* colorings, which form the primary topic of this paper. We begin by recalling the definition of subadditivity, splitting the definition into its two constituent parts.

Definition 3.1. A coloring $c : [\kappa]^2 \to \theta$ is *subadditive* iff the following two statements hold:

(1) c is *subadditive of the first kind*, that is, for all $\alpha < \beta < \gamma < \kappa$,

$$c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\};$$

(2) c is *subadditive of the second kind*, that is, for all $\alpha < \beta < \gamma < \kappa$,

$$c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}.$$

We shall write $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \mu, \theta, \chi)$ to assert that $\mathrm{U}(\kappa, \mu, \theta, \chi)$ holds and that it moreover admits a subadditive witness. Note that the function $c : [\kappa]^2 \to \kappa$

defined by letting $c(\alpha, \beta) := \alpha$ for all $(\alpha, \beta) \in [\kappa]^2$ is a closed, subadditive witness to $\mathrm{U}(\kappa, \kappa, \kappa, \kappa)$, so, in all situations of interest, we will have $\theta < \kappa$.

Subadditivity allows us to show that witnesses to certain instances of $\mathrm{U}(\dots)$ in fact satisfy stronger instances, as in the following lemma.

Lemma 3.2. *Suppose that $c : [\kappa]^2 \to \theta$ is a witness to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$, with $\theta < \kappa$. Then the following statements all hold.*

(1) *For every $\mu < \kappa$, c witnesses $\mathrm{U}(\kappa, \mu, \theta, 2)$. If there exist no κ-Souslin trees, then c moreover witnesses $\mathrm{U}(\kappa, \kappa, \theta, 2)$.*

(2) *For every $\mu \leq \kappa$, if c witnesses $\mathrm{U}(\kappa, \mu, \theta, 2)$, then c moreover witnesses $\mathrm{U}(\kappa, \mu, \theta, \mathrm{cf}(\theta))$.*

(3) *For every $\chi \in \mathrm{Reg}(\kappa)$, if c is Σ-closed for some stationary $\Sigma \subseteq E^\kappa_{\geq \chi}$, then c witnesses $\mathrm{U}(\kappa, \kappa, \theta, \chi)$.*

Proof. For each $i < \theta$, we define an ordering $<_i$ on κ by letting $\alpha <_i \beta$ iff $\alpha < \beta$ and $c(\alpha, \beta) \leq i$. The fact that $<_i$ is transitive follows from the fact that c is subadditive of the first kind.

Claim 3.2.1. *For every $i < \theta$, $(\kappa, <_i)$ is a tree with no branches of size κ.*

Proof. Let $i < \theta$ and $\gamma < \kappa$. To see that set $\{\alpha < \kappa \mid \alpha <_i \gamma\}$ is well-ordered by $<_i$, fix $\alpha < \beta$ such that $\alpha, \beta <_i \gamma$. As c is subadditive of the second kind, $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\} \leq i$, so $\alpha <_i \beta$.

In addition, by the hypothesis on c, for every $A \in [\kappa]^\kappa$, there is a pair $(\alpha, \beta) \in [A]^2$ such that $c(\alpha, \beta) > i$, so $\neg(\alpha <_i \beta)$. That is, $(\kappa, <_i)$ has no chains of size κ. ⊠

(1) Fix $\mu \leq \kappa$, and suppose that we are given some $A \in [\kappa]^\kappa$ and $i < \theta$. We would like to find $B \in [A]^\mu$ such that $c(\alpha, \beta) > i$ for all $(\alpha, \beta) \in [B]^2$. In particular, if there exists $B \in [\kappa]^\mu$ which is an antichain with respect to $<_i$, then we are done. Hereafter, suppose this is not the case.

▶ If $\mu < \kappa$, then $(A, <_i)$ is a tree of size κ all of whose levels have size $< \mu$. Since $\mu < \kappa$ and κ is regular, a result of Kurepa [Kur77] implies that $(\kappa, <_i)$ has a branch of size κ, contradicting the preceding claim.

▶ If $\mu = \kappa$, then $(A, <_i)$ forms a κ-Souslin tree.

(2) Suppose that $\mu \leq \kappa$ and c witnesses $\mathrm{U}(\kappa, \mu, \theta, 2)$. Suppose also that $\mathcal{A} \subseteq [\kappa]^{<\mathrm{cf}(\theta)}$ consists of κ-many pairwise disjoint sets and $i < \theta$ is a prescribed color. We will find $\mathcal{B} \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) > i$ for all $(a, b) \in [\mathcal{B}]^2$. For each $\gamma < \kappa$, pick $a_\gamma \in \mathcal{A}$ with $\gamma < a_\gamma$. Define $f : \kappa \to \theta$ and $g : \kappa \to \kappa$ by setting, for all $\gamma < \kappa$,

- $f(\gamma) := \sup\{c(\gamma, \alpha) \mid \alpha \in a_\gamma\}$;
- $g(\gamma) := \sup(a_\gamma)$.

Fix $\varepsilon < \theta$ for which $A := \{\gamma < \kappa \mid f(\gamma) = \varepsilon \ \& \ g[\gamma] \subseteq \gamma\}$ is stationary. Since c witnesses $\mathrm{U}(\kappa, \mu, \theta, 2)$, we can pick $B \in [A]^\mu$ such that $c(\gamma, \delta) > \max\{\varepsilon, i\}$ for all $(\gamma, \delta) \in [B]^2$. We claim that $\mathcal{B} := \{a_\gamma \mid \gamma \in B\}$ is a subfamily of $\mathcal{A}$ as sought. To see this, pick a pair $(\gamma, \delta) \in [B]^2$ along with $(\alpha, \beta) \in a_\gamma \times a_\delta$. Clearly, $\gamma < \alpha < \delta < \beta$. By the subadditivity of c and the choice of ε, we have:

$$c(\gamma, \delta) \leq \max\{c(\gamma, \alpha), c(\alpha, \delta)\} \leq \max\{c(\gamma, \alpha), c(\alpha, \beta), c(\delta, \beta)\} \leq \max\{\varepsilon, c(\alpha, \beta)\}.$$

Recalling that $\max\{\varepsilon, i\} < c(\gamma, \delta)$, we infer that $i < c(\gamma, \delta) \leq c(\alpha, \beta)$, as sought.

(3) Suppose that $\chi \in \mathrm{Reg}(\kappa)$, $\Sigma \subseteq E^\kappa_{\geq \chi}$ is stationary, and c is Σ-closed.

Claim 3.2.2. *Suppose that $\mathcal{A} \subseteq [\kappa]^{<\chi}$ is a family consisting of κ-many pairwise disjoint sets, D is a club in κ, and $i < \theta$ is a prescribed color. Then there exist $\gamma \in D \cap \Sigma$ and $a \in \mathcal{A}$ such that*

  (a) *$\gamma < a$;*
  (b) *for all $\beta \in a$, we have $c(\gamma, \beta) > i$.*

Proof. Suppose not. Let $T := D \cap \Sigma$, and note that, for all $\gamma \in T$ and all $a \in \mathcal{A}$ with $a > \gamma$, for some $\beta \in a$, we have $c(\gamma, \beta) \leq i$. Consider the tree $(T, <_i)$. We claim that it has no antichains of size χ. To see this, fix an arbitrary $X \subseteq T$ of order type χ, and let $\delta := \sup(X)$. Fix an arbitrary $a \in \mathcal{A}$ with $a > \delta$. For all $\gamma \in X$, since $a > \gamma$, we may find some $\beta \in a$ with $c(\gamma, \beta) \leq i$. Since $|X| = \chi > |a|$, we may then pick $(\gamma, \gamma') \in [X]^2$ and $\beta \in a$ such that $c(\gamma, \beta) \leq i$ and $c(\gamma', \beta) \leq i$. It follows that $c(\gamma, \gamma') \leq \max\{c(\gamma, \beta), c(\gamma', \beta)\} \leq i$. In particular, X is not an antichain. Now, since $\chi < \kappa$ regular cardinals, the aforementioned result of Kurepa [Kur77] entails the existence of $B \in [T]^\kappa$ which is a chain with respect to $<_i$. By the hypothesis on c, we can pick $(\alpha, \beta) \in [B]^2$ such that $c(\alpha, \beta) > i$. Then $\neg(\alpha <_i \beta)$ which is a contradiction to the fact that B is a chain in $(T, <_i)$.   ⊠

As c is Σ-closed and $\Sigma \subseteq E^\kappa_{\geq \chi}$, Clause (b) of the preceding is equivalent to:

  (b') there is $\epsilon < \gamma$ such that, for all $\alpha \in (\epsilon, \gamma)$ and all $\beta \in a$, we have $c(\alpha, \beta) > i$.

Thus, by the implication $(2) \implies (3)$ of Fact 2.2, we can conclude that c witnesses $\mathrm{U}(\kappa, \kappa, \theta, \chi)$, as desired.   □

3.1. **Narrow systems and trees with ascent paths.** Given a binary relation R on a set X, for $a, b \in X$, we say that a and b are *R-comparable* iff $a = b$, $a \mathrel{R} b$, or $b \mathrel{R} a$. R is *tree-like* iff, for all $a, b, c \in X$, if $a \mathrel{R} c$ and $b \mathrel{R} c$, then a and b are R-comparable.

Definition 3.3 (Magidor-Shelah, [MS96]). $S = \langle \bigcup_{\alpha \in I} \{\alpha\} \times \theta_\alpha, \mathcal{R} \rangle$ is a *κ-system* if all of the following hold:

  (1) $I \subseteq \kappa$ is unbounded and, for all $\alpha \in I$, θ_α is a cardinal such that $0 < \theta_\alpha < \kappa$;
  (2) $\mathcal{R}$ is a set of binary, transitive, tree-like relations on $\bigcup_{\alpha \in I} \{\alpha\} \times \theta_\alpha$ and $0 < |\mathcal{R}| < \kappa$;
  (3) for all $R \in \mathcal{R}$, $\alpha_0, \alpha_1 \in I$, $\beta_0 < \theta_{\alpha_0}$, and $\beta_1 < \theta_{\alpha_1}$, if $(\alpha_0, \beta_0) \mathrel{R} (\alpha_1, \beta_1)$, then $\alpha_0 < \alpha_1$;
  (4) for every $(\alpha_0, \alpha_1) \in [I]^2$. there are $(\beta_0, \beta_1) \in \theta_{\alpha_0} \times \theta_{\alpha_1}$ and $R \in \mathcal{R}$ such that $(\alpha_0, \beta_0) \mathrel{R} (\alpha_1, \beta_1)$.

Define width$(S) := \sup\{|\mathcal{R}|, \theta_\alpha \mid \alpha \in I\}$. A κ-system S is *narrow* if width$(S)^+ < \kappa$. For $R \in \mathcal{R}$, a *branch of S through R* is a set $B \subseteq \bigcup_{\alpha \in I} \{\alpha\} \times \theta_\alpha$ such that for all $a, b \in B$, a and b are R-comparable. A branch B is *cofinal* iff $\sup\{\alpha \in I \mid \exists \tau < \theta_\alpha \ (\alpha, \tau) \in B\} = \kappa$.

Definition 3.4 ([LH17b]). The *(θ, κ)-narrow system property*, which is abbreviated $\mathsf{NSP}(\theta, \kappa)$, asserts that every narrow κ-system of width $< \theta$ has a cofinal branch.

By [LH17b, §10], PFA implies $\mathsf{NSP}(\omega_1, \kappa)$ for all regular $\kappa \geq \omega_2$.

Lemma 3.5. *Suppose that $\theta < \theta^+ < \kappa$ and $c : [\kappa]^2 \to \theta$ is a subadditive coloring. If $\mathsf{NSP}(\theta^+, \kappa)$ holds, then c fails to witness $\mathrm{U}(\kappa, 2, \theta, 2)$.*

Proof. Define a binary relation R on $\kappa \times \theta$ by letting $(\alpha, i) \mathrel{R} (\beta, j)$ iff $\alpha < \beta$, $i = j$, and $c(\alpha, \beta) \leq i$.

Claim 3.5.1. *Let $\alpha < \beta < \gamma < \kappa$. Then:*
  (1) *there exists $i < \theta$ such that $(\alpha, i)\ R\ (\beta, i)$;*
  (2) *for all $i < \theta$, if $(\alpha, i)\ R\ (\beta, i)$ and $(\beta, i)\ R\ (\gamma, i)$, then $(\alpha, i)\ R\ (\gamma, i)$;*
  (3) *for all $i < \theta$, if $(\alpha, i)\ R\ (\gamma, i)$ and $(\beta, i)\ R\ (\gamma, i)$, then $(\alpha, i)\ R\ (\beta, i)$.*

Proof. (1) Just take $i := c(\alpha, \beta)$.
  (2) By subadditivity of the first kind.
  (3) By subadditivity of the second kind.   ⊠

It thus follows that $S := \langle \kappa \times \theta, \{R\} \rangle$ is a narrow κ-system. So, assuming that $\mathsf{NSP}(\theta^+, \kappa)$ holds, we may fix $B \subseteq \kappa \times \theta$ that forms a cofinal branch of S through R. Pick $A \in [\kappa]^\kappa$ and $i < \theta$ such that $B = A \times \{i\}$. Now, if c were to witness $\mathrm{U}(\kappa, 2, \theta, 2)$, then we could fix $(\alpha, \beta) \in [A]^2$ such that $c(\alpha, \beta) > i$. But then $(\alpha, i)\ R\ (\beta, i)$ would fail to hold, contradicting the fact that $\{(\alpha, i), (\beta, i)\} \subseteq B$.   □

Next, we prove a pair of lemmas establishing an equivalency between instances of $\mathrm{U}^{\mathrm{subadditive}}(\ldots)$ and the existence of trees with ascent paths but no cofinal branches. We first recall the following definition.

Definition 3.6. Suppose that $\mathcal{T} = (T, <_T)$ is a tree of height κ.
  (1) For all $\alpha < \kappa$, T_α denotes level α of $\mathcal{T}$.
  (2) A θ-*ascent path* through $\mathcal{T}$ is a sequence of functions $\langle f_\alpha \mid \alpha < \kappa \rangle$ such that
      (a) for all $\alpha < \kappa$, $f_\alpha : \theta \to T_\alpha$;
      (b) for all $\alpha < \beta < \kappa$, there is $i < \theta$ such that, for all $j \in [i, \theta)$, we have $f_\alpha(j) <_T f_\beta(j)$.

Lemma 3.7. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$, and $\mathcal{T}$ is a tree of height κ admitting a θ-ascent path but no branch of size κ. Then $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$ holds.*

Proof. Write $\mathcal{T}$ as $(T, <_T)$. Fix a θ-ascent path $\langle f_\alpha \mid \alpha < \kappa \rangle$ through $\mathcal{T}$, and derive a coloring $c : [\kappa]^2 \to \theta$ via

$$c(\alpha, \beta) := \min\{i < \theta \mid \forall j \in [i, \theta)\ f_\alpha(j) <_T f_\beta(j)\}.$$

We shall show that c witnesses $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$.

Claim 3.7.1. *c is subadditive.*

Proof. Suppose that $\alpha < \beta < \gamma < \kappa$.
  ▶ Let $i := \max\{c(\alpha, \beta), c(\beta, \gamma)\}$. Then, for all $j \in [i, \theta)$, we have

$$f_\alpha(j) <_T f_\beta(j) <_T f_\gamma(j),$$

and hence $f_\alpha(j) <_T f_\gamma(j)$. Consequently, $c(\alpha, \gamma) \le i$.
  ▶ Let $i := \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$. Then, for all $j \in [i, \theta)$, we have $f_\alpha(j) <_T f_\gamma(j)$ and $f_\beta(j) <_T f_\gamma(j)$. But $\mathcal{T}$ is a tree, and hence $f_\alpha(j) <_T f_\beta(j)$. Consequently, $c(\alpha, \beta) \le i$.   ⊠

By Lemma 3.2, it remains to verify that c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$. Suppose this is not the case. Then for some $A \in [\kappa]^\kappa$, $i := \sup(c``[A]^2)$ is $< \theta$. But then the $<_T$-downward closure of $\{f_\alpha(i) \mid \alpha \in A\}$ is a branch of size κ through $\mathcal{T}$, contradicting our assumptions.   □

We now prove the converse.

Lemma 3.8. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$ and $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$ holds. Then there exists a tree $\mathcal{T}$ of height κ admitting a θ-ascent path but no branch of size κ.*

Proof. Let $c : [\kappa]^2 \to \theta$ be a witness for $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$. First, put

$$T := \{(i, \alpha, \beta) \mid i < \theta, \ \alpha \le \beta < \kappa, \ \mathrm{ssup}(D^c_{\le i}(\beta)) \le \alpha \le \beta\},$$

and then let $(i, \alpha, \beta) <_T (i', \alpha', \beta')$ iff $i = i'$, $\alpha < \alpha'$ and either $\beta = \beta'$ or $\beta < \beta'$ and $c(\beta, \beta') \le i$. Finally, let $\mathcal{T} := (T, <_T)$.

Claim 3.8.1. $\mathcal{T}$ *is a tree.*

Proof. It is clear that $<_T$ is anti-symmetric. To see that $<_T$ is transitive, suppose that $(i_0, \alpha_0, \beta_0) <_T (i_1, \alpha_1, \beta_1)$ and $(i_1, \alpha_1, \beta_1) <_T (i_2, \alpha_2, \beta_2)$. Then $i_0 = i_1 = i_2$ and $\alpha_0 < \alpha_1 < \alpha_2$. Now, if $\beta_1 \in \{\beta_0, \beta_2\}$, then clearly $(i_0, \alpha_0, \beta_0) <_T (i_2, \alpha_2, \beta_2)$. Otherwise $\beta_0 < \beta_1 < \beta_2$ and $c(\beta_0, \beta_2) \le \max\{c(\beta_0, \beta_1), c(\beta_1, \beta_2)\} \le i_0$, as sought.

It is clear that $<_T$ is well-founded, so to see that $<_T$ is a tree ordering, suppose that $(i_0, \alpha_0, \beta_0) <_T (i_2, \alpha_2, \beta_2)$ and $(i_1, \alpha_1, \beta_1) < (i_2, \alpha_2, \beta_2)$. Then $i_0 = i_2 = i_1$ and to avoid trivialities, we may also assume that $|\{\beta_0, \beta_1, \beta_2\}| = 3$, say, $\beta_0 < \beta_1 < \beta_2$. But then $c(\beta_0, \beta_1) \le \max\{c(\beta_0, \beta_2), c(\beta_1, \beta_2)\} \le i$, as sought.   ⊠

Claim 3.8.2. *For every* $\alpha < \kappa$, $T_\alpha := \{(i, \epsilon, \beta) \in T \mid \epsilon = \alpha\}$ *is nonempty and constitutes the* α^{th}*-level of* $\mathcal{T}$.

Proof. By induction on $\alpha < \kappa$.

▶ For any set of ordinals x, $\mathrm{ssup}(x) = 0$ iff x is empty, and hence $\{(i, 0, 0) \mid i < \theta\} \subseteq T_0$. In addition, it is clear that any element of T of the form $(i, 0, \beta)$ is minimal in $\mathcal{T}$.

▶ For every $\alpha < \kappa$ such that T_α constitutes the αth-level of $\mathcal{T}$, given $(i, \alpha+1, \beta) \in T_{\alpha+1}$, we shall find an immediate predecessor of it, lying in T_α. There are two cases to consider:

▶▶ If (i, α, β) is in T, then it is clearly an immediate predecessor of $(i, \alpha+1, \beta)$.

▶▶ Otherwise, we have $\alpha < \mathrm{ssup}(D^c_{\le i}(\beta)) \le \alpha+1$, i.e., $\mathrm{ssup}(D^c_{\le i}(\beta)) = \alpha+1$, so $\alpha = \max(D^c_{\le i}(\beta))$. Therefore, (i, α, α) is an immediate predecessor of $(i, \alpha+1, \beta)$.

▶ For every $\alpha \in \mathrm{acc}(\kappa)$ such that, for every $\bar{\alpha} < \alpha$, $T_{\bar{\alpha}}$ constitutes the $\bar{\alpha}^{th}$-level of $\mathcal{T}$, given $(i, \alpha, \beta) \in T_\alpha$, we shall show that for every $\bar{\alpha} < \alpha$, there is $\bar{\beta} \le \beta$ such that $(i, \bar{\alpha}, \bar{\beta})$ belongs to T and is $<_T$-below (i, α, β).

▶▶ If $(i, \bar{\alpha}, \beta)$ is in T, then $(i, \bar{\alpha}, \beta) <_T (i, \alpha, \beta)$, and we are done.

▶▶ Otherwise, $\mathrm{ssup}(D^c_{\le i}(\beta)) > \bar{\alpha}$. Let $\bar{\beta} := \min(D^c_{\le i}(\beta) \setminus \bar{\alpha})$. Since c is subadditive, we have $D^c_{\le i}(\bar{\beta}) = D^c_{\le i}(\beta) \cap \bar{\beta}$, and hence, by our choice of $\bar{\beta}$, it follows that $\mathrm{ssup}(D^c_{\le i}(\bar{\beta})) \le \alpha$. Therefore, $(i, \bar{\alpha}, \bar{\beta}) \in T$ and $(i, \bar{\alpha}, \bar{\beta}) <_T (i, \alpha, \beta)$.   ⊠

The fact that $\mathcal{T}$ has no branch of size κ follows immediately from the fact that c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, as in the proof of Claim 3.2.1. Finally, to see that $\mathcal{T}$ admits a θ-ascent path, for every α, define $f_\alpha : \theta \to T_\alpha$ by letting $f_\alpha(i) := (i, \alpha, \alpha)$.   □

Under an additional assumption, we may infer the existence of a tree with the preceding features that is moreover κ-Aronszajn. Before stating the next result, we recall the notion of a *C-sequence*, which will be used in its proof.

Definition 3.9. A *C-sequence* over κ is a sequence $\langle C_\beta \mid \beta < \kappa \rangle$ such that, for all $\beta < \kappa$, C_β is a closed subset of β with $\mathrm{sup}(C_\beta) = \mathrm{sup}(\beta)$.

Lemma 3.10. *Suppose that* $\theta \le \chi \le \kappa$ *are infinite regular cardinals such that* κ *is* $<\chi$*-inaccessible and* $\theta < \kappa$. *If there exists an* $E^\kappa_{\ge \chi}$*-closed witness to* $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$, *then there exists a* κ*-Aronszajn tree* $\mathcal{T}$ *admitting a* θ*-ascent path.*

Proof. As there is an ω_1-Aronszajn tree, and as any ω_1-Aronszajn tree admits an ω-ascent path, we may assume that $\kappa \geq \aleph_2$. Fix an $E^\kappa_{\geq \chi}$-closed coloring $c : [\kappa]^2 \to \theta$ witnessing $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$. For every $\beta \in \mathrm{acc}(\kappa)$, let $i(\beta)$ denote the least $i \leq \theta$ such that $\sup(D^c_{\leq i}(\beta)) = \beta$. Note that if $\mathrm{cf}(\beta) \neq \theta$, then $i(\beta) < \theta$. Now, for every $i < \theta$, define a C-sequence $\vec{C}^i := \langle C^i_\beta \mid \beta < \kappa \rangle$ as follows.

▶ Let $C^i_0 := \emptyset$.
▶ For all $\beta < \kappa$, let $C^i_{\beta+1} := \{\beta\}$.
▶ For all $\beta \in \mathrm{acc}(\kappa)$ such that $i \geq i(\beta)$, let $C^i_\beta := \mathrm{cl}(D^c_{\leq i}(\beta))$.
▶ For all $\beta \in \mathrm{acc}(\kappa)$ such that $i < i(\beta) < \theta$, let $C^i_\beta := \mathrm{cl}(D^c_{\leq i(\beta)}(\beta))$.
▶ For any other β, let C^i_β be a club in β of order-type θ.

Claim 3.10.1. *For every $\alpha < \kappa$, $|\{C^i_\beta \cap \alpha \mid \beta < \kappa, \ i < \theta\}| < \kappa$.*

Proof. Suppose not, and let α be the least counterexample. As κ is $<\chi$-inaccessible, $\mathrm{cf}(\alpha) \geq \chi$. By minimality of α,

$$|\{C^i_\beta \cap \alpha \mid \beta < \kappa, \ i < \theta, \ \sup(C^i_\beta \cap \alpha) = \alpha\}| = \kappa.$$

As κ is in particular $<\theta$-inaccessible, moreover

$$|\{C^i_\beta \cap \alpha \mid \beta < \kappa, \ i < \theta, \ \sup(C^i_\beta \cap \alpha) = \alpha, \ \mathrm{otp}(C^i_\beta) > \theta\}| = \kappa.$$

It follows that we may fix $B \in [\kappa \setminus (\alpha+1)]^\kappa$ and $i < \theta$ such that:

• for every $\beta \in B$, $i(\beta) \leq i$, and
• $\beta \mapsto C^i_\beta \cap \alpha$ is an injection from B to the family of cofinal subsets of α.

Consequently, $\beta \mapsto D^c_{\leq i}(\beta) \cap \alpha$ is an injection from B to the family of cofinal subsets of α. Now, as $\mathrm{cf}(\alpha) \geq \chi$ and c is $E^\kappa_{\geq \chi}$-closed, it follows that for every $\beta \in B$, $c(\alpha, \beta) \leq i$. But, then, by subadditivity, for every $\beta \in B$, $D^c_{\leq i}(\beta) \cap \alpha = D^c_{\leq i}(\alpha)$. This is a contradiction.   ⊠

For each $i < \theta$, let $\rho^i_2 : [\kappa]^2 \to \omega$ be the corresponding *number of steps* function derived from walking along $\vec{C}^i$, as in [Tod07, §6.3]. Then, for all $i < \theta$ and $\beta < \kappa$, define a function $\rho^i_{2\beta} : \beta \to \omega$ via $\rho^i_{2\beta}(\alpha) := \rho^i_2(\alpha, \beta)$. Finally, let

$$T := \{\rho^i_{2\beta} \restriction \alpha \mid i < \theta, \ \alpha \leq \beta < \kappa\}.$$

By Claim 3.10.1, $\mathcal{T} := (T, \subseteq)$ is a κ-tree.

Claim 3.10.2. *$\mathcal{T}$ has no branch of size κ.*

Proof. Otherwise, by a standard argument (e.g., the proof of [Rin14, Corollary 2.6]), there exists $i < \theta$ for which ρ^i_2 admits a homogeneous set of size κ. Fix such an i. By [Tod07, Theorem 6.3.2], then, we may fix a club $C \subseteq \kappa$ such that, for every $\alpha < \kappa$, there exists $\beta < \kappa$ such that $C \cap \alpha \subseteq C^i_\beta$. By the definition of $\vec{C}^i$, for every $\alpha < \kappa$ with $\mathrm{otp}(C \cap \alpha) > \theta$, it follows that there exist $j < \theta$ and $\beta < \kappa$ such that $C \cap \alpha \subseteq \mathrm{cl}(D^c_{\leq j}(\beta))$. By the pigeonhole principle, we may now fix $j < \kappa$ such that for every $\alpha < \kappa$, there exists $\beta_\alpha \geq \alpha$ such that $C \cap \alpha \subseteq \mathrm{cl}(D^c_{\leq j}(\beta_\alpha))$.

As c in particular witnesses $\mathrm{U}(\kappa, 2, \theta, 3)$, we may find $(\gamma, \alpha) \in [C \cap E^\kappa_{\geq \chi}]^2$ such that $c(\gamma, \beta_\alpha) > j$. As $\gamma \in C \cap \alpha$, we infer that $\gamma \in \mathrm{cl}(D^c_{\leq j}(\beta_\alpha))$. As $\gamma \in E^\kappa_{\geq \chi}$ and c is $E^\kappa_{\geq \chi}$-closed, it follows that $\gamma \in D^c_{\leq j}(\beta_\alpha)$, contradicting the fact $c(\gamma, \beta_\alpha) > j$.   ⊠

Claim 3.10.3. *$\mathcal{T}$ admits a θ-ascent path.*

Proof. As $\kappa \geq \aleph_2$, $S := \mathrm{acc}(\kappa) \setminus E_\theta^\kappa$ is stationary. For every $\alpha < \kappa$, define $f_\alpha :$ $\theta \to T_\alpha$ via $f_\alpha(i) := \rho_{2\min(S\setminus\alpha)}^i \restriction \alpha$. To see that $\langle f_\alpha \mid \alpha < \kappa \rangle$ forms a θ-ascent path through our tree, fix arbitrary $\alpha < \beta < \kappa$. Write $\gamma := \min(S\setminus\alpha)$ and $\delta := \min(S\setminus\beta)$. To avoid trivialities, suppose that $\gamma \neq \delta$, so that $\alpha \leq \gamma < \beta \leq \delta$. As $(\gamma, \delta) \in [S]^2$, $i := \max\{i(\gamma), i(\delta), c(\gamma, \delta)\}$ is $< \theta$, and, for all $j \in [i, \theta)$, $C_\gamma^j := \mathrm{cl}(D_{\leq j}^c(\gamma))$ and $C_\delta^j := \mathrm{cl}(D_{\leq j}^c(\delta))$. By subadditivity, for every $\epsilon < \gamma$, $c(\epsilon, \gamma) \leq \max\{c(\epsilon, \delta), c(\gamma, \delta)\}$ and $c(\epsilon, \delta) \leq \max\{c(\epsilon, \gamma), c(\gamma, \delta)\}$. So, for every $j \in [i, \theta)$, from $c(\gamma, \delta) \leq i \leq j$, we infer that $C_\gamma^j = C_\delta^j \cap \gamma$, $\rho_{2\gamma}^j = \rho_{2\delta}^j$ and $f_\alpha(j) \subseteq f_\beta(j)$, as sought. $\boxtimes$

This completes the proof. $\square$

3.2. **Locally small colorings.**

Definition 3.11. A coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ is *locally small* iff $|D_{\leq i}^c(\beta)| < \lambda$ for all $i < \mathrm{cf}(\lambda)$ and $\beta < \lambda^+$.

By Corollary 3.27 below, if λ is regular, then any locally small coloring is a witness to $\mathrm{U}(\lambda^+, \lambda^+, \lambda, \lambda)$. In the general case, a locally small coloring witnesses an unbalanced strengthening of $\mathrm{U}(\lambda^+, 2, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$, as follows.

Lemma 3.12. *Suppose that* $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ *is a locally small coloring.*
  (1) *For every* $\nu < \mathrm{cf}(\lambda)$, *c witnesses* $\mathrm{U}(\lambda^+, \lambda \circledast \nu, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$;
  (2) *If c is subadditive, then c moreover witnesses* $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$.

Proof. (1) Suppose that we are given $\sigma < \mathrm{cf}(\lambda)$, a pairwise disjoint subfamily $\mathcal{A} \subseteq [\lambda^+]^\sigma$ of size λ^+, and some $i < \mathrm{cf}(\lambda)$. Find a large enough $\epsilon < \lambda^+$ such that $\mathcal{A} \cap \mathcal{P}(\epsilon)$ has size λ. Now, given $\nu < \mathrm{cf}(\lambda)$, fix any $\mathcal{B}' \in [\mathcal{A}]^\nu$ with $\min(\bigcup \mathcal{B}') > \epsilon$. Since c is locally small and $|\bigcup \mathcal{B}'| < \mathrm{cf}(\lambda)$, $X := \bigcup_{b \in \mathcal{B}'} \bigcup_{\beta \in b} D_{\leq i}^c(\beta)$ has size $< \lambda$. As $\mathcal{A} \cap \mathcal{P}(\epsilon)$ consists of λ-many pairwise disjoint sets, it follows that $\mathcal{A}' := \{a \in \mathcal{A} \cap \mathcal{P}(\epsilon) \mid a \cap X = \emptyset\}$ has size λ. Evidently, for every $(a, b) \in \mathcal{A}' \times \mathcal{B}'$, we have $a < b$ and $\min(c[a \times b]) > i$.

(2) Suppose that c is subadditive. By Lemma 3.2(2), it suffices to prove that c witnesses $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), 2)$. So, let $A \in [\lambda^+]^{\lambda^+}$ and $i < \mathrm{cf}(\lambda)$ be given; we need to find $B \in [A]^{\lambda^+}$ such that $c(\alpha, \beta) > i$ for all $(\alpha, \beta) \in [B]^{\lambda^+}$.

As in the proof of Lemma 3.2, we define a tree ordering $<_i$ on λ^+ by letting $\alpha <_i \beta$ iff $\alpha < \beta$ and $c(\alpha, \beta) \leq i$. By Clause (1), c in particular witnesses $\mathrm{U}(\lambda^+, 2, \mathrm{cf}(\lambda), 2)$, and hence $(\lambda^+, <_i)$ admits no chains of size λ^+. Assuming that the sought-after $B \in [A]^{\lambda^+}$ does not exist, it follows that $(A, <_i)$ has no antichains of size λ^+, so $(A, <_i)$ is a λ^+-Souslin tree. Forcing with this tree, we arrive at a λ^+-distributive forcing extension $V[G]$ in which $(A, <_i)$ does admit a chain of size λ^+. But $V[G]$ is a λ^+-distributive forcing extension of V, and hence c remains locally small in $V[G]$, and in particular, it still witnesses $\mathrm{U}(\lambda^+, 2, \mathrm{cf}(\lambda), 2)$. This is a contradiction. $\square$

Lemma 3.13. *For every infinite cardinal λ, there exists a locally small witness to* $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \lambda)$ *which is subadditive of the first kind.*

Proof. We focus on the case in which λ is singular, since, if λ is regular, a better result is given by Lemma 3.14(1) below. Let $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ be a strictly increasing sequence of regular uncountable cardinals converging to λ. By [She91, Lemma 4.1], there exists a coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ which is subadditive of the first kind and locally small in the following strong sense: for all $i < \mathrm{cf}(\lambda)$ and $\beta < \lambda^+$, $|D_{\leq i}^c(\beta)| \leq \lambda_i$.

Claim 3.13.1. *For every pairwise disjoint subfamily $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ of size λ^+, for every club $D \subseteq \lambda^+$, and for every $i < \mathrm{cf}(\lambda)$, there exist $\gamma \in D$, $a \in \mathcal{A}$, and $\epsilon < \gamma$ such that:*

- *$\gamma < a$;*
- *for all $\alpha \in (\epsilon, \gamma)$ and $\beta \in a$, we have $c(\alpha, \beta) > i$.*

Proof. Given $\mathcal{A}$ and i as above, fix a large enough $j < \mathrm{cf}(\lambda)$ such that $j \geq i$ and $\mathcal{A}_j := \{a \in \mathcal{A} \mid |a| \leq \lambda_j\}$ has size λ^+. Now, given a club $D \subseteq \lambda^+$, fix $\gamma \in D \cap E^{\lambda^+}_{>\lambda_j}$, and pick any $a \in \mathcal{A}_j$ with $\gamma < a$.

For all $\beta \in a$, we have $|D^c_{\leq j}(\beta)| \leq \lambda_j < \mathrm{cf}(\gamma)$, so $\epsilon := \sup(\bigcup\{D^c_{\leq j}(\beta) \cap \gamma \mid \beta \in a\})$ is $< \gamma$. Then for all $\alpha \in (\epsilon, \gamma)$ and $\beta \in a$, we have $c(\alpha, \beta) > j \geq i$.　　⊠

It now follows from Fact 2.2 that c witnesses $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \lambda)$. □

Lemma 3.14 (Todorcevic, [Tod07]). (1) *If λ is regular, then there exists a locally small and subadditive witness to $\mathrm{U}(\lambda^+, \lambda^+, \lambda, \lambda)$.*
  (2) *If $\square_\lambda$ holds, then there exists a locally small and subadditive witness to $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \lambda)$ which is moreover closed.*

Proof. (1) Consider the function $\rho : [\lambda^+]^2 \to \lambda$ defined in [Tod07, §9.1]. By [Tod07, Lemma 9.1.1], ρ is locally small. By [Tod07, Lemma 9.1.2], ρ is subadditive. By [Tod07, Theorem 6.2.7], ρ is a witness to $\mathrm{U}(\lambda^+, \lambda^+, \lambda, \lambda)$.

(2) This follows from Lemmas 7.3.7, 7.3.8, 7.3.11 and 7.3.12 of [Tod07], together with Lemma 3.2(3). For λ singular, a slightly better result is proved in [LH17a, Theorem 5.8]. □

The following result, due independently to Shani and Lambie-Hanson, shows that the hypothesis of $\square_\lambda$ cannot be weakened to $\square_{\lambda,2}$ in Lemma 3.14(2). (We note that GCH is not explicitly mentioned in the quoted results, but it is evident from their proofs that, if GCH holds in the relevant ground models, then it continues to hold in the forcing extensions witnessing the conclusion of the result.)

Fact 3.15 (Shani, [Sha16, Theorem 1], Lambie-Hanson, [LH17a, Corollaries 5.13 and 5.14]). *Relative to the existence of large cardinals, it is consistent with GCH that there is an uncountable cardinal λ such that $\square_{\lambda,2}$ holds, and, for every $\theta < \lambda$, $\mathrm{U}^{\mathrm{subadditive}}(\lambda^+, 2, \theta, 2)$ fails. λ can be either regular or singular here, though attaining the result for singular λ requires significantly larger cardinals than attaining it for regular λ.*

The principle SAP^*_λ was introduced in [Rin10, Definition 2.12] as a weakening of Jensen's weak square principle $\square^*_\lambda$. By [Rin10, Theorem 2.6], assuming $2^\lambda = \lambda^+$, SAP^*_λ implies that $\diamondsuit(S)$ holds for every stationary subset $S \subseteq \lambda^+$ that reflects stationarily often. By [Rin10, Corollary 2.16], if λ is a singular cardinal such that $2^{<\lambda} < 2^\lambda = \lambda^+$ and every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects, then SAP^*_λ moreover implies $\diamondsuit^*(\lambda^+)$.

Proposition 3.16. *Suppose that λ is a singular cardinal and there exists a locally small and subadditive coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$. Then SAP^*_λ holds and $E^{\lambda^+}_{>\mathrm{cf}(\lambda)} \in I[\lambda^+; \lambda]$.*

Proof. Let c be as above. By [Rin10, Definitions 2.4 and 2.12], as c is locally small and subadditive of the first kind,[2] to verify SAP^*_λ, it suffices to verify that for every stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ and every $\gamma \in \mathrm{Tr}(S)$, there exists a stationary $S_\gamma \subseteq S \cap \gamma$ such that $\sup(c``[S_\gamma]^2) < \mathrm{cf}(\lambda)$.

To this end, fix arbitrary $\gamma \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ and a stationary $s \subseteq \gamma$. As $\mathrm{cf}(\gamma) > \mathrm{cf}(\lambda)$, there exists a large enough $i < \mathrm{cf}(\lambda)$ such that $S_\gamma := D^c_{\leq i}(\gamma) \cap s$ is stationary in γ. Since c is subadditive of the second kind, for any pair $(\alpha, \beta) \in [S_\gamma]^2$, we have that $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\} \leq i$. Therefore, $\sup(c``[S_\gamma]^2) \leq i$. Recalling [Rin10, Definitions 2.3], the very same argument shows that $E^{\lambda^+}_{>\mathrm{cf}(\lambda)} \in I[\lambda^+; \lambda]$. $\square$

Corollary 3.17. *If λ is a singular strong limit and there exists a locally small and subadditive coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$, then AP_λ holds.*

Proof. By the preceding proposition, the hypothesis imply that $E^{\lambda^+}_{>\mathrm{cf}(\lambda)} \in I[\lambda^+; \lambda]$. Now, given that λ is a strong limit, we moreover get that $\lambda^+ \in I[\lambda^+; \lambda] = I[\lambda^+]$, meaning that AP_λ holds. $\square$

3.3. **Forms of coherence and levels of divergence.** We will also be interested in variants of subadditivity, as captured by the next definition and by Definition 3.24 below.

Definition 3.18. A coloring $c : [\kappa]^2 \to \theta$ is *weakly subadditive* iff the following two statements hold:

(1) c is *weakly subadditive of the first kind*, that is, for all $\beta < \gamma < \kappa$ and $i < \theta$, there is $j < \theta$ such that $D^c_{\leq i}(\beta) \subseteq D^c_{\leq j}(\gamma)$;
(2) c is *weakly subadditive of the second kind*, that is, for all $\beta < \gamma < \kappa$ and $i < \theta$, there is $j < \theta$ such that $D^c_{\leq i}(\gamma) \cap \beta \subseteq D^c_{\leq j}(\beta)$.

Remark 3.19. The statement of [LH17a, Lemma 5.11], together with Fact 3.15, makes it clear that it is consistent with $\mathsf{GCH} + \square_{\lambda,2}$ (for either a regular or a singular cardinal λ), that, for every $\theta < \lambda$, there exists no witness to $\mathrm{U}(\lambda^+, 2, \theta, 2)$ which is weakly subadditive of the first kind. In particular, very good scales for $\aleph_\omega$, and hence also the weaker combinatorial principle $\mathrm{ADS}_{\aleph_\omega}$, both of which are consequences of $\square_{\aleph_\omega,2}$, are not sufficient to get a weakly subadditive witness to $\mathrm{U}(\aleph_{\omega+1}, 2, \aleph_n, 2)$.

Proposition 3.20. *Suppose that $\chi \leq \theta$ is a pair of infinite regular cardinals $< \kappa$, and $c : [\kappa]^2 \to \theta$ is a witness to $\mathrm{U}(\kappa, 2, \theta, 2)$ that is weakly subadditive of the second kind. If c is Σ-closed for some stationary $\Sigma \subseteq E^\kappa_{\geq \chi}$, then c witnesses $\mathrm{U}(\kappa, \kappa, \theta, \chi)$.*

Proof. Let $\Sigma \subseteq E^\kappa_{\geq \chi}$ be stationary.

Claim 3.20.1. *Suppose that $\mathcal{A} \subseteq [\kappa]^{<\chi}$ is a family consisting of κ-many pairwise disjoint sets, D is a club in κ, and $i < \theta$ is a prescribed color. Then there exist $\gamma \in D \cap \Sigma$ and $a \in \mathcal{A}$ such that*

(a) $\gamma < a$;
(b) *for all $\beta \in a$, we have $c(\gamma, \beta) > i$.*

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[2]The terminology in [Rin10] is slightly different; there *locally small* is dubbed *normal*, and *subadditive of the first kind* is dubbed *subadditive*.

Proof. For each $\delta < \kappa$, pick $a_\delta \in \mathcal{A}$ with $\delta < a_\delta$. As $\mathcal{A} \subseteq [\kappa]^{<\mathrm{cf}(\theta)}$, we may define a function $f : \kappa \to \theta$ via:

$$f(\delta) := \min\{j < \theta \mid \forall \beta \in a_\delta \, [D^c_{\leq i}(\beta) \cap \delta \subseteq D^c_{\leq j}(\delta)]\}.$$

Fix $j < \theta$ for which $A := \{\delta \in D \cap \Sigma \mid f(\delta) = j\}$ is cofinal. Since c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, we can now pick $(\gamma, \delta) \in [A]^2$ such that $c(\gamma, \delta) > j$. Clearly, $\gamma < a_\delta$. Now, if there exists $\beta \in a_\delta$ such that $c(\gamma, \beta) \leq i$, then $\gamma \in D^c_{\leq i}(\beta) \cap \delta \subseteq D^c_{\leq j}(\delta)$ which is not the case. Therefore, both clauses of the claim hold for $a := a_\delta$. $\boxtimes$

Evidently, if c is Σ-closed, then Clause (b) of the preceding is equivalent to:

(b') there is $\epsilon < \gamma$ such that, for all $\alpha \in (\epsilon, \gamma)$ and all $\beta \in a$, we have $c(\alpha, \beta) > i$.

Thus, the implication (2) $\implies$ (3) of Fact 2.2 gives the desired conclusion. $\square$

Motivated by the proof of Lemma 3.10, we introduce the following definition.

Definition 3.21 (Levels of divergence). For a coloring $c : [\kappa]^2 \to \theta$, let

$$\partial(c) := \{\beta \in \mathrm{acc}(\kappa) \mid \forall \gamma < \kappa \forall i < \theta \, \sup(D^c_{\leq i}(\gamma) \cap \beta) < \beta\}.$$

Note that $\partial(c) \subseteq E^\kappa_{\mathrm{cf}(\theta)}$ and that c is vacuously $\partial(c)$-closed.

Lemma 3.22. *Suppose that $c : [\kappa]^2 \to \theta$ is a coloring. If c is weakly subadditive of the second kind, then*

$$\partial(c) = \{\beta \in \mathrm{acc}(\kappa) \mid \forall i < \theta \, \sup(D^c_{\leq i}(\beta)) < \beta\}.$$

Proof. If c is weakly subadditive of the second kind, then for all $\beta < \gamma < \kappa$ and $i < \theta$ such that $\sup(D^c_{\leq i}(\gamma) \cap \beta) = \beta$, there exists $j < \theta$ such that $D^c_{\leq i}(\gamma) \cap \beta \subseteq D^c_{\leq j}(\beta)$, and hence $\sup(D^c_{\leq j}(\beta)) = \beta$. $\square$

We will be particularly interested in situations in which $\partial(c)$ is stationary in κ; one reason for this is the following lemma, indicating that colorings c for which $\partial(c)$ is stationary automatically witness an instance of $\mathrm{U}(\ldots)$.

Lemma 3.23. *Suppose that $c : [\kappa]^2 \to \theta$ is a coloring for which $\partial(c)$ is stationary. Then c witnesses $\mathrm{U}(\kappa, \kappa, \theta, \mathrm{cf}(\theta))$.*

Proof. We prove that the coloring c satisfies Clause (2) of Fact 2.2 with $\chi := \mathrm{cf}(\theta)$, which will yield our desired conclusion. Fix a family $\mathcal{A} \subseteq [\kappa]^{<\mathrm{cf}(\theta)}$ consisting of κ-many pairwise disjoint sets, a club $D \subseteq \kappa$, and an $i < \theta$. Since $\partial(c)$ is stationary, we can fix $\gamma \in \partial(c) \cap D$. Also fix an $a \in \mathcal{A}$ such that $\gamma < a$. As $\gamma \in \partial(c)$, for all $\beta \in a$, we have $\sup(D^c_{\leq i}(\beta) \cap \gamma) < \gamma$. Since $|a| < \mathrm{cf}(\theta) = \mathrm{cf}(\gamma)$, we can find $\epsilon < \gamma$ such that $\sup(D^c_{\leq i}(\beta) \cap \gamma) < \epsilon$ for all $\beta \in a$. Now, for all $\alpha \in (\epsilon, \gamma)$ and all $\beta \in a$, we have $\alpha \notin D^c_{\leq i}(\beta)$, so $c(\alpha, \beta) > i$, as desired. $\square$

Definition 3.24 (Forms of coherence). Let $c : [\kappa]^2 \to \theta$ be a coloring.
  (1) c is ℓ_∞-*coherent* iff for all $\gamma < \delta < \kappa$, there is $j < \theta$ such that, for all $i < \theta$, $D^c_{\leq i}(\gamma) \subseteq D^c_{\leq i+j}(\delta)$ and $D^c_{\leq i}(\delta) \cap \gamma \subseteq D^c_{\leq i+j}(\gamma)$;
  (2) For a cardinal $\lambda < \kappa$, c is λ-*coherent* iff for every $(\gamma, \delta) \in [\kappa]^2$,

$$|\{\alpha < \gamma \mid c(\alpha, \gamma) \neq c(\alpha, \delta)\}| < \lambda;$$

(3) For $S \subseteq \mathrm{acc}(\kappa),^3$ c is S-coherent iff for all $\beta \leq \gamma < \delta < \kappa$ with $\beta \in S$,

$$\sup\{\alpha < \beta \mid c(\alpha, \gamma) \neq c(\alpha, \delta)\} < \beta.$$

Remark 3.25. For every $\lambda \in \mathrm{Reg}(\kappa)$, c is λ-coherent iff it is E_λ^κ-coherent iff it is $E_{\geq \lambda}^\kappa$-coherent.

Lemma 3.26. *Let $c : [\kappa]^2 \to \theta$ be a coloring, with $\theta \in \mathrm{Reg}(\kappa)$.*
   (1) *If c is θ-coherent, then it is ℓ_∞-coherent;*
   (2) *If c is subadditive, then it is $\partial(c)$-coherent and ℓ_∞-coherent;*
   (3) *If c is ℓ_∞-coherent, then it is weakly subadditive.*

Proof. (1) If c is θ-coherent, then for all $\gamma < \delta < \kappa$,

$$j := \sup\{c(\alpha, \gamma), c(\alpha, \delta) \mid \alpha < \gamma, \ c(\alpha, \gamma) \neq c(\alpha, \delta)\}$$

is $< \theta$, and, for every $i \in [j, \theta)$, $D_{\leq i}^c(\gamma) = D_{\leq i}^c(\delta) \cap \gamma$.

(2) Suppose that c is subadditive. To see that c is ℓ_∞-coherent, let $\gamma < \delta < \kappa$ be arbitrary. Set $j := c(\gamma, \delta)$. For all $i < \theta$ and $\alpha \in D_{\leq i}^c(\gamma)$, $c(\alpha, \delta) \leq \max\{c(\alpha, \gamma), c(\gamma, \delta)\} \leq \max\{i, j\}$, and, for all $i < \theta$ and $\alpha \in D_{\leq i}^c(\delta) \cap \gamma$, $c(\alpha, \gamma) \leq \max\{c(\alpha, \delta), c(\gamma, \delta)\} \leq \max\{i, j\}$. Altogether, $D_{\leq i}^c(\gamma) = D_{\leq i}^c(\delta) \cap \gamma$ for every $i \in [j, \theta)$.

Next, we show that c is $\partial(c)$-coherent. To this end, fix $\beta \leq \gamma < \delta < \kappa$ with $\beta \in \partial(S)$. Set $i := c(\gamma, \delta)$. By the subadditivity of c, it follows that $D_{\leq j}^c(\gamma) = D_{\leq j}^c(\delta) \cap \gamma$ for all $j \in [i, \theta)$. In particular, $\{\alpha < \beta \mid c(\alpha, \gamma) \neq c(\alpha, \delta)\} \subseteq D_{\leq i}^c(\gamma) \cap \beta$. Since $\beta \in \partial(c)$, $\sup(D_{\leq i}^c(\gamma) \cap \beta) < \beta$, so we are done.

(3) This is clear. $\square$

Corollary 3.27. *If λ is regular and $c : [\lambda^+]^2 \to \lambda$ is locally small, then $\partial(c) = E_\lambda^{\lambda^+}$, and hence:*
   (1) *c is a witness to $\mathrm{U}(\lambda^+, \lambda^+, \lambda, \lambda)$;*
   (2) *If c is subadditive, then c is λ-coherent.*

Proof. The fact that $\partial(c) = E_\lambda^{\lambda^+}$ is immediate. Now, Clause (1) follows from Lemma 3.23, and Clause (2) follows from Lemma 3.26(2). $\square$

Lemma 3.28. *Let $\theta \in \mathrm{Reg}(\kappa)$. Either of the following implies that $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ holds:*
   (1) *There exists a θ-coherent witness to $\mathrm{U}(\kappa, 2, \theta, 2)$;*
   (2) *θ is uncountable, and there exists an ℓ_∞-coherent witness to $\mathrm{U}(\kappa, 2, \theta, 2)$.*

Proof. (1) By Clause (2) and Lemma 3.26(1), we may assume here that $\theta = \omega$; however, the proof works uniformly. Suppose that $c : [\kappa]^2 \to \theta$ is a θ-coherent witness to $\mathrm{U}(\kappa, 2, \theta, 2)$. Define a coloring $d : [\kappa]^2 \to \theta$ by letting, for all $\gamma < \delta < \kappa$, $d(\gamma, \delta)$ be the least $j < \theta$ such that $D_{\leq i}^c(\gamma) = D_{\leq i}^c(\delta) \cap \gamma$ for every $i \in [j, \theta)$. It is clear that d is subadditive. Now, if d fails to witness $\mathrm{U}(\kappa, 2, \theta, 2)$, then we may fix $A \in [\kappa]^\kappa$ and $j < \theta$ such that $d(\gamma, \delta) \leq j$ for every pair $(\gamma, \delta) \in [A]^2$. For every $\gamma \in A$, let $\gamma' := \min(A \setminus (\gamma + 1))$ and $i_\gamma := c(\gamma, \gamma')$. Fix $A' \in [A]^\kappa$ and $i < \theta$ such that $i = \max\{j, i_\gamma\}$ for all $\gamma \in A'$.

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[3]Strictly speaking, to avoid ambiguity with Clause (2), we need to assume that $|S| \geq 2$, but in all cases of interest S will in fact be stationary in κ.

Then, for every $(\gamma, \delta) \in [A']^2$, $\gamma < \gamma' \leq \delta$ and $\gamma \in D^c_{\leq i_\gamma}(\gamma') \subseteq D^c_{\leq i}(\gamma') = D^c_{\leq i}(\delta) \cap \gamma'$, so $c(\gamma, \delta) \leq i$. It follows that $c``[A']^2$ is bounded in θ, contradicting the fact that c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$.

(2) Suppose that $c : [\kappa]^2 \to \theta$ is an ℓ_∞-coherent witness to $\mathrm{U}(\kappa, 2, \theta, 2)$ and $\theta > \aleph_0$. For every $j < \theta$, let τ_j denote the least indecomposable ordinal greater than j. For every $(\gamma, \delta) \in [\kappa]^2$, let $j_{\gamma,\delta}$ denote the least $j < \theta$ such that, for all $i < \theta$, $D^c_{\leq i}(\gamma + 1) \subseteq D^c_{\leq i+j}(\delta + 1)$ and $D^c_{\leq i}(\delta + 1) \cap (\gamma + 1) \subseteq D^c_{\leq i+j}(\gamma + 1)$. Finally, define a coloring $d : [\kappa]^2 \to \theta$ via $d(\gamma, \delta) := \tau_{j_{\gamma,\delta}}$.

Claim 3.28.1. *Let $\beta < \gamma < \delta < \kappa$. Then:*
   (i) $d(\beta, \delta) \leq \max\{d(\beta, \gamma), d(\gamma, \delta)\}$;
   (ii) $d(\beta, \gamma) \leq \max\{d(\beta, \delta), d(\gamma, \delta)\}$.

Proof. (i) For all $i < \theta$, $D^c_{\leq i}(\beta + 1) \subseteq D^c_{\leq i+j_{\beta,\gamma}}(\gamma + 1) \subseteq D^c_{\leq i+j_{\beta,\gamma}+j_{\gamma,\delta}}(\delta + 1)$ and $D^c_{\leq i}(\delta + 1) \cap (\beta + 1) \subseteq D^c_{\leq i+j_{\gamma,\delta}}(\gamma + 1) \cap (\beta + 1) \subseteq D^c_{\leq i+j_{\gamma,\delta}+j_{\beta,\gamma}}(\beta + 1)$. Therefore, $j_{\beta,\delta}$ is no more than $j := \max\{j_{\beta,\gamma} + j_{\gamma,\delta}, j_{\gamma,\delta} + j_{\beta,\gamma}\}$. As $\tau := \max\{d(\beta, \gamma), d(\gamma, \delta)\}$ is an indecomposable ordinal greater than both $j_{\beta,\gamma}$ and $j_{\gamma,\delta}$, it is moreover greater than j. Altogether, $d(\beta, \delta) = \tau_{j_{\beta,\delta}} \leq \tau_j \leq \tau$.

(ii) For all $i < \theta$, $D^c_{\leq i}(\beta + 1) \subseteq D^c_{\leq i+j_{\beta,\delta}}(\delta + 1) \cap (\gamma + 1) \subseteq D^c_{\leq i+j_{\beta,\delta}+j_{\gamma,\delta}}(\gamma + 1)$ and $D^c_{\leq i}(\gamma + 1) \cap (\beta + 1) \subseteq D^c_{\leq i+j_{\gamma,\delta}}(\delta + 1) \cap (\beta + 1) \subseteq D^c_{\leq i+j_{\gamma,\delta}+j_{\beta,\delta}}(\beta + 1)$. Therefore, $j_{\beta,\gamma}$ is no more than $j := \max\{j_{\beta,\delta} + j_{\gamma,\delta}, j_{\gamma,\delta} + j_{\beta,\delta}\}$. As $\tau := \max\{d(\beta, \delta), d(\gamma, \delta)\}$ is an indecomposable ordinal greater than both $j_{\beta,\delta}$ and $j_{\gamma,\delta}$, it is moreover greater than j. Altogether, $d(\beta, \gamma) = \tau_{j_{\beta,\gamma}} \leq \tau_j \leq \tau$. ⊠

Now, if d fails to witness $\mathrm{U}(\kappa, 2, \theta, 2)$, then we may fix $A \in [\kappa]^\kappa$ and $j < \theta$ such that $d(\gamma, \delta) \leq j$ for every pair $(\gamma, \delta) \in [A]^2$. By possibly increasing j and thinning out A, we may moreover assume that for every $\gamma \in A$, $c(\gamma, \gamma + 1) \leq j$ and $D^c_{\leq i}(\gamma + 1) \cap \gamma \subseteq D^c_{\leq i+j}(\gamma)$ for all $i < \theta$. It follows that for every $(\gamma, \delta) \in [A]^2$, $\gamma \in D^c_{\leq j}(\gamma + 1) \cap \delta \subseteq D^c_{\leq j+j}(\delta + 1) \cap \delta \subseteq D^c_{\leq j+j+j}(\delta)$, so that $c``[A]^2 \leq j \cdot 3 < \theta$, contradicting the fact that c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$. □

Corollary 3.29. *If there exists a uniformly coherent κ-Souslin tree, then, for every $\theta \in \mathrm{Reg}(\kappa)$, $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$ holds.*

Proof. Suppose that there exists a uniformly coherent κ-Souslin tree. This means that there exists a downward closed subfamily $T \subseteq {}^{<\kappa}2$ such that:
   (a) $(T, \subseteq)$ is a κ-Souslin tree;
   (b) for all $s, t \in T$, $\{\epsilon \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\epsilon) \neq t(\epsilon)\}$ is finite;
   (c) for all $s, t \in T$, if $\mathrm{dom}(s) < \mathrm{dom}(t)$, then $s * t := s \cup (t \restriction (\mathrm{dom}(t) \setminus \mathrm{dom}(s)))$ is in T.

Claim 3.29.1. *There exists a downward closed subfamily $\hat{T} \subseteq {}^{<\kappa}\kappa$ satisfying (a)–(c), in addition to the following:*
   (d) *For all $t \in \hat{T} \cap {}^\alpha\kappa$ and $i < \alpha$, $t^\frown\langle i\rangle \in \hat{T}$.*

Proof. The proof is similar to that of [Kön03, Theorem 3.6]. Denote $T_\alpha := T \cap {}^\alpha\kappa$ for all $\alpha < \kappa$. By a standard fact, we may fix a club $E \subseteq \kappa$ such that, for every $(\alpha, \beta) \in [E]^2$, every node in T_α admits at least $|\alpha|$-many extensions in T_β. We may also assume that $0 \in E$. Let $\pi : \kappa \leftrightarrow E$ denote the order-preserving bijection, and denote $T' := \bigcup_{\alpha < \kappa} T_{\pi(\alpha)}$. Our next goal is to define a map $\Pi : T' \to {}^{<\kappa}\kappa$ such that all of the following hold:

(1) for all $\alpha < \kappa$ and $t \in T_{\pi(\alpha)}$, $\Pi(t) \in {}^{\alpha}\kappa$;
(2) for all $s \subsetneq t$ from T', $\Pi(s) \subsetneq \Pi(t)$;
(3) for all $s, t \in T'$, if $\mathrm{dom}(s) < \mathrm{dom}(t)$, then $\Pi(s * t) = \Pi(s) * \Pi(t)$.

We shall define $\Pi \restriction T_{\pi(\alpha)}$ by recursion on $\alpha < \kappa$:

▶ For $\alpha = 0$, we have $T_{\pi(\alpha)} = T_0 = \{\emptyset\}$, so we set $\Pi(\emptyset) := \emptyset$.
▶ For $\alpha = \bar{\alpha} + 1$ such that $\Pi \restriction T_{\pi(\bar{\alpha})}$ has been successfully defined, we first fix $\bar{t} \in T_{\pi(\bar{\alpha})}$. Find a cardinal $\mu \geq |\bar{\alpha}|$ and an injective enumeration $\langle t^i \mid i < \mu \rangle$ of all the extensions of $\bar{t}$ in $T_{\pi(\alpha)}$. Finally, for every $t \in T_{\pi(\alpha)}$, find the unique $i < \mu$ such that $\bar{t} * t = t^i$, and then let $\Pi(t) := \Pi(t \restriction \bar{\alpha})^{\frown}\langle i \rangle$. It is clear that Properties (1)–(3) are preserved.
▶ For $\alpha \in \mathrm{acc}(\kappa)$ such that $\Pi \restriction T_{\pi(\bar{\alpha})}$ has been successfully defined for all $\bar{\alpha} < \alpha$, we just let $\Pi(t) := \bigcup \{\Pi(t \restriction \pi(\bar{\alpha})) \mid \bar{\alpha} < \alpha\}$ for every $t \in T_{\pi(\alpha)}$. It is clear that Properties (1)–(3) are preserved.

Set $\hat{T} := \mathrm{Im}(\Pi)$. By Property (2), $(\hat{T}, \subseteq)$ is order-isomorphic to $(T', \subseteq)$ which is a κ-sized subtree of $(T, \subseteq)$, so $(\hat{T}, \subseteq)$ is indeed a κ-Souslin tree. By Properties (3) and (b), for all $s, t \in \hat{T}$, $\{\epsilon \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\epsilon) \neq t(\epsilon)\}$ is finite. By Properties (3) and (c), for all $s, t \in \hat{T}$, if $\mathrm{dom}(s) < \mathrm{dom}(t)$, then $s * t := s \cup (t \restriction (\mathrm{dom}(t) \setminus \mathrm{dom}(s)))$ is in $\hat{T}$. Finally, by the definition of $\Pi \restriction T_{\pi(\alpha)}$ for successor ordinals $\alpha < \kappa$, and by Property (c), it is indeed the case that, for all $t \in \hat{T} \cap {}^{\alpha}\kappa$ and $i < \alpha$, $t^{\frown}\langle i \rangle \in \hat{T}$. ⊠

Let $\theta \in \mathrm{Reg}(\kappa)$. By Lemma 3.28 and Lemma 3.2(2), in order to show that $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, \theta)$ holds, it suffices to find a θ-coherent witness to $\mathrm{U}(\kappa, 2, \theta, 2)$. To this end, fix $\hat{T}$ as in the preceding claim, and then fix some sequence $\langle t_\beta \mid \beta < \kappa \rangle$ such that $t_\beta \in \hat{T} \cap {}^{\beta}\kappa$ for all $\beta < \kappa$. Define a coloring $c : [\kappa]^2 \to \theta$ via:

$$c(\alpha, \beta) := \begin{cases} t_\beta(\alpha), & \text{if } t_\beta(\alpha) < \theta; \\ 0, & \text{otherwise.} \end{cases}$$

Evidently, c is ω-coherent. Now, given $A \in [\kappa]^{\kappa}$, we claim that $c``[A]^2 = \theta$. To see this, let $i < \theta$, and note that $S := \{t_\alpha^{\frown}\langle i \rangle \mid \alpha \in A \setminus \theta\}$ forms a subset of $\hat{T}$ of size κ, and hence it cannot be an antichain. Pick $s, t \in S$ such that $s \subsetneq t$. Let $(\alpha, \beta) \in [A]^2$ be such that $s = t_\alpha^{\frown}\langle i \rangle$ and $t = t_\beta^{\frown}\langle i \rangle$. As $t_\alpha^{\frown}\langle i \rangle$ and t_β are both initial segments of t, we infer that $t_\beta(\alpha) = i$, and hence $c(\alpha, \beta) = i$, as sought. □

Remark 3.30. By [BR19, Theorem C], for every singular cardinal λ, $\square(\lambda^+) + \mathsf{GCH}$ entails the existence of a uniformly coherent λ^+-Souslin tree. By Fact 3.15 and the preceding Corollary, the same conclusion does not follow from $\square(\lambda^+, 2) + \mathsf{GCH}$.

We now show that the existence of a coloring c for which $\partial(c)$ is stationary is in fact equivalent to the existence of a nonreflecting stationary subset of E^{κ}_θ.

Lemma 3.31. *For a subset $S \subseteq E^{\kappa}_\theta$, the following are equivalent:*

(1) *for every $\gamma \in E^{\kappa}_{>\omega}$, $S \cap \gamma$ is nonstationary in γ;*
(2) *there exists a coloring $c : [\kappa]^2 \to \theta$ for which $\partial(c) \supseteq S$;*
(3) *there exists an S-coherent, closed coloring $c : [\kappa]^2 \to \theta$ for which $\partial(c) \supseteq S$.*

Proof. (3) $\implies$ (2): This is trivial.
(2) $\implies$ (1): Suppose that $c : [\kappa]^2 \to \theta$ is a coloring for which $\partial(c) \supseteq S$. Towards a contradiction, suppose that we are given $\gamma \in E^{\kappa}_{>\omega}$ such that $S \cap \gamma$ is stationary in γ. As $S \subseteq E^{\kappa}_\theta$, it follows that $\mathrm{cf}(\gamma) > \theta$, and hence $\gamma \notin \partial(c)$. As $\mathrm{cf}(\gamma) \neq \theta$, we

may pick $i < \theta$ such that $\sup(D^c_{\leq i}(\gamma)) = \gamma$. Now, pick $\beta \in \mathrm{acc}^+(D^c_{\leq i}(\gamma)) \cap S$. It follows that $\sup(D^c_{\leq i}(\gamma) \cap \beta) = \beta$, contradicting the fact that $\beta \in S \subseteq \partial(c)$.

(1) $\implies$ (3): Pick a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that, for all $\alpha < \kappa$, $\mathrm{otp}(C_\alpha) = \mathrm{cf}(\alpha)$ and $\mathrm{acc}(C_\alpha) \cap S = \emptyset$. Let tr be the function derived from $\vec{C}$ as in [LHR18, Definition 4.4]. Define a coloring $c : [\kappa]^2 \to \theta$ via

$$c(\alpha, \gamma) := \sup(\{\mathrm{otp}(C_\eta \cap \alpha) \mid \eta \in \mathrm{Im}(\mathrm{tr}(\alpha, \gamma))\} \cap \theta).$$

By [LHR18, Lemma 4.7], c is closed.

Claim 3.31.1. *Let $\beta < \gamma < \kappa$ with $\beta \in S$. Then there exists $\epsilon < \beta$ such that, for every $\alpha \in (\epsilon, \beta)$, $c(\alpha, \gamma) = c(\alpha, \beta) \geq \mathrm{otp}(C_\beta \cap \alpha)$.*

Proof. For every $\alpha < \beta$, since $\beta \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta))$ and $\mathrm{otp}(C_\beta \cap \alpha) < \mathrm{otp}(C_\beta) = \theta$, it is immediate that $c(\alpha, \beta) \geq \mathrm{otp}(C_\beta \cap \alpha)$. Set $j := c(\beta, \gamma)$, and then fix a large enough $\epsilon < \beta$ such that $j \leq \mathrm{otp}(C_\beta \cap \epsilon)$ and $\lambda(\beta, \gamma) \leq \epsilon$.[4]

Let $\alpha \in (\epsilon, \beta)$. Then $\mathrm{tr}(\alpha, \gamma) = \mathrm{tr}(\beta, \gamma)^\frown \mathrm{tr}(\alpha, \beta)$, and hence

$$c(\alpha, \gamma) = \sup(\{\mathrm{otp}(C_\eta \cap \alpha) \mid \eta \in \mathrm{Im}(\mathrm{tr}(\beta, \gamma)) \cup \mathrm{Im}(\mathrm{tr}(\alpha, \beta))\} \cap \theta).$$

In particular, $c(\alpha, \gamma) \geq c(\alpha, \beta) \geq \mathrm{otp}(C_\beta \cap \alpha) \geq j$. Now, as

$$\begin{aligned}
j = c(\beta, \gamma) &= \sup(\{\mathrm{otp}(C_\eta \cap \beta) \mid \eta \in \mathrm{Im}(\mathrm{tr}(\beta, \gamma))\} \cap \theta) \\
&\geq \sup(\{\mathrm{otp}(C_\eta \cap \alpha) \mid \eta \in \mathrm{Im}(\mathrm{tr}(\beta, \gamma))\} \cap \theta),
\end{aligned}$$

it follows that

$$c(\alpha, \gamma) = \sup(\{\mathrm{otp}(C_\eta \cap \alpha) \mid \eta \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta))\} \cap \theta) = c(\alpha, \beta),$$

as sought. ⊠

It now immediately follows that $\partial(c) \supseteq S$ and that c is S-coherent. $\square$

Remark 3.32. By Fact 3.15, the preceding lemma cannot be strengthened to assert that the existence of a nonreflecting stationary subset of E^κ_θ gives rise to a *subadditive* coloring $c : [\kappa]^2 \to \theta$ for which $\partial(c)$ is stationary. In fact, a nonreflecting stationary subset of E^κ_θ is not even enough to imply the existence of a coloring $c : [\kappa]^2 \to \theta$ such that $\partial(c)$ is stationary and c is weakly subadditive of the first kind. This is because, by Theorem 3.45 below, PFA implies that, for example, any witness to $\mathrm{U}(\omega_3, 2, \omega, 2)$ is not weakly subadditive of the first kind, whereas, by a result of Beaudoin (see the remark at the end of [Bea91, §2]), PFA is consistent with the existence of a nonreflecting stationary subset of $E^{\omega_3}_\omega$.

By Lemma 3.14(1) and Lemma 3.27(2), for every infinite regular cardinal λ, there exists a locally small coloring $c : [\lambda^+]^2 \to \lambda$ that is λ-coherent. We shall now prove that for every singular cardinal λ, a locally small coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ is never $\mathrm{cf}(\lambda)$-coherent. Assuming that c is subadditive of the first kind (which is indeed possible, by Lemma 3.13), even weaker forms of coherence are not feasible.

Lemma 3.33. *Suppose that $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ is a locally small coloring.*

(1) *If λ is regular or if c is subadditive of the first kind, then for every cardinal $\theta < \lambda$, c is not θ-coherent.*

(2) *If λ is singular, then c is not $\mathrm{cf}(\lambda)$-coherent.*

---

[4]The function $\lambda(\cdot, \cdot)$ is defined on [Tod07, p. 258].

Proof. The proof is similar to that of [Kön03, Theorem 3.7]. Suppose for sake of contradiction that c is θ-coherent for some fixed cardinal $\theta < \lambda$ and that either c is subadditive of the first kind or $\theta \leq \mathrm{cf}(\lambda)$.

Claim 3.33.1. *For every $\gamma < \lambda^+$, there exists $i < \mathrm{cf}(\lambda)$ such that*

$$\mathrm{otp}(D^c_{\leq i}(\gamma)) + \theta < \mathrm{otp}(D^c_{\leq i}(\gamma + \lambda)).$$

Proof. Let $\gamma < \lambda^+$. Denote $\delta := \gamma + \lambda$. First, since $\mathrm{otp}([\gamma, \delta)) = \lambda > \theta$, we may let

$$i_0 := \min\{i < \mathrm{cf}(\lambda) \mid \mathrm{otp}(D^c_{\leq i}(\delta) \setminus \gamma) > \theta\}.$$

Second, if $\theta \leq \mathrm{cf}(\lambda)$, then, since c is θ-coherent,

$$i_1 := \mathrm{ssup}\{c(\alpha, \gamma), c(\alpha, \delta) \mid \alpha < \gamma, c(\alpha, \gamma) \neq c(\alpha, \delta)\}$$

is an ordinal $< \mathrm{cf}(\lambda)$. If $\theta > \mathrm{cf}(\lambda)$, then we instead let $i_1 := c(\gamma, \delta)$.

We claim that $i := \max\{i_0, i_1\}$ is as sought.

▶ If $\theta \leq \mathrm{cf}(\lambda)$, then for every $\alpha \in D^c_{\leq i}(\gamma)$, either $c(\alpha, \delta) \leq i_1 \leq i$ or $c(\alpha, \delta) = c(\alpha, \gamma) \leq i$. Therefore, $D^c_{\leq i}(\gamma) \subseteq D^c_{\leq i}(\delta)$.

▶ Otherwise, c is subadditive of the first kind and $i_i = c(\gamma, \delta)$. Then, for every $\alpha \in D^c_{\leq i}(\gamma)$, we have $c(\alpha, \delta) \leq \max\{c(\alpha, \gamma), c(\gamma, \delta)\} \leq \max\{c(\alpha, \gamma), i_1\} \leq i$, so that $\alpha \in D^c_{\leq i}(\delta)$. Thus, again, $D^c_{\leq i}(\gamma) \subseteq D^c_{\leq i}(\delta)$.

Now, set $x := D^c_{\leq i}(\delta) \setminus \gamma$, and notice that $\mathrm{otp}(x) \geq \mathrm{otp}(D^c_{\leq i_0}(\delta) \setminus \gamma) \geq \theta + 1$. Altogether $D^c_{\leq i}(\gamma) \uplus x \subseteq D^c_{\leq i}(\delta)$ with $D^c_{\leq i}(\gamma) \subseteq \min(x)$, and hence $\mathrm{otp}(D^c_{\leq i}(\gamma)) + \theta + 1 \leq \mathrm{otp}(D^c_{\leq i}(\delta))$.   ⊠

By the claim, for each $\gamma < \lambda^+$, we may fix $i_\gamma < \mathrm{cf}(\lambda)$ such that $\mathrm{otp}(D^c_{\leq i_\gamma}(\gamma)) + \theta < \mathrm{otp}(D^c_{\leq i_\gamma}(\gamma + \lambda))$. Fix a sparse enough stationary subset $S \subseteq \lambda^+$ along with $i < \mathrm{cf}(\lambda)$ such that $i_\gamma = i$ for all $\gamma \in S$, and such that $\beta + \lambda < \gamma$ for all $(\beta, \gamma) \in [S]^2$. Define a map $f : S \to \lambda$ via

$$f(\beta) := \mathrm{otp}(D^c_{\leq i}(\beta + \lambda)).$$

Let $(\beta, \gamma) \in [S]^2$. As $\beta + \lambda < \gamma$ and c is θ-coherent,

$$\mathrm{otp}(D^c_{\leq i}(\beta + \lambda)) < \mathrm{otp}(D^c_{\leq i}(\gamma)) + \theta.$$

Altogether,

$$f(\beta) = \mathrm{otp}(D^c_{\leq i}(\beta + \lambda)) < \mathrm{otp}(D^c_{\leq i}(\gamma)) + \theta < \mathrm{otp}(D^c_{\leq i}(\gamma + \lambda)) = f(\gamma).$$

Therefore, f is an injection from a set of size λ^+ to λ, which is a contradiction.   □

We conclude this subsection by introducing a notion of forcing that adds a subadditive coloring $c : [\kappa]^2 \to \theta$ whose $\partial(c)$ is stationary. This will prove Clause (1) of Theorem C.

Theorem 3.34. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$. Then:*

(1) *There exists a θ^+-directed closed, κ-strategically closed forcing notion $\mathbb{P}$ that adds an $E^\kappa_{\geq \theta}$-closed, subadditive coloring $c : [\kappa]^2 \to \theta$ for which $\partial(c)$ is stationary. In particular, $\Vdash_\mathbb{P}$ "$\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$".*

(2) *If $\kappa = \lambda^+$ and $\mathrm{cf}(\lambda) = \theta < \lambda$, then there is also a θ^+-directed closed, $(\lambda + 1)$-strategically closed (hence $(<\kappa)$-distributive) forcing notion $\mathbb{P}$ that adds a coloring as above which is moreover locally small.*

Proof. In Case (2), let $\langle \lambda_i \mid i < \theta \rangle$ be an increasing sequence of regular cardinals converging to λ, with $\lambda_i > \theta$. In Case (1), simply let $\lambda_i := \kappa$ for all $i < \theta$.

We will define a forcing notion $\mathbb{P}$ whose generic object will generate a coloring c as above. Our poset $\mathbb{P}$ consists of all subadditive colorings of the form $p : [\gamma_p + 1]^2 \to \theta$ such that

- $\gamma_p < \kappa$;
- p is $E_{\geq \theta}^{\gamma_p + 1}$-closed;
- for all $\beta \leq \gamma_p$ and all $i < \theta$, we have $|D_{\leq i}^p(\beta)| < \lambda_i$.

$\mathbb{P}$ is ordered by reverse inclusion. We also include the empty set as the unique maximal condition in $\mathbb{P}$.

Claim 3.34.1. $\mathbb{P}$ *is θ^+-directed closed.*

Proof. Note that $\mathbb{P}$ is *tree-like*, i.e., if $p, q, r \in \mathbb{P}$ and r extends both p and q, then p and q are $\leq_{\mathbb{P}}$-comparable. It therefore suffices to prove that $\mathbb{P}$ is θ^+-closed. To this end, suppose that $\xi < \theta^+$ and $\vec{p} = \langle p_\eta \mid \eta < \xi \rangle$ is a decreasing sequence of conditions in $\mathbb{P}$. We may assume without loss of generality that ξ is an infinite regular cardinal and $\vec{p}$ is strictly decreasing, i.e., $\langle \gamma_\eta \mid \eta < \xi \rangle$ is strictly increasing, where γ_η denotes γ_{p_η}. For all $\eta < \xi$, by possibly extending p_η to copy some information from $p_{\eta+1}$, we may also assume that γ_η is a successor ordinal. Let $\gamma^* := \sup\{\gamma_\eta \mid \eta < \xi\}$, and let $q^* := \bigcup_{\eta < \xi} p_\eta$. Note that q^* is not a condition in $\mathbb{P}$, since its domain is not the square of a successor ordinal. We will extend it to a condition $q : [\gamma^* + 1]^2 \to \theta$, which will then be a lower bound for $\vec{p}$. To do so, it suffices to specify $q(\alpha, \gamma^*)$ for all $\alpha < \gamma^*$. There are two cases to consider:

▶ Assume that $\xi < \theta$. We can then fix an $i^* < \theta$ such that $q^*(\gamma_\eta, \gamma_{\eta'}) \leq i^*$ for all $\eta < \eta' < \xi$. Now, given $\alpha < \gamma^*$, let $\eta_\alpha < \xi$ be the least η for which $\alpha < \gamma_\eta$ and then set $q(\alpha, \gamma^*) := \max\{i^*, q^*(\alpha, \gamma_{\eta_\alpha})\}$. It is straightforward to prove, using our choice of i^* and the fact that each p_η is subadditive, that the coloring q thus defined is also subadditive.

Since each p_η is $E_{\geq \theta}^{\gamma_\eta + 1}$-closed, in order to show that q is $E_{\geq \theta}^{\gamma^* + 1}$-closed, it suffices to prove that for all $A \subseteq \gamma^*$ and all $i < \theta$ such that $A \subseteq D_{\leq i}^q(\gamma^*)$ and $\sup(A) \in E_{\geq \theta}^{\gamma^*}$, we have $\sup(A) \in D_{\leq i}^q(\gamma^*)$. To this end, fix such an A and i. Let $\beta := \sup(A)$. By our choice of i^*, we know that $i \geq i^*$ and $q(\alpha, \gamma^*) = \max\{i^*, p_{\eta_\beta}(\alpha, \gamma_{\eta_\beta})\}$ for all $\alpha \in A \cup \{\beta\}$. By the fact that p_{η_β} is $E_{\geq \theta}^{\gamma_{\eta_\beta} + 1}$-closed, we know that $p_{\eta_\beta}(\beta) \leq i$, so $\beta \in D_{\leq i}^q(\gamma^*)$.

To show that q is a condition, it remains only to verify that $|D_{\leq i}^q(\gamma^*)| < \lambda_i$ for all $i < \theta$. To this end, fix $i < \theta$. By our construction, we have $D_{\leq i}^q(\gamma^*) \subseteq \bigcup_{\eta < \xi} D_{\leq i}^{p_\eta}(\gamma_\eta)$. Since $\xi < \lambda_i$ and λ_i is regular, the fact that each p_η is a condition in $\mathbb{P}$ then implies that $|D_{\leq i}^q(\gamma^*)| < \lambda_i$.

▶ Assume that $\xi = \theta$. Fix a strictly increasing sequence $\langle i_\eta \mid \eta < \theta \rangle$ of ordinals below θ such that, for all $\eta < \eta' < \theta$, we have $q^*(\gamma_\eta, \gamma_{\eta'}) \leq i_{\eta'}$. Now, given $\alpha < \gamma^*$, let $\eta_\alpha < \theta$ be the least η for which $\alpha < \gamma_\eta$ and then set $q(\alpha, \gamma^*) := \max\{i_{\eta_\alpha}, q^*(\alpha, \gamma_{\eta_\alpha})\}$. It is again straightforward to prove that the coloring q thus defined is subadditive. The verification involves a case analysis; to illustrate the type of argument involved, we go through the proof of one of the required inequalities in one of the cases, leaving the other similar arguments to the reader.

Suppose that $\alpha < \beta < \gamma^*$ and we have $\gamma_{\eta_\alpha} < \beta$. We will prove that $q(\alpha, \gamma^*) \leq \max\{q(\alpha, \beta), q(\beta, \gamma^*)\}$. If $q(\alpha, \gamma^*) = i_{\eta_\alpha}$, then this is trivial, so assume that

$q(\alpha, \gamma^*) = q^*(\alpha, \gamma_{\eta_\alpha})$. Now, since each p_η is subadditive (and hence q^* is subadditive), we have

$$
\begin{aligned}
q^*(\alpha, \gamma_{\eta_\alpha}) &\leq \max\{q^*(\alpha, \beta),\ q^*(\gamma_{\eta_\alpha}, \beta)\} \\
&\leq \max\{q^*(\alpha, \beta),\ q^*(\gamma_{\eta_\alpha}, \gamma_{\eta_\beta}),\ q^*(\beta, \gamma_{\eta_\beta})\} \\
&\leq \max\{q^*(\alpha, \beta),\ i_{\eta_\beta},\ q^*(\beta, \gamma_{\eta_\beta})\} \\
&= \max\{q(\alpha, \beta),\ q(\beta, \gamma^*)\}.
\end{aligned}
$$

Note that the final equality above holds because $q(\alpha, \beta) = q^*(\alpha, \beta)$ and $q(\beta, \gamma^*) = \max\{i_{\eta_\beta}, q^*(\beta, \gamma_{\eta_\beta})\}$.

We now verify that q is $E_{\geq \theta}^{\gamma^*+1}$-closed. As in the previous case, we fix an $A \subseteq \gamma^*$ and an $i < \theta$ such that $A \subseteq D_{\leq i}^q(\gamma^*)$ and $\beta := \sup(A)$ is in $E_{\geq \theta}^{\gamma^*}$. It will suffice to show that $\beta \in D_{\leq i}^q(\gamma^*)$. To avoid triviality, assume that $\beta \notin A$. Since β is a limit ordinal we know that β is not equal to γ_η for any $\eta < \theta$. It follows that, by passing to a tail of A if necessary, we may assume that $\eta_\alpha = \eta_\beta$ for all $\alpha \in A$. Then, for all $\alpha \in A \cup \{\beta\}$, we have $q(\alpha, \gamma^*) = \max\{i_{\eta_\beta}, p_{\gamma_\beta}(\alpha, \gamma_{\eta_\beta})$. It follows that $i \geq i_{\eta_\beta}$ and, since p_{γ_β} is $E_{\geq \theta}^{\gamma_\beta+1}$-closed, that $p_{\eta_\beta}(\beta) \leq i$, so $\beta \in D_{\leq i}^q(\gamma^*)$.

The fact that $|D_{\leq i}^q(\gamma^*)| < \lambda_i$ for all $i < \theta$ follows by exactly the same reasoning as in the previous case.   ⊠

Let $\dot{c}$ be the canonical $\mathbb{P}$-name for the union of the generic filter. Then $\dot{c}$ is forced to be a subadditive function from an initial segment of $[\kappa]^2$ to θ (we will see shortly that its domain is forced to be all of $[\kappa]^2$).

Note that in the $\xi = \theta$ case of the above claim, we actually proved something stronger that will be useful later: if $\vec{p} = \langle p_\eta \mid \eta < \theta \rangle$ is a strictly decreasing sequence of conditions in $\mathbb{P}$ and $\gamma := \sup\{\gamma_{p_\eta} \mid \eta < \theta\}$, then there is a lower bound q for $\vec{p}$ such that $q \Vdash_{\mathbb{P}} ``\gamma \in \partial(\dot{c})"$.

The next claim will show that $\mathbb{P}$ is $(<\kappa)$-distributive.

Claim 3.34.2. *In Case (1), $\mathbb{P}$ is κ-strategically closed. In Case (2), $\mathbb{P}$ is $(\lambda+1)$-strategically closed.*

Proof. In Case (1), denote $\chi := \kappa$. In Case (2), denote $\chi := \lambda$. We describe a winning strategy for Player II in $\partial_\chi(\mathbb{P})$. (Note that, if $\chi = \lambda$, then it appears that we are just showing that $\mathbb{P}$ is λ-strategically closed, but the fact that $\mathbb{P}$ is θ^+-closed will then show that $\mathbb{P}$ is in fact $(\lambda+1)$-strategically closed.) We will arrange so that, if $\langle p_\eta \mid \eta < \chi \rangle$ is a play of the game in which Player II plays according to their prescribed strategy, then, letting $\gamma_\eta := \gamma_{p_\eta}$ for all $\eta < \chi$,

(a) $\langle \gamma_\eta \mid \eta < \chi$ is a nonzero even ordinal$\rangle$ is a continuous, strictly increasing sequence;

(b) for all even ordinals $\eta < \xi < \chi$, we have $p_\xi(\gamma_\eta, \gamma_\xi) = \min\{i < \theta \mid \xi < \lambda_i\}$.

Now suppose that $\xi < \chi$ is an even ordinal and $\langle p_\eta \mid \eta < \xi \rangle$ is a partial run of $\partial_\chi(\mathbb{P})$ that thus far satisfies requirements (a) and (b) above. We will describe a strategy for Player II to choose the next play, p_ξ, while maintaining (a) and (b).

▶ If $\xi = 0$, then we are required to set $p_\xi = 1_{\mathbb{P}} = \emptyset$.

▶ If $\xi = \xi' + 1$ is a successor ordinal, then, since ξ is even, there is another even ordinal ξ'' such that $\xi = \xi'' + 2$. Let $\gamma^* := \gamma_{\xi'} + 1$. We will define p_ξ so that $\gamma_\xi = \gamma^*$. To do so, we must define $p_\xi(\alpha, \gamma^*)$ for all $\alpha < \gamma^*$. We assume that $\gamma_{\xi''} < \gamma_{\xi'}$ (if they are equal, the construction is similar but easier). Let $i^* := \min\{i < \theta \mid$

$\xi < \lambda_i\}$. First, to satisfy (b), we must let $p_\xi(\gamma_{\xi''}, \gamma^*) := i^*$. Next, for all $\alpha < \gamma_{\xi''}$, let $p_\xi(\alpha, \gamma^*) := \max\{p_{\xi''}(\alpha, \gamma_{\xi''}), i^*\}$. Let $i^{**} := \max\{i^*, p_{\xi'}(\gamma_{\xi''}, \gamma_{\xi'})\}$, and set $p(\gamma_{\xi'}, \gamma^*) := i^{**}$. Finally, for all $\alpha \in (\gamma_{\xi''}, \gamma_{\xi'})$, let $p_\xi(\alpha, \gamma^*) := \max\{i^{**}, p_{\xi'}(\alpha, \gamma_{\xi'})\}$. It is easily verified that p_ξ thus defined is a condition in $\mathbb{P}$ and that we have continued to satisfy requirements (a) and (b).

▶ If ξ is a nonzero limit ordinal, then let $p^* := \bigcup_{\eta < \xi} p_\eta$, and let $\gamma^* := \sup\{\gamma_\eta \mid \eta < \xi\}$. We will define a lower bound p_ξ for the run of the game so far with $\gamma_\xi = \gamma^*$. To do so, it suffices to define $p_\xi(\alpha, \gamma^*)$ for all $\alpha < \gamma^*$. Let $i^* := \min\{i < \theta \mid \xi < \lambda_i\}$. For all $\alpha < \gamma^*$, let $\eta_\alpha < \xi$ be the least even ordinal η such that $\alpha < \gamma_\eta$, and then set $p_\xi(\alpha, \gamma^*) := \max\{i^*, p^*(\alpha, \gamma_{\eta_\alpha})\}$. By the fact that the play of the game thus far satisfied (b), we know that $p^*(\gamma_\eta, \gamma_{\eta'}) \leq i^*$ for all even ordinals $\eta < \eta' < \xi$, so this definition does in fact ensure that $p_\xi(\gamma_\eta, \gamma_\xi) = i^*$, so we have satisfied (b). The fact that the play of the game thus far satisfied (a) and (b) also implies that p_ξ is subadditive and $E_{\geq \theta}^{\gamma_\xi + 1}$-closed and that we have continued to satisfy requirement (a). Finally, to show that $|D_{\leq i}^{p_\xi}(\gamma_\xi)| < \lambda_i$ for all $i < \theta$, note firstly that $D_{\leq i}^{p_\xi}(\gamma_\xi) = \emptyset$ for all $i < i^*$ and, secondly, that for all $i \in [i^*, \theta)$, we have $D_{\leq i}^{p_\xi}(\gamma_\xi) \subseteq \bigcup_{\eta < \xi} D_{\leq i}^{p_\eta}(\gamma_\eta)$. For each $i \in [i^*, \theta)$, we know that $\xi < \lambda_i$ and λ_i is regular, so the fact that p_η is a condition for all $\eta < \xi$ implies that $|D_{\leq i}^{p_\xi}(\gamma_\xi)| < \lambda_i$.

This completes the description of Player II's winning strategy and hence the proof of the claim. ⊠

By the argument of the proof of the above claim, it follows that, for every $\alpha < \kappa$, the set E_α of $p \in \mathbb{P}$ for which $\gamma_p \geq \alpha$ is dense in $\mathbb{P}$. Therefore, the domain of $\dot{c}$ is forced to be $[\kappa]^2$. The definition of $\mathbb{P}$ also immediately implies that $\dot{c}$ is forced to be $E_{\geq \theta}^\kappa$-closed and, in Case (2), $\dot{c}$ is also forced to be locally small. We now finish the proof of the theorem by showing that $\partial(\dot{c})$ is forced to be stationary. (By Lemma 3.23, this will imply that $\dot{c}$ witnesses $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$.) To this end, fix a condition p and a $\mathbb{P}$-name $\dot{D}$ forced to be a club in κ. We will find $q \leq p$ and $\gamma < \kappa$ such that $q \Vdash_\mathbb{P}$ "$\gamma \in \dot{D} \cap \partial(\dot{c})$".

Using the fact that $\mathbb{P}$ is θ^+-closed and the fact that each E_α as defined above is dense, fix a decreasing sequence $\langle p_\eta \mid \eta < \theta \rangle$ of conditions in $\mathbb{P}$ together with an increasing sequence $\langle \alpha_\eta \mid \eta < \theta \rangle$ of ordinals below κ such that

    (1) $p_0 = p$;
    (2) for all $\eta < \theta$, $p_{\eta+1} \Vdash_\mathbb{P}$ "$\alpha_\eta \in \dot{D}$";
    (3) for all $\eta < \theta$, $\gamma_\eta < \alpha_\eta < \gamma_{\eta+1}$.

Now let $\gamma := \sup\{\gamma_\eta \mid \eta < \theta\}$, so that $\gamma = \sup\{\alpha_\eta \mid \eta < \theta\}$. By the proof of Claim 3.34.1, we can find a lower bound q for $\langle p_\eta \mid \eta < \theta \rangle$ such that $q \Vdash_\mathbb{P}$ "$\gamma \in \partial(\dot{c})$". For all $\eta < \theta$, since q extends $p_{\eta+1}$, q also forces α_η to be in $\dot{D}$. Since $\gamma = \sup\{\alpha_\eta \mid \eta < \theta\}$ and $\dot{D}$ is forced to be a club, it follows that q forces γ to be in $\dot{D}$. We then have $q \Vdash_\mathbb{P}$ "$\gamma \in \dot{D} \cap \partial(\dot{c})$", as desired. □

3.4. **Large cardinals and consistency results.** In this subsection, we investigate the effect on $\mathrm{U}^{\mathrm{subadditive}}(\dots)$ of certain compactness principles, including stationary reflection, the existence of highly complete or indecomposable ultrafilters, and the P-ideal dichotomy

Recall that, for stationary subsets S, T of κ, $\mathrm{Refl}(\theta, S, T)$ asserts that, for every θ-sized collection $\mathcal{S}$ of stationary subsets of S, there exists $\beta \in T \cap E_{>\omega}^\kappa$ such that $S \cap \beta$ is stationary in β for every $S \in \mathcal{S}$. We write $\mathrm{Refl}(\theta, S)$ for $\mathrm{Refl}(\theta, S, \kappa)$.

Theorem 3.35. *Suppose that Σ is some stationary subset of κ, $c : [\kappa]^2 \to \theta$ is a Σ-closed witness to $\mathrm{U}(\kappa, 2, \theta, 2)$, and $\mathrm{Refl}(\theta, \Sigma, \kappa \setminus \partial(c))$ holds. Then:*

    (1) *If κ is θ-inaccessible, then c is not subadditive of the first kind.*
    (2) *If $\partial(c)$ is stationary, then c is not subadditive.*
    (3) *If c is locally small, then c is not subadditive.*

Proof. Suppose that c is subadditive of the first kind. For each $\alpha < \kappa$, pick $i_\alpha < \theta$, for which the following set is stationary:

$$S_\alpha^{i_\alpha} := \{\sigma \in \Sigma \mid \alpha < \sigma \ \& \ c(\alpha, \sigma) \le i_\alpha\}.$$

Next, using the pigeonhole principle, fix $H \in [\kappa]^\kappa$ and $i < \theta$ such that $i_\alpha = i$ for all $\alpha \in H$.

Claim 3.35.1. *For every $A \in [H]^\theta$, there is $\beta_A < \kappa$ above $\sup(A)$ such that $\sup_{\alpha \in A} c(\alpha, \beta_A) < \theta$.*

Proof. Given $A \in [H]^\theta$, as $\mathrm{Refl}(\theta, \Sigma, \kappa \setminus \partial(c))$ holds, we may pick $\beta \in \kappa \setminus \partial(c)$ such that $S_\alpha^i \cap \beta$ is stationary in β for all $\alpha \in A$. Now, as $\beta \notin \partial(c)$, we may find some $j < \theta$ and $\gamma \in [\beta, \kappa)$ such that $\sup(D_{\le j}^c(\gamma)) \cap \beta = \beta$. As c is Σ-closed, it follows that there exists a club E in β such that $E \cap \Sigma \subseteq D_{\le j}^c(\gamma)$. For every $\alpha \in A$, fix $\sigma_\alpha \in S_\alpha^i \cap E$. Since c is subadditive of the first kind, we have $c(\alpha, \gamma) \le \max\{c(\alpha, \sigma_\alpha), c(\sigma_\alpha, \gamma)\} \le \max\{i, j\}$. So, setting $\beta_A = \gamma$, we have $\sup_{\alpha \in A} c(\alpha, \beta_A) < \theta$. $\boxtimes$

    (1) Using Lemma 2.4(1), fix $\epsilon < \kappa$ such that, for cofinally many $\beta < \kappa$, $\{c(\alpha, \beta) \mid \alpha \in H \cap \epsilon\}$ is unbounded in θ. Assuming that κ is θ-inaccessible, we may find some $A \in [H \cap \epsilon]^\theta$ such that, for cofinally many $\beta < \kappa$, $\{c(\alpha, \beta) \mid \alpha \in A\}$ is unbounded in θ. In particular, we may find such a $\beta < \kappa$ above β_A. Set $j := \max\{\sup_{\alpha \in A} c(\alpha, \beta_A), c(\beta_A, \beta)\}$. Pick $\alpha \in A$ such that $c(\alpha, \beta) > j$. As c is subadditive of the first kind, $c(\alpha, \beta) \le \max\{c(\alpha, \beta_A), c(\beta_A, \beta)\} \le j$. This is a contradiction.

    (2) Suppose that $\partial(c)$ is stationary, and pick $\beta \in \mathrm{acc}^+(H) \cap \partial(c)$. Fix a cofinal subset A of $H \cap \beta$ of size θ. As $\beta \in \partial(c)$, $\{c(\alpha, \beta) \mid \alpha \in A\}$ is unbounded in θ. Note that $\beta_A > \sup(A) = \beta$. Set $i := \sup_{\alpha \in A} c(\alpha, \beta_A)$. If c were weakly subadditive of the second kind, then we could find $j < \theta$ such that $D_{\le i}^c(\beta_A) \cap \beta \subseteq D_{\le j}^c(\beta)$, contradicting the fact that for every $j < \theta$, there exists $\alpha \in A \subseteq D_{\le i}^c(\beta_A) \cap \beta$ with $c(\alpha, \beta) > j$.

    (3) Suppose that $\kappa = \lambda^+$, $\theta = \mathrm{cf}(\lambda)$, and c is locally small. Fix the least $\beta < \kappa$ such that $\mathrm{otp}(H \cap \beta) = \lambda$. For all $\epsilon < \beta$ and $i < \mathrm{cf}(\lambda)$, $|H \cap [\epsilon, \beta)| = \lambda > |D_{\le i}^c(\beta)|$, and hence there exists a cofinal subset A of $H \cap \beta$ of size θ such that $\{c(\alpha, \beta) \mid \alpha \in A\}$ is unbounded in θ. Now, as in the proof of Clause (3), c cannot be weakly subadditive of the second kind. $\square$

Corollary 3.36. *For every $\theta \in \mathrm{Reg}(\kappa)$ and stationary $\Sigma \subseteq E_{\ge \theta}^\kappa$, if $\mathrm{Refl}(\theta, \Sigma)$ holds, then there exists no Σ-closed witness to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$.* $\square$

    We next show that the existence of subadditive witnesses to $\mathrm{U}(\ldots)$ is ruled out by the existence of certain ultrafilters.

Definition 3.37. An ultrafilter U over κ is *θ-indecomposable* if it is uniform and, for every sequence of sets $\langle A_i \mid i < \theta \rangle$ satisfying $\bigcup_{i < \theta} A_i \in U$, there is $B \in [\theta]^{<\theta}$ such that $\bigcup_{i \in B} A_i \in U$.

Lemma 3.38. *Suppose that $c : [\kappa]^2 \to \theta$ is a witness to $U(\kappa, 2, \theta, 2)$.*

(1) *If there exists a θ^+-complete uniform ultrafilter over κ, then c is not weakly subadditive;*

(2) *If there exists a θ^+-complete uniform ultrafilter over κ and κ is θ-inaccessible, then c is not weakly subadditive of the first kind;*

(3) *If there exists a θ-indecomposable ultrafilter over κ, then c is not subadditive of the second kind.*

Proof. (1) Suppose that U is a θ^+-complete ultrafilter over κ. For all $\alpha < \kappa$ and $i < \theta$, let

$$A_\alpha^i := \{\beta < \kappa \mid \alpha < \beta \ \& \ c(\alpha, \beta) \le i\},$$

so that $\langle A_\alpha^i \mid i < \theta \rangle$ is a $\subseteq$-increasing sequence, converging to $\kappa \setminus (\alpha + 1)$. Since U is, in particular, a θ-indecomposable ultrafilter over κ, we may find some $i_\alpha < \theta$ such that $A_\alpha^{i_\alpha} \in U$.

Next, using the pigeonhole principle, let us fix $H \in [\kappa]^\kappa$ and $i < \theta$ such that $i_\alpha = i$ for all $\alpha \in H$. As U is closed under intersections of length θ, for every $A \in [H]^\theta$, we may let $\beta_A := \min(\bigcap_{\alpha \in A} A_\alpha^i)$, so that $\sup_{\alpha \in A} c(\alpha, \beta_A) \le i$.

By Lemma 2.4(2), fix $\beta \in H$ such that $\{c(\alpha, \beta) \mid \alpha \in H \cap \beta\}$ is unbounded in θ. Now, pick $A \in [H \cap \beta]^\theta$ such that $\{c(\alpha, \beta) \mid \alpha \in A\}$ is unbounded in θ. If c were weakly subadditive, then we could pick $j < \theta$ such that $D_{\le i+1}^c(\beta_A) \cap \beta \subseteq D_{\le j}^c(\beta)$. Then $A \subseteq D_{\le j}^c(\beta)$, contradicting the choice of A.

(2) Let H, i, and the notation β_A be as in the proof of Clause (1). Assuming that κ is θ-inaccessible, and using Lemma 2.4(1), we may find some $A \in [H]^\theta$ such that, for cofinally many $\beta < \kappa$, $\{c(\alpha, \beta) \mid \alpha \in A\}$ is unbounded in θ. In particular, we may find such a $\beta < \kappa$ above β_A. If c were weakly subadditive of the first kind, then we could find $j < \theta$ such that $D_{\le i}^c(\beta_A) \subseteq D_{\le j}^c(\beta)$. However, for every $j < \theta$, there exists $\alpha \in A$ such that $c(\alpha, \beta) > j$, so that $\alpha \in D_{\le i}^c(\beta_A) \setminus D_{\le j}^c(\beta)$. This is a contradiction.

(3) Suppose that U is a θ-indecomposable ultrafilter over κ. As in the proof of Clause (1), we may fix $H \in [\kappa]^\kappa$ and $i < \theta$ such that, for all $\alpha \in H$, $A_\alpha^i := \{\beta < \kappa \mid \alpha < \beta \ \& \ c(\alpha, \beta) \le i\}$ is in U. Towards a contradiction, suppose that c is subadditive of the second kind. Then, for all $(\alpha, \beta) \in [H]^2$, we may pick some $\gamma \in A_\alpha^i \cap A_\beta^i$, and infer that $c(\alpha, \beta) \le \max\{c(\alpha, \gamma), c(\beta, \gamma)\} \le i$. So, $\sup(c``[H]^2) \le i$, contradicting the fact that c witnesses $U(\kappa, 2, \theta, 2)$. □

Corollary 3.39. *Under a suitable large cardinal hypothesis, each of the following statements is consistent:*

(1) *For every $n < \omega$, there is no weakly subadditive witness to $U(\aleph_{\omega+1}, 2, \aleph_n, 2)$.*

(2) *For every positive $n < \omega$, there is no witness to $U(\aleph_{\omega+1}, 2, \aleph_n, 2)$ that is subadditive of the second kind.*

Proof. (1) By [LH17a, Corollary 5.13], relative to the consistency of the existence of infinitely many supercompact cardinals, it is consistent that, for every $n < \omega$, any witness to $U(\aleph_{\omega+1}, 2, \aleph_n, 2)$ is not subadditive. An inspection of [LH17a, Lemma 5.11], on which [LH17a, Corollary 5.13] relies, makes it clear that one obtains a model in which, furthermore, for every $n < \omega$, any witness to $U(\aleph_{\omega+1}, 2, \aleph_n, 2)$ is not weakly subadditive.

(2) Starting from the consistency of the existence of a cardinal κ that is κ^+-supercompact, Ben-David and Magidor produced in [BDM86] a model of ZFC in

which there exists an ultrafilter over $\aleph_{\omega+1}$ that is $\aleph_n$-indecomposable for all positive $n < \omega$. Now appeal to Lemma 3.38. $\qquad\square$

Lemma 3.40. *Suppose that $\theta \leq \mu < \kappa$, and $c : [\kappa]^2 \to \theta$ witnesses $\mathrm{U}(\kappa, \mu \circledast 1, \theta, 2)$. If there exists a μ^+-complete uniform ultrafilter over κ, then c is not μ-coherent.*

Proof. As in the proof of Lemma 3.38(1), a μ^+-complete uniform ultrafilter over κ gives rise to $H \in [\kappa]^\kappa$ and $i < \theta$ such that, for every $A \in [H]^\mu$, there exists $\beta_A < \kappa$ above $\sup(A)$ such that $\sup_{\alpha \in A} c(\alpha, \beta_A) \leq i$. Now, as c witnesses $\mathrm{U}(\kappa, \mu \circledast 1, \theta, 2)$, we may find $A \in [H]^\mu$ and $\beta \in H$ above $\sup(A)$ such that $c(\alpha, \beta) > i$ for all $\alpha \in A$. So
$$\{\alpha < \min\{\beta, \beta_A\} \mid c(\alpha, \beta) \neq c(\alpha, \beta_A)\}$$
covers A, which is a set of size μ. Thereby, c is not μ-coherent. $\qquad\square$

Corollary 3.41 (Todorcevic, [Tod07, Remark 6.2.3]). *Suppose that λ is a singular limit of strongly compact cardinals. Then there exists no locally small coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ that is λ-coherent.*

Proof. Fix a strictly increasing sequence of strongly compact cardinals $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ converging to λ. Let $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ be a locally small coloring. Towards a contradiction, suppose that c is λ-coherent, and then define $d : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ via
$$d(\gamma, \delta) := \min\{i < \mathrm{cf}(\lambda) \mid |\{\alpha < \gamma \mid c(\alpha, \gamma) \neq c(\alpha, \delta)\}| \leq \lambda_i\}.$$
It is not hard to see that d is subadditive, and so, by Lemma 3.38(3), d fails to witness $\mathrm{U}(\lambda^+, 2, \mathrm{cf}(\lambda), 2)$. This means that there exist $H \in [\lambda^+]^{\lambda^+}$ and $i < \mathrm{cf}(\lambda)$ such that $d``[H]^2 \subseteq i$. Define $e : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ via $e(\alpha, \beta) := c(\alpha, \min(H \setminus \beta))$. Then e is a locally small coloring which is moreover λ_i-coherent. By Lemma 3.12(1), e in particular witnesses $\mathrm{U}(\lambda^+, \lambda_i \circledast 1, \mathrm{cf}(\lambda), 2)$, and then Lemma 3.40 implies that there exists no $(\lambda_i)^+$-complete uniform ultrafilter over λ^+, contradicting the facts that $\lambda_i < \lambda_{i+1} < \lambda^+$ and λ_{i+1} is strongly compact. $\qquad\square$

Complementary to Lemma 3.13, we obtain the following.

Corollary 3.42. *Suppose that $c : [\kappa]^2 \to \theta$ is a witness to $\mathrm{U}(\kappa, 2, \theta, 2)$ and there exists a strongly compact cardinal in the interval $(\theta, \kappa]$.*

(1) *c is not weakly subadditive;*
(2) *If κ is not the successor a singular cardinal of cofinality θ, then c is not weakly subadditive of the first kind;*
(3) *c is not subadditive of the second kind.*

Proof. The hypothesis entails the existence of a θ^+-complete uniform ultrafilter over κ, and in particular, the existence of a θ-indecomposable uniform ultrafilter over κ. In addition, if κ is not the successor a singular cardinal of cofinality θ, then by Solovay's theorem stating that SCH holds above a strongly compact cardinal [Sol74], κ is θ-inaccessible. Now appeal to Lemma 3.38. $\qquad\square$

Remark 3.43. A similar statement can consistently hold at small cardinals. By [Lüc17, Lemma 3.2(v)] and Lemma 3.5, if κ is weakly compact and $\lambda < \kappa$ is any regular uncountable cardinal, then in the forcing extension by $\mathrm{Coll}(\lambda, <\kappa)$, for every $\theta < \lambda$, $\mathrm{U}^{\mathrm{subadditive}}(\lambda^+, 2, \theta, 2)$ fails.

Lemma 3.14(1) implies that the restriction "$\theta < \lambda$" in the preceding remark cannot be waived.

Corollary 3.44. *It is consistent that all of the following hold simultaneously:*

- GCH*;*
- $\mathrm{U}^{\mathrm{subadditive}}(\omega_2, \omega_2, \omega_1, \omega_1)$ *holds;*
- $\mathrm{U}^{\mathrm{subadditive}}(\omega_2, 2, \omega, 2)$ *fails.*

Proof. By Lemma 3.14(1) and Fact 3.15. $\qquad\square$

In [Via08, Definition 8], Viale defined the covering property $\mathrm{CP}(\kappa, \theta)$. In [LH17a, Lemma 5.11], the first author proved that for infinite regular cardinals $\theta < \kappa$, $\mathrm{CP}(\kappa, \theta)$ implies that $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ fails. By [Via08, §6], the P-ideal dichotomy (PID), which is a consequence of the proper forcing axiom (PFA), implies that $\mathrm{CP}(\kappa, \omega)$ holds for every regular $\kappa \geq \aleph_2$ (the relevant result in [Via08] is only stated for $\kappa > \mathfrak{c}$, but its proof works without any modifications for any regular $\kappa \geq \aleph_2$). Putting this all together, we see that the failure of $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \omega, 2)$ for all regular $\kappa \geq \aleph_2$ already follows from PID.

Here, by combining the arguments of [Tod07, §7],[Via08, §6] and [LH17a, §5], we provide a self-contained proof of a slightly more informative result:

Theorem 3.45 (Todorcevic, Viale)**.** *Suppose that* PID *holds and* $c : [\kappa]^2 \to \omega$ *witnesses* $\mathrm{U}(\kappa, 2, \omega, 2)$.

(1) *If* $\kappa \geq \aleph_2$*, then* c *is not weakly subadditive;*
(2) *If* $\kappa > 2^{\aleph_0}$ *is not the successor of a singular cardinal of countable cofinality, then* c *is not weakly subadditive of the first kind.*

Proof. Suppose not. For all $X \in [\kappa]^{\leq \aleph_0}$ and $\beta < \kappa$, define a function $f_X^\beta : X \cap \beta \to \omega$ by letting $f_X^\beta(\alpha) := c(\alpha, \beta)$. Note that if $\kappa > 2^{\aleph_0}$, then for every $X \in [\kappa]^{\leq \aleph_0}$, there exists some $\Gamma \in [\kappa]^\kappa$ such that $f_X^\gamma = f_X^\delta$ for all $(\gamma, \delta) \in [\Gamma]^2$. In this case, we shall denote such a set Γ by Γ_X and $\min(\Gamma_X)$ by γ_X.

Now, let $\mathcal{I}$ be the collection of all $X \in [\kappa]^{\leq \aleph_0}$ such that, for every $\beta < \kappa$, f_X^β is finite-to-one. It is clear that $\mathcal{I}$ is an ideal.

Claim 3.45.1. *Let* $Z \in [\kappa]^{<\kappa}$*. Then there exists an ordinal* $\epsilon_Z \in [\mathrm{ssup}(Z), \kappa)$ *such that, for every* $X \in [Z]^{\leq \aleph_0}$*,* $X \in \mathcal{I}$ *iff there exists* $\gamma \in [\epsilon_Z, \kappa)$ *for which* f_X^γ *is finite-to-one.*

Proof. ▶ If c is weakly subadditive, then set $\epsilon_Z := \mathrm{ssup}(Z)$. Towards a contradiction, suppose that there exist $X \in [Z]^{\leq \aleph_0}$ and $\gamma \in [\epsilon_Z, \kappa)$ such that f_X^γ is finite-to-one, yet $X \notin \mathcal{I}$. Fix $\beta < \kappa$ such that f_X^β is not finite-to-one, and then fix $i < \omega$ for which $Y := X \cap D_{\leq i}^c(\beta)$ is infinite. Find $j < \theta$ such that $D_{\leq i}^c(\beta) \cap \gamma \subseteq D_{\leq j}^c(\gamma)$. Then $X \cap D_{\leq j}^c(\gamma)$ covers the infinite set Y, contradicting the fact that f_X^γ is finite-to-one.

▶ If $\kappa > 2^{\aleph_0}$ is not the successor of a singular cardinal of countable cofinality, then by Viale's theorem [Via08] stating that PID implies SCH, the fact that $|Z| < \kappa$ implies that $|Z|^{\aleph_0} < \kappa$. It follows that $\epsilon_Z := \sup\{\gamma_X \mid X \in [Z]^{\leq \aleph_0}\} + 1$ is $< \kappa$. Suppose that c is weakly subadditive of the first kind, yet, there exist $X \in [Z]^{\leq \aleph_0} \setminus \mathcal{I}$ and $\gamma \in [\epsilon_Z, \kappa)$ such that f_X^γ is finite-to-one. Fix $\beta < \kappa$ and $i < \omega$ such that $Y := X \cap D_{\leq i}^c(\beta)$ is infinite. Pick $\delta \in \Gamma_X$ above β, and use weak subadditivity of the first kind to find $j < \theta$ such that $D_{\leq i}^c(\beta) \subseteq D_{\leq j}^c(\delta)$. As $\delta \in \Gamma_X$, $X \cap D_{\leq j}^c(\delta) = X \cap D_{\leq j}^c(\gamma_X)$, so that $Y \subseteq D_{\leq j}^c(\gamma_X)$. As $\gamma_X < \epsilon_Z \leq \gamma$, we use weak subadditivity of the first kind to find $k < \theta$ such that $D_{\leq j}^c(\gamma_X) \subseteq D_{\leq k}^c(\gamma)$. Altogether, $Y \subseteq D_{\leq k}^c(\gamma)$, so that $f_X^\gamma[Y] \subseteq k + 1$, contradicting the fact that f_X^γ is finite-to-one. $\qquad\boxtimes$

To see that $\mathcal{I}$ is a P-ideal, suppose that $\vec{X} = \langle X_n \mid n < \omega \rangle$ is a sequence of sets in $\mathcal{I}$. Set $\gamma := \epsilon_Z$, for $Z := \bigcup_{n<\omega} X_n$. Evidently $Y := \bigcup_{n<\omega}(X_n \setminus D^c_{\leq n}(\gamma))$ is a pseudo-union for $\vec{X}$. In addition, for every $i < \omega$, $Y \cap D^c_{\leq i}(\gamma)$ is covered by the finite set $\bigcup_{n<i}(X_n \cap D^c_{\leq i}(\gamma))$, so, by the preceding claim, $Y \in \mathcal{I}$.

Finally, by PID, one of the following alternatives must hold:

(1) There exists $A \in [\kappa]^\kappa$ such that $[A]^{\aleph_0} \cap \mathcal{I} = \emptyset$.

(2) There exists $B \in [\kappa]^{\aleph_1}$ such that $[B]^{\aleph_0} \subseteq \mathcal{I}$.

In Case (1), given $A \in [\kappa]^\kappa$, pick some strictly increasing function $g : \kappa \to A$ such that $g(\alpha) > \epsilon_{A \cap \mathrm{ssup}(g[\alpha])}$ for all $\alpha < \kappa$. It follows that $A' := \mathrm{Im}(g)$ is a cofinal subset of A such that $\epsilon_{A' \cap \gamma} < \gamma$ for every $\gamma \in A'$. Next, by Lemma 2.4(2), we may fix $\gamma \in A'$ such that $\{c(\alpha, \gamma) \mid \alpha \in A' \cap \gamma\}$ is infinite. So, we may find $X \in [A' \cap \gamma]^{\aleph_0}$ such that f^γ_X is one-to-one. As $\epsilon_{A' \cap \gamma} < \gamma$, the preceding claim implies that $X \in \mathcal{I}$. In particular, $[A]^{\aleph_0} \cap \mathcal{I} \neq \emptyset$.

In Case (2), given $B \in [\kappa]^{\aleph_1}$, let $\beta := \mathrm{ssup}(B)$. As $B \cap \bigcup_{i<\omega} D^c_{\leq i}(\beta)$ is uncountable, we may find some $i < \omega$ such that $B \cap D^c_{\leq i}(\beta)$ is uncountable. In particular, there exist $X \in [B]^{\aleph_0}$ such that $X \cap D^c_{\leq i}(\beta)$ is infinite. So, $[B]^{\aleph_0} \nsubseteq \mathcal{I}$. $\square$

On the other hand, since MM is preserved by ω_2-directed closed set forcings, Theorem 3.34 implies that MM does not refute $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$ for regular uncountable cardinals $\theta < \kappa$.

Corollary 3.46. *In the model of* [Tod00], *for every regular uncountable cardinal κ, the following are equivalent:*

- *There is a witness to $\mathrm{U}(\kappa, \kappa, \omega, \omega)$ that is subadditive of the first kind;*
- *There is a witness to $\mathrm{U}(\kappa, 2, \omega, 2)$ that is weakly subadditive of the first kind;*
- *κ is the successor of a cardinal of countable cofinality.*

Proof. By Lemma 3.13, if $\kappa = \lambda^+$ for an infinite cardinal λ of countable cofinality, then there exists a witness to $\mathrm{U}(\kappa, \kappa, \omega, \omega)$ that is subadditive of the first kind. For any other κ, since in the model of [Tod00], CH and PID both hold, Theorem 3.45(2) implies that no witness to $\mathrm{U}(\kappa, 2, \omega, 2)$ is weakly subadditive of the first kind. $\square$

We conclude this section by pointing out a corollary to Theorem 3.45 and the arguments from the proofs of Lemma 3.33 and Corollary 3.41.

Corollary 3.47. *Assuming* PID, *for every singular cardinal λ of countable cofinality, every locally small coloring $c : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ that is subadditive of the first kind is not λ-coherent.* $\square$

4. Indexed square sequences

Recall that the principle $\square(\kappa, \theta, \sqsubseteq_\chi)$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ satisfying all of the following:

- for every limit ordinal $\alpha < \kappa$, $\mathcal{C}_\alpha$ is a collection of clubs in α, and $1 \leq |\mathcal{C}_\alpha| \leq \theta$;
- for all $\alpha < \kappa$, $C \in \mathcal{C}_\alpha$ and $\bar{\alpha} \in \mathrm{acc}(C)$, if $\mathrm{otp}(C) \geq \chi$, then $C \cap \bar{\alpha} \in \mathcal{C}_{\bar{\alpha}}$;
- for every club D in κ, there exists some $\alpha \in \mathrm{acc}(D)$ such that $D \cap \alpha \notin \mathcal{C}_\alpha$.

Convention 4.1. If we omit θ, then we mean that $\theta = 1$. If we omit $\sqsubseteq_\chi$, then we mean that $\chi = \omega$.

A strengthening of $\square(\kappa, \theta)$, denoted by $\square^{\mathrm{ind}}(\kappa, \theta)$, was introduced in [LH17b]. In this section, we identify a weakening of $\square^{\mathrm{ind}}(\kappa, \theta)$, denoted by $\boxminus^{\mathrm{ind}}(\kappa, \theta)$, that is tightly related to the existence of closed, subadditive witnesses to $\mathrm{U}(\kappa, \ldots)$.

Definition 4.2. $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ asserts the existence of a matrix

$$\langle C_{\alpha, i} \mid \alpha \in \Gamma, \ i(\alpha) \leq i < \theta \rangle$$

satisfying the following statements.

(1) $(E^{\kappa}_{\neq \theta} \cap \mathrm{acc}(\kappa)) \subseteq \Gamma \subseteq \mathrm{acc}(\kappa)$.
(2) For all $\alpha \in \Gamma$, we have $i(\alpha) < \theta$, and $\langle C_{\alpha, i} \mid i(\alpha) \leq i < \theta \rangle$ is a $\subseteq$-increasing sequence of clubs in α, with $\Gamma \cap \alpha = \bigcup_{i(\alpha) \leq i < \theta} \mathrm{acc}(C_{\alpha, i})$.
(3) For all $\alpha \in \Gamma$, $i(\alpha) \leq i < \theta$, and $\bar{\alpha} \in \mathrm{acc}(C_{\alpha, i})$, we have $i(\bar{\alpha}) \leq i$ and $C_{\bar{\alpha}, i} = C_{\alpha, i} \cap \bar{\alpha}$.
(4) For every club D in κ, there exists $\alpha \in \mathrm{acc}(D) \cap \Gamma$ such that, for all $i < \theta$, $D \cap \alpha \neq C_{\alpha, i}$.

Remark 4.3. The principle $\square^{\mathrm{ind}}(\kappa, \theta)$ is the strengthening of $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ obtained by requiring that $\Gamma = \mathrm{acc}(\kappa)$.

We now turn to prove Theorem A.

Theorem 4.4. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$. Then the following are equivalent.*

(1) $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ *holds;*
(2) *There exists a closed, subadditive witness to* $\mathrm{U}(\kappa, 2, \theta, 2)$;
(3) *There exists a closed, subadditive witness to* $\mathrm{U}(\kappa, \kappa, \theta, \sup(\mathrm{Reg}(\kappa)))$;
(4) *For every stationary $S \subseteq \kappa$, there exists a $\boxminus^{\mathrm{ind}}(\kappa, \theta)$-sequence $\langle C_{\alpha, i} \mid \alpha \in \Gamma, i(\alpha) \leq i < \theta \rangle$ such that either $S \cap \Gamma$ is nonstationary or, for all $i < \theta$, $\{\alpha \in S \cap \Gamma \mid i(\alpha) > i\}$ is stationary.*

Proof. (2) $\iff$ (3) By Lemma 3.2(3).

(4) $\implies$ (1) This is trivial.

(1) $\implies$ (2): Fix a $\boxminus^{\mathrm{ind}}(\kappa, \theta)$-sequence, $\vec{C} = \langle C_{\alpha, i} \mid \alpha \in \Gamma, \ i(\alpha) \leq i < \theta \rangle$. For each $\alpha \in \kappa$, let $\tilde{\alpha} := \min(\Gamma \setminus \alpha)$. Define $c : [\kappa]^2 \to \theta$ by setting, for all $(\alpha, \beta) \in [\kappa]^2$,

$$c(\alpha, \beta) := \begin{cases} \min\{i \in [i(\tilde{\beta}), \theta) \mid \tilde{\alpha} \in \mathrm{acc}(C_{\tilde{\beta}, i})\} & \text{if } \tilde{\alpha} < \tilde{\beta}; \\ 0 & \text{otherwise.} \end{cases}$$

Claim 4.4.1. *c is closed.*

Proof. Fix $\beta < \kappa$, $i < \theta$, and $A \subseteq D^c_{\leq i}(\beta)$ with $\gamma := \sup(A)$ in $\beta \setminus A$. To show that $\gamma \in D^c_{\leq i}(\beta)$, there are two cases to consider.

▶ If $\tilde{\alpha} < \gamma$ for all $\alpha \in A$, then, by our definition of c, it follows that $\gamma \in \mathrm{acc}(C_{\tilde{\beta}, i})$, and hence $\gamma \in \Gamma$, $\gamma = \tilde{\gamma}$, and $c(\gamma, \beta) \leq i$.

▶ Otherwise, there is $\alpha \in A$ such that $\tilde{\alpha} \geq \gamma$. But then $\tilde{\gamma} = \tilde{\alpha}$, and hence $c(\gamma, \beta) = c(\alpha, \beta) \leq i$. ⊠

Claim 4.4.2. *c is subadditive.*

Proof. Suppose that $\alpha < \beta < \gamma < \kappa$. To prove subadditivity, there are three cases to consider.

▶ If $\tilde{\alpha} = \tilde{\beta}$, then $c(\alpha, \beta) = 0$ and $c(\alpha, \gamma) = c(\beta, \gamma)$.
▶ If $\tilde{\beta} = \tilde{\gamma}$, then $c(\alpha, \beta) = c(\alpha, \gamma)$.

▶ Otherwise, we have $|\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}| = 3$ in which case it is easy to verify that $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$ and $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$.   ⊠

To finish the proof, suppose towards a contradiction that c fails to witness $U(\kappa, 2, \theta, 2)$. Then there exist $A \in [\kappa]^{\kappa}$ and $i < \theta$ such that $\sup(c``[A]^2) \leq i$. Set $S := \mathrm{acc}^+(A) \cap \Gamma$, note that S is stationary, and let $D := \bigcup\{C_{\alpha, i} \mid \alpha \in S\}$. Then D is a club and $D \cap \alpha = C_{\alpha, i}$ for all $\alpha \in \mathrm{acc}(D)$, contradicting the hypothesis that $\vec{C}$ is a $\boxminus^{\mathrm{ind}}(\kappa, \theta)$-sequence.

(2) $\implies$ (4): Fix a closed, subadditive coloring c witnessing $U(\kappa, 2, \theta, 2)$. Set $\Gamma := \mathrm{acc}(\kappa) \setminus \partial(c)$, so that $\mathrm{acc}(\kappa) \cap E^{\kappa}_{\neq \theta} \subseteq \Gamma \subseteq \mathrm{acc}(\kappa)$. By Lemma 3.22,

$$\Gamma = \{\alpha \in \mathrm{acc}(\kappa) \mid \text{for some } i < \theta,\ \sup(D^c_{\leq i}(\alpha)) = \alpha\}.$$

Now, let S be a given stationary subset of κ. If $S \cap \Gamma$ is stationary, then set $S' := S \cap \Gamma$; otherwise, set $S' := \kappa$. Using Lemma 2.4(3), let us fix $\epsilon < \kappa$ such that, for every $i < \theta$, $\{\beta \in S' \mid \epsilon < \beta,\ c(\epsilon, \beta) > i\}$ is stationary.

Let $\alpha \in \Gamma$ be arbitrary. Let $i'(\alpha)$ be the least $i < \theta$ for which $\sup(D^c_{\leq i}(\alpha)) = \alpha$. Next, if $\alpha \leq \epsilon$, then let $i(\alpha) := i'(\alpha)$, and otherwise, let $i(\alpha) := \max\{i'(\alpha), c(\epsilon, \alpha)\}$. For all $i \in [i(\alpha), \theta)$, let $C_{\alpha, i} := D^c_{\leq i}(\alpha)$. Clearly, $\langle C_{\alpha, i} \mid i(\alpha) \leq i < \theta \rangle$ is a $\subseteq$-increasing sequence of clubs in α.

We claim that $\langle C_{\alpha, i} \mid \alpha \in \Gamma,\ i(\alpha) \leq i < \theta \rangle$ is a $\boxminus^{\mathrm{ind}}(\kappa, \theta)$-sequence.

Claim 4.4.3. *Let $\beta \in \Gamma$. Then $\Gamma \cap \beta = \bigcup_{i(\beta) \leq i < \theta} \mathrm{acc}(C_{\beta, i})$, and, for all $i \in [i(\beta), \theta)$ and $\alpha \in \mathrm{acc}(C_{\beta, i})$, we have $i(\alpha) \leq i$ and $C_{\alpha, i} = C_{\beta, i} \cap \alpha$.*

Proof. To show the forward inclusion, let $\alpha \in \Gamma \cap \beta$ be arbitrary. Put $i := \max\{i(\alpha), i(\beta), c(\alpha, \beta)\}$. For all $\gamma \in C_{\alpha, i}$, we have $c(\gamma, \alpha) \leq i$ and $c(\alpha, \beta) \leq i$, so, by subadditivity, $c(\gamma, \beta) \leq i$. It follows that $\alpha \in \mathrm{acc}(C_{\beta, i})$.

To show the reverse inclusion and the second statement, let $i \in [i(\beta), \theta)$ and $\alpha \in \mathrm{acc}(C_{\beta, i})$ be arbitrary. For all $\gamma \in C_{\beta, i} \cap \alpha$, we have $c(\gamma, \beta) \leq i$ and $c(\alpha, \beta) \leq i$, so, by subadditivity, $c(\gamma, \alpha) \leq i$. It follows that $C_{\beta, i} \cap \alpha \subseteq D^c_{\leq i}(\alpha)$, so $\sup(D^c_{\leq i}(\alpha)) = \alpha$, and hence $\alpha \in \Gamma$, $i'(\alpha) \leq i$, and $C_{\alpha, i} \supseteq C_{\beta, i} \cap \alpha$. To see that $i(\alpha) \leq i$, suppose that $\alpha > \epsilon$, and we shall show that $c(\epsilon, \alpha) \leq i$. We have already observed that $c(\alpha, \beta) \leq i$. From $i \geq i(\beta)$, we infer $c(\epsilon, \beta) \leq i$. So, by subadditivity, $c(\epsilon, \alpha) \leq i$.

It remains to show that $C_{\alpha, i} \subseteq C_{\beta, i} \cap \alpha$. But, if $\gamma \in C_{\alpha, i}$, then $c(\gamma, \alpha) \leq i$ and $c(\alpha, \beta) \leq i$, so, again by subadditivity, we have $c(\gamma, \beta) \leq i$, and we are done.   ⊠

Claim 4.4.4. *Suppose that D is a club in κ. Then there is $\alpha \in \mathrm{acc}(D) \cap \Gamma$ such that, for all $i \in [i(\alpha), \theta)$, $D \cap \alpha \neq C_{\alpha, i}$.*

Proof. Suppose not. For all $\alpha \in \mathrm{acc}(D)$, fix $j_\alpha \in [i(\alpha), \theta)$ such that $D \cap \alpha = C_{\alpha, j_\alpha}$, and find $j < \theta$ and $A \in [\mathrm{acc}(D)]^\kappa$ such that $j_\alpha = j$ for all $\alpha \in A$. As c witnesses $U(\kappa, 2, \theta, 2)$, we can find $(\alpha, \beta) \in [A]^2$ such that $c(\alpha, \beta) > j$. But this contradicts the fact that $\alpha \in D \cap \beta = C_{\beta, j}$, and hence $c(\alpha, \beta) \leq j$.   ⊠

Finally, by the choice of ϵ, it is clear that if $S \cap \Gamma$ is stationary, then for all $i < \theta$, $\{\alpha \in S \cap \Gamma \mid i(\alpha) > i\}$ is stationary.   □

The proof of the preceding theorem together with Remark 4.3 makes it clear that the following holds as well.

Theorem 4.5. *For every $\theta \in \mathrm{Reg}(\kappa)$, the following are equivalent:*

(1) $\square^{\mathrm{ind}}(\kappa, \theta)$ *holds;*

(2) *There exists a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ for which $\partial(c)$ is nonstationary.* $\qquad\square$

Theorem 4.6. *If $\boxminus^{\mathrm{ind}}(\kappa, \omega)$ holds, then so does $\square^{\mathrm{ind}}(\kappa, \omega)$.*

Proof. Suppose that $\boxminus^{\mathrm{ind}}(\kappa, \omega)$ holds. By Theorem 4.4, we may fix a closed, subadditive coloring $c : [\kappa]^2 \to \omega$ witnessing $\mathrm{U}(\kappa, 2, \omega, 2)$. Let $\Gamma := \{\alpha \in \mathrm{acc}(\kappa) \mid \exists i < \omega[\sup(D^c_{\leq i}(\alpha)) = \alpha]\}$, so that $\Gamma \supseteq \mathrm{acc}(\kappa) \cap E^\kappa_{\neq \omega}$. For all $\delta \in E^\kappa_\omega \setminus \Gamma$, let $a_\delta := \{\max(D^c_{\leq i}(\delta)) \mid i < \omega \ \& \ D^c_{\leq i}(\delta) \neq \emptyset\}$. Clearly, a_δ is a cofinal subset of δ of order-type ω.

For all $\alpha \in E^\kappa_\omega \setminus \Gamma$, let $i(\alpha) := 0$. For all $\alpha \in \Gamma$, let $i(\alpha)$ be the least $i < \omega$ for which $\sup(D^c_{\leq i}(\alpha)) = \alpha$. Then, for all $\alpha \in \mathrm{acc}(\kappa)$ and $i \in [i(\alpha), \omega)$, let

$$C_{\alpha,i} := D^c_{\leq i}(\alpha) \cup \bigcup\{a_\delta \mid \delta \in (D^c_{\leq i}(\alpha) \cup \{\alpha\}) \setminus \Gamma\}.$$

Note that $C_{\alpha,i} \subseteq C_{\alpha,j}$ for all $j \geq i$.

Claim 4.6.1. *Let $\beta \in \mathrm{acc}(\kappa)$. Then:*
- *(1) $\mathrm{acc}(\beta) = \bigcup_{i(\beta) \leq i < \omega} \mathrm{acc}(C_{\beta,i})$;*
- *(2) for all $i \in [i(\beta), \omega)$, $C_{\beta,i}$ is a club in β, and $\mathrm{acc}(C_{\beta,i}) \subseteq D^c_{\leq i}(\beta)$;*
- *(3) for all $i \in [i(\beta), \omega)$ and $\alpha \in \mathrm{acc}(C_{\beta,i})$, we have $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$.*

Proof. (1) Let $\alpha \in \mathrm{acc}(\beta)$ be arbitrary. Fix a large enough $i \geq \max\{i(\alpha), i(\beta)\}$ such that $c(\alpha, \beta) \leq i$. If $\alpha \notin \Gamma$, then $a_\alpha \subseteq C_{\beta,i}$. If $\alpha \in \Gamma$, then for all $\gamma \in C_{\alpha,i}$, we have $c(\gamma, \alpha) \leq i$ and $c(\alpha, \beta) \leq i$, so, by subadditivity, $c(\gamma, \beta) \leq i$. Thus, in both cases, $\alpha \in \mathrm{acc}(C_{\beta,i})$.

(2) Let $i \in [i(\beta), \omega)$ be arbitrary. Suppose first that $\beta \in \Gamma$. Since $D^c_{\leq i}(\beta)$ is a club in β, to show that $C_{\beta,i}$ is a club and $\mathrm{acc}(C_{\beta,i}) \subseteq D^c_{\leq i}(\beta)$, it suffices to show that for any pair $\gamma < \delta$ of successive elements of $D^c_{\leq i}(\beta)$, if $(\gamma, \delta) \cap C_{\beta,i} \neq \emptyset$, then $\delta \notin \Gamma$ and $(\gamma, \delta) \cap C_{\beta,i} \subseteq a_\delta$.

Fix γ, δ as above along with $\alpha \in (\gamma, \delta) \cap C_{\beta,i}$. Using the definition of $C_{\beta,i}$, let us fix some $\delta' \in D^c_{\leq i}(\beta) \setminus \Gamma$ such that $\alpha \in a_{\delta'}$. We have $c(\delta', \beta) \leq i$ and $c(\alpha, \beta) > i$. So, by subadditivity, $j := c(\alpha, \delta')$ is greater than i. As $\alpha \in a_{\delta'}$, it follows that $\alpha = \max(D^c_{\leq j}(\delta'))$.

Notice that if $\delta < \delta'$, then from $\delta, \delta' \in D^c_{\leq i}$ and subadditivity, we have $c(\delta, \delta') \leq i$, and so $\delta \leq \max(D^c_{\leq i}(\delta')) \leq \max(D^c_{\leq j}(\delta')) = \alpha$, which is a contradiction. So $\delta = \delta'$ and $\alpha \in a_\delta$, as desired.

Next, suppose that $\beta \in \mathrm{acc}(\kappa) \setminus \Gamma$. Then the very same argument as before shows that for any pair $\gamma < \delta$ of successive elements of $D^c_{\leq i}(\beta)$, if the interval $(\gamma, \delta) \cap C_{\beta,i}$ is nonempty, then it is covered by a_δ. Moreover, by the definition of $C_{\beta,i}$, we have $C_{\beta,i} \setminus \max(D^c_{\leq i}) \subseteq a_\beta$. It follows that $C_{\beta,i}$ is a club in β and $\mathrm{acc}(C_{\beta,i}) \subseteq D^c_{\leq i}(\beta)$.

(3) Fix $i \in [i(\beta), \omega)$ and $\alpha \in \mathrm{acc}(C_{\beta,i})$. In particular, $\alpha \in D^c_{\leq i}(\beta)$, so, by subadditivity, $D^c_{\leq i}(\alpha) = D^c_{\leq i}(\beta) \cap \alpha$.

▶ If $\alpha \in \Gamma$, then $\alpha \in \mathrm{acc}(D^c_{\leq i}(\beta))$, so that $i(\alpha) \leq i$, and it is clear from the definition that $C_{\alpha,i} = C_{\beta,i} \cap \alpha$.

▶ If $\alpha \notin \Gamma$, then $i(\alpha) = 0 \leq i$, and it is clear from the definition that $C_{\alpha,i} = C_{\beta,i} \cap \alpha$. $\qquad\boxtimes$

The following claim will now finish our proof.

Claim 4.6.2. *Suppose that D is a club in κ. Then there is $\alpha \in \mathrm{acc}(D)$ such that, for all $i \in [i(\alpha), \omega)$, $D \cap \alpha \neq C_{\alpha,i}$.*

Proof. Suppose not. Then, for all $\alpha \in \mathrm{acc}(D)$, for some $j_\alpha \in [i(\alpha), \omega)$, $D \cap \alpha = C_{\alpha, j_\alpha}$. Find $j < \omega$ and $A \in [\mathrm{acc}(D)]^\kappa$ such that $j_\alpha = j$ for all $\alpha \in A$. As c witnesses $\mathrm{U}(\kappa, 2, \omega, 2)$, we may pick $(\alpha, \beta) \in [A]^2$ such that $c(\alpha, \beta) > j$. However $\alpha \in \mathrm{acc}(D \cap \beta) = \mathrm{acc}(C_{\beta, j}) \subseteq D^c_{\leq j}(\beta)$, meaning that $c(\alpha, \beta) \leq j$. This is a contradiction.   ⊠ □

An analogue of the preceding result holds for uncountable θ under the additional assumption of stationary reflection.

Corollary 4.7. *Suppose that $\theta \in \mathrm{Reg}(\kappa)$. If every stationary subset of E^κ_θ reflects, then $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ is equivalent to $\square^{\mathrm{ind}}(\kappa, \theta)$.*

Proof. Suppose that $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ holds. By Theorem 4.4, we may then fix a closed, subadditive witness c to $\mathrm{U}(\kappa, 2, \theta, 2)$. As $\partial(c)$ is a subset of E^κ_θ and every stationary subset of E^κ_θ reflects, it follows from Lemma 3.31 that $\partial(c)$ is nonstationary. So, by Theorem 4.5, $\square^{\mathrm{ind}}(\kappa, \theta)$ holds.   □

We shall now turn to prove Clause (3) of Theorem C, in particular, establishing that, in general, for uncountable θ, $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ is not equivalent to $\square^{\mathrm{ind}}(\kappa, \theta)$. This will follow from the following two theorems; these are fairly straightforward modifications of results of Cummings and Schimmerling [CS02] and Levine and Sinapova [LS21], respectively, but we provide some details for completeness.

Theorem 4.8. *Suppose that $V \subseteq W$ are models of ZFC and $\theta < \lambda$ are regular cardinals in V such that*

(1) *λ is inaccessible in V;*
(2) *λ is singular and $\mathrm{cf}(\lambda) = \theta$ in W;*
(3) *$(\lambda^+)^V = (\lambda^+)^W$.*

Then $\boxminus^{\mathrm{ind}}(\lambda^+, \theta)$ holds in W. Moreover, in W there is a closed, locally small witness c to $\mathrm{U}^{\mathrm{subadditive}}(\lambda^+, 2, \theta, 2)$ such that $\partial(c) \supseteq (E^{\lambda^+}_\lambda)^V$.

Proof. Work first in V, and let $\Gamma := E^{\lambda^+}_{<\lambda}$. Recall that a subset $X \subseteq \lambda$ is $(>\omega)$-*club* if there is a club $C \subseteq \lambda$ such that $C \cap E^\lambda_{>\omega} \subseteq X$. By [CS02, Lemma 4.4], we can fix a matrix $\vec{D} = \langle D_{\alpha, \eta} \mid \alpha \in \Gamma, \ \eta \in X_\alpha \rangle$ such that, for all $\alpha \in \Gamma$,

(1) X_α is a $(>\omega)$-club subset of λ;
(2) for all $\eta \in X_\alpha$, $D_{\alpha, \eta}$ is a club in α and $|D_{\alpha, \eta}| < \lambda$;
(3) $\langle D_{\alpha, \eta} \mid \eta \in X_\alpha \rangle$ is $\subseteq$-increasing and $\Gamma \cap \alpha = \bigcup_{\eta \in X_\alpha} \mathrm{acc}(D_{\alpha, \eta})$;
(4) for all $\eta \in X_\alpha$ and all $\bar{\alpha} \in \mathrm{acc}(D_{\alpha, \eta})$, we have $\eta \in X_{\bar{\alpha}}$ and $D_{\bar{\alpha}, \eta} = D_{\alpha, \eta} \cap \bar{\alpha}$.

Now move to W and note that, since $\mathrm{cf}(\lambda) = \theta$, we have $(E^{\lambda^+}_{\neq\theta} \cap \mathrm{acc}(\lambda^+)) \subseteq \Gamma \subseteq \mathrm{acc}(\lambda^+)$. By assumptions (1)–(3) in the statement of the theorem, [LH18, Corollary 4.2] implies that we can find an increasing sequence of ordinals $\langle \eta_i \mid i < \theta \rangle$ that is cofinal in λ and such that, for all $\alpha \in \Gamma$, for all sufficiently large $i < \theta$, we have $\delta_i \in X_\alpha$. (This fact is also implicit in the earlier [DS95, Theorem 2.0])

For each $\alpha \in \Gamma$, let $i(\alpha)$ be the least $j < \theta$ such that $\delta_i \in X_\alpha$ for all $i \in [j, \theta)$. Now define a matrix $\vec{C} = \langle C_{\alpha, i} \mid \alpha \in \Gamma, i(\alpha) \leq i < \theta \rangle$ by setting $C_{\alpha, i} := D_{\alpha, \delta_i}$ for all $\alpha \in \Gamma$ and all $i \in [i(\alpha), \theta)$. Notice that, for all $\alpha \in \Gamma$, $i \in [i(\alpha), \theta)$, and $\bar{\alpha} \in \mathrm{acc}(C_{\alpha, i})$, the properties of $\vec{D}$ imply that $X_\alpha \setminus i \subseteq X_{\bar{\alpha}}$, and therefore, since $i \geq i(\alpha)$ we also have $i \geq i(\bar{\alpha})$ and $C_{\bar{\alpha}, i} = C_{\alpha, i} \cap \bar{\alpha}$. It is then straightforward to verify that $\vec{C}$ satisfies Clauses (1)–(3) of Definition 4.2. To verify that $\vec{C}$ satisfies Clause (4) of Definition 4.2, fix a club $D \subseteq \lambda^+$, and let $\alpha \in \mathrm{acc}(D) \cap \Gamma$ be such

that $|D \cap \alpha| = \lambda$. Then, for all $i \in [i(\alpha), \theta)$, the fact that $|C_{\alpha,i}| < \lambda$ implies that $D \cap \alpha \neq C_{\alpha,i}$. It follows that $\vec{C}$ is an $\boxminus^{\mathrm{ind}}(\lambda^+, \theta)$-sequence in W.

To prove the "moreover" statement, let $c : [\lambda^+]^2 \to \theta$ be the closed witness to $\mathrm{U}^{\mathrm{subadditive}}(\lambda^+, 2, \theta, 2)$ derived from $\vec{C}$ as in the proof of $(1) \implies (2)$ of Theorem 4.4. The fact that $\partial(c) \supseteq (E_\lambda^{\lambda^+})^V$ follows immediately from our definition of Γ, and the fact that c is locally small follows immediately from the fact that $|C_{\alpha,i}| < \lambda$ for all $\alpha \in \Gamma$ and $i \in [i(\alpha), \theta)$. $\qquad \square$

Theorem 4.9. *Suppose that $\theta < \lambda < \kappa$ are regular uncountable cardinals such that λ is supercompact and κ is weakly compact. Then there is a forcing extension in which λ is a singular strong limit of cofinality θ, $\kappa = \lambda^+$, $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ holds, and $\square(\kappa, \tau)$ fails for all $\tau < \lambda$.*

Proof. By forcing with the Laver preparation forcing if necessary, we may assume that the supercompactness of λ is indestructible under λ-directed closed forcing. Following [LS21, §4], let $\mathbb{C} := \mathrm{Coll}(\lambda, <\kappa)$, and let $\dot{\mathbb{M}}$ be a $\mathbb{C}$-name for the Magidor forcing that turns λ into a singular cardinal of cofinality θ.

For every inaccessible $\delta < \kappa$ above λ, let $\mathbb{C}_\delta := \mathrm{Coll}(\lambda, <\delta)$. By [LS21, Proposition 4.3], there is a club $C \subseteq \kappa$ such that, for every inaccessible $\delta \in C$, $\dot{\mathbb{M}}_\delta := \dot{\mathbb{M}} \cap V_\delta$ is a $\mathbb{C}_\delta$-name for a Magidor forcing to turn λ into a singular cardinal of cofinality θ such that there is a complete embedding of $\mathbb{C}_\delta * \dot{\mathbb{M}}_\delta$ into $\mathbb{C} * \dot{\mathbb{M}}$.

In $V^{\mathbb{C}}$, $\mathbb{M}$ has the λ^+-cc and thus preserves λ^+. Therefore, applying Theorem 4.8 to the models $V^{\mathbb{C}}$ and $V^{\mathbb{C}*\dot{\mathbb{M}}}$ shows that $\boxminus^{\mathrm{ind}}(\kappa, \theta)$ holds in $V^{\mathbb{C}*\dot{\mathbb{M}}}$. It remains to show that $\square(\kappa, \tau)$ fails for every $\tau < \lambda$.

Suppose for sake of contradiction that $\tau < \lambda$ and $\dot{\mathcal{D}} = \langle \dot{\mathcal{D}}_\alpha \mid \alpha \in \mathrm{acc}(\kappa) \rangle$ is a $\mathbb{C} * \dot{\mathbb{M}}$-name for a $\square(\kappa, \tau)$-sequence. This is a Π_1^1 statement about the structure $(V_\kappa, \in, \mathbb{C} * \dot{\mathbb{M}}, \dot{\mathcal{D}})$ (the sole universal quantification over subsets of V_κ is the assertion that there exists no $\mathbb{C} * \dot{\mathbb{M}}$-name for a thread through $\dot{\mathcal{D}}$). Therefore, by the weak compactness of κ, we can find an inaccessible $\delta \in C$ such that $\dot{\mathcal{D}}^* := \dot{\mathcal{D}} \cap V_\delta$ is a $\mathbb{C}_\delta * \dot{\mathbb{M}}_\delta$-name for a $\square(\delta, \tau)$-sequence.

Note that $\delta = \lambda^+$ in $V^{\mathbb{C}_\delta * \dot{\mathbb{M}}_\delta}$. In $V^{\mathbb{C}*\dot{\mathbb{M}}}$, we have $\mathrm{cf}(\delta) = \theta > \omega$. Therefore, in $V^{\mathbb{C}*\dot{\mathbb{M}}}$, any element of $\mathcal{D}_\delta$ is a thread through $\mathcal{D}^*$. However, the proof of [LS21, Lemma 4.7] shows that forcing over $V^{\mathbb{C}_\delta * \dot{\mathbb{M}}_\delta}$ with $(\mathbb{C} * \dot{\mathbb{M}})/(\mathbb{C}_\delta * \dot{\mathbb{M}}_\delta)$ cannot add a thread to a $\square(\delta, \tau)$-sequence. ([LS21, Lemma 4.7] is about $\square_{\lambda,\tau}$-sequences, but the exact same proof still works for $\square(\delta, \tau)$-sequences.) This is a contradiction, thus completing the proof. $\qquad \square$

Remark 4.10. Suppose that $\theta \in \mathrm{Reg}(\kappa)$ is such that there exists a closed subadditive witness to $\mathrm{U}(\kappa, 2, \theta, 2)$. By Lemma 3.10, if κ is $<\theta$-inaccessible, then there exists a κ-Aronszajn tree with a θ-ascent path. In particular, such a tree exists in the model of Theorem 4.9. We note that by a combination of Theorem 4.4, [BR21, Theorem 6.11] and a minor variation of [BR21, Theorem 4.44], if $\kappa = \kappa^{<\kappa}$ is a successor cardinal which is θ-inaccessible, then there moreover exists a κ-Souslin tree with a θ-ascent path.

The *Mapping Reflection Principle* (MRP), introduced by Moore in [Moo05], is a useful consequence of PFA.

Corollary 4.11. MRP *implies that for every regular cardinal $\kappa \geq \aleph_2$, there exists no closed, subadditive witness to $\mathrm{U}(\kappa, 2, \omega, 2)$.*

Proof. By Theorems 4.4 and 4.6, if there exists a closed subadditive witness to $U(\kappa, 2, \omega, 2)$, then $\square^{\mathrm{ind}}(\kappa, \omega)$ holds. In particular, $\square(\kappa, \omega)$ holds. However, by [Str11, Theorem 1.8], for every regular cardinal $\kappa \geq \aleph_2$, $\square(\kappa, \omega)$ is refuted by MRP. This is also implicit in [Via08, §7]. $\square$

Remark 4.12. In [Str11], Strullu proves that $\mathsf{MRP} + \mathsf{MA}_{\omega_1}$ refutes $\square(\kappa, \omega_1)$ for all regular $\kappa \geq \aleph_2$. In light of this fact and Theorem 3.45, it is natural to ask whether $\mathsf{MRP} + \mathsf{MA}_{\omega_1}$ moreover refutes $\boxminus^{\mathrm{ind}}(\kappa, \omega_1)$. In an upcoming paper, it will be shown that MM is compatible with $\boxminus^{\mathrm{ind}}(\kappa, \omega_1)$ for any choice of a regular cardinal $\kappa \geq \aleph_2$. It follows that $\boxminus^{\mathrm{ind}}(\omega_2, \omega_1)$ does not imply the existence of an ω_2-Aronszajn tree. Thus, the arithmetic hypothesis in Lemma 3.10 cannot be waived.

By [LHL18, Theorem 3.4], for every $\theta \in \mathrm{Reg}(\kappa)$, $\square(\kappa)$ implies $\square^{\mathrm{ind}}(\kappa, \theta)$. Essentially the same proof of that theorem establishes that for every $\theta \in \mathrm{Reg}(\kappa)$, $\square(\kappa, \sqsubseteq_\theta)$ implies $\boxminus^{\mathrm{ind}}(\kappa, \theta)$. Here, we shall prove a generalization that also yields the second part of Theorem A.

Theorem 4.13. *Suppose that $\kappa \geq \aleph_2$, $\theta \in \mathrm{Reg}(\kappa)$, $T \subseteq E_\theta^\kappa$ and there is a $\square(\kappa, \sqsubseteq_\theta)$-sequence that avoids T. Then:*

(1) *There is a $\boxminus^{\mathrm{ind}}(\kappa, \theta)$-sequence $\langle C_{\alpha,i} \mid \alpha \in \Gamma,\ i(\alpha) \leq i < \theta \rangle$ such that $\Gamma \cap T = \emptyset$;*
(2) *There is a closed witness c to $U^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ such that $T \subseteq \partial(c)$.*

Proof. The proof of Theorem 4.4 makes it clear that (1) $\implies$ (2), so we focus on proving Clause (1). Fix a $\square(\kappa, \sqsubseteq_\theta)$-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ that avoids T, i.e., for all $\beta < \kappa$, we have $\mathrm{acc}(C_\beta) \cap T = \emptyset$. Set $\Delta := \{\delta \in \mathrm{acc}(\kappa) \mid \forall \bar{\delta} \in \mathrm{acc}(C_\delta)[C_{\bar{\delta}} = C_\delta \cap \bar{\delta}]\}$. By the $\sqsubseteq_\theta$-coherence of $\vec{C}$, we have $E_{\geq\theta}^\kappa \subseteq \Delta$. Set $\Omega := E_{>\theta}^\kappa \cup \bigcup \{\mathrm{acc}(C_\delta) \mid \delta \in \Delta\}$, and note that $\Omega \subseteq \Delta \setminus T$. As $\kappa \geq \aleph_2$, Ω is stationary. So, by [BR19, Lemma 1.23 and Lemma 1.7], we may fix $\zeta < \kappa$ such that $\{\beta \in \Omega \mid \mathrm{otp}(C_\beta) > \zeta \text{ and } C_\beta(\zeta) \geq \tau\}$ is stationary for all $\tau < \kappa$. By successive applications of Fodor's Lemma, we may then recursively construct a strictly increasing sequence of ordinals $\langle \tau_i \mid i < \theta \rangle$ such that $\tau_0 > \min(\Omega)$ and such that, for every $i < \theta$, $\{\beta \in \Omega \mid \mathrm{otp}(C_\beta) > \zeta \text{ and } C_\beta(\zeta) = \tau_i\}$ is stationary.

Next, define $\vec{D} = \langle D_\beta \mid \beta < \kappa \rangle$, as follows:

- Let $D_0 := \emptyset$;
- For every $\beta < \kappa$, let $D_{\beta+1} := \{\beta\}$;
- For every $\beta \in \Delta$ such that $\mathrm{otp}(C_\beta) \leq \zeta$, let $D_\beta := C_\beta$;
- For every $\beta \in \Delta$ such that $\mathrm{otp}(C_\beta) > \zeta$, let $D_\beta := C_\beta \setminus C_\beta(\zeta)$;
- For every $\beta \in \mathrm{acc}(\kappa) \setminus \Delta$, pick a club D_β in β of order-type $\mathrm{cf}(\beta)$ such that $D_\beta \cap T = \emptyset$.

For every ordinal $\beta < \kappa$, let

$$j(\beta) := \begin{cases} i, & \text{if } \min(D_\beta) = \delta_i; \\ 0, & \text{otherwise.} \end{cases}$$

For every $i < \theta$, set $\Gamma_i := \{\beta \in \Omega \mid j(\beta) = i\}$. Then set $\Gamma := \Omega \cup (\mathrm{acc}(\kappa) \cap E_{<\theta}^\kappa)$.

Claim 4.13.1. (1) $(E_{\neq\theta}^\kappa \cap \mathrm{acc}(\kappa)) \subseteq \Gamma \subseteq \mathrm{acc}(\kappa)$ *and* $\Gamma \cap T = \emptyset$.

(2) *For all $\beta \in \Delta$ and $\alpha \in \mathrm{acc}(D_\beta)$, we have:*
- $\alpha \in \Omega$,
- $D_\alpha = D_\beta \cap \alpha$, *and*

- $j(\alpha) = j(\beta)$.

(3) *For every $\beta \in \Gamma \setminus \mathrm{acc}^+(\Gamma)$, we have $\mathrm{otp}(D_\beta \setminus \sup(\Gamma \cap \beta)) = \omega$.*

(4) *For every $i < \theta$, Γ_i is stationary.*

(5) $\min(\Omega) \in \Gamma_0$.

Proof. (1) This follows directly from the definition of Γ.

(2) Let $\beta \in \Delta$ and $\alpha \in \mathrm{acc}(D_\beta)$ be arbitrary. There two cases to consider:

▶ If $\mathrm{otp}(C_\beta) \leq \zeta$, then $\alpha \in \mathrm{acc}(D_\beta) = \mathrm{acc}(C_\beta)$, so $C_\alpha = C_\beta \cap \alpha$ and $\mathrm{otp}(C_\alpha) < \zeta$. Since $\beta \in \Delta$ and $\alpha \in \mathrm{acc}(C_\beta)$, the definition of Ω implies that $\alpha \in \Omega \subseteq \Delta$. We therefore have $D_\alpha = C_\alpha = C_\beta \cap \alpha = D_\beta \cap \alpha$. In particular, $\min(D_\alpha) = \min(D_\beta)$, so $j(\alpha) = j(\beta)$.

▶ If $\mathrm{otp}(C_\beta) > \zeta$, then $\alpha \in \mathrm{acc}(D_\beta) = \mathrm{acc}(C_\beta \setminus C_\beta(\zeta))$, so $C_\alpha = C_\beta \cap \alpha$ and $\mathrm{otp}(C_\alpha) > \zeta$. As in the previous case, we have $\alpha \in \Omega \subseteq \Delta$, and therefore $D_\alpha = C_\alpha \setminus C_\alpha(\zeta) = (C_\beta \setminus C_\beta(\zeta)) \cap \alpha = D_\beta \cap \alpha$. In particular, $j(\alpha) = j(\beta)$.

(3) Let $\beta \in \Gamma$. Note the following:

▶ If $\omega < \mathrm{cf}(\beta) \leq \theta$, then $E^\beta_\omega \subseteq \mathrm{acc}(\kappa) \cap E^\kappa_{<\theta} \subseteq \Gamma$, and hence $\beta \in \mathrm{acc}^+(\Gamma)$.

▶ If $\mathrm{cf}(\beta) > \theta$, then $\mathrm{acc}(C_\beta) \subseteq \Omega \subseteq \Gamma$, so again $\beta \in \mathrm{acc}^+(\Gamma)$.

Therefore, if $\beta \in \Gamma \setminus \mathrm{acc}^+(\Gamma)$, then $\mathrm{cf}(\beta) = \omega$. So, if $\mathrm{otp}(D_\beta) > \omega$, then $\beta \in \Delta$, and then $\mathrm{acc}(C_\beta) \subseteq \Omega \subseteq \Gamma$. As we assume that $\beta \notin \mathrm{acc}^+(\Gamma)$, it in particular follows that $\sup(\mathrm{acc}(C_\beta)) < \beta$, and $\mathrm{otp}(C_\beta \setminus \sup(\Gamma \cap \beta)) = \omega$. As D_β is a final segment of C_β, it also follows that $\mathrm{otp}(D_\beta \setminus \sup(\Gamma \cap \beta)) = \omega$.

(4) This follows directly from the choice of $\langle \tau_i \mid i < \theta \rangle$.

(5) For every $i < \theta$, we have $\tau_i \geq \tau_0 > \min(\Omega)$, so it is impossible for $j(\min(\Omega))$ to be greater than 0. ⊠

We now construct a $\boxdot^{\mathrm{ind}}(\kappa, \theta)$-sequence $\langle C_{\alpha,i} \mid \alpha \in \Gamma, \ i(\alpha) \leq i < \theta \rangle$. We will maintain the requirement that, for all $\alpha \in \Omega$, we have $i(\alpha) = j(\alpha)$ and $\mathrm{acc}(D_\alpha) \subseteq \mathrm{acc}(C_{\alpha,i(\alpha)})$.

As a base case, if $\beta = \min(\Gamma)$, then by Claim 4.13.1(3), we have $\mathrm{otp}(D_\beta) = \omega$. Set $i(\beta) := 0$, and let $C_{\beta,i} := D_\beta$ for all $i < \theta$. Note that if $\beta \in \Omega$, then by Claim 4.13.1(5), indeed $i(\beta) = j(\beta)$.

Suppose now that $\beta \in \Gamma \setminus \{\min(\Gamma)\}$ and we have defined $\langle C_{\alpha,i} \mid \alpha \in \Gamma \cap \beta, \ i(\alpha) \leq i < \theta \rangle$ satisfying all relevant instances of Clauses (2) and (3) of Definition 4.2 as well as our recursive requirement. The construction breaks into a number of different cases based on the identity of β. In all cases, the verification that our sequence satisfies our recursive requirement and Clauses (2) and (3) of Definition 4.2 at β is routine and therefore largely left to the reader.

Case 1: $\beta \in \Gamma \setminus \Omega$. In particular, we have $\mathrm{cf}(\beta) < \theta$. We now split into subcases depending on the behavior of $\Gamma \cap \beta$.

Case 1a: $\sup(\Gamma \cap \beta) < \beta$ **and** $\max(\Gamma \cap \beta)$ **exists.** Let $\alpha := \max(\Gamma \cap \beta)$. By Claim 4.13.1(3), we know that $D^-_\beta := D_\beta \setminus \alpha$ has order type ω. Let $i(\beta) := i(\alpha)$ and, for all $i \in [i(\beta), \theta)$, let $C_{\beta,i} = C_{\alpha,i} \cup \{\alpha\} \cup D^-_\beta$.

Case 1b: $\sup(\Gamma \cap \beta) < \beta$ **but** $\max(\Gamma \cap \beta)$ **does not exist.** Let $\alpha := \sup(\Gamma \cap \beta)$. As in Case (1a), we know that $D^-_\beta := D_\beta \setminus \alpha$ has order type ω. Since $\alpha \notin \Gamma$, it follows that $\alpha \in E^\kappa_\theta$. Since $\mathrm{cf}(\beta) < \theta$, we know that $\theta > \omega$, and hence $\mathrm{acc}(D_\alpha)$ is unbounded in α. Let $\langle \alpha_i \mid i < \theta \rangle$ be a strictly increasing sequence of elements of $\mathrm{acc}(D_\alpha)$ converging to α. As $\alpha \in E^\kappa_\theta \subseteq \Delta$, it follows from Claim 4.13.1(2) that, for all $i < \theta$, $\alpha_i \in \Omega \subseteq \Gamma$ and $j(\alpha_i) = j(\alpha)$. So,

by the fact that our construction thus far satisfies all of our requirements,
we know that, for all $i < \theta$, we have $i(\alpha_i) = j(\alpha)$ and $\mathrm{acc}(D_\alpha) \cap \alpha_i = \mathrm{acc}(D_{\alpha_i}) \subseteq \mathrm{acc}(C_{\alpha_i, j(\alpha)})$. Set $i(\beta) := j(\alpha)$ and $C_{\beta,i} := C_{\alpha_i, i} \cup \{\alpha_i\} \cup D_{\bar{\beta}}^-$
for all $i \in [i(\beta), \theta)$.

Case 1c: $\sup(\Gamma \cap \beta) = \beta$. In this case, let $\langle \alpha_\eta \mid \eta < \mathrm{cf}(\beta) \rangle$ be an increasing
sequence of elements of Γ converging to β. Let $i(\beta)$ be the least ordinal
$i < \theta$ such that, for all $\eta < \xi < \mathrm{cf}(\beta)$, we have $i(\alpha_\eta), i(\alpha_\xi) \le i$ and $\alpha_\eta \in \mathrm{acc}(C_{\alpha_\xi, i})$. Such an ordinal exists because $\mathrm{cf}(\beta) < \theta$ and our sequence so
far satisfies Clauses (2) and (3) of Definition 4.2. Then, for all $i \in [i(\beta), \theta)$,
let $C_{\beta,i} = \bigcup_{\eta < \mathrm{cf}(\beta)} C_{\alpha_\eta, i}$.

Case 2: $\beta \in \Omega$. In this case, we are required to set $i(\beta) := j(\beta)$. We again
split into subcases depending on the behavior of $\Gamma \cap \beta$ and $\mathrm{acc}(D_\beta)$.

Case 2a: $\sup(\Gamma \cap \beta) < \beta$ **and** $\max(\Gamma \cap \beta)$ **exists.** Let $\alpha := \max(\Gamma \cap \beta)$. As
in Case (1a), we know that $D_{\bar{\beta}}^- := D_\beta \setminus \alpha$ has order type ω.

▶ If $\alpha \in \mathrm{acc}(D_\beta)$, then $i(\alpha) = j(\alpha) = j(\beta) = i(\beta)$, so we let $C_{\beta,i} := C_{\alpha,i} \cup \{\alpha\} \cup D_{\bar{\beta}}^-$ for all $i \in [i(\beta), \theta)$. Note that

$$\mathrm{acc}(D_\beta) = \mathrm{acc}(D_\alpha) \cup \{\alpha\} \subseteq \mathrm{acc}(C_{\alpha, i(\alpha)}) \cup \{\alpha\} \subseteq \mathrm{acc}(C_{\beta, i(\beta)}),$$

so we have satisfied our recursive hypothesis.

▶ If $\mathrm{acc}(D_\beta) \ne \emptyset$ but $\alpha > \max(\mathrm{acc}(D_\beta))$, then let $\alpha^* := \max(\mathrm{acc}(D_\beta))$
and let $i^* \in [i(\alpha), \theta)$ be least such that $\alpha^* \in \mathrm{acc}(C_{\alpha, i^*})$. Note that $i(\beta) = i(\alpha^*) \le i^*$. Let $C_{\beta,i} := C_{\alpha^*, i} \cup \{\alpha^*\} \cup D_{\bar{\beta}}^-$ for all $i \in [i(\beta), i^*)$, and let
$C_{\beta,i} := C_{\alpha,i} \cup \{\alpha\} \cup D_{\bar{\beta}}^-$ for all $i \in [i^*, \theta)$.

Note that we have satisfied our recursive hypothesis.

▶ If $\mathrm{acc}(D_\beta) = \emptyset$, then let $C_{\beta,i} := D_{\bar{\beta}}^-$ for all $i \in [i(\beta), i(\alpha))$, and let
$C_{\beta,i} := C_{\alpha,i} \cup \{\alpha\} \cup D_{\bar{\beta}}^-$ for all $i \in [\max\{i(\beta), i(\alpha)\}, \theta)$.

Case 2b: $\sup(\Gamma \cap \beta) < \beta$ **but** $\max(\Gamma \cap \beta)$ **does not exist.** Let $\alpha := \sup(\Gamma \cap \beta)$. Once again, it follows that $D_{\bar{\beta}}^- := D_\beta \setminus \alpha$ has order type ω. As in
Case (1b), we have $\alpha \in E_\theta^\kappa$, and, for all $\bar{\alpha} \in \mathrm{acc}(D_\alpha)$, we have $\bar{\alpha} \in \Omega$ and
$j(\bar{\alpha}) = j(\alpha)$. Let $\alpha^* := \max(\mathrm{acc}(D_\beta))$ if $\mathrm{acc}(D_\beta) \ne \emptyset$, and $\alpha^* := \min(\Gamma)$
otherwise. In either case, note that $\alpha^* < \alpha$ and $i(\alpha^*) \le i(\beta)$.

Our construction now depends on whether or not $\theta = \omega$.

▶ If $\theta = \omega$, then let $\langle \alpha_n \mid n < \omega \rangle$ be a strictly increasing sequence of
elements of $\Gamma \cap \alpha$ converging to α such that $\alpha_0 = \alpha^*$. Let $\langle i_n \mid n < \omega \rangle$ be
a strictly increasing sequence of natural numbers such that, for all $n < \omega$,
$\max\{i(\alpha_m) \mid m \le n\} \le i_n$ and $\{\alpha_m \mid m < n\} \subseteq \mathrm{acc}(C_{\alpha_n, i_n})$.

Finally, for all $i \in [i(\beta), \omega)$, let $n < \omega$ be such that $i_n \le i < i_{n+1}$, and
set $C_{\beta,i} := C_{\alpha_n, i} \cup \{\alpha_n\} \cup D_{\bar{\beta}}^-$.

▶ If $\theta > \omega$, then let $\langle \alpha_i \mid i < \theta \rangle$ be a strictly increasing sequence of
elements of $\mathrm{acc}(D_\alpha)$ converging to α such that $\alpha_0 > \alpha^*$. As noted earlier,
we have $i(\alpha_i) = j(\alpha)$. Let $i^* \in [j(\alpha), \theta)$ be least such that $\alpha^* \in \mathrm{acc}(C_{\alpha_0, i^*})$.
In particular, for all $i \in [i^*, \theta)$, we have $\alpha^* \in \mathrm{acc}(C_{\alpha_i, i})$, so that $\mathrm{acc}(D_\beta) \subseteq \mathrm{acc}(C_{\alpha_i, i})$.

Now, let $C_{\beta,i} := C_{\alpha^*, i} \cup \{\alpha^*\} \cup D_{\bar{\beta}}^-$ for all $i \in [i(\beta), i^*)$, and let $C_{\beta,i} := C_{\alpha_i, i} \cup \{\alpha_i\} \cup D_{\bar{\beta}}^-$ for all $i \in [i^*, \theta)$.

Case 2c: $\sup(\Gamma \cap \beta) = \beta$ **but** $\sup(\mathrm{acc}(D_\beta)) < \beta$. In this case, $\mathrm{otp}(D_\beta \setminus \sup(\mathrm{acc}(D_\beta))) = \omega$. Let $\alpha^* := \max(\mathrm{acc}(D_\beta))$ if $\mathrm{acc}(D_\beta) \ne \emptyset$, and $\alpha^* :=$

$\min(\Gamma)$ otherwise. In either case, $\alpha^* < \alpha$ and $i(\alpha^*) \leq i(\beta)$. Let $\langle \alpha_n \mid n < \omega \rangle$ be an increasing sequence of elements of Γ converging to β with $\alpha_0 := \alpha^*$. Fix an increasing sequence $\langle i_n \mid n < \omega \rangle$ of ordinals below θ such that for all $n < \omega$, $\max\{i(\alpha_m) \mid m \leq n\} \leq i_n$ and $\{\alpha_m \mid m < n\} \subseteq \operatorname{acc}(C_{\alpha_n, i_n})$.

Finally, fix $i \in [i(\beta), \theta)$. If there is $k < \omega$ such that $i_k \leq i < i_{k+1}$, then set $C_{\beta,i} := C_{\alpha_k,i} \cup \{\alpha_n \mid k \leq n < \omega\}$ for this unique k. Otherwise, set $C_{\beta,i} := \bigcup_{n<\omega} C_{\alpha_n,i}$.

Case 2d: $\sup(\operatorname{acc}(D_\beta)) = \beta$**.** Note that, for all $\alpha \in \operatorname{acc}(D_\beta)$, we have $\alpha \in \Omega$, $i(\alpha) = i(\beta)$ and $\operatorname{acc}(D_\beta) \cap \alpha = \operatorname{acc}(D_\alpha) \subseteq C_{\alpha,i(\alpha)}$. Therefore, for all $i \in [i(\beta), \theta)$, we can simply set $C_{\beta,i} := \bigcup_{\alpha \in \operatorname{acc}(D_\beta)} C_{\alpha,i}$.

Our construction has yielded a matrix satisfying clauses (1)–(3) of Definition 4.2 and such that $\Gamma \cap T = \emptyset$. It remains to verify Clause (4) of Definition 4.2. Towards a contradiction, suppose that D is a club in κ satisfying that, for every $\alpha \in \operatorname{acc}(D) \cap \Gamma$, there exists $i < \theta$ such that $D \cap \alpha \neq C_{\alpha,i}$. Fix $i < \theta$ for which $G := \{\gamma \in \Gamma \mid D \cap \gamma = C_{\gamma,i}\}$ is stationary. Recalling Clause 4.13.1(4), let us now fix $\beta \in \Gamma_{i+1} \cap \operatorname{acc}(D)$. Pick $\gamma \in G$ above β. Then $\beta \in \operatorname{acc}(D \cap \gamma) = \operatorname{acc}(C_{\gamma,i})$, so $i(\beta) \leq i$, contradicting the fact that $i(\beta) = j(\beta) = i + 1$. $\square$

Corollary 4.14. $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *is compatible with* $\mathrm{U}^{\mathrm{subadditive}}(\aleph_{\omega+1}, \aleph_{\omega+1}, \theta, \aleph_\omega)$ *holding for every infinite cardinal* $\theta < \aleph_\omega$.

Proof. Starting with a ground model in which $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds, one can add a $\square(\aleph_{\omega+1})$-sequence via an $\aleph_{\omega+1}$-strategically closed forcing, hence preserving the principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. By Theorems 4.4 and 4.13, in the extension, $\mathrm{U}^{\mathrm{subadditive}}(\aleph_{\omega+1}, \aleph_{\omega+1}, \theta, \aleph_\omega)$ holds for every infinite cardinal $\theta < \aleph_\omega$. $\square$

By Theorem 3.34, for any $\theta \in \operatorname{Reg}(\kappa)$, there exists a θ^+-directed closed, κ-strategically closed forcing notion that introduces a somewhere-closed witness to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$. In order to force a fully closed witness, it seems that we must decrease the degree of closure of the poset by one cardinal. The next theorem forms Clause (2) of Theorem C.

Theorem 4.15. *Suppose that* $\theta \in \operatorname{Reg}(\kappa)$*. Then there exists a* θ*-directed closed, κ-strategically closed forcing notion that introduces a closed witness* $c : [\kappa]^2 \to \theta$ *to* $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$ *for which* $\partial(c)$ *is stationary.*

Proof. By [LH17b, §7], there is a θ-directed closed, κ-strategically closed forcing notion that adds an $\square^{\mathrm{ind}}(\kappa, \theta)$-sequence and therefore, by Theorem 4.4, it adds a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$. However, unless $\theta = \omega$, it is unclear whether $\partial(c)$ can be made to be stationary in this case. Thus, instead, we appeal to the forcing $\mathbb{P}(\kappa, \theta)$ from [LHR19, §3.3]. This is a θ-directed closed, κ-strategically closed forcing notion that introduces a witness $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ to $\mathrm{P}^-(\kappa, 2, \sqsubseteq_\theta, 1, \{\kappa\}, 2, \sigma)$ for every $\sigma < \kappa$. Now, utilizing the instance $\sigma := \theta$, the proof of [LH17a, Theorem 4.1] makes it clear that the set $\{\alpha \in E^\kappa_\theta \mid \exists \eta < \alpha[\operatorname{otp}(C_\alpha \setminus \eta) = \theta]\}$ is stationary. Fix some $\eta < \kappa$ for which $T := \{\alpha \in E^\kappa_\theta \mid \operatorname{otp}(C_\alpha \setminus \eta) = \theta\}$ is stationary. Note that, by the $\sqsubseteq_\theta$-coherence of $\vec{C}$, the set $\Gamma := \{\alpha \in \operatorname{acc}(\kappa) \mid \forall \bar{\alpha} \in \operatorname{acc}(C_\alpha)[C_{\bar{\alpha}} \sqsubseteq C_\alpha]\}$ covers $E^\kappa_{\geq \theta}$. Now, define a C-sequence $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ as follows:

- Let $D_0 := \emptyset$.
- For every $\alpha < \kappa$, let $D_{\alpha+1} := \{\alpha\}$.

- For every $\alpha \in \mathrm{acc}(\kappa) \setminus \Gamma$, let D_α be some club in α of order-type $\mathrm{cf}(\alpha)$ such that $\mathrm{nacc}(D_\alpha) \subseteq \mathrm{nacc}(\alpha)$.
- For every $\alpha \in \Gamma$ such that $\mathrm{otp}(C_\alpha \setminus \eta) \leq \theta$, let $D_\alpha := C_\alpha$.
- For any other α, let $D_\alpha := \{\beta \in C_\alpha \mid \mathrm{otp}((C_\alpha \setminus \eta) \cap \beta) > \theta\}$.

Note that, for every $\alpha \in \kappa \setminus \Gamma$, $\mathrm{otp}(D_\alpha) = \mathrm{cf}(\alpha) < \theta$, so $\mathrm{acc}(D_\alpha) \cap T = \emptyset$.

Claim 4.15.1. $\vec{D}$ *is a* $\square(\kappa, \sqsubseteq_\theta)$*-sequence that avoids* T.

Proof. As any $\sqsubseteq_\theta$-coherent C-sequence that avoids a stationary set is a $\square(\kappa, \sqsubseteq_\theta)$-sequence, we shall focus on verifying that $\vec{D}$ is $\sqsubseteq_\theta$-coherent and avoids T. By the definition of $\vec{D}$, it suffices to verify that for all $\alpha \in \Gamma$ and $\bar{\alpha} \in \mathrm{acc}(D_\alpha)$, $D_{\bar{\alpha}} \sqsubseteq D_\alpha$ and $\bar{\alpha} \notin T$. Now, given such a pair $\bar{\alpha} < \alpha$, since $\alpha \in \Gamma$, we infer that $C_{\bar{\alpha}} \sqsubseteq C_\alpha$ and $\bar{\alpha} \in \Gamma$. We shall verify that $D_{\bar{\alpha}} \sqsubseteq D_\alpha$ and that $\bar{\alpha} \notin T$.

▶ If $\mathrm{otp}(C_\alpha \setminus \eta) \leq \theta$, then $\mathrm{otp}(C_{\bar{\alpha}} \setminus \eta) < \theta$, so $D_{\bar{\alpha}} = C_{\bar{\alpha}} \sqsubseteq C_\alpha = D_\alpha$ and $\bar{\alpha} \notin T$.

▶ If $\mathrm{otp}(C_\alpha \setminus \eta) > \theta$, then $D_\alpha = \{\beta \in C_\alpha \mid \mathrm{otp}((C_\alpha \setminus \eta) \cap \beta) > \theta\}$, and so, from $\bar{\alpha} \in \mathrm{acc}(D_\alpha)$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$, it follows that $\mathrm{otp}(C_{\bar{\alpha}} \setminus \eta) > \theta$, so again $D_{\bar{\alpha}} \sqsubseteq D_\alpha$ and $\bar{\alpha} \notin T$. $\boxtimes$

Now, by Theorem 4.13(2) and Lemma 3.2(3), there is a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$ such that $T \subseteq \partial(c)$. $\square$

It follows that there can be dramatic failures of monotonicity in the third coordinate of $\mathrm{U}^{\mathrm{subadditive}}(\ldots)$.

Corollary 4.16. *Suppose that* θ *is an indestructible supercompact cardinal below* κ. *Then there is a forcing extension in which all cofinalities* $\leq \kappa$ *are preserved and all of the following hold:*

- $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$ *holds and is witnessed by a closed coloring;*
- *for all* $\theta' < \theta$, *no witness to* $\mathrm{U}(\kappa, 2, \theta', 2)$ *is subadditive of the second kind;*
- *for all* $\theta' < \theta$ *such that* κ *is not the successor of a singular cardinal of cofinality* θ', *no witness to* $\mathrm{U}(\kappa, 2, \theta', 2)$ *is weakly subadditive of the first kind.*

Proof. By Theorem 4.15, there is a θ-directed closed, κ-strategically closed forcing notion $\mathbb{P}$ that adds a closed witness to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, \kappa, \theta, \theta)$. Since $\mathbb{P}$ is θ-directed closed, θ remains supercompact, so, we may appeal to Corollary 3.42. $\square$

5. Successors of singular cardinals

Definition 5.1. Suppose that $\vec{\lambda} = \langle \lambda_j \mid j < \nu \rangle$ is a sequence of infinite cardinals, each greater than ν. The principle $\boxtimes(\vec{\lambda})$ asserts the existence of a sequence $\langle X_\alpha \mid \alpha < \sup(\vec{\lambda}) \rangle$, and, for all $j < \nu$, a sequence $\vec{C}^j = \langle C_\alpha^j \mid \alpha < \lambda_j^+ \rangle$ such that

- for all $j < \nu$ and $\alpha \in \mathrm{acc}(\lambda_j^+)$, C_α^j is a club in α, and for all $\bar{\alpha} \in \mathrm{acc}(C_\alpha^j)$,
  - $C_{\bar{\alpha}}^j = C_\alpha^j \cap \bar{\alpha}$;
  - $X_{\bar{\alpha}} = X_\alpha \cap \bar{\alpha}$, provided that $\alpha > \lambda_j$.
- for arbitrary large regular cardinals Υ, for all $p \in H_\Upsilon$ and $X \subseteq \sup(\vec{\lambda})$, there exists $N \prec H_\Upsilon$ such that
  - $p \in N$;
  - $|N| < \sup(\vec{\lambda})$;
  - N is internally approachable of length ν^+;
  - for all sufficiently large $j < \nu$, we have $X \cap \sup(N \cap \lambda_j^+) = X_{\sup(N \cap \lambda_j^+)}$.

Remark 5.2. Note that, if $\boxtimes(\vec{\lambda})$ holds for a sequence $\vec{\lambda} = \langle \lambda_j \mid j < \nu \rangle$, then $2^\mu \leq \lambda$ for all $\mu < \lambda$. Moreover, as we will see below in Claim 5.4.1, $\square(\lambda_j^+)$ holds for all sufficiently large $j < \nu$, as witnessed by the sequence $\vec{C^j}$ from the above definition.

We will soon show that $\boxtimes(\vec{\lambda})$ entails certain instances of $\mathrm{U}^{\mathrm{subadditive}}(\lambda^+, \ldots)$. We first need the following lemma.

Lemma 5.3. *Suppose that* $\langle C_\alpha \mid \alpha < \kappa \rangle$ *is a* $\square(\kappa)$-*sequence,* $\theta \in \mathrm{Reg}(\kappa)$, *and* $h : \kappa \to \theta$ *is any function such that*

(1) *for unboundedly many* $i < \theta$, $h^{-1}\{i\}$ *is stationary;*
(2) *for all* $\alpha < \kappa$ *and all* $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, *we have* $h(\bar{\alpha}) = h(\alpha)$.

Then there is a closed, subadditive witness c *to* $\mathrm{U}(\kappa, \kappa, \theta, \sup(\mathrm{Reg}(\kappa)))$ *such that, for all* $(\alpha, \beta) \in [\kappa]^2$ *with* $\alpha \in \mathrm{acc}(\kappa)$, *we have* $c(\alpha, \beta) \geq h(\alpha)$.

Proof. By the proof of [LHL18, Theorem 3.4], the hypotheses of the lemma entail the existence of a $\square^{\mathrm{ind}}(\kappa, \theta)$-sequence $\langle C_{\alpha,i} \mid \alpha \in \mathrm{acc}(\kappa), \; i(\alpha) \leq i < \theta \rangle$ such that $i(\alpha) = h(\alpha)$ for all $\alpha \in \mathrm{acc}(\kappa)$. The construction in the proof of the implication $(1) \implies (2)$ of Theorem 4.4 then yields a closed subadditive function $c : [\kappa]^2 \to \theta$ witnessing $\mathrm{U}(\kappa, 2, \theta, 2)$ such that, for all $(\alpha, \beta) \in [\kappa]^2$, if $\alpha \in \mathrm{acc}(\kappa)$, then $c(\alpha, \beta) \geq i(\alpha) = h(\alpha)$. By Fact 3.2, c is moreover a witness to $\mathrm{U}(\kappa, \kappa, \theta, \sup(\mathrm{Reg}(\kappa)))$. $\square$

Our next result is Theorem B. It is very much in the spirit of [CFM04, Corollary 3.10]; both results take instances of incompactness below a singular cardinal λ and use them together with a scale of length λ^+ to produce an instance of incompactness at λ^+.

Theorem 5.4. *Suppose that* λ *is a singular cardinal,* $\theta \in \mathrm{Reg}(\lambda) \setminus (\mathrm{cf}(\lambda) + 1)$, *and* $\vec{\lambda} = \langle \lambda_j \mid j < \mathrm{cf}(\lambda) \rangle$ *is an increasing sequence of cardinals, converging to* λ, *such that*

- $\boxtimes(\vec{\lambda})$ *holds;*
- $\prod_{j < \mathrm{cf}(\lambda)} \lambda_j^+$ *carries a scale* $\vec{f}$ *of length* λ^+.

Then there is a Σ-*closed, subadditive witness to* $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$, *where* $\Sigma \subseteq E_{\neq \mathrm{cf}(\lambda)}^{\lambda^+}$ *denotes the set of good points for* $\vec{f}$.

Proof. Without loss of generality, assume that $\lambda_0 > \theta$. Fix sequences $\langle X_\alpha \mid \alpha < \lambda \rangle$ and $\vec{C^j} = \langle C_\alpha^j \mid \alpha < \lambda_j^+ \rangle$ for $j < \mathrm{cf}(\lambda)$ witnessing $\boxtimes(\vec{\lambda})$.

Let $j < \mathrm{cf}(\lambda)$ be arbitrary. It is clear from the definition of $\boxtimes(\vec{\lambda})$ that we may assume that $\min(C_\alpha^j) \geq \lambda_j$ whenever $\alpha \in (\lambda_j, \lambda_j^+)$. Now, define a function $h_j : \lambda_j^+ \to \theta$ by letting, for all $\alpha < \lambda_j^+$,

$$h_j(\alpha) := \begin{cases} X_\alpha & \text{if } \alpha > \lambda_j \; \& \; X_\alpha \in \theta; \\ 0 & \text{otherwise.} \end{cases}$$

Claim 5.4.1. *For all sufficiently large* $j < \mathrm{cf}(\lambda)$, $\vec{C^j}$ *and* h_j *satisfy the hypotheses of Lemma 5.3.*

Proof. We first show that $\vec{C^j}$ is a $\square(\lambda_j^+)$-sequence for all sufficiently large $j < \mathrm{cf}(\lambda)$. Suppose for sake of contradiction that there is an unbounded set $J \subseteq \mathrm{cf}(\lambda)$ such that, for all $j \in J$, $\vec{C^j}$ is not a $\square(\lambda_j^+)$-sequence. Each $\vec{C^j}$ is evidently a coherent C-sequence; therefore, for every $j \in J$, we can fix a club $D_j \subseteq \lambda_j^+$ such that

$D_j \cap \alpha = C_\alpha^j$ for all $\alpha \in \mathrm{acc}(D_j)$. By our assumption from the previous paragraph, this implies that $D_j \subseteq [\lambda_j, \lambda_j^+)$. Fix a regular cardinal Υ sufficiently large to ensure that all objects of interest are in H_Υ. Let $p := \langle D_j \mid j \in J \rangle$, $X^0 := 0$ and $X^1 := 1$. Apply $\boxtimes(\vec{\lambda})$ twice to find $N_0, N_1 \prec H_\Upsilon$ such that, for $i < 2$:

- $p \in N_i$;
- $|N_i| < \lambda$;
- N_i is internally approachable of length $\mathrm{cf}(\lambda)^+$;
- for all sufficiently large $j < \mathrm{cf}(\lambda)$, we have $X^i = X_{\sup(N_i \cap \lambda_j^+)}$.

Now fix $j^* \in J$ sufficiently large so that, for $i < 2$, we have

- $|N_i| < \lambda_{j^*}$;
- $X^i = X_{\sup(N_i \cap \lambda_{j^*}^+)}$.

For each $i < 2$, the ordinal $\beta_i := \sup(N_i \cap \lambda_j^+)$ is less than λ_j^+. Since $D_{j^*} \in N_i$ and D_{j^*} is club in λ_j^+, we have $\beta_i \in \mathrm{acc}(D_{j^*})$. In particular, $\beta_i > \lambda_{j^*}$. By our choice of D_{j^*}, it follows that $D_{j^*} \cap \beta_i = C_{\beta_i}^{j^*}$. If $\beta_0 < \beta_1$, then $\beta_0 \in \mathrm{acc}(D_{j^*}) \cap \beta_1 = \mathrm{acc}(C_{\beta_1}^{j^*})$ and $\beta_1 > \lambda_{j^*}$, so $X_{\beta_0} = X_{\beta_1} \cap \beta_0$. Likewise, if $\beta_1 < \beta_0$, then $X_{\beta_1} = X_{\beta_0} \cap \beta_1$. However, by our choice of j^*, $X_{\beta_0} = X^0 = 0$ and $X_{\beta_1} = X^1 = 1$, which is a contradiction.

By the definition of h_j and the fact that $\lambda_0 > \theta$, it is immediate that $\vec{C}_j$ and h_j satisfy Clause (2) in the statement of Lemma 5.3. It remains to show that, for all sufficiently large $j < \mathrm{cf}(\lambda)$, for unboundedly many $i < \theta$, $h_j^{-1}\{i\}$ is stationary in λ_j^+. Suppose for sake of contradiction that there is an unbounded $J \subseteq \mathrm{cf}(\lambda)$ and, for each $j \in J$, an $i_j < \theta$ such that $h_j^{-1}\{i\}$ is nonstationary for all $i \in [i_j, \theta)$. Since $\theta > \mathrm{cf}(\lambda)$ is regular, $i^* := \sup\{i_j \mid j < \mathrm{cf}(\lambda)\}$ is less than θ. It follows that for all $j \in J$, $h_j^{-1}[i^*, \theta)$ is nonstationary in λ_j^+; fix a club $D_j \subseteq \lambda_j^+$ disjoint from it. Fix a regular cardinal Υ sufficiently large to ensure that all objects of interest are in H_Υ. Let $p := \{\theta, \vec{\lambda}, \langle D_j \mid j \in J \rangle\}$ and $X := i^*$, and apply $\boxtimes(\vec{\lambda})$ to find $N \prec H_\Upsilon$ such that:

- $p \in N$;
- $|N| < \lambda$;
- N is internally approachable of length $\mathrm{cf}(\lambda)^+$;
- for all sufficiently large $j < \mathrm{cf}(\lambda)$, we have $X = X_{\sup(N \cap \lambda_j^+)}$.

Fix a sufficiently large $j^* \in J$ so that $|N| < \lambda_{j^*}$ and $X = i^* = X_{\sup(N \cap \lambda_{j^*}^+)}$. Let $\beta := \sup(N \cap \lambda_{j^*}^+)$. Since $\vec{\lambda} \in N$, we have $\beta > \lambda_{j^*}$, and hence $h_{j^*}(\beta) = i^*$. However, since D_{j^*} is a club in $\lambda_{j^*}^+$ lying in N, we have $\beta \in D_{j^*}$, and hence $h_{j^*}(\beta) < i^*$. This is a contradiction, completing the proof of the claim. $\boxtimes$

By the claim, by discarding an initial segment of $\vec{\lambda}$ if necessary, we may assume that, for all $j < \mathrm{cf}(\lambda)$, $\vec{C}^j$ and h_j satisfy the hypotheses of Lemma 5.3. We may therefore fix a closed, subadditive witness $c_j : [\lambda_j^+]^2 \to \theta$ to $\mathrm{U}(\lambda_j^+, \lambda_j^+, \theta, \lambda_j)$ such that $c_j(\alpha, \beta) \geq h_j(\alpha)$ for all $(\alpha, \beta) \in [\lambda_j^+]^2$ with $\alpha \in \mathrm{acc}(\lambda_j^+)$.

Next, let $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ be a continuous scale in $\prod_{j < \mathrm{cf}(\lambda)} \lambda_j^+$, and let $\Sigma \subseteq E_{\neq \mathrm{cf}(\lambda)}^{\lambda^+}$ denote the set of good points for $\vec{f}$. Define a coloring $c : [\lambda^+]^2 \to \theta$ by

setting, for all $\beta < \gamma < \lambda^+$,

$$c(\beta, \gamma) := \limsup_{j \to \mathrm{cf}(\lambda)} c_j(f_\beta(j), f_\gamma(j)).$$

Note that, for all $(\beta, \gamma) \in [\lambda^+]^2$ and all sufficiently large $j < \mathrm{cf}(\lambda)$, we have $f_\beta(j) < f_\gamma(j)$, so the above expression is well-defined. We claim that c is a Σ-closed, subadditive witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$. Let us verify each of these requirements in turn.

Claim 5.4.2. c is Σ-closed.

Proof. Suppose that $\gamma < \lambda^+$, $i < \theta$, and $B \subseteq D^c_{\leq i}(\gamma)$, with $\beta := \sup(B)$ in $(\gamma \cap \Sigma) \setminus B$. We will show that $\beta \in D^c_{\leq i}(\gamma)$.

Since $\beta \in \Sigma$, we can assume, by thinning out B if necessary, that there is $j_0 < \mathrm{cf}(\lambda)$ such that, for all $(\alpha, \alpha') \in [B]^2$, we have $f_\alpha <_{j_0} f_{\alpha'}$. Since $\vec{f}$ is continuous, there is $j_1 \in [j_0, \mathrm{cf}(\lambda))$ such that, for all $j \in [j_1, \mathrm{cf}(\lambda))$, we have $f_\beta(j) = \sup\{f_\alpha(j) \mid \alpha \in B\}$. Finally, since $B \subseteq D^c_{\leq i}(\gamma)$, we can assume, by thinning out B again if necessary, that there is j_2 with $j_1 \leq j_2 < \mathrm{cf}(\lambda)$ such that, for all $j \in [j_2, \mathrm{cf}(\lambda))$,

- $f_\beta(j) < f_\gamma(j)$;
- for all $\alpha \in B$, $c_j(f_\alpha(j), f_\gamma(j)) \leq i$.

Since each c_j is closed, it follows that, for all $j \in [j_2, \mathrm{cf}(\lambda))$, we have $\sup\{f_\alpha(j) \mid \alpha \in B\} = f_\beta(j)$ and $c_j(f_\beta(j), f_\gamma(j)) \leq i$, and hence $c(\beta, \gamma) \leq i$. $\boxtimes$

Claim 5.4.3. c is subadditive.

Proof. Let $\alpha < \beta < \gamma < \lambda^+$ be arbitrary. For all sufficiently large $j < \mathrm{cf}(\lambda)$, we have $f_\alpha(j) < f_\beta(j) < f_\gamma(j)$ and hence, since each c_j is subadditive, we have

- $c_j(f_\alpha(j), f_\gamma(j)) \leq \max\{c_j(f_\alpha(j), f_\beta(j)), c_j(f_\beta(j), f_\gamma(j))\}$; and
- $c_j(f_\alpha(j), f_\beta(j)) \leq \max\{c_j(f_\alpha(j), f_\gamma(j)), c_j(f_\beta(j), f_\gamma(j))\}$.

It follows immediately from the definition of c that $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$ and $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$. $\boxtimes$

Claim 5.4.4. c witnesses $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$.

Proof. By a theorem of Shelah, $\Sigma \cap E^{\lambda^+}_{\geq \chi}$ is stationary for all $\chi < \lambda$. So, since c is Σ-closed and subadditive, Lemma 3.2(3) implies that it suffices to verify that c witnesses $\mathrm{U}(\lambda^+, 2, \theta, 2)$. To this end, fix $A \in [\lambda^+]^{\lambda^+}$ and a color $i < \theta$. We will find $(\alpha, \gamma) \in [A]^2$ such that $c(\alpha, \gamma) > i$.

Fix a regular cardinal Υ sufficiently large to ensure that all objects of interest are in H_Υ. Apply $\boxtimes(\vec{\lambda})$ with $p := \{\vec{\lambda}, \vec{f}, A\}$ and $X := i + 1$ to find $N \prec H_\Upsilon$ such that

- $p \in N$;
- $|N| < \lambda$;
- N is internally approachable of length $\mathrm{cf}(\lambda)^+$;
- for all sufficiently large $j < \mathrm{cf}(\lambda)$, we have $X_{\sup(N \cap \lambda^+_j)} = i + 1$.

As $|N| < \lambda$ and $\vec{\lambda} \in N$, there exists a function $\chi_N \in \prod_{j < \mathrm{cf}(\lambda)} \mathrm{acc}(\lambda^+_j)$ such that $\chi_N(j) = \sup(N \cap \lambda^+_j)$ for all sufficiently large $j < \mathrm{cf}(\lambda)$. Let $\beta := \sup(N \cap \lambda^+)$. As N is internally approachable of length $\mathrm{cf}(\lambda)^+$, we know that $\beta \in \Sigma \cap E^{\lambda^+}_{\mathrm{cf}(\lambda)^+}$. Since $\vec{f} \in N$, $\vec{f}$ is continuous, and N is internally approachable, we know that

$f_\beta(j) = \chi_N(j)$ for all sufficiently large $j < \mathrm{cf}(\lambda)$. Thus, for all sufficiently large $j < \mathrm{cf}(\lambda)$ and all $\eta \in (f_\beta(j), \lambda_j^+)$,

$$c_j(f_\beta(j), \eta) \geq h_j(f_\beta(j)) = h_j(\chi_N(j)) = X_{\sup(N \cap \lambda_j^+)} = i + 1.$$

Consequently, $c(\beta, \gamma) > i$ for all $\gamma > \beta$.

Fix $\gamma \in A \setminus (\beta + 1)$. Since $\beta \in \Sigma$ and c is Σ-closed, there is $\epsilon < \beta$ such that $c(\alpha, \gamma) > i$ for all $\alpha \in (\epsilon, \beta)$. As $A \in N$, the elementarity of N entails that $\sup(A \cap \beta) = \beta$, so we may fix $\alpha \in A \cap (\epsilon, \beta)$. Then $c(\alpha, \gamma) > i$, as desired. $\boxtimes \square$

We now connect the above finding with the notion of the *C-sequence spectrum*, introduced in Part II of this series.

Definition 5.5 ([LHR21]).   (1) For every C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$, $\chi(\vec{C})$ is the least cardinal $\chi \leq \kappa$ such that there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

(2) $\mathrm{Cspec}(\kappa) := \{\chi(\vec{C}) \mid \vec{C} \text{ is a } C\text{-sequence over } \kappa\} \setminus \omega$.

The next result yields the "In particular" part of Theorem B.

Corollary 5.6. *Suppose that λ is a singular cardinal, $\vec{\lambda}$ is an increasing $\mathrm{cf}(\lambda)$-sequence of cardinals, converging to λ, such that $\boxtimes(\vec{\lambda})$ holds, and $\mathrm{tcf}(\prod \vec{\lambda}, <^*) = \lambda^+$. Then $\mathrm{Reg}(\lambda) \subseteq \mathrm{Cspec}(\lambda^+)$.*

Proof. ▶ By Remark 5.2, $2^{\mathrm{cf}(\lambda)} < \lambda$. So, by [LHR21, Theorem 5.29(1)], $\mathrm{Reg}(\mathrm{cf}(\lambda)) \subseteq \mathrm{Cspec}(\lambda^+)$.

▶ By [LHR21, Lemma 4.11], $\mathrm{cf}(\lambda) \in \mathrm{Cspec}(\lambda^+)$.

▶ By Theorem 5.4 and [LHR21, Corollary 5.21], $\mathrm{Reg}(\lambda) \setminus (\mathrm{cf}(\lambda)+1) \subseteq \mathrm{Cspec}(\lambda^+)$.
$\square$

Our next goal is to improve the following fact from Part II, and present a weaker sufficient condition for $\mathrm{Cspec}(\lambda^+)$ to cover $\mathrm{Reg}(\mathrm{cf}(\lambda))$.

Fact 5.7 ([LHR21, Theorem 5.29(2)]). *Suppose that λ is a singular cardinal of successor cofinality μ^+. Then:*

- $\mathrm{Reg}(\mu) \subseteq \mathrm{Cspec}(\lambda^+)$;
- *If $2^\mu \leq \lambda$, then $\mathrm{Reg}(\mathrm{cf}(\lambda)) \subseteq \mathrm{Cspec}(\lambda^+)$.*

Theorem 5.8. *Suppose that λ is a singular cardinal. If λ is of uncountable cofinality, suppose also that $\chi(\mathrm{cf}(\lambda)) > 1$. Then, for every infinite regular $\theta \leq \mathrm{cf}(\lambda)$, there exists a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$.*

Proof. By [LHR18, Corollary 4.17], there exists a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$, so assume that λ has uncountable cofinality. Denote $\nu := \mathrm{cf}(\lambda)$, so that $\chi(\nu) > 1$. Recalling Claim 4.21.3 from the proof of [LHR18, Theorem 4.21], it suffices to prove that the ideal $\mathcal{I}$ defined there in "Case 1: Uncountable cofinality" is not weakly ν-saturated.

Let us first remind the reader that the definition of the ideal $\mathcal{I}$ goes through first fixing a stationary subset $\Delta \subseteq E_{\mathrm{cf}(\lambda)}^{\lambda^+}$ and a sequence $\vec{e} = \langle e_\delta \mid \delta \in \Delta \rangle$ such that

- for every $\delta \in \Delta$, e_δ is a club in δ of order type $\mathrm{cf}(\lambda)$;
- for every $\delta \in \Delta$, $\langle \mathrm{cf}(\gamma) \mid \gamma \in \mathrm{nacc}(e_\delta) \rangle$ is strictly increasing and converging to λ;
- for every club D in λ^+, there exists $\delta \in \Delta$ such that $e_\delta \subseteq D$.

Then, the ideal $\mathcal{I}$ consists of all subsets $\Gamma \subseteq \lambda^+$ for which there exists a club $D \subseteq \lambda^+$ such that $\sup(\mathrm{nacc}(e_\delta) \cap D \cap \Gamma) < \delta$ for every $\delta \in \Delta \cap D$.

Now, since $\chi(\nu) > 1$, by [IR22, Theorem A], $\mathsf{unbounded}^+(J^{\mathrm{bd}}[\nu], \nu)$ holds. That is, we may fix a coloring $c : [\nu]^2 \to \nu$ satisfying that, for every cofinal $B \subseteq \nu$, there exist $(\eta, T) \in (\nu, [\nu]^\nu)$ such that, for all $\tau \in T$, $\sup\{\beta \in B \mid c(\eta, \beta) = \tau\} = \nu$.

Fix a club Λ in λ of order-type ν. For every $\eta < \nu$ and $\tau < \nu$, let

$$\Gamma_\eta^\tau := \{\gamma < \lambda^+ \mid c(\eta, \mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma))) = \tau\}.$$

Claim 5.8.1. *There exists $\eta < \nu$ such that $|\{\tau < \nu \mid \Gamma_\eta^\tau \in \mathcal{I}^+\}| = \nu$.*

Proof. Suppose not. Then, for every $\eta < \nu$, the set $T_\eta := \{\tau < \nu \mid \Gamma_\eta^\tau \in \mathcal{I}^+\}$ is bounded in ν. For each $\eta < \nu$ and $\tau \in \nu \setminus T_\eta$, fix a club $D_\eta^\tau \subseteq \lambda^+$ such that $\sup(\mathrm{nacc}(e_\delta) \cap D_\eta^\tau \cap \Gamma_\eta^\tau) < \delta$ for every $\delta \in \Delta \cap D_\tau^\eta$. Let $D := \bigcap\{D_\eta^\tau \mid \eta < \nu, \tau \in \nu \setminus T_\eta\}$. Now, using the choice of $\vec{e}$, let us fix $\delta \in \Delta$ such that $e_\delta \subseteq D$. In particular, $\delta \in D$. As $\langle \mathrm{cf}(\gamma) \mid \gamma \in \mathrm{nacc}(e_\delta)\rangle$ is strictly increasing and converging to λ, it follows that $B := \{\mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma)) \mid \gamma \in \mathrm{nacc}(e_\delta)\}$ is cofinal in ν. Thus, by the choice of c, we may now find $(\eta, T) \in (\nu, [\nu]^\nu)$ such that, for all $\tau \in T$, $\sup\{\beta \in B \mid c(\eta, \beta) = \tau\} = \nu$. Pick $\tau \in T \setminus T_\eta$. As $\delta \in \Delta \cap D \subseteq \Delta \cap D_\eta^\tau$, we infer that $\sup(\mathrm{nacc}(e_\delta) \cap D_\eta^\tau \cap \Gamma_\eta^\tau) < \delta$. In particular, $A := \{\mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma)) \mid \gamma \in \mathrm{nacc}(e_\delta) \cap D_\eta^\tau \cap \Gamma_\eta^\tau\}$ is bounded in ν. Pick $\beta \in B$ above $\sup(A \cup (\eta + 1))$ such that $c(\eta, \beta) = \tau$. Then, find $\gamma \in \mathrm{nacc}(e_\delta)$ such that $\beta = \mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma))$. Altogether:

- $\gamma \in \mathrm{nacc}(e_\delta)$;
- $\gamma \in e_\delta$, so that $\gamma \in D \subseteq D_\eta^\tau$;
- $c(\eta, \mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma)) = c(\eta, \beta) = \tau$, so that $\gamma \in \Gamma_\eta^\tau$;
- $\mathrm{otp}(\Lambda \cap \mathrm{cf}(\gamma)) = \beta > \sup(A)$, so that $\gamma \notin \mathrm{nacc}(e_\delta) \cap D_\eta^\tau \cap \Gamma_\eta^\tau$.

This is a contradiction.   ⊠

It follows that $\mathcal{I}$ is indeed not weakly ν-saturated, so we are done.   □

As successor cardinals are non-Mahlo, the following indeed improves Fact 5.7.

Corollary 5.9. *Suppose that λ is a singular cardinal of cofinality ν. If ν is not greatly Mahlo, then $\mathrm{Reg}(\mathrm{cf}(\lambda)) \subseteq \mathrm{Cspec}(\lambda^+)$.*

Proof. By [LHR21, Corollary 5.21], to show that an infinite regular cardinal θ is in $\mathrm{Cspec}(\lambda^+)$, it suffices to prove that there exists a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, \theta)$. By the preceding theorem, if $\mathrm{Reg}(\mathrm{cf}(\lambda))$ is not a subset of $\mathrm{Cspec}(\lambda^+)$, then $\mathrm{cf}(\lambda)$ is an uncountable cardinal and $\chi(\mathrm{cf}(\lambda)) \leq 1$. In this case, by [LHR21, Theorem A(2)], $\mathrm{cf}(\lambda)$ is greatly Mahlo.   □

6. Stationarily layered posets

In Part I of this project [LHR18], we motivated the study of $\mathrm{U}(\kappa, \mu, \theta, \chi)$ by showing that it places limits on the infinite productivity of the κ-Knaster condition. Here, we present an analogous result, indicating that closed witnesses to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ place limits on the infinite productivity of the property of being κ-stationarily layered, which is a strengthening of the κ-Knaster condition.

Definition 6.1 ([Cox18])**.** A partial order $\mathbb{P}$ is κ-*stationarily layered* if the collection of regular suborders of $\mathbb{P}$ of size less than κ is stationary in $\mathcal{P}_\kappa(\mathbb{P})$.

By [CL17, Lemma 1.5], any κ-stationarily layered poset is also κ-Knaster. We now recall a useful equivalence.

Fact 6.2 ([CL17, Lemma 2.3]). *Given a poset $\mathbb{P}$, the following are equivalent.*

(1) $\mathbb{P}$ *is κ-stationarily layered.*
(2) *There is a regular cardinal Υ with $\mathcal{P}_\kappa(\mathbb{P}) \in H_\Upsilon$ and an elementary sub-structure $M \prec H_\Upsilon$ such that*
   (a) $\kappa, \mathbb{P} \in M$;
   (b) $\kappa \cap M \in \kappa$;
   (c) $\mathbb{P} \cap M$ *is a regular suborder of $\mathbb{P}$.*

The following is Theorem D:

Theorem 6.3. *Suppose that $\theta \leq \chi < \kappa$ are infinite, regular cardinals, κ is $<\chi$-inaccessible, and $c : [\kappa]^2 \to \theta$ is a closed witness to $\mathrm{U}(\kappa, 2, \theta, 2)$ that is subadditive of the second kind. Then:*

- *There is a sequence of posets $\langle \mathbb{P}_i \mid i < \theta \rangle$ such that:*
  (1) *for all $i < \theta$, $\mathbb{P}_i$ is well-met and χ-directed closed with greatest lower bounds;*
  (2) *for all $j < \theta$, $\prod_{i<j} \mathbb{P}_i$ is κ-stationarily layered;*
  (3) $\prod_{i<\theta} \mathbb{P}_i$ *is not κ-cc.*
- *There is a poset $\mathbb{P}$ such that, if $\partial(c) \cap E_\chi^\kappa$ is stationary, then*
  (1) $\mathbb{P}$ *is well-met and χ-directed closed with greatest lower bounds;*
  (2) *for all $j < \theta$, $\mathbb{P}^j$ is κ-stationarily layered;*
  (3) $\mathbb{P}^\theta$ *is not κ-cc.*

Proof. Using Lemma 2.4(3) (with $S := E_\chi^\kappa$), let us fix $\epsilon < \kappa$ such that, for every $j < \theta$, $\{\beta \in E_\chi^\kappa \mid c(\epsilon, \beta) > j\}$ is stationary. For every $i < \theta$, let

- $\Gamma_i := \{\gamma < \kappa \mid \epsilon < \gamma \text{ and } c(\epsilon, \gamma) \leq i\}$,
- P_i denote the collection of all pairs (i, f) where $f : \kappa \to 2$ is a partial function of size less than χ.

Set $P := \{\emptyset\} \cup \biguplus_{i<\theta} P_i$. Define an ordering $\leq$ of P as follows:

(1) $\emptyset$ is the top element of $(P, \leq)$.
(2) For all $(i, f), (j, g) \in P \setminus \{\emptyset\}$, we let $(j, g) \leq (i, f)$ iff
    - $i = j$,
    - $g \supseteq f$, and
    - for all $\gamma \in \mathrm{dom}(f) \cap \Gamma_i$ and $\alpha \in \gamma \cap \mathrm{dom}(g) \setminus \mathrm{dom}(f)$, if $c(\alpha, \gamma) \leq i$, then $g(\alpha) = 0$.

We then let $\mathbb{P} := (P, \leq)$ and $\mathbb{P}_i := (\{\emptyset\} \cup P_i, \leq)$ for all $i < \theta$.

Claim 6.3.1. $\leq$ *is transitive.*

Proof. Suppose $(i, h) \leq (i, g)$ and $(i, g) \leq (i, f)$. It is clear that $h \supseteq f$. Let $\gamma \in \mathrm{dom}(f) \cap \Gamma_i$ and $\alpha \in \gamma \cap \mathrm{dom}(h) \setminus \mathrm{dom}(f)$ with $c(\alpha, \gamma) \leq i$. Clearly, $\gamma \in \mathrm{dom}(g) \cap \Gamma_i$.

▶ If $\alpha \notin \mathrm{dom}(g)$, then from $(i, h) \leq (i, g)$, it follows that $h(\alpha) = 0$, as sought.

▶ If $\alpha \in \mathrm{dom}(g)$, then from $(i, g) \leq (i, f)$, it follows that $h(\alpha) = g(\alpha) = 0$, as sought. ⊠

Clearly, $\mathbb{P}$ and each of the $\mathbb{P}_i$'s is a well-met poset which is χ-directed closed with greatest lower bounds.

Claim 6.3.2.   (1) $\prod_{i<\theta} \mathbb{P}_i$ *does not satisfy the κ-cc;*
(2) $\mathbb{P}^\theta$ *does not satisfy the κ-cc.*

Proof. (1) For every $\alpha < \kappa$, define $p_\alpha \in \prod_{i<\theta} \mathbb{P}_i$ by setting $p_\alpha(i) := (i, \{(\alpha, 1)\})$ for all $i < \theta$. To see that $\{p_\alpha \mid \alpha < \kappa\}$ is an antichain (of size κ) in $\prod_{i<\theta} \mathbb{P}_i$, fix arbitrary $(\alpha, \gamma) \in [\kappa]^2$. Let $i := \max\{c(\epsilon, \gamma), c(\alpha, \gamma)\}$. Then $p_\alpha(i)$ and $p_\gamma(i)$ are incompatible in $\mathbb{P}_i$, so p_α and p_γ are incompatible in $\prod_{i<\theta} \mathbb{P}_i$.

(2) This follows from Clause (1). ⊠

Claim 6.3.3. *Let $j < \theta$.*

    (1) $\prod_{i<j} \mathbb{P}_i$ *is κ-stationarily layered;*

    (2) *If $\partial(c) \cap E_\chi^\kappa$ is stationary, then $\mathbb{P}^j$ is κ-stationarily layered.*

Proof. Let $\mathbb{Q}$ denote $\prod_{i<j} \mathbb{P}_i$ (resp. $\mathbb{P}^j$). Let Υ be a regular cardinal with $\mathcal{P}_\kappa(\mathbb{Q}) \in H_\Upsilon$. Using the fact that κ is $<\chi$-inaccessible, construct an $\in$-increasing, continuous sequence $\vec{M} = \langle M_\beta \mid \beta < \kappa \rangle$ such that, for all $\beta < \kappa$,

- $M_\beta \prec H_\Upsilon$;
- $|M_\beta| < \kappa$;
- $\kappa, \mathbb{Q} \in M_\beta$;
- $^{<\chi}M_\beta \subseteq M_{\beta+1}$.

    ▶ If $\mathbb{Q} = \prod_{i<j} \mathbb{P}_i$, then, using our choice of ϵ, we fix $\beta \in E_\chi^\kappa$ with $\kappa \cap M_\beta = \beta > \epsilon$ such that $c(\epsilon, \beta) > j$.

    ▶ If $\partial(c) \cap E_\chi^\kappa$ is stationary, then we fix $\beta \in \partial(c) \cap E_\chi^\kappa$ with $\kappa \cap M_\beta = \beta > \epsilon$.

Note that by the continuity of the sequence $\vec{M}$, $^{<\chi}M_\beta \subseteq M_\beta$. Now, by Fact 6.2, it suffices to prove that $\mathbb{Q} \cap M_\beta$ is a regular suborder of $\mathbb{Q}$. To this end, we will define, for each $p \in \mathbb{Q}$, a *reduction* of p to M_β, i.e., a condition $p|M_\beta \in \mathbb{Q} \cap M_\beta$ such that, for all $q \leq_\mathbb{Q} p|M_\beta$ with $q \in M_\beta$, q is compatible with p.

Fix $p \in \mathbb{Q}$. For every $\eta < j$, write $p(\eta)$ as (i_η, f_η), and set $x_\eta := \mathrm{dom}(f_\eta) \cap \Gamma_{i_\eta} \setminus \beta$.

Subclaim 6.3.3.1. *Let $\eta < j$ and $\gamma \in x_\eta$. Then $D_{\leq i_\eta}^c(\gamma) \cap \beta$ is a closed bounded subset of β.*

Proof. As c is closed, it suffices to prove that $\sup(D_{\leq i_\eta}^c(\gamma) \cap \beta) < \beta$. To avoid trivialities, suppose that $\beta \notin \partial(c)$. So, $\mathbb{Q} = \prod_{i<j} \mathbb{P}_i$ and $i_\eta = \eta$. Now, as c is subadditive of the second kind,

$$j < c(\epsilon, \beta) \leq \max\{c(\epsilon, \gamma), c(\beta, \gamma)\}.$$

So, since $c(\epsilon, \gamma) \leq i_\eta = \eta < j$, it follows that $c(\beta, \gamma) \geq j > i_\eta$. As c is closed, it thus follows that $\{\alpha < \beta \mid c(\alpha, \gamma) \leq i_\eta\}$ is bounded below β. ⊟

For every $\eta < j$, by the subclaim and since $|x_\eta| < \chi$,

$$y_\eta := \{\max(D_{\leq i_\eta}^c(\gamma) \cap \beta) \mid \gamma \in x_\eta \ \& \ D_{\leq i_\eta}^c(\gamma) \cap \beta \neq \emptyset\}$$

is a well-defined element of $[\beta]^{<\chi}$.

For each $\eta < j$, define a partial function $g_\eta : \beta \to 2$ by letting

$$g_\eta := (f_\eta \restriction \beta) \cup ((y_\eta \setminus \mathrm{dom}(f_\eta)) \times \{0\}).$$

Clearly, g_η has size less than χ, so as $^{<\chi}M_\beta \subseteq M_\beta$, we have $(i_\eta, g_\eta) \in M_\beta \cap P_{i_\eta}$. Define a condition $p|M_\beta$ in $\mathbb{Q}$ by letting $(p|M_\beta)(\eta) := (i_\eta, g_\eta)$ for all $\eta < j$.

As $^{<\theta}M_\beta \subseteq M_\beta$, we have $p|M_\beta \in M_\beta \cap \mathbb{Q}$. To see that $p|M_\beta$ is a reduction of p to M_β, fix $q \leq_\mathbb{Q} p|M_\beta$ in M_β. The only way that q can fail to be compatible with p is if, for some $\eta < j$, there are $\gamma \in x_\eta$ and $\alpha \in \mathrm{dom}(q(\eta)) \setminus \mathrm{dom}(f_\eta)$ such that $c(\alpha, \gamma) \leq i_\eta$, but $q(\eta)(\alpha) = 1$. So suppose η, α and γ are as described. In

particular, α is smaller than $\gamma_\eta := \max(D^c_{\leq i_\eta}(\gamma) \cap \beta)$. As $\gamma_\eta \in \mathrm{dom}(g_\eta)$, it thus follows that either $\gamma_\eta \notin \Gamma_{i_\eta}$ or $c(\alpha, \gamma_\eta) > i_\eta$. But c is subadditive of the second kind, so $c(\epsilon, \gamma_\eta) \leq \max\{c(\epsilon, \gamma), c(\gamma_\eta, \gamma)\} \leq i_\eta$ and $c(\alpha, \gamma_\eta) \leq \max\{c(\alpha, \gamma), c(\gamma_\eta, \gamma)\} \leq i_\eta$. This is a contradiction. $\boxtimes$ $\square$

Remark 6.4. For infinite regular cardinals $\theta \leq \chi < \kappa$ such that κ is $<\chi$-inaccessible, for any coloring $c : [\kappa]^2 \to \theta$, the poset $\mathbb{P}$ derived from c in the preceding proof has the following features:

(1) $\mathbb{P}$ is well-met and χ-directed closed with greatest lower bounds;
(2) if c witnesses $\mathrm{U}(\kappa, 2, \theta, 2)$, then for all $j < \theta$, $\mathbb{P}^j$ is κ-cc;
(3) if c witnesses $\mathrm{U}(\kappa, \kappa, \theta, 2)$, then for all $j < \theta$, $\mathbb{P}^j$ has precaliber κ;
(4) if c is closed, subadditive of the second kind, and $\partial(c) \cap E^\kappa_\chi$ is stationary, then for all $j < \theta$, $\mathbb{P}^j$ is κ-stationarily layered;
(5) $\mathbb{P}^\theta$ is not κ-cc.

Corollary 6.5. *Suppose that κ is $<\theta$-inaccessible and any one of the following six statements holds.*

(1) $\kappa = \lambda^+$ *and* $\square_\lambda$ *holds.*
(2) $V = L$ *and* κ *is not weakly compact.*
(3) $\square(\kappa)$ *holds after being added generically.*[5]
(4) *There exists a* $\square(\kappa)$*-sequence that avoids a stationary subset of* E^κ_θ.
(5) $\kappa = \lambda^+$ *where* λ *is a former inaccessible that changed its cofinality to* θ *via Prikry/Magidor forcing.*
(6) $\theta = \omega$ *and* $\square^{\mathrm{ind}}(\kappa, \omega)$ *holds after being added generically.*

Then there is a poset $\mathbb{P}$ *such that:*

- $\mathbb{P}$ *is well-met and* θ*-directed closed with greatest lower bounds;*
- *for all* $j < \theta$, $\mathbb{P}^j$ *is* κ*-stationarily layered;*
- $\mathbb{P}^\theta$ *is not* κ*-cc.*

Proof. We will show that any one of the assumptions entails the existence of a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ such that $\partial(c)$ is stationary. Then the second bullet point of Theorem 6.3 will furnish the desired poset $\mathbb{P}$.

Note first that (1) $\implies$ (4) and (2) $\implies$ (4) (cf. [Jen72, Theorem 6.1] or [Dev84, Theorem VII.1.2']). In addition, as established in the proof of Theorem 4.15, (3) $\implies$ (4). Next, by Theorem 4.13(2), if (4) holds, then there is a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \theta, 2)$ for which $\partial(c)$ is stationary.

The fact that Clause (5) implies the desired conclusion is a consequence of Theorem 4.8 and the fact that Prikry forcing and Magidor forcing do not kill the stationarity of the ground model's $E^{\lambda^+}_\lambda$.

It remains to deal with (6). To this end, suppose that $\vec{C} = \langle C_{\alpha,i} \mid \alpha \in \mathrm{acc}(\kappa), i(\alpha) \leq i < \omega \rangle$ is a generically-added $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence, i.e., V is an extension of some ground model by the forcing from [LH17b, §7] to add a $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence, and $\vec{C}$ is the sequence generated by the generic filter. Let

$$S := \{\alpha \in E^\kappa_\omega \mid \forall i \in [i(\alpha), \omega)[\sup(\mathrm{acc}(C_\alpha, i)) < \alpha]\}.$$

By [LHR21, Claim 3.4.1], S is stationary. Let $\Gamma := \mathrm{acc}(\kappa) \setminus S$. We now define an $\boxminus^{\mathrm{ind}}(\kappa, \omega)$-sequence $\vec{D} := \langle D_{\alpha,i} \mid \alpha \in \Gamma, \ j(\alpha) \leq i < \omega \rangle$ as follows. For each

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[5]More precisely, V is a generic extension of some ground model by $\mathbb{S}(\kappa)$, the standard poset for adding a $\square(\kappa)$-sequence; see [LH17a, Definition 2.4] for the definition of $\mathbb{S}(\kappa)$.

$\alpha \in S$, let $j(\alpha)$ be the least $j \in [i(\alpha), \omega)$ such that $\sup(\mathrm{acc}(C_\alpha, i)) = \alpha$ and, for all $j \in [j(\alpha), \omega)$, let $D_{\alpha,j} := \mathrm{acc}(C_{\alpha,j})$. Using our choice of S and the fact that $\vec{C}$ is an $\square^{\mathrm{ind}}(\kappa, \omega)$ sequence, it is straightforward to verify that $\vec{D}$ is an $\boxminus^{\mathrm{ind}}(\kappa, \omega)$-sequence. By the proof of Theorem 4.4 (cf. also Theorem 4.13) and the fact that $\Gamma \cap S = \emptyset$, it follows that there is a closed witness c to $\mathrm{U}^{\mathrm{subadditive}}(\kappa, 2, \omega, 2)$ such that $S \subseteq \partial(c)$. $\square$

Acknowledgments

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