

FORCING A $\square(\kappa)$ -LIKE PRINCIPLE TO HOLD AT A WEAKLY COMPACT CARDINAL

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ABSTRACT. Hellsten [Hel03a] proved that when κ is Π_n^1 -indescribable, the n -club subsets of κ provide a filter base for the Π_n^1 -indescribability ideal, and hence can also be used to give a characterization of Π_n^1 -indescribable sets which resembles the definition of stationarity: a set $S \subseteq \kappa$ is Π_n^1 -indescribable if and only if $S \cap C \neq \emptyset$ for every n -club $C \subseteq \kappa$. By replacing clubs with n -clubs in the definition of $\square(\kappa)$, one obtains a $\square(\kappa)$ -like principle $\square_n(\kappa)$, a version of which was first considered by Brickhill and Welch [BW]. The principle $\square_n(\kappa)$ is consistent with the Π_n^1 -indescribability of κ but inconsistent with the Π_{n+1}^1 -indescribability of κ . By generalizing the standard forcing to add a $\square(\kappa)$ -sequence, we show that if κ is κ^+ -weakly compact and GCH holds then there is a cofinality-preserving forcing extension in which κ remains κ^+ -weakly compact and $\square_1(\kappa)$ holds. If κ is Π_2^1 -indescribable and GCH holds then there is a cofinality-preserving forcing extension in which κ is κ^+ -weakly compact, $\square_1(\kappa)$ holds and every weakly compact subset of κ has a weakly compact proper initial segment. As an application, we prove that, relative to a Π_2^1 -indescribable cardinal, it is consistent that κ is κ^+ -weakly compact, every weakly compact subset of κ has a weakly compact proper initial segment, and there exist two weakly compact subsets S^0 and S^1 of κ such that there is no $\beta < \kappa$ for which both $S^0 \cap \beta$ and $S^1 \cap \beta$ are weakly compact.

1. INTRODUCTION

In this paper, we investigate an incompactness principle $\square_1(\kappa)$, which is closely related to $\square(\kappa)$ but is consistent with weak compactness. Let us begin by recalling the basic facts about $\square(\kappa)$.

The principle $\square(\kappa)$ asserts that there is a κ -length coherent sequence of clubs $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa) \rangle$ that cannot be threaded. For an uncountable cardinal κ , a sequence $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa) \rangle$ of clubs $C_\alpha \subseteq \alpha$ is called *coherent* if whenever β is a limit point of C_α we have $C_\beta = C_\alpha \cap \beta$. Given a coherent sequence \vec{C} , we say that C is a *thread* through \vec{C} if C is a club subset of κ and $C \cap \alpha = C_\alpha$ for every limit point α of C . A coherent sequence \vec{C} is called a $\square(\kappa)$ -sequence if it cannot be threaded, and $\square(\kappa)$ holds if there is a $\square(\kappa)$ -sequence. It is easy to see that $\square(\kappa)$ implies that κ is not weakly compact, and thus $\square(\kappa)$ can be viewed as asserting that κ exhibits a certain amount of incompactness. The principle $\square(\kappa)$ was isolated by Todorćević [Tod87], building on work of Jensen [Jen72], who showed that, if $V = L$, then $\square(\kappa)$ holds for every regular uncountable κ that is not weakly compact.

The natural $\leq \kappa$ -strategically closed forcing to add a $\square(\kappa)$ -sequence [LH14, Lemma 35] preserves the inaccessibility as well as the Mahloness of κ , but kills the weak

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compactness of κ and indeed adds a non-reflecting stationary set. However, if κ is weakly compact, there is a forcing [HLH17] which adds a $\square(\kappa)$ -sequence and also preserves the fact that every stationary subset of κ reflects. Thus, relative to the existence of a weakly compact cardinal, $\square(\kappa)$ is consistent with $\text{Refl}(\kappa)$, the principle that every stationary set reflects. However, $\square(\kappa)$ implies the failure of the simultaneous stationary reflection principle $\text{Refl}(\kappa, 2)$ which states that if S and T are any two stationary subsets of κ , then there is some $\alpha < \kappa$ with $\text{cf}(\alpha) > \omega$ such that $S \cap \alpha$ and $T \cap \alpha$ are both stationary in α . In fact, $\square(\kappa)$ implies that every stationary subset of κ can be partitioned into two stationary sets that do not simultaneously reflect [HLH17, Theorem 2.1].

If κ is a weakly compact cardinal, then the collection of non- Π_1^1 -indescribable subsets of κ forms a natural normal ideal called the Π_1^1 -indescribability ideal:

$$\Pi_1^1(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi_1^1\text{-indescribable}\}.$$

A set $S \subseteq \kappa$ is Π_1^1 -*indescribable* if for every $A \subseteq V_\kappa$ and every Π_1^1 -sentence φ , whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. More generally, a Π_n^1 -indescribable cardinal κ carries the analogously defined Π_n^1 -indescribability ideal. It is natural to ask the question: which results concerning the nonstationary ideal can be generalized to the various ideals associated to large cardinals, such as the Π_n^1 -indescribability ideals? It follows from the work of Sun [Sun93] and Hellsten [Hel03a] that when κ is Π_n^1 -indescribable the collection of *n-club* subsets of κ (see the next section for definitions) is a filter-base for the filter $\Pi_n^1(\kappa)^*$ dual to the Π_n^1 -indescribability ideal, yielding a characterization of Π_n^1 -indescribable sets that resembles the definition of stationarity: when κ is Π_n^1 -indescribable, a set $S \subseteq \kappa$ is Π_n^1 -indescribable if and only if $S \cap C \neq \emptyset$ for every *n-club* $C \subseteq \kappa$. Several recent results have used this characterization ([Hel06], [Hel10], [Cod19] and [CS20]) to generalize theorems concerning the nonstationary ideal to the Π_1^1 -indescribability ideal. For technical reasons discussed below in Section 8, there has been less success with the Π_n^1 -indescribability ideals for $n > 1$. In this article we continue this line of research: by replacing “clubs” with “1-clubs” we obtain a $\square(\kappa)$ -like principle $\square_1(\kappa)$ (see Definition 2.1) that is consistent with weak compactness but not with Π_2^1 -indescribability. Brickhill and Welch [BW] showed that a slightly different version of $\square_1(\kappa)$, which they call $\square^1(\kappa)$, can hold at a weakly compact cardinal in L . See Remark 2.3 for a discussion of the relationship between $\square^1(\kappa)$ and $\square_1(\kappa)$. In this article we consider the extent to which principles such as $\square_1(\kappa)$ can be forced to hold at large cardinals.

We will see that the principle $\square_1(\kappa)$ holds trivially at weakly compact cardinals κ below which stationary reflection fails. (This is analogous to the fact that $\square(\kappa)$ holds trivially for every κ of cofinality ω_1 .) Thus, the task at hand is not just to force $\square_1(\kappa)$ to hold at a weakly compact cardinal, but to show that one can force $\square_1(\kappa)$ to hold at a weakly compact cardinal κ even when stationary reflection holds at many cardinals below κ , so that nontrivial coherence of the sequence is obtained. Recall that when κ is κ^+ -weakly compact (see Definition 5.8 below), the set of weakly compact cardinals below κ is weakly compact and much more, so, in particular, the set of inaccessible $\alpha < \kappa$ at which stationary reflection holds is weakly compact. By [BW, Theorem 3.24], assuming $V = L$, if κ is κ^+ -weakly compact and κ is not Π_2^1 -indescribable then $\square_1(\kappa)$ holds. We show that the same can be forced.

Theorem 1.1. *If κ is κ^+ -weakly compact and the GCH holds, then there is a cofinality-preserving forcing extension in which*

- (1) κ remains κ^+ -weakly compact and
- (2) $\square_1(\kappa)$ holds.

We will also investigate the relationship between $\square_1(\kappa)$ and weakly compact reflection principles. The *weakly compact reflection principle* $\text{Refl}_1(\kappa)$ states that κ is weakly compact and for every weakly compact $S \subseteq \kappa$ there is an $\alpha < \kappa$ such that $S \cap \alpha$ is weakly compact. It is straightforward to see that if κ is Π_2^1 -indescribable, then $\text{Refl}_1(\kappa)$ holds, and if $\text{Refl}_1(\kappa)$ holds, then κ is ω -weakly compact (see [Cod19, Section 2]). However, the following results show that neither of these implications can be reversed. The first author [Cod19] showed that if $\text{Refl}_1(\kappa)$ holds then there is a forcing which adds a non-reflecting weakly compact subset of κ and preserves the ω -weak compactness of κ , hence the ω -weak compactness of κ does not imply $\text{Refl}_1(\kappa)$. The first author and Hiroshi Sakai [CS20] showed that $\text{Refl}_1(\kappa)$ can hold at the least ω -weakly compact cardinal, and hence $\text{Refl}_1(\kappa)$ does not imply the Π_2^1 -indescribability of κ . Just as $\square(\kappa)$ and $\text{Refl}(\kappa)$ can hold simultaneously relative to a weakly compact cardinal, we will prove that $\square_1(\kappa)$ and $\text{Refl}_1(\kappa)$ can hold simultaneously relative to a Π_2^1 -indescribable cardinal; this provides a new consistency result which does not follow from the results in [BW] due to the fact that, if $V = L$, then $\text{Refl}_1(\kappa)$ holds at a weakly compact cardinal if and only if κ is Π_2^1 -indescribable.

Theorem 1.2. *Suppose that κ is Π_2^1 -indescribable and the GCH holds. Then there is a cofinality-preserving forcing extension in which*

- (1) $\square_1(\kappa)$ holds,
- (2) $\text{Refl}_1(\kappa)$ holds and
- (3) κ is κ^+ -weakly compact.

In Section 2, using n -club subsets of κ , we formulate a generalization of $\square_1(\kappa)$ to higher degrees of indescribability. It is easily seen that $\square_n(\kappa)$ implies that κ is not Π_{n+1}^1 -indescribable (see Proposition 2.9 below). However, for technical reasons outlined in Section 8, our methods do not seem to show that $\square_n(\kappa)$ can hold nontrivially (see Definition 2.10) when κ is Π_n^1 -indescribable. Our methods do allow for a generalization of Hellsten's 1-club shooting forcing to n -club shooting, and we also show that, if S is a Π_n^1 -indescribable set, a 1-club can be shot through S while preserving the Π_n^1 -indescribability of all Π_n^1 -indescribable subsets of S .

Finally, we consider the influence of $\square_n(\kappa)$ on *simultaneous* reflection of Π_n^1 -indescribable sets. We let $\text{Refl}_n(\kappa, \mu)$ denote the following simultaneous reflection principle: κ is Π_n^1 -indescribable and whenever $\{S_\alpha : \alpha < \mu\}$ is a collection of Π_n^1 -indescribable sets, there is a $\beta < \kappa$ such that $S_\alpha \cap \beta$ is Π_n^1 -indescribable for all $\alpha < \mu$. In Section 7, we show that for $n \geq 1$, if $\square_n(\kappa)$ holds at a Π_n^1 -indescribable cardinal, then the simultaneous reflection principle $\text{Refl}_n(\kappa, 2)$ fails (see Theorem 7.1). As a consequence, we show that relative to a Π_2^1 -indescribable cardinal, it is consistent that $\text{Refl}_1(\kappa)$ holds and $\text{Refl}_1(\kappa, 2)$ fails (see Corollary 7.4).

2. THE PRINCIPLES $\square_n(\kappa)$

Suppose that κ is a cardinal. A set $S \subseteq \kappa$ is Π_n^1 -*indescribable* if for every $A \subseteq V_\kappa$ and every Π_n^1 -sentence φ , whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. The cardinal κ is said to be Π_n^1 -*indescribable* if κ is

a Π_n^1 -inaccessible subset of κ . The Π_0^1 -inaccessible cardinals are precisely the inaccessible cardinals, and, if κ is inaccessible, then $S \subseteq \kappa$ is Π_0^1 -inaccessible if and only if it is stationary. The Π_1^1 -inaccessible cardinals are precisely the weakly compact cardinals.

The Π_n^1 -*inaccessibility ideal* on κ is

$$\Pi_n^1(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi_n^1\text{-inaccessible}\},$$

the corresponding collection of positive sets is

$$\Pi_n^1(\kappa)^+ = \{X \subseteq \kappa : X \text{ is } \Pi_n^1\text{-inaccessible}\}$$

and the dual filter is

$$\Pi_n^1(\kappa)^* = \{\kappa \setminus X : X \in \Pi_n^1(\kappa)\}.$$

Clearly, if κ is not Π_n^1 -inaccessible, then $\Pi_n^1(\kappa) = P(\kappa)$. Lévy proved [Lév71] that if κ is Π_n^1 -inaccessible, then $\Pi_n^1(\kappa)$ is a nontrivial normal ideal on κ .

A set $C \subseteq \kappa$ is called *0-club* if it is a club. A set $X \subseteq \kappa$ is said to be *n-closed* if it contains all of its Π_{n-1}^1 -inaccessible reflection points: whenever $\alpha < \kappa$ and $X \cap \alpha$ is Π_{n-1}^1 -inaccessible, then $\alpha \in X$ (note that such α must be Π_{n-1}^1 -inaccessible). If a set $C \subseteq \kappa$ is both *n-closed* and Π_{n-1}^1 -inaccessible, then C is said to be an *n-club* subset of κ . For example, $C \subseteq \kappa$ is 1-club if and only if it is stationary and contains all of its inaccessible stationary reflection points (a stationary reflection point that occurs at an inaccessible cardinal), and $C \subseteq \kappa$ is 2-club if and only if it is weakly compact and contains all of its weakly compact reflection points. Building on work of Sun [Sun93], Hellsten showed [Hel03b] that when κ is a Π_n^1 -inaccessible cardinal, a set $S \subseteq \kappa$ is Π_n^1 -inaccessible if and only if $S \cap C \neq \emptyset$ for every *n-club* $C \subseteq \kappa$. Thus, when κ is Π_n^1 -inaccessible, the collection of *n-club* subsets of κ generates the filter $\Pi_n^1(\kappa)^*$. In particular, this implies that *n-club* sets are themselves Π_n^1 -inaccessible.

For $n < \omega$ and $X \subseteq \kappa$, we define the *n-trace* of X to be

$$\text{Tr}_n(X) = \{\alpha < \kappa : X \cap \alpha \in \Pi_n^1(\alpha)^+\}.$$

Notice that when $X = \kappa$, $\text{Tr}_n(\kappa)$ is the set of Π_n^1 -inaccessible cardinals below κ , and in particular $\text{Tr}_0(\kappa)$ is the set of inaccessible cardinals less than κ . For uniformity of notation, let us say that an ordinal α is Π_{n-1}^1 -*inaccessible* if it is a limit ordinal, and if α is a limit ordinal, $S \subseteq \alpha$ is Π_{n-1}^1 -*inaccessible* if it is unbounded in α . Thus, if $X \subseteq \kappa$, then $\text{Tr}_{-1}(X) = \{\alpha < \kappa : \sup(X \cap \alpha) = \alpha\}$.

Definition 2.1. Suppose $n < \omega$ and $\text{Tr}_{n-1}(\kappa)$ is cofinal in κ . A sequence $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ is called a *coherent sequence of n-clubs* if

- (1) for all $\alpha \in \text{Tr}_{n-1}(\kappa)$, C_α is an *n-club* subset of α and
- (2) for all $\alpha < \beta$ in $\text{Tr}_{n-1}(\kappa)$, $C_\beta \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$ implies $C_\alpha = C_\beta \cap \alpha$.

We say that a set $C \subseteq \kappa$ is a *thread* through a coherent sequence of *n-clubs*

$$\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$$

if C is *n-club* and for all $\alpha \in \text{Tr}_{n-1}(\kappa)$, $C \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$ implies $C_\alpha = C \cap \alpha$. A coherent sequence of *n-clubs* $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ is called a $\square_n(\kappa)$ -*sequence* if there is no thread through \vec{C} . We say that $\square_n(\kappa)$ holds if there is a $\square_n(\kappa)$ -sequence $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$.

Remark 2.2. Note that $\square_0(\kappa)$ is simply $\square(\kappa)$. For $n = 1$, the principle $\square_1(\kappa)$ states that there is a coherent sequence of 1-clubs

$$\langle C_\alpha : \alpha < \kappa \text{ is inaccessible} \rangle$$

that cannot be threaded.

Remark 2.3. Before we prove some basic results about $\square_n(\kappa)$, let us consider a similar principle due to Brickhill and Welch [BW]. First, let us note that the notion of n -club, for all n , used in [BW] was first introduced by Beklemishev [Bek09] in 2009 (see also [BG14]) in the context of Generalized Provability Logics, and independently introduced and studied in full generality for all ordinals by Bagaria [Bag12] in 2012 (see [Bag19] for more information on the history of the notion).

We will consider the Brickhill-Welch principle in detail in the case $n = 1$. We will refer to the notion of 1-club used in [BW] as *strong 1-club* in order to avoid confusion. A set $C \subseteq \kappa$ is a *strong 1-club* if it is stationary in κ and whenever $C \cap \alpha$ is stationary in α then $\alpha \in C$. This notion of strong 1-club is precisely the same notion considered in [Sun93]. Thus, it follows from the results of [Sun93] that when κ is weakly Π_1^1 -indescribable¹ the collection of strong 1-clubs generates the weak Π_1^1 -indescribable filter. Moreover, when κ is weakly compact, the collection of strong 1-club subsets of κ generates the weakly compact filter $\Pi_1^1(\kappa)$. For $S \subseteq \kappa$, Brickhill and Welch [BW] define $\square^1(\kappa)$ to be the principle asserting the existence of a sequence $\langle C_\alpha : \text{cf}(\alpha) > \omega \rangle$ such that

- (1) C_α is a strong 1-club in α ,
- (2) whenever $C_\alpha \cap \beta$ is stationary in β we have $C_\beta = C_\alpha \cap \beta$ and
- (3) there is no $C \subseteq \kappa$ that is strong 1-club in κ such that whenever $C \cap \alpha$ is stationary in α we have $C_\alpha = C \cap \alpha$.

Such a sequence is called a $\square^1(\kappa)$ -sequence. Let us show that when κ is weakly compact, the Brickhill-Welch principle $\square^1(\kappa)$ implies our principle $\square_1(\kappa)$. Suppose $\langle C_\alpha : \text{cf}(\alpha) > \omega \rangle$ is a $\square^1(\kappa)$ -sequence. Clearly, $\langle C_\alpha : \alpha \in \text{Tr}_0(\kappa) \rangle$ is a coherent sequence of 1-clubs. For the sake of contradiction, suppose $C \subseteq \kappa$ is a 1-club thread through $\langle C_\alpha : \alpha \in \text{Tr}_0(\kappa) \rangle$. Using the coherence of $\langle C_\alpha : \text{cf}(\alpha) > \omega \rangle$ it is straightforward to check that C is a strong 1-club and whenever $C \cap \alpha$ is stationary in α we have $C_\alpha = C \cap \alpha$. This contradicts $\square^1(\kappa)$. Thus $\square^1(\kappa)$ implies $\square_1(\kappa)$. It is not known whether $\square_1(\kappa)$ implies $\square^1(\kappa)$.

Brickhill and Welch also generalized their definition to obtain the principles $\square^n(\kappa)$, and again it is not difficult to see that $\square^n(\kappa)$ implies our principle $\square_n(\kappa)$.

Generalizing the fact that $\square(\kappa)$ implies κ is not weakly compact, let us show that $\square_n(\kappa)$ implies κ is not Π_{n+1}^1 -indescribable. To do this, we first recall the Hauser characterization of Π_n^1 -indescribability.

We say that a transitive model $\langle M, \in \rangle$ is a κ -*model* if $|M| = \kappa$, $\kappa \in M$, $M^{<\kappa} \subseteq M$, and $M \models \text{ZFC}^-$ (ZFC without the power set axiom). It is not difficult to see that if κ is inaccessible, then V_κ is an element of every κ -model M .

Definition 2.4 (Hauser [Hau91]). Suppose κ is inaccessible. For $n \geq 0$, a κ -model N is Π_n^1 -*correct at κ* if and only if

$$V_\kappa \models \varphi \iff (V_\kappa \models \varphi)^N$$

¹Recall that a set $S \subseteq \kappa$ is *weakly Π_1^1 -indescribable* if for all $A \subseteq \kappa$ and all Π_1^1 sentences φ , $(\kappa, \in, A) \models \varphi$ implies that there is and $\alpha \in S$ such that $(\alpha, \in, A \cap \alpha) \models \varphi$.

for all Π_n^1 -formulas φ whose parameters are contained in $N \cap V_{\kappa+1}$.

Remark 2.5. Notice that every κ -model is Π_0^1 -correct at κ .

Theorem 2.6 (Hauser [Hau91]). *The following statements are equivalent for every inaccessible cardinal κ , every subset $S \subseteq \kappa$, and all $0 < n < \omega$.*

- (1) S is Π_n^1 -indescribable.
- (2) For every κ -model M with $S \in M$, there is a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with $\text{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
- (3) For every $A \subseteq \kappa$ there is a κ -model M with $A, S \in M$ for which there is a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with $\text{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
- (4) For every $A \subseteq \kappa$ there is a κ -model M with $A, S \in M$ for which there is a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with $\text{crit}(j) = \kappa$ such that $\kappa \in j(S)$ and $j, M \in N$.

Lemma 2.7. *Suppose κ is a cardinal. If $S \in \Pi_n^1(\kappa)^+$ and $S_\alpha \in \Pi_n^1(\alpha)^+$ for each $\alpha \in S$, then $\bigcup_{\alpha \in S} S_\alpha \in \Pi_n^1(\kappa)^+$.*

Proof. Fix an n -club C in κ . The set $\text{Tr}_{n-1}(C)$ is n -closed because if $\text{Tr}_{n-1}(C) \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$, then $C \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$ since, by n -closure of C , $\text{Tr}_{n-1}(C) \subseteq C$. Also, $\text{Tr}_{n-1}(C)$ meets every n -club D because the intersection $C \cap D$ is an n -club. Thus, $\text{Tr}_{n-1}(C)$ is an n -club. It follows that there is an $\alpha \in S \cap \text{Tr}_{n-1}(C)$. Since S_α is Π_n^1 -indescribable in α and $C \cap \alpha$ is an n -club in α , we have $S_\alpha \cap C \cap \alpha \neq \emptyset$, and hence $(\bigcup_{\alpha \in S} S_\alpha) \cap C \neq \emptyset$. \square

A simple complexity calculation shows that for every $n < \omega$, there is a Π_{n+1}^1 -formula $\chi_n(X)$ such that for every κ and every $S \subseteq \kappa$, $(V_\kappa, \in) \models \chi_n(S)$ if and only if S is Π_n^1 -indescribable (see [Kan03, Corollary 6.9]). It therefore follows that there is a Π_n^1 -formula $\psi_n(X)$ such that for every κ and every $C \subseteq \kappa$, $(V_\kappa, \in) \models \psi_n(C)$ if and only if C is an n -club subset of κ . Thus, in particular, a Π_n^1 -correct model N is going to be correct about Π_{n-1}^1 -indescribable sets as well as n -clubs.

Corollary 2.8. *Suppose κ is Π_n^1 -indescribable. If $S \in \Pi_n^1(\kappa)^+$, then*

$$\text{Tr}_{n-1}(S) = \{\alpha < \kappa : S \cap \alpha \in \Pi_{n-1}^1(\alpha)^+\}$$

is an n -club.

Proof. Suppose S is Π_n^1 -indescribable. First, let us argue that $\text{Tr}_{n-1}(S)$ is Π_n^1 -indescribable. Let M be a κ -model with $S, \text{Tr}_{n-1}(S) \in M$ and let $j : M \rightarrow N$ be an elementary embedding with critical point κ such that N is Π_{n-1}^1 -correct and $\kappa \in j(S)$. The Π_{n-1}^1 -correctness of N implies that $j(S) \cap \kappa = S$ is a Π_{n-1}^1 -indescribable subset of κ in N . Thus, $\kappa \in j(\text{Tr}_{n-1}(S))$. Hence $\text{Tr}_{n-1}(S)$ is Π_n^1 -indescribable.

It remains to show that $\text{Tr}_{n-1}(S)$ is n -closed, which is equivalent to showing that if $\text{Tr}_{n-1}(S) \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$, then $S \cap \alpha \in \Pi_{n-1}^1(\alpha)^+$. More generally, observe that if $X \subseteq \alpha$ and $\text{Tr}_m(X)$ is Π_m^1 -indescribable, then $X = \bigcup_{\beta \in \text{Tr}_m(X)} X \cap \beta$ must be Π_m^1 -indescribable by Lemma 2.7. \square

Proposition 2.9. *For every $n < \omega$, $\square_n(\kappa)$ implies that κ is not Π_{n+1}^1 -indescribable.*

Proof. Suppose $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_n(\kappa) \rangle$ is a $\square_n(\kappa)$ -sequence and κ is Π_{n+1}^1 -indescribable. Let M be a κ -model with $\vec{C} \in M$. Since κ is Π_{n+1}^1 -indescribable, we may let $j : M \rightarrow N$ be an elementary embedding with critical point κ and a Π_n^1 -correct N as in Theorem 2.6 (2). By elementarity, it follows that $j(\vec{C}) = \langle \bar{C}_\alpha : \alpha \in \text{Tr}_n^N(j(\kappa)) \rangle$ is a $\square_n(j(\kappa))$ -sequence in N . Since N is Π_n^1 -correct, we know that $\kappa \in \text{Tr}_n^N(j(\kappa))$ and \bar{C}_κ must also be n -club in V . Since $j(\vec{C})$ is a $\square_n(j(\kappa))$ -sequence in N , it follows that for every Π_n^1 -indescribable $\alpha < \kappa$ if $\bar{C}_\kappa \cap \alpha \in \Pi_n^1(\alpha)^+$, then $\bar{C}_\kappa \cap \alpha = C_\alpha$, and hence \bar{C}_κ is a thread through \vec{C} , a contradiction. \square

Let us now describe the sense in which $\square_n(\kappa)$ can hold trivially when κ is Π_n^1 -indescribable and certain reflection principles fail often below κ .

Definition 2.10. Suppose $n < \omega$ and $\text{Tr}_{n-1}(\kappa)$ is cofinal in κ . We say that $\square_n(\kappa)$ holds trivially if there is a $\square_n(\kappa)$ -sequence $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ and a club $E \subseteq \kappa$ such that for all $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n -club subset of α in the sense that C_α is a Π_{n-1}^1 -indescribable subset of α and has no Π_{n-1}^1 -indescribable proper initial segment (see Remark 2.12 below for the reason for adopting this notion of triviality).

Notice that $\square(\kappa)$ holds trivially if $\text{cf}(\kappa) = \omega_1$. In this case we can find a club $E \subseteq \kappa$ consisting of ordinals of countable cofinality. For all $\alpha \in E$, we can let C_α be a cofinal subset of α of order type ω . Then, for every limit ordinal $\beta \in \kappa \setminus E$, we can let $\alpha_\beta = 0$ if $\beta < \min(E)$ and $\alpha_\beta = \max(E \cap \beta)$ otherwise, and set C_β to be the interval (α_β, β) . It is easily verified that a sequence thus defined is a $\square(\kappa)$ -sequence.

Recall that the principle $\text{Refl}_n(\kappa)$ holds if and only if κ is Π_n^1 -indescribable and for every Π_n^1 -indescribable subset X of κ , there is an $\alpha < \kappa$ such that $X \cap \alpha$ is Π_n^1 -indescribable (see [Cod19] and [CS20] for more details).

Proposition 2.11. *Suppose $1 \leq n < \omega$ and κ is Π_n^1 -indescribable. Then $\square_n(\kappa)$ holds trivially if and only if there is a club $E \subseteq \kappa$ such that $\neg \text{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$.*

Proof. If $\square_n(\kappa)$ holds trivially, then there is a $\square_n(\kappa)$ -sequence $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ and a club $E \subseteq \kappa$ such that for $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$, C_α is a Π_{n-1}^1 -indescribable set with no Π_{n-1}^1 -indescribable initial segment, in which case C_α is a witness to the fact that $\text{Refl}_{n-1}(\alpha)$ fails.

Conversely, suppose that $E \subseteq \kappa$ is a club and $\neg \text{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$. For each $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$, let C_α be a Π_{n-1}^1 -indescribable subset of α which has no Π_{n-1}^1 -indescribable proper initial segment. Then each C_α is trivially n -club in α . For all $\beta \in \text{Tr}_{n-1}(\kappa) \setminus E$, let $\alpha_\beta = \max(E \cap \beta)$, and let C_β be the interval (α_β, β) . Then $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ is easily seen to be a coherent sequence of n -clubs, since there are no points at which coherence needs to be checked for indices in E and coherence is easily checked for indices outside of E because of the uniformity of the definition. We must argue that \vec{C} has no thread. Suppose there is a thread $C \subseteq \kappa$ through \vec{C} . Since κ is Π_n^1 -indescribable and C is an n -club subset of κ it follows, by Corollary 2.8, that $\text{Tr}_{n-1}(C)$ is an n -club in κ . Thus we can choose $\alpha, \beta \in \text{Tr}_{n-1}(C) \cap E$ with $\alpha < \beta$. Since C is a thread we have $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$, which contradicts the fact that C_β has no Π_{n-1}^1 -indescribable proper initial segment. This shows that $\square_n(\kappa)$ holds trivially. \square

Remark 2.12. It seems like it might be more optimal to change Definition 2.10 to instead say that $\square_n(\kappa)$ holds trivially if there is a $\square_n(\kappa)$ -sequence \vec{C} and an n -club $E \subseteq \kappa$ such that for all $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n -club subset of α . However, we were not able to prove the analogue of Proposition 2.11 corresponding to this alternative definition, namely that $\square_n(\kappa)$ holds trivially if and only if there is an n -club $E \subseteq \kappa$ such that $\neg \text{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \text{Tr}_{n-1}(\kappa) \cap E$.

Corollary 2.13. *In L , if κ is the least Π_n^1 -indescribable cardinal, then $\square_n(\kappa)$ holds trivially.*

Proof. Generalizing a result of Jensen [Jen72], Bagaria, Magidor and Sakai proved [BMS15] that in L a cardinal κ is Π_n^1 -indescribable if and only if $\text{Refl}_{n-1}(\kappa)$ holds. Suppose $V = L$ and κ is the least Π_n^1 -indescribable cardinal. Then $\text{Refl}_{n-1}(\alpha)$ fails for all $\alpha < \kappa$. Hence by Proposition 2.11, $\square_n(\kappa)$ holds trivially. \square

Brickhill and Welch showed more generally that in L , if κ is Π_n^1 -indescribable and not Π_{n+1}^1 -indescribable, then their principle $\square^n(\kappa)$ holds. Since L can have Π_n^1 -indescribable, but not Π_{n+1}^1 -indescribable, cardinals below which, for instance, the set of Π_{n-1}^1 -indescribable cardinals is Π_n^1 -indescribable, it follows from reasonable assumptions that in L our principle $\square_n(\kappa)$ can hold nontrivially at a Π_n^1 -indescribable cardinal. We do not know how to force $\square_n(\kappa)$ to hold non-trivially at a Π_n^1 -indescribable cardinal.

Another consequence of Proposition 2.11 is that we can force $\square_1(\kappa)$ to hold *trivially* at a Π_1^1 -indescribable cardinal by killing certain stationary reflection principles below κ .

Recall that a partial order \mathbb{P} is said to be α -*strategically closed*, for an ordinal α , if Player II has a winning strategy in the following two-player game $\mathcal{G}_\alpha(\mathbb{P})$ of perfect information. In a run of $\mathcal{G}_\alpha(\mathbb{P})$, the two players take turns playing elements of a decreasing sequence $\langle p_\beta : \beta < \alpha \rangle$ of conditions from \mathbb{P} . Player I plays at all odd ordinal stages, and Player II plays at all even ordinal stages (in particular, at limits). Player II goes first and must play $\mathbb{1}_\mathbb{P}$ on their first move. Player I wins if there is a limit ordinal $\gamma < \alpha$ such that $\langle p_\beta : \beta < \gamma \rangle$ has no lower bound (i.e., if Player II is unable to play at stage γ). If the game continues successfully for α -many moves, then Player II wins. Clearly, for a cardinal α , if \mathbb{P} is α -strategically closed, then \mathbb{P} is $<\alpha$ -distributive, and hence adds no new $<\alpha$ -sequences of ground model sets.

We will use the following general proposition about indestructibility of weakly compact cardinals.

Definition 2.14. Suppose κ is an inaccessible cardinal. We say that a forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$$

is *good* if it has Easton support and, for all $\alpha < \kappa$, if α is inaccessible, then \dot{Q}_α is a \mathbb{P}_α -name for a poset such that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{Q}_\alpha \in \dot{V}_\kappa$, where \dot{V}_κ is a \mathbb{P}_α -name for $(V_\kappa)^{V^{\mathbb{P}_\alpha}}$ and, otherwise, \dot{Q}_α is a \mathbb{P}_α -name for trivial forcing.

If \mathbb{P}_κ is a good iteration, then we can argue by induction on α that $\mathbb{P}_\alpha \in V_\kappa$ for every $\alpha < \kappa$. This is because, if $\mathbb{P}_\alpha \in V_\kappa$ and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \dot{Q}_\alpha \in \dot{V}_\kappa$, then $\mathbb{P}_\alpha * \dot{Q}_\alpha \in V_\kappa$. The following standard proposition about good iterations can be found, for example, in [Cum10].

Proposition 2.15. *Suppose κ is a Mahlo cardinal. Then a good iteration \mathbb{P}_κ has size κ and is κ -c.c.*

Lemma 2.16. *Suppose κ is weakly compact and $\langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is a good iteration which at non-trivial stages α has $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{Q}_\alpha \text{ is } \alpha\text{-strategically closed”}$, and let G be \mathbb{P} -generic over V . Then κ remains weakly compact in $V[G]$.*

Proof. By Proposition 2.15, we can assume without loss that $\mathbb{P} \subseteq V_\kappa$. Since κ is weakly compact, there are κ -models M and N with $A \in M$ for which there is an elementary embedding $j : M \rightarrow N$ with critical point κ . A nice-name counting argument, using the κ -c.c. and the fact that the tails of the forcing iteration are eventually α -distributive for every $\alpha < \kappa$, shows that κ is inaccessible in $V[G]$.

Suppose $A \in P(\kappa)^{V[G]}$ and let $\dot{A} \in H(\kappa^+)^V$ be a \mathbb{P}_κ -name such that $\dot{A}_G = A$. Let M be a κ -model with $\mathbb{P}_\kappa, \dot{A} \in M$ for which there are a κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ . Since $N^{<\kappa} \cap V \subseteq N$, we have $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{Q}_\kappa * \dot{\mathbb{P}}_{\kappa, j(\kappa)}$, where N believes that $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash \text{“}\dot{Q}_\kappa \text{ is } \kappa\text{-strategically closed”}$, and $\dot{\mathbb{P}}_{\kappa, j(\kappa)}$ is a $\mathbb{P}_\kappa * \dot{Q}_\kappa$ -name for N 's version of the tail of the iteration $j(\mathbb{P}_\kappa)$ of length $j(\kappa)$. By the generic closure criterion (Lemma 3.2), since \mathbb{P}_κ has the κ -c.c., $N[G]$ is a κ -model in $V[G]$. The poset $(\dot{Q}_\kappa * \dot{\mathbb{P}}_{\kappa, j(\kappa)})_G$ is κ -strategically closed in $N[G]$, so, by diagonalizing, we can build an $N[G]$ -generic filter $H * G' \in V[G]$ for $(\dot{Q}_\kappa * \dot{\mathbb{P}}_{\kappa, j(\kappa)})_G$. Since conditions in \mathbb{P}_κ have supports of size less than the critical point of j we have $j \restriction G \subseteq \hat{G} =_{\text{def}} G * H * G'$. Thus j lifts to $j : M[G] \rightarrow N[\hat{G}]$. Since $A = \dot{A}_G \in M[G]$, this shows that κ remains weakly compact in $V[G]$. \square

Proposition 2.17. *If κ is Π_1^1 -indescribable (weakly compact), then there is a forcing extension in which $\square_1(\kappa)$ holds trivially and κ remains Π_1^1 -indescribable.*

Proof. For regular $\alpha > \omega$, let \mathbb{S}_α denote the usual forcing to add a nonreflecting stationary subset of $\alpha \cap \text{cof}(\omega)$ (see Example 6.5 in [Cum10]). Recall that conditions in \mathbb{S}_α are bounded subsets p of $\alpha \cap \text{cof}(\omega)$ such that for every $\beta \leq \sup(p)$ with $\text{cf}(\beta) > \omega$, the set $p \cap \beta$ is nonstationary in β . It is not difficult to see that the poset \mathbb{S}_α is α -strategically closed.

Now we let $\langle \mathbb{P}_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton-support iteration of length κ such that if $\alpha < \kappa$ is inaccessible, then \dot{Q}_α is a \mathbb{P}_α -name for $\mathbb{S}_\alpha^{V^{\mathbb{P}_\alpha}}$, and otherwise \dot{Q}_α is a \mathbb{P}_α -name for trivial forcing.

Suppose G is generic for \mathbb{P}_κ over V . By Lemma 2.16, since \mathbb{P}_κ has all the right properties, κ remains weakly compact in $V[G]$. Also, in $V[G]$, for each inaccessible $\alpha < \kappa$, by a routine genericity argument and the fact that the tail of the forcing iteration from stage $\alpha + 1$ to κ is α^+ -strategically closed, the stage α generic H_α obtained from G yields a nonreflecting stationary subset of α : $S_\alpha = \bigcup H_\alpha$. Thus in $V[G]$, $\text{Refl}_0(\alpha)$ fails for all inaccessible $\alpha < \kappa$, and hence $\square_1(\kappa)$ holds trivially by Proposition 2.11. \square

In Section 5 we will show that $\square_1(\kappa)$ can hold non-trivially at a weakly compact cardinal.

3. PRESERVING Π_n^1 -INDESCRIBABILITY BY FORCING

In this section, we will provide some results to be used in indestructibility arguments for Π_n^1 -indescribable cardinals in later sections.

The following two folklore lemmas (and their variants) are often used in indestructibility arguments for large cardinals characterized by the existence of elementary embeddings between κ -models (see [Cum10, Proposition 8.3–8.4] for more details).

Lemma 3.1 (Ground closure criterion). *Suppose κ is a cardinal, M is a κ -model, $\mathbb{P} \in M$ is a forcing notion, and $G \in V$ is generic for \mathbb{P} over M . Then $M[G]$ is a κ -model.*

Lemma 3.2 (Generic closure criterion). *Suppose κ is a cardinal, M is a κ -model, $\mathbb{P} \in M$ is a forcing notion with the κ -c.c., and G is generic for \mathbb{P} over V . Then $M[G]$ is a κ -model in $V[G]$.*

Lemma 3.3. *Suppose κ is inaccessible, \mathbb{P} is a κ -strategically closed forcing and G is generic for \mathbb{P} over V . Then $(V_\kappa, \in, A) \models \forall X \psi(X, A)$ implies $((V_\kappa, \in, A) \models \forall X \psi(X, A))^{V[G]}$ for all $A \in V_{\kappa+1}^V$ and all first order ψ .*

Proof. First, observe that since \mathbb{P} is $<\kappa$ -distributive, κ remains inaccessible in $V[G]$ and $V_\kappa = V_\kappa^{V[G]}$. Suppose towards a contradiction that $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, but for some $B \subseteq V_\kappa$ in $V[G]$, $(V_\kappa, \in, A) \models \neg \psi(B, A)$. Let \hat{B} be a \mathbb{P} -name for B . Since κ is inaccessible in $V[G]$, the set

$$C = \{\alpha < \kappa : (V_\alpha, \in, A \cap \alpha, B \cap \alpha) \models \neg \psi(B \cap \alpha, A \cap \alpha)\}$$

contains a club in $V[G]$. Let \hat{C} be a \mathbb{P} -name for such a club. In V , we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with the names \hat{B} and \hat{C} to build \hat{B} and \hat{C} such that \hat{C} is club in κ and for each $\alpha \in \hat{C}$ we have

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \neg \psi(\hat{B} \cap V_\alpha, A \cap V_\alpha).$$

Since $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, we have $(V_\kappa, \in, A, \hat{B}) \models \psi(\hat{B}, A)$, and since κ is inaccessible, the set

$$\{\alpha < \kappa : (V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap \alpha, A \cap \alpha)\}$$

contains a club. Thus, there is an $\alpha \in \hat{C}$ such that

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap V_\alpha, A \cap V_\alpha),$$

a contradiction. \square

Corollary 3.4. *Suppose κ is inaccessible, \mathbb{P} is a κ -strategically closed forcing notion and G is generic for \mathbb{P} over V . If N is a Π_1^1 -correct κ -model in V , then N remains a Π_1^1 -correct κ -model in $V[G]$.*

Proof. Clearly N remains a κ -model because \mathbb{P} is $<\kappa$ -distributive. Let φ be a Π_1^1 -statement, and suppose first that $(V_\kappa \models \varphi)^N$. By Π_1^1 -correctness, $V_\kappa \models \varphi$, and so by Lemma 3.3, $(V_\kappa \models \varphi)^{V[G]}$. On the other hand, if $(V_\kappa \models \neg \varphi)^N$, then there is a $B \subseteq V_\kappa$ in N witnessing this failure. Since N , V , and $V[G]$ all have the same V_κ , B witnesses the failure of φ in both V and $V[G]$ as well, so $(V_\kappa \models \neg \varphi)^{V[G]}$. \square

Proposition 3.5. *Suppose κ is inaccessible, \mathbb{P} is κ -strategically closed, and G is generic for \mathbb{P} over V . If $S \in P(\kappa)^V$ is Π_1^1 -indescribable in $V[G]$, then S is Π_1^1 -indescribable in V .*

Proof. Suppose towards a contradiction that there is $S \in P(\kappa)^V$ that is Π_1^1 -indescribable in $V[G]$ but not Π_1^1 -indescribable in V . In V , find a subset $A \subseteq V_\kappa$ and a Π_1^1 statement $\varphi = \forall X \psi(X, A)$ such that $(V_\kappa, \in, A) \models \varphi$ and for all $\alpha \in S$ we have $(V_\alpha, \in, A \cap V_\alpha) \models \neg \varphi$. Since \mathbb{P} is $<\kappa$ -distributive, V and $V[G]$ have the same V_κ , so it follows that in $V[G]$, by the Π_1^1 -indescribability of S , it must be the case that $(V_\kappa, \in, A) \models \exists X \neg \psi(X, A)$. Working in $V[G]$, we fix $B \subseteq V_\kappa$ such that $(V_\kappa, \in, A) \models \neg \psi(B, A)$ and observe that the set

$$C = \{\alpha < \kappa : V_\alpha \models \neg \psi(B \cap V_\alpha, A \cap V_\alpha)\}$$

contains a club. Let \dot{C} be a \mathbb{P} -name for such a club, and let \dot{B} be a \mathbb{P} -name for B . In V , we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with \dot{B} and \dot{C} to build \hat{B} and \hat{C} such that $\hat{C} \subseteq \hat{\kappa}$ is club and $\forall \alpha \in \hat{C}$, $V_\alpha \models \neg \psi(\hat{B} \cap V_\alpha, A \cap V_\alpha)$. But this implies that $V_\kappa \models \neg \psi(\hat{B}, A)$, a contradiction. \square

Hamkins showed [Ham98] that if κ is weakly compact, any forcing of size less than κ will produce an extension in which κ is *superdestructible* in the sense that any further $<\kappa$ -closed forcing which adds a subset of κ will kill the weak compactness of κ . Thus, the converse of Proposition 3.5 is clearly false because the forcing $\text{Add}(\kappa, 1)$ to add a Cohen subset to κ with bounded conditions can destroy the weak compactness of κ and it is $<\kappa$ -closed and therefore κ -strategically closed. We will see in Section 6 (Remark 6.4) that Proposition 3.5 can fail for Π_2^1 -indescribable sets.

A good iteration \mathbb{P}_κ of length κ is said to be *progressively closed* if for every $\alpha < \kappa$, there is $\alpha \leq \beta_\alpha < \kappa$ such that every stage after β_α is forced to be α -strategically closed. In this case, it is not difficult to see that $\mathbb{P}_{\beta_\alpha}$ forces that the tail of the iteration is α -strategically closed. Next, we will show that good progressively closed κ -length iterations preserve Π_n^1 -correctness of κ -models.

Let \mathbb{P} be a forcing notion and suppose σ is a \mathbb{P} -name. Recall that τ is a *nice name for a subset of σ* if

$$\tau = \bigcup (\{\pi\} \times A_\pi : \pi \in \text{dom}(\sigma)\},$$

where each A_π is an antichain of \mathbb{P} . It is well known and easy to verify that for every \mathbb{P} -name μ , there is a nice name τ for a subset of σ such that $\mathbb{1}_\mathbb{P} \Vdash \mu \subseteq \sigma \rightarrow \mu = \tau$. We call such τ the *nice replacement for μ* .

Lemma 3.6. *Suppose σ is a \mathbb{P} -name and $n \geq 0$. Let X_σ be the set of nice names for subsets of σ , let p be a condition in \mathbb{P} and let φ be any Π_n -assertion in the forcing language of the form*

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \dots, x_n).$$

Then $p \Vdash \varphi$ if and only if

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \dots, \tau_n).$$

The analogous statement holds for Σ_n -assertions in the forcing language.

Proof. We will prove the lemma simultaneously for Π_n and Σ_n statements by induction on n . Clearly the lemma holds for $n = 0$. Assume inductively that the lemma holds for some n , and suppose φ is an assertion in the forcing language of complexity Π_{n+1} . Let

$$\varphi = (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \dots, x_{n+1}) = (\forall x_1 \subseteq \sigma) \bar{\varphi}(x_1),$$

where ψ is Δ_0 and $\bar{\varphi}(x)$ is Σ_n . For the forward direction, clearly $p \Vdash \varphi$ implies $(\forall \tau_1 \in X_\sigma)(p \Vdash \bar{\varphi}(\tau_1))$. By the inductive hypothesis applied to $p \Vdash \bar{\varphi}(\tau_1)$, we conclude that $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \dots, \tau_n)$. For the converse, suppose $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \dots, \tau_n)$ holds. Let us argue that $p \Vdash (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \dots, x_n)$. If not, there is some $q \leq p$ and some \mathbb{P} -name μ for a subset of σ such that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg \psi(\mu, x_2, \dots, x_n)$. Let τ be a nice replacement for μ so that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg \psi(\tau, x_2, \dots, x_n)$, or in other words, $q \Vdash \neg \bar{\varphi}(\tau)$. By assumption $(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau, \tau_2, \dots, \tau_n)$, so applying the inductive hypothesis, we obtain $p \Vdash (\exists x_2 \subseteq \sigma) \cdots \psi(\tau, x_2, \dots, x_n)$ and hence $p \Vdash \bar{\varphi}(\tau)$, a contradiction. The proof of the lemma for Σ_{n+1} statements is similar. \square

Theorem 3.7. *Suppose κ is a Mahlo cardinal, N is a Π_n^1 -correct κ -model and $\mathbb{P} \in N$ is a progressively closed good Easton-support iteration of length κ . If $G \subseteq \mathbb{P}$ is generic over V , then $N[G]$ is a Π_n^1 -correct κ -model in $V[G]$.*

Proof. By Proposition 2.15, \mathbb{P} has the κ -c.c. and without loss of generality $\mathbb{P} \subseteq V_\kappa$. Thus, by the generic closure criterion Lemma 3.2, $N[G]$ remains a κ -model in $V[G]$. By the progressive closure of the iteration, $V_\kappa^{V[G]} = V_\kappa[G]$. Thus, $V_\kappa^{N[G]} = V_\kappa^{V[G]}$. Let $\sigma \in N$ be a \mathbb{P} -name such that $\sigma_G = V_\kappa^{N[G]} = V_\kappa^{V[G]}$ and $\text{dom}(\sigma) \subseteq V_\kappa$.

Let us argue that $N[G]$ is Π_n^1 -correct. Suppose $(V_\kappa^{N[G]}, \in, A) \models \varphi$ in $N[G]$, where

$$\varphi = \forall X_1 \exists X_2 \cdots \psi(X_1, \dots, X_n, A)$$

is Π_n^1 and all quantifiers appearing in ψ are first-order over $V_\kappa^{N[G]}$. Let \dot{A} be a \mathbb{P} -name for A such that $\text{dom}(\dot{A}) \subseteq V_\kappa$. Let $\bar{\psi}(x_1, \dots, x_n, \dot{A})$ be a formula in the forcing language obtained from ψ by replacing all parameters with \mathbb{P} -names and all first-order quantifiers “ Qx ” with “ $Qx \in \sigma$ ” for $Q = \forall, \exists$. Let $\bar{\varphi}(\sigma, \dot{A})$ denote the following formula in the forcing language:

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \bar{\psi}(x_1, \dots, x_n, \dot{A}).$$

Since $(V_\kappa^{N[G]}, \in, A) \models \varphi$ holds in $N[G]$, it follows that $N[G] \models \bar{\varphi}(\sigma_G, \dot{A}_G)$. Thus, we may choose $p \in G$ with $(p \Vdash \bar{\varphi}(\sigma, \dot{A}))^N$. By Lemma 3.6,

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \dots, \tau_n, \dot{A}) \quad (3.1)$$

holds in N . The statement $p \Vdash \psi(\tau_1, \dots, \tau_n, \dot{A})$ is first-order in the structure $(V_\kappa, \in, \tau_1, \dots, \tau_n, \sigma, \dot{A}, \mathbb{P})$. This requires that the forcing relation for \mathbb{P} is definable over V_κ , and in general would not hold for all class partial orders, but does hold for pretame partial orders (see [Fri00] for definition and proof) and progressively closed Easton-support iterations are pretame (see for example, section “Reverse Easton Forcing” in Chapter 2 of [Fri00]). Furthermore, since “ $\tau \in X_\sigma$ ” can be expressed by a first-order formula $\chi(\tau, \sigma)$ over $(V_\kappa, \in, \sigma, \tau, \mathbb{P})$, it follows that the statement in (3.1) is Π_n^1 over $(V_\kappa, \in, \sigma, \dot{A})$. Since $N \models$ “(3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$ ” and N is Π_n^1 -correct at κ , it follows that (3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$. Hence by Lemma 3.6, $p \Vdash \bar{\varphi}(\sigma, \dot{A})$ over V , and since $p \in G$, we conclude that $V[G] \models \bar{\varphi}(\sigma, \dot{A}_G)$, which implies $(V_\kappa^{V[G]}, \in, A) \models \varphi$ in $V[G]$.

An analogous argument establishes the converse, verifying that, if

$$(V_\kappa^{V[G]}, \in, A) \models \varphi$$

for a Π_n^1 -assertion φ and $A \in N[G]$, then the same assertion holds in $N[G]$. \square

A similar argument yields the following result.

Corollary 3.8. *Suppose κ is an inaccessible cardinal, N is a Π_n^1 -correct κ -model and $\mathbb{P} \in N$ is a $<\kappa$ -distributive forcing notion of size κ . If $G \subseteq \mathbb{P}$ is generic over V , then $N[G]$ remains a Π_n^1 -correct κ -model in $V[G]$.*

Proof. Without loss of generality we assume that $\mathbb{P} \subseteq V_\kappa$. Since \mathbb{P} is $<\kappa$ -distributive, N remains a κ -model in $V[G]$, and, since $G \in V[G]$, it follows that $N[G]$ is a κ -model in $V[G]$ by the ground closure criterion, Lemma 3.1, applied in $V[G]$. The $<\kappa$ -distributivity of \mathbb{P} entails that $V_\kappa^{N[G]} = V_\kappa^{V[G]} = V_\kappa$ and that \mathbb{P} is a pretame class forcing over V_κ (see [Fri00] for proof that $<\text{Ord}$ -distributivity implies pretameness) ensuring the definability of the forcing relation. Since the statement “ τ is a nice name for a subset of \check{V}_κ ” is first-order over the structure $(V_\kappa, \in, \tau, \mathbb{P})$, the rest of the argument can be carried out as in the proof of Theorem 3.7. \square

The conclusion of Corollary 3.8 need not hold if the N -generic filter G is not fully V -generic (see Remark 5.7).

4. SHOOTING n -CLUBS

Hellsten [Hel10] showed that if $W \subseteq \kappa$ is any Π_1^1 -indescribable (i.e., weakly compact) subset of κ , then there is a forcing extension in which W contains a 1-club and all weakly compact subsets of W remain weakly compact. We will define a generalization of Hellsten’s forcing to shoot an n -club through a Π_n^1 -indescribable subset of a cardinal κ while preserving the Π_n^1 -indescribability of all its subsets, so that, in particular, κ remains Π_n^1 -indescribable in the forcing extension.

Suppose γ is an inaccessible cardinal and $A \subseteq \gamma$ is cofinal. For $n \geq 1$, we define a poset $T^n(A)$ consisting of all bounded n -closed $c \subseteq A$ ordered by end extension: $c \leq d$ if and only if $d = c \cap \sup_{\alpha \in d} (\alpha + 1)$.

We now argue that, if $n \geq 1$, γ is inaccessible, and $A \subseteq \gamma$ is cofinal, then $T^n(A)$ is γ -strategically closed. We will actually prove a stronger statement, for which we need to introduce some new terminology. Given an ordinal γ , a subset $X \subseteq \gamma$, and a poset \mathbb{P} , let $\mathcal{G}_{\gamma, X}(\mathbb{P})$ be the modification of $\mathcal{G}_\gamma(\mathbb{P})$ in which Player I plays at all stages indexed by an ordinal in X and Player II plays elsewhere, and it is still the case that Player I wins if and only if there is a limit ordinal $\beta < \gamma$ such that $\langle p_\alpha : \alpha < \beta \rangle$ has no lower bound in \mathbb{P} . So $\mathcal{G}_\gamma(\mathbb{P})$ is precisely the game $\mathcal{G}_{\gamma, X}(\mathbb{P})$, where X is the set of odd ordinals less than γ .

Lemma 4.1. *Suppose that $n \geq 1$, γ is inaccessible, $A \subseteq \gamma$ is cofinal, and $X \subseteq \gamma$ is such that, for all $\beta < \gamma$, $X \cap \beta$ is not a Π_{n-1}^1 -indescribable subset of β . Then Player II has a winning strategy in the game $\mathcal{G}_{\gamma, X}(T^n(A))$.*

Proof. We describe Player II’s winning strategy in $\mathcal{G}_{\gamma, X}(T^n(A))$. Suppose that $\alpha \in \gamma \setminus X$ and $\langle c_\beta : \beta < \alpha \rangle$ is a partial run of the game in which Player II has thus far played according to the prescribed winning strategy. We describe how Player II chooses her next play, c_α .

If $\alpha = \alpha_0 + 1$ is a successor ordinal, then Player II simply plays any condition $c_\alpha \not\leq c_{\alpha_0}$. If α is a limit ordinal, then Player II records an ordinal $\gamma_\alpha = \bigcup_{\beta < \alpha} c_\beta$, chooses an element $\eta_\alpha \in A \setminus (\gamma_\alpha + 1)$, and plays $c_\alpha = \left(\bigcup_{\beta < \alpha} c_\beta \right) \cup \{\eta_\alpha\}$. Notice, in particular, that $\gamma_\alpha \notin c_\alpha$. In order to argue that c_α is a condition in $T^n(A)$, we need to verify, letting $c = \bigcup_{\beta < \alpha} c_\beta$, that c is not a Π_{n-1}^1 -indescribable subset of γ_α .

We can assume that γ_α is Π_{n-1}^1 -indescribable, as otherwise c is clearly not Π_{n-1}^1 -indescribable in γ_α . In particular, we have $\alpha \geq \gamma_\alpha$, since $\{\sup(c_\beta) : \beta < \alpha\}$ is cofinal in γ_α . But, since $X \cap \gamma_\alpha$ is not Π_{n-1}^1 -indescribable by assumption, we know that $\gamma_\alpha \setminus X$ is cofinal in γ_α , so, by the definition of our strategy for Player II, $\langle \sup(c_\beta) : \beta \in \gamma_\alpha \setminus X \rangle$ is strictly increasing and thus cofinal in γ_α . Therefore, if $\alpha > \gamma_\alpha$, then it must be the case that $c = c_{\gamma_\alpha}$, and hence c is not Π_{n-1}^1 -indescribable in γ_α due to the fact that $c_{\gamma_\alpha} \in T^n(A)$ is n -closed in γ and $\gamma_\alpha \notin c_{\gamma_\alpha}$.

We may therefore assume that $\alpha = \gamma_\alpha$. For every limit ordinal $\xi < \alpha$, let $\gamma_\xi = \bigcup_{\beta < \xi} c_\beta$. Then $\{\gamma_\xi : \xi < \alpha \text{ is a limit ordinal}\}$ is a club in γ_α . Moreover, if ξ is a limit ordinal in $\alpha \setminus X$, then, since Player II played according to their winning strategy at stage ξ , we have $\gamma_\xi \notin c$.

Let $D = \{\xi < \alpha : \xi \text{ is a limit ordinal and } \xi = \gamma_\xi\}$. Then D is a club (and hence an $(n-1)$ -club) in $\alpha = \gamma_\alpha$. Since γ_α is Π_{n-1}^1 -indescribable and, by assumption, $X \cap \gamma_\alpha$ is not Π_{n-1}^1 -indescribable, we can fix an $(n-1)$ -club E in γ_α such that $E \cap X = \emptyset$. Then $E \cap D$ is an $(n-1)$ -club in γ_α and $E \cap D \cap c = \emptyset$, so c is not a Π_{n-1}^1 -indescribable subset of γ_α . Thus, c_α is a valid play by Player II, and we have described a winning strategy in $\mathcal{G}_\kappa(T^n(A))$. \square

Remark 4.2. The set of odd ordinals less than γ clearly satisfies the hypothesis of Lemma 4.1, so the lemma indeed implies that $T^n(A)$ is γ -strategically closed. But the lemma also applies to larger sets X , such as the set of all $\alpha < \gamma$ such that α is not Π_{n-2}^1 -indescribable.

Theorem 4.3. *Suppose that $n \geq 1$ and $S \subseteq \kappa$ is Π_n^1 -indescribable. Then there is a forcing extension in which S contains a 1-club and all Π_n^1 -indescribable subsets of S from V remain Π_n^1 -indescribable.*

Proof. Let $\mathbb{P}_{\kappa+1} = \langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$ be an Easton-support iteration such that

- if $\gamma \leq \kappa$ is inaccessible and $S \cap \gamma$ is cofinal in γ , then $\dot{\mathbb{Q}}_\gamma = (T^1(S \cap \gamma))^{V^{\mathbb{P}_\gamma}}$;
- otherwise, $\dot{\mathbb{Q}}_\gamma$ is a \mathbb{P}_γ -name for trivial forcing.

Since κ is Π_n^1 -indescribable, Proposition 2.15 implies that \mathbb{P}_κ has size κ and the κ -c.c.. Forcing with $\mathbb{P}_{\kappa+1}$ therefore preserves the inaccessibility of κ because \mathbb{P}_κ has the κ -c.c. and is progressively closed and $\dot{\mathbb{Q}}_\kappa$ is forced to be $<\kappa$ -distributive.

Suppose $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ is generic over V . Let $C(\kappa) =_{\text{def}} \bigcup H$. We will show that, in $V[G * H]$, $C(\kappa)$ is a 1-club subset of κ ; notice that it suffices to show that $C(\kappa)$ is a stationary subset of κ in $V[G * H]$ because then $C(\kappa)$ is clearly 1-closed since it is a union of a filter contained in $\dot{\mathbb{Q}}_\kappa = T^1(S)^{V^{\mathbb{P}_\kappa}}$.

Suppose $T \subseteq S$ is Π_n^1 -indescribable in V . We will simultaneously show that in $V[G * H]$, $C(\kappa)$ has nonempty intersection with every club subset of κ and T remains Π_n^1 -indescribable (in particular, κ remains Π_n^1 -indescribable). Fix $A, C \in P(\kappa)^{V[G * H]}$ such that C is a club subset of κ in $V[G * H]$. Let $\dot{A}, \dot{C}, \dot{C}(\kappa) \in H(\kappa^+)$ be $\mathbb{P}_{\kappa+1}$ -names such $\dot{A}_{G * H} = A$, $\dot{C}_{G * H} = C$ and $\dot{C}(\kappa)_{G * H} = C(\kappa)$. In V , let M be a κ -model with $\dot{A}, \dot{C}, \dot{C}(\kappa), \mathbb{P}_{\kappa+1}, T, S \in M$. Since T is Π_n^1 -indescribable in V , it follows by Theorem 2.6 that there is a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $\kappa \in j(T)$.

Since $N^{<\kappa} \cap V \subseteq N$ and $j(S) \cap \kappa = S$, it follows that $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{T}^1(S) * \dot{\mathbb{P}}_{\kappa, j(\kappa)}$, where $\dot{\mathbb{P}}_{\kappa, j(\kappa)}$ is a $\mathbb{P}_{\kappa+1}$ -name for the tail of the iteration $j(\mathbb{P}_\kappa)$. Since \mathbb{P}_κ has the κ -c.c., by the generic closure criterion (Lemma 3.2), $N[G]$ is a κ -model in $V[G]$. Since

$T^1(S)$ is κ -strategically closed, $N[G]$ remains a κ -model in $V[G * H]$, and hence by the ground closure criterion (Lemma 3.1), $N[G * H]$ is a κ -model in $V[G * H]$. Since $\mathbb{P}_{\kappa, j(\kappa)} = (\dot{\mathbb{P}}_{\kappa, j(\kappa)})_{G * H}$ is κ -strategically closed in $N[G * H]$ and $N[G * H]$ is a κ -model in $V[G * H]$, it follows that there is a filter $G' \in V[G * H]$ which is generic for $\mathbb{P}_{\kappa, j(\kappa)}$ over $N[G * H]$ and the embedding j lifts to $j : M[G] \rightarrow N[\hat{G}]$, where $\hat{G} \cong G * H * G'$.

Notice that $p = C(\kappa) \cup \{\kappa\} = \bigcup H \cup \{\kappa\} \in N[\hat{G}]$. Since $\kappa \in j(T) \subseteq j(S)$, we see that $N[\hat{G}] \models "p \text{ is a closed subset of } j(S)"$. Thus, $p \in j(T^1(S))$. Since $j(T^1(S))$ is $j(\kappa)$ -strategically closed in $N[\hat{G}]$ and $N[\hat{G}]$ is a κ -model in $V[G * H]$ by the ground closure criterion, there is a filter $\hat{H} \in V[G * H]$ generic for $j(T^1(S))$ over $N[\hat{G}]$ with $p \in \hat{H}$. Since p is below every condition in $j" H$, we have $j" H \subseteq \hat{H}$, and thus j lifts to $j : M[G * H] \rightarrow N[\hat{G} * \hat{H}]$, where $\kappa \in j(C(\kappa))$. By Theorem 3.7 and Corollary 3.8, $N[G * H]$ is a Π_{n-1}^1 -correct κ -model in $V[G * H]$. Since $\mathbb{P}_{\kappa, j(\kappa)}$ and $j(T^1(S))$ are $(\kappa + 1)$ -strategically closed in $N[G * H]$, it follows that $N[G * H]$ and $N[\hat{G} * \hat{H}]$ have the same subsets of V_κ , so, in particular, $N[\hat{G} * \hat{H}]$ is a Π_{n-1}^1 -correct κ -model in $V[G * H]$. Thus, by Theorem 2.6, we have verified that T remains Π_n^1 -indescribable in $V[G * H]$.

It remains to show that $C(\kappa) \cap C \neq \emptyset$. Recall that C is a club subset of κ in $V[G * H]$, so $j(C)$ is a club subset of $j(\kappa)$ in $N[\hat{G} * \hat{H}]$. Since $j(C) \cap \kappa = C$, it follows that $\kappa \in j(C)$, and hence $\kappa \in j(C(\kappa) \cap C)$. By elementarity, $C(\kappa) \cap C \neq \emptyset$, so $C(\kappa)$ is a stationary and hence 1-club subset of κ in $V[G * H]$. \square

Remark 4.4. In the proof of Theorem 4.3, for any $m \leq n$ we can force with $T^m(S \cap \gamma)$ at every relevant $\gamma \leq \kappa$ instead of $T^1(S \cap \gamma)$. This iteration will still preserve the Π_n^1 -indescribability of every subset of S that is Π_n^1 -indescribable in V , and it will shoot an m -club through S . If $m > 1$, then this forcing will have slightly better closure properties than $T^1(S \cap \gamma)$ (see Lemma 4.1), which could be useful for certain applications, though we have not found any such applications as of yet.

5. $\square_1(\kappa)$ CAN HOLD NONTRIVIALY AT A WEAKLY COMPACT CARDINAL

In this section, we will prove Theorem 1.1, which states that if κ is κ^+ -weakly compact then the principle $\square_1(\kappa)$ can be forced to hold at κ while preserving the κ^+ -weak compactness of κ . Let us remind the reader that the corresponding relative consistency result was first obtained by Brickhill and Welch [BW] assuming $V = L$.

First, we define a forcing to add a generic coherent sequence of 1-clubs to a Mahlo cardinal κ .

Definition 5.1. Suppose κ is a Mahlo cardinal. We define a forcing $\mathbb{Q}(\kappa)$ such that q is a condition in $\mathbb{Q}(\kappa)$ if and only if

- q is a sequence with $\text{dom}(q) = \text{inacc}(\kappa) \cap (\gamma^q + 1)$ for some $\gamma^q < \kappa$,
- $q(\alpha) = C_\alpha^q$ is a 1-club subset of α for each $\alpha \in \text{dom}(q)$ and
- for all $\alpha, \beta \in \text{dom}(q)$, if $C_\beta^q \cap \alpha \in \Pi_0^1(\alpha)^+$, then $C_\alpha^q = C_\beta^q \cap \alpha$.²

The ordering on $\mathbb{Q}(\kappa)$ is defined by letting $p \leq q$ if and only if p is an end extension of q .

²Equivalently, for all $\alpha, \beta \in \text{dom}(q)$, if α is inaccessible and $C_\beta^q \cap \alpha$ is stationary, then $C_\alpha^q = C_\beta^q \cap \alpha$.

Proposition 5.2. *Suppose κ is a Mahlo cardinal. The poset $\mathbb{Q}(\kappa)$ is κ -strategically closed.*

Proof. We describe a winning strategy for Player II in the game $\mathcal{G}_\kappa(\mathbb{Q}(\kappa))$. We will recursively arrange so that, if $\delta < \kappa$ and $\langle q_\alpha : \alpha < \delta \rangle$ is a partial play of the game with Player II playing according to her winning strategy, then, for all limit ordinals $\beta < \delta$, we have $\{\gamma^{q_\alpha} : \alpha < \beta, \alpha \text{ even}\}$ is a club in its supremum and, if γ^{q_β} is inaccessible, is a subset of $C_{\gamma^{q_\beta}}^{q_\beta}$. We will also arrange that, for all even successor ordinals $\alpha < \beta < \delta$, γ^{q_α} and γ^{q_β} are inaccessible cardinals and $C_{\gamma^{q_\beta}}^{q_\beta} \cap \gamma^{q_\alpha} = C_{\gamma^{q_\alpha}}^{q_\alpha}$.

We first deal with successor ordinals. Suppose that $\delta < \kappa$ is an even ordinal and $\langle q_\alpha : \alpha \leq \delta + 1 \rangle$ has been played. Suppose first that γ^{q_δ} is an inaccessible cardinal (in particular, by our recursion hypotheses, this must be the case if δ is a successor ordinal). In this case, let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$ and let $q_{\delta+2}$ be the condition extending $q_{\delta+1}$ by setting

$$C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}} = C_{\gamma^{q_\delta}}^{q_\delta} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}}, \gamma^{q_{\delta+2}}).$$

The fact that $C_{\gamma^{q_\delta}}^{q_\delta} \cup \{\gamma^{q_\delta}\} \subseteq C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}}$ ensures that the recursion hypothesis is maintained. The set $C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}}$ is stationary in $\gamma^{q_{\delta+2}}$ because it contains a tail, and it has all of its inaccessible stationary reflection points because those are $\leq \gamma^{q_\delta}$. The coherence property holds because we have omitted the interval $(\gamma^{q_\delta}, \gamma^{q_{\delta+1}})$ from $C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}}$ ensuring that for no α in that interval is $C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}} \cap \alpha$ stationary. It follows that $q_{\delta+2}$ is a condition and a valid play for Player II.

If γ^{q_δ} is not inaccessible, then δ is a limit ordinal (by our recursion hypothesis). In this case, again let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$, and define $q_{\delta+2}$ by setting

$$C_{\gamma^{q_{\delta+2}}}^{q_{\delta+2}} = \bigcup_{\substack{\alpha < \delta \\ \alpha \text{ even}}} C_{\gamma^{q_{\alpha+2}}}^{q_\alpha} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}}, \gamma^{q_{\delta+2}}).$$

A similar argument as above verifies that $q_{\delta+2}$ is a valid play in the game and maintains our recursion hypotheses.

Finally, suppose that $\delta < \kappa$ is a limit ordinal and $\langle q_\alpha : \alpha < \delta \rangle$ has been played. Let $\gamma^{q_\delta} = \sup\{\gamma^{q_\alpha} : \alpha < \delta\}$. If γ^{q_δ} is not inaccessible, then we can simply set $q_\delta = \bigcup_{\alpha < \delta} q_\alpha$. If γ^{q_δ} is inaccessible, then we must additionally define $C_{\gamma^{q_\delta}}^{q_\delta}$. We do this by setting

$$C_{\gamma^{q_\delta}}^{q_\delta} = \bigcup_{\substack{\alpha < \delta \\ \alpha \text{ even}}} C_{\gamma^{q_{\alpha+2}}}^{q_\alpha}.$$

It is easy to verify that this is as desired. The fact that $C_{\gamma^{q_\delta}}^{q_\delta}$ is stationary in γ^{q_δ} follows from the fact that $\{\gamma^{q_\alpha} : \alpha < \delta, \alpha \text{ even}\} \subseteq C_{\gamma^{q_\delta}}^{q_\delta}$, so it in fact contains a club in γ^{q_δ} . \square

It follows from Proposition 5.2 that $\mathbb{Q}(\kappa)$ is $<\kappa$ -distributive. In particular, if $G \subseteq \mathbb{Q}(\kappa)$ is a generic filter, then $\bigcup G$ is a coherent sequence of 1-clubs of length κ , since V_κ remains unchanged.

Next, we define a forcing which will be used to generically thread a coherent sequence of 1-clubs.

Definition 5.3. Suppose that $\vec{C}(\kappa) = \langle C_\alpha(\kappa) : \alpha \in \text{inacc}(\kappa) \rangle$ is a coherent sequence of 1-clubs. The poset $\mathbb{T}(\vec{C}(\kappa))$ consists of all conditions t such that

- t is a 1-closed bounded subset of κ and
- for every $\alpha < \kappa$, if $t \cap \alpha \in \Pi_0^1(\alpha)^+$, then $C_\alpha(\kappa) = t \cap \alpha$.³

The ordering on $\mathbb{T}(\vec{C}(\kappa))$ is defined by letting $t \leq s$ if and only if t end-extends s .

Lemma 5.4. *Suppose κ is a regular cardinal and $\vec{C}(\kappa)$ is a coherent sequence of 1-clubs. Then the poset $\mathbb{T}(\vec{C}(\kappa))$ is κ -strategically closed.*⁴

Proof. We describe a winning strategy for player II in $\mathcal{G}_\kappa(\mathbb{T}(\vec{C}(\kappa)))$. Player II's strategy at successor ordinal stages can be arbitrary provided that Player II chooses conditions properly extending Player I's previous play.

So let δ be a limit stage and let $\langle t_\alpha : \alpha < \delta \rangle$ be the sequence of conditions played at previous stages of the game. Player II then plays $t_\delta = (\bigcup_{\alpha < \delta} t_\alpha) \cup \{\kappa_\delta + 1\}$, where $\kappa_\delta = \sup(\bigcup_{\alpha < \delta} t_\alpha)$. We will also assume recursively that Player II has played according to this strategy successfully at previous limit stages of the game, so that, if $\lambda < \delta$ is a limit ordinal, then $\kappa_\lambda \notin t_\delta$. It remains to show that $t_\delta \in \mathbb{T}(\vec{C}(\kappa))$.

To argue that t_δ is a 1-closed subset of κ , it suffices to see that $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ . By our recursive assumption, $\{\kappa_\lambda : \lambda < \delta\}$ is a club subset of κ_δ disjoint from t_δ , and hence $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ . The coherence condition follows easily. \square

Lemma 5.5. *Suppose κ is a regular cardinal and $\vec{C}(\kappa) = \langle C_\alpha(\kappa) : \alpha \in \text{inacc}(\kappa) \rangle$ is a coherent sequence of 1-clubs. If $G \subseteq \mathbb{T}(\vec{C}(\kappa))$ is generic over V , then $C_\kappa = \bigcup G$ threads $\vec{C}(\kappa)$ in $V[G]$.*

Proof. By the $<\kappa$ -distributivity of $\mathbb{T}(\vec{C}(\kappa))$ and the definition of its conditions, C_κ meets the coherence requirements and contains all of its inaccessible stationary reflection points. So it remains to check that C_κ is stationary.

Fix a club $C \subseteq \kappa$ in $V[G]$ and let \dot{C} be a $\mathbb{T}(\vec{C}(\kappa))$ -name for C . Assume towards a contradiction that $C \cap C_\kappa = \emptyset$. Fix $t_0 \in \mathbb{T}(\vec{C}(\kappa))$ forcing that \dot{C} is a club and $\dot{C} \cap \dot{C}_\kappa = \emptyset$, where \dot{C}_κ is the canonical $\mathbb{T}(\vec{C}(\kappa))$ -name for C_κ , and let β_0 be the supremum of t_0 . Recursively define a decreasing sequence $\langle t_n : n < \omega \rangle$ of conditions from $\mathbb{T}(\vec{C}(\kappa))$ as follows, letting β_n denote $\sup(t_n)$. Given $n < \omega$, if t_n is defined, find an ordinal α_n with $\beta_n < \alpha_n < \kappa$ and a condition $t_{n+1} \leq t_n$ such that $t_{n+1} \Vdash \check{\alpha}_n \in \dot{C}$. Let $\alpha = \bigcup_{n < \omega} \alpha_n = \bigcup_{n < \omega} \beta_n$, and let $t = \bigcup_{n < \omega} t_n \cup \{\alpha\}$. Clearly t is a condition in $\mathbb{T}(\vec{C}(\kappa))$ and $t \Vdash \alpha \in \dot{C} \cap \dot{C}_\kappa$, which is the desired contradiction. \square

Theorem 5.6. *Suppose κ is weakly compact and the GCH holds. There is a cofinality-preserving forcing extension in which*

- (1) *for all $\gamma \leq \kappa$, every set $W \in P(\gamma)^V$ which is weakly compact in V remains weakly compact and*
- (2) $\square_1(\kappa)$ *holds.*

Proof. Define an Easton-support iteration $\langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$ as follows.

- If $\gamma < \kappa$ is Mahlo, let $\dot{\mathbb{Q}}_\gamma = (\mathbb{Q}(\gamma) * \dot{\mathbb{T}}(\vec{C}(\gamma)))^{V^{\mathbb{P}^\gamma}}$, where $\vec{C}(\gamma)$ is the generic coherent sequence of 1-clubs of length γ added by $\mathbb{Q}(\gamma)$.
- If $\gamma = \kappa$, let $\dot{\mathbb{Q}}_\kappa = (\mathbb{Q}(\gamma))^{V^{\mathbb{P}^\kappa}}$.

³Equivalently, for every inaccessible cardinal $\alpha < \kappa$, if $t \cap \alpha$ is stationary in α then $C_\alpha(\kappa) = t \cap \alpha$.

⁴Note that the forcing to thread a $\square(\kappa)$ -sequence is never κ -strategically closed.

- Otherwise, let $\dot{\mathbb{Q}}_\gamma$ be a \mathbb{P}_γ -name for trivial forcing.

Let $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ be generic over V . $V[G * H]$ is our desired model. Standard arguments using progressive closure of the iteration \mathbb{P}_κ together with the GCH (see [Ham94]) show that cofinalities are preserved in $V[G * H]$.

The argument for the preservation of weakly compact subsets of $\gamma < \kappa$ is similar to and easier than the argument for the preservation of weakly compact subsets of κ , which is given next.

Recall that $\vec{C}(\kappa) = \bigcup H$ is a coherent sequence of 1-clubs of length κ . Fix $W \in P(\kappa)^V$ which is weakly compact in V . It remains to argue that in $V[G * H]$, W is weakly compact and $\vec{C}(\kappa)$ has no thread.

Fix a set $C \in P(\kappa)^{V[G * H]}$ which is a 1-club subset of κ in $V[G * H]$. We will simultaneously show that C is not a thread through $\vec{C}(\kappa)$ and that W remains weakly compact in $V[G * H]$. Fix $A \in P(\kappa)^{V[G * H]}$ and let $\dot{C}, \dot{A}, \tau \in H(\kappa^+)^V$ be $\mathbb{P}_{\kappa+1}$ -names with $\dot{C}_{G * H} = C$, $\dot{A}_{G * H} = A$ and $\tau_{G * H} = \vec{C}(\kappa)$. Let M be a κ -model with $W, \dot{C}, \dot{A}, \tau, \mathbb{P}_{\kappa+1} \in M$. Since W is weakly compact in V , there is a κ -model N and an elementary embedding $j : M \rightarrow N$ such that $\text{crit}(j) = \kappa$ and $\kappa \in j(W)$.

Since $N^{<\kappa} \cap V \subseteq N$, we have, in N ,

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}(\kappa) * \dot{\mathbb{T}}(\vec{C}(\kappa))) * \dot{\mathbb{P}}_{\kappa, j(\kappa)},$$

where $\dot{\mathbb{P}}_{\kappa, j(\kappa)}$ is a $\mathbb{P}_{\kappa+1} * \dot{\mathbb{T}}(\vec{C}(\kappa))$ -name for the iteration from $\kappa + 1$ to $j(\kappa)$. By Lemma 5.4, $\mathbb{T}(\vec{C}(\kappa))$ is κ -strategically closed in $N[G * H]$, and hence, using standard arguments, we can build a filter $h \in V[G * H]$ for $\mathbb{T}(\vec{C}(\kappa))$ which is generic over $N[G * H]$. Let $C_\kappa = \bigcup h$ and notice that $C_\kappa \neq C$ because $C \in N[G * H]$ and C_κ is generic over $N[G * H]$. Similarly, we can build a filter $G' \in V[G * H]$ which is generic for $\mathbb{P}_{\kappa, j(\kappa)} = (\dot{\mathbb{P}}_{\kappa, j(\kappa)})_{G * H * h}$ over $N[G * H * h]$. Since $j'' G \subseteq G * H * h * G'$, the embedding can be extended to $j : M[G] \rightarrow N[\hat{G}]$, where $\hat{G} = G * H * h * G'$.

Let $\mathbb{Q}(\kappa) = (\dot{\mathbb{Q}}(\kappa))_G$. Working in $N[\hat{G}]$, since

$$\vec{C}(\kappa) = \bigcup H = \langle C_\alpha(\kappa) : \alpha \in \text{inacc}(\kappa) \rangle$$

is a coherent sequence of 1-clubs and C_κ is a thread through $\vec{C}(\kappa)$ by Lemma 5.5, it follows that the function

$$q = \langle C_\alpha(\kappa) : \alpha \in \text{inacc}(\kappa) \rangle \cup \{(\kappa, C_\kappa)\}$$

is a condition in $j(\mathbb{Q}(\kappa))$ below every element of $j'' H$. We may build a filter $\hat{H} \in V[G * H]$ which is generic for $j(\mathbb{Q}(\kappa))$ over $N[\hat{G}]$ with $q \in \hat{H}$. Since $j'' H \subseteq \hat{H}$, it follows that j extends to $j : M[G * H] \rightarrow N[\hat{G} * \hat{H}]$. Now $A \in M[G * H]$ and $\kappa \in j(W)$, so W is weakly compact in $V[G * H]$.

It remains to show that C is not a thread through $\vec{C}(\kappa)$. For the sake of contradiction, assume C is a thread through $\vec{C}(\kappa)$. By elementarity we see that in $N[\hat{G} * \hat{H}]$,

$$j(\vec{C}(\kappa)) = \langle \vec{C}_\alpha(j(\kappa)) : \alpha \in \text{inacc}(j(\kappa)) \rangle$$

is a coherent sequence of 1-clubs. Since $q = \vec{C}(\kappa) \wedge \langle C_\kappa \rangle \in \hat{H}$ we have $\vec{C}_\kappa(j(\kappa)) = C_\kappa$. Now since C is a thread for $\vec{C}(\kappa)$ in $M[G * H]$, by elementarity, $j(C)$ is a thread for $j(\vec{C}(\kappa)) = \langle \vec{C}_\alpha(j(\kappa)) : \alpha \in \text{inacc}(j(\kappa)) \rangle$. Since κ is inaccessible in $N[\hat{G} * \hat{H}]$ and $\kappa \in \text{Tr}_0(j(C))$, it follows that $C_\kappa = \vec{C}_\kappa(j(\kappa)) = j(C) \cap \kappa = C$, a contradiction. \square

Remark 5.7. Observe that in the proof of Theorem 5.6, if we assume that κ is Π_2^1 -indescribable and that the target N of the embedding $j : M \rightarrow N$ we start with is Π_1^1 -correct, then the κ -model $N[G * H]$ from the proof of Theorem 5.6 is Π_1^1 -correct by Theorem 3.7 and Corollary 3.8. However, the κ -model $N[G * H * h]$ cannot be Π_1^1 -correct because otherwise we would have shown that, in the extension $V[G * H]$, κ is Π_2^1 -indescribable, contradicting Proposition 2.9. Thus, a forcing extension of a Π_1^1 -correct κ -model, even by a κ -strategically closed forcing notion, need not be Π_1^1 -correct if the generic filter is not fully V -generic.

For the next theorem, let us recall what it means for a cardinal κ to be α -weakly compact, where $\alpha \leq \kappa^+$. Suppose κ is a weakly compact cardinal. It is not difficult to see that if sets $X, Y \in P(\kappa)$ are equivalent modulo the ideal $\Pi_1^1(\kappa)$, then their traces $\text{Tr}_1(X)$ and $\text{Tr}_1(Y)$ are equivalent as well. Thus, the trace operation $\text{Tr}_1 : P(\kappa) \rightarrow P(\kappa)$ leads to a well defined operation $\text{Tr}_1 : P(\kappa)/\Pi_1^1(\kappa) \rightarrow P(\kappa)/\Pi_1^1(\kappa)$ on the collection $P(\kappa)/\Pi_1^1(\kappa)$ of equivalence classes of subsets of κ modulo the ideal $\Pi_1^1(\kappa)$. By taking diagonal intersections at limit ordinals, we can iterate the trace operation on the equivalence classes κ^+ -many times (see [BTW77, Section 4] for a similar construction related to Mahloness). To be more precise, fix a sequence $\langle e_\beta \mid \kappa \leq \beta < \kappa^+, \beta \text{ limit} \rangle$, where $e_\beta : \kappa \rightarrow \beta$ is a bijection for all relevant β . To start, let $\text{Tr}_1^0([S]) = [S]$ for all $[S] \in P(\kappa)/\Pi_1^1(\kappa)$. Given $\alpha < \kappa^+$, if $\text{Tr}_1^\alpha : P(\kappa)/\Pi_1^1(\kappa) \rightarrow P(\kappa)/\Pi_1^1(\kappa)$ has been defined, let $\text{Tr}_1^{\alpha+1} = \text{Tr}_1 \circ \text{Tr}_1^\alpha$. If $\beta < \kappa$ is a limit ordinal and Tr_1^α has been defined for all $\alpha < \beta$, then define Tr_1^β by letting $\text{Tr}_1^\beta([S]) = [\bigcap_{\alpha < \beta} S_\alpha]$, where S_α is a representative element of $\text{Tr}_1^\alpha([S])$ for all $\alpha < \beta$. Finally, if β is a limit ordinal and $\kappa \leq \beta < \kappa^+$, then let $\text{Tr}_1^\beta([S]) = [\Delta_{\eta < \kappa} S_{e_\beta(\eta)}]$.

It is straightforward to verify that each of these functions is well-defined and does not depend on our choice of e_β for limit β .

Definition 5.8. Suppose κ is a regular cardinal. For $0 < \alpha < \kappa^+$ we say that κ is α -weakly compact if $\text{Tr}_1^\beta([\kappa]) \neq [\emptyset]$ for all $\beta < \alpha$, and κ is κ^+ -weakly compact if it is α -weakly compact for all $\alpha \in \kappa^+ \setminus \{0\}$.

Notice that κ is 1-weakly compact if and only if it is weakly compact and κ is ω -weakly compact if and only if for all $n < \omega$ the set $\{\gamma < \kappa : \gamma \text{ is } n\text{-weakly compact}\}$ is weakly compact in κ . For more details regarding α -weak compactness the reader is referred to [Cod19].

If $0 < \alpha \leq \kappa^+$ and we start with an α -weakly compact cardinal κ in Theorem 5.6, then it will remain α -weakly compact in the extension $V[G * H]$. This is the case because weakly compact subsets of all cardinals $\gamma \leq \kappa$ are preserved to $V[G * H]$, so it is easy to show by induction on $0 < \beta \leq \alpha$ that if a set X is in the equivalence class $\text{Tr}^\beta([\kappa])$ as computed in V , then the equivalence class $\text{Tr}^\beta([\kappa])$ as computed in $V[G * H]$ contains some $Y \supseteq X$. Thus, we get the following, which immediately implies Theorem 1.1 in the special case $\alpha = \kappa^+$.

Theorem 5.9. *If $0 < \alpha \leq \kappa^+$, κ is α -weakly compact and GCH holds, then there is a cofinality preserving forcing extension in which*

- (1) κ remains α -weakly compact and
- (2) $\square_1(\kappa)$ holds.

Next we will show that, if κ is Mahlo, one can characterize precisely when $\square_1(\kappa)$ holds after forcing with $\mathbb{Q}(\kappa)$. Notice that, if there is a stationary subset of κ that

does not reflect at an inaccessible cardinal (i.e., if $\text{Refl}_0(\kappa)$ fails), then $\square_1(\kappa)$ must fail, since any such non-reflecting stationary set of κ would then trivially be a thread through any coherent sequence of 1-clubs of length κ . We will see in Theorem 5.11 that $\text{Refl}_0(\kappa)$ holding in the extension by $\mathbb{Q}(\kappa)$ is in fact sufficient for $\square_1(\kappa)$ to hold. First, we need the following general proposition. Recall that $\text{Refl}_n(\kappa)$ holds if and only if κ is Π_n^1 -indescribable and, for every Π_n^1 -indescribable subset S of κ , there is an $\alpha < \kappa$ such that $S \cap \alpha$ is Π_n^1 -indescribable.

Proposition 5.10. *Fix $n < \omega$. If κ is a cardinal, $\text{Refl}_n(\kappa)$ holds and $S \in \Pi_n^1(\kappa)^+$, then the set*

$$T = \{\alpha < \kappa : (S \cap \alpha \in \Pi_n^1(\alpha)^+) \wedge (\text{Refl}_n(\alpha) \text{ fails})\}$$

is Π_n^1 -indescribable.

Proof. We proceed by induction on κ . Suppose the proposition holds for all cardinals $\alpha < \kappa$, $\text{Refl}_n(\kappa)$ holds and $S \in \Pi_n^1(\kappa)^+$. It suffices to show that $T \cap C \neq \emptyset$ for every n -club subset C of κ . Fix an n -club set C and note that $S \cap C$ is Π_n^1 -indescribable. Thus, by $\text{Refl}_n(\kappa)$, there is some $\alpha_0 < \kappa$ such that $S \cap C \cap \alpha_0 \in \Pi_n^1(\alpha_0)^+$. It follows that $\alpha_0 \in \text{Tr}_n(S)$, but also $\alpha_0 \in C$ because C contains all of its Π_{n-1}^1 -reflection points. If $\alpha_0 \in T$, we have shown that $T \cap C \neq \emptyset$. So suppose that $\alpha_0 \notin T$, so $\text{Refl}_n(\alpha_0)$ holds. We can now appeal to the inductive hypothesis at α_0 , applied to the Π_n^1 -indescribable set $S \cap \alpha_0$ and the n -club $C \cap \alpha_0$, to find a cardinal $\alpha_1 \in T \cap C$. \square

Theorem 5.11. *Suppose κ is Mahlo and $p \in \mathbb{Q}(\kappa)$. The following are equivalent:*

- (1) $p \Vdash_{\mathbb{Q}(\kappa)} \text{Refl}_0(\kappa)$
- (2) $p \Vdash_{\mathbb{Q}(\kappa)} \square_1(\kappa)$

Proof. The implication (2) \Rightarrow (1) follows immediately from the observation that a stationary subset of κ that does not reflect at any inaccessible cardinal is a thread through any putative $\square_1(\kappa)$ -sequence.

We now show (1) \Rightarrow (2). Suppose for the sake of contradiction that $p \Vdash_{\mathbb{Q}(\kappa)} \text{Refl}_0(\kappa)$ and there is $p_1 \leq_{\mathbb{Q}(\kappa)} p$ such that $p_1 \Vdash_{\mathbb{Q}(\kappa)} \neg \square_1(\kappa)$. In particular, p_1 forces that $\bigcup \dot{G}$ is not a $\square_1(\kappa)$ -sequence, so there is a $\mathbb{Q}(\kappa)$ -name \dot{C} that is forced by p_1 to be a thread through $\bigcup \dot{G}$.

Let G be $\mathbb{Q}(\kappa)$ -generic over V with $p_1 \in G$, and move to $V[G]$. Let $C = \dot{C}_G$. Since C is stationary in κ and $\text{Refl}_0(\kappa)$ holds, Proposition 5.10 implies that there are stationarily many inaccessible $\lambda < \kappa$ such that C reflects at λ and $\text{Refl}_0(\lambda)$ fails.

Next, observe that every sequence of elements of G of size less than κ has a lower bound in G . Suppose that $\beta < \kappa$, and fix in $V[G]$ a sequence $\vec{p} = \langle p_\xi : \xi < \beta \rangle$ of elements of G . The sequence \vec{p} must be in V by the $< \kappa$ -distributivity of $\mathbb{Q}(\kappa)$, and so there is a condition $p \in G$ forcing that \vec{p} is contained in G . But then p is a lower bound for \vec{p} . Observe also that, for all $\gamma < \kappa$, the initial segment $C^{(\gamma)} = C \cap \gamma$ of C is in V .

Now, in $V[G]$, we build a strictly decreasing sequence of conditions $\langle q_\alpha : \alpha < \kappa \rangle$ from G such that

- (1) $q_0 = p_1$,
- (2) $\{\gamma^{q_\alpha} : \alpha < \kappa\}$, the set of suprema of the domains of the conditions, is a club and
- (3) for all $\alpha < \kappa$, $q_{\alpha+1} \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \gamma^{q_\alpha} = \check{C}^{(\alpha)}$.

We can ensure that (2) holds as follows. At a limit stage $\lambda < \kappa$, given that we have already constructed $\langle q_\alpha : \alpha < \lambda \rangle$, we know that there is some $q \in G$ below our sequence. So we let $\gamma_\lambda = \bigcup_{\alpha < \lambda} \gamma_\alpha$ and take $q_\lambda = q \upharpoonright \gamma_\lambda + 1$.

Thus, we can find an inaccessible cardinal λ such that $\lambda = \gamma^{q_\lambda}$, C reflects at λ , and $\text{Refl}_0(\lambda)$ fails. Since $\text{Refl}_0(\lambda)$ fails (in $V[G]$ and hence also in V , since forcing with $\mathbb{Q}(\kappa)$ did not add any bounded subsets to κ), we can fix in V a stationary $C_\lambda \subseteq \lambda$ that is different from $C^{(\lambda)} = C \cap \lambda$ and that does not reflect at any inaccessible cardinal below λ . Now form a condition $q_\lambda^* \in \mathbb{Q}$ with $\gamma^{q_\lambda^*} = \gamma^{q_\lambda} = \lambda$ by letting $q_\lambda^* \upharpoonright \lambda = \bigcup_{\alpha < \lambda} q_\alpha$ and $C_\lambda^{q_\lambda^*} = C_\lambda$. This is easily seen to be a valid condition, because everything needed to construct it is in V and since C_λ does not reflect at any inaccessible cardinal. Since $q_\lambda^* \leq_{\mathbb{Q}(\kappa)} q_\alpha$ for all $\alpha < \lambda$, we have

$$q_\lambda^* \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \lambda = \check{C}^{(\lambda)}.$$

In particular, since $C^{(\gamma)} = C \cap \lambda$ is stationary in λ , and since q_λ^* extends p_1 and thus forces that \dot{C} is a thread through $\bigcup \dot{G}$, it must be the case that q_λ^* forces that the λ -th entry in $\bigcup \dot{G}$ is $C^{(\lambda)}$. However, q_λ^* forces the λ -th entry in $\bigcup \dot{G}$ to be C_λ , which is different from $C \cap \lambda$. This gives the desired contradiction. \square

Remark 5.12. Since the weak compactness of κ implies $\text{Refl}_0(\kappa)$, by Theorem 5.11 it follows that in the proof of Theorem 5.6, in order to show that $\square_1(\kappa)$ holds in $V[G * H]$ it suffices to show that κ remains weakly compact.

6. CONSISTENCY OF $\square_1(\kappa)$ WITH $\text{Refl}_1(\kappa)$

In this section, we will show that the principle $\square_1(\kappa)$ is consistent with $\text{Refl}_1(\kappa)$. First, we will need a lemma showing that we can force the existence of a fast function while preserving Π_2^1 -indescribability.

The fast function forcing \mathbb{F}_κ , introduced by Woodin, consists of conditions that are partial functions $p : \kappa \rightarrow \kappa$ such that for every $\gamma \in \text{dom}(p)$, the following statements hold:

- γ is inaccessible,
- $p \restriction \gamma \subseteq \gamma$, and
- $|p \upharpoonright \gamma| < \gamma$.

The union $f : \kappa \rightarrow \kappa$ of a generic filter G for \mathbb{F}_κ is called a *fast function*. Since G can clearly be recovered from f , we will often conflate the two. For example, $V[f]$ and $V[G]$ are the same model, and, if $\tau \in V$ is an \mathbb{F}_κ -name, then we will sometimes write τ_f for the interpretation of τ in $V[G]$. Let $\mathbb{F}_{[\gamma, \kappa]}$ denote the subset of \mathbb{F}_κ consisting of conditions p with $\text{dom}(p) \subseteq [\gamma, \kappa)$ and observe that $\mathbb{F}_{[\gamma, \kappa]}$ is $\leq \gamma$ -closed. It is not difficult to see that for any condition $p \in \mathbb{F}_\kappa$ and $\gamma \in \text{dom}(p)$, the forcing \mathbb{F}_κ factors below p as

$$\mathbb{F}_\kappa \upharpoonright p \cong \mathbb{F}_\kappa \upharpoonright (p \upharpoonright \gamma) \times \mathbb{F}_{[\gamma, \kappa]} \upharpoonright (p \upharpoonright [\gamma, \kappa)).$$

Lemma 6.1. *Suppose κ is Π_n^1 -indescribable. In a generic extension $V[f]$ by fast function forcing, κ remains Π_n^1 -indescribable and the fast function f has the following property. For every $A \in H(\kappa^+)$ and $\alpha < \kappa^+$, there are a κ -model M with $f, A \in M$, a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $j(f)(\kappa) = \alpha$ and $j, M \in N$.*

Proof. The cardinal κ remains inaccessible in $V[f]$ because for unboundedly many inaccessible $\alpha < \kappa$, there is a condition $p \in G$ with $\alpha \in \text{dom } p$, so \mathbb{F}_κ below p factors with a first factor of size α and a second factor that is $\leq \alpha$ -closed.

Fix $A \in H(\kappa^+)^{V[f]}$ and $\alpha < \kappa^+$ (note that V and $V[f]$ have the same κ^+). Let \dot{A} be an \mathbb{F}_κ -name for A and let $B \subseteq \kappa$ code α . By Theorem 2.6 (4), there are a κ -model M with $\mathbb{F}_\kappa, \dot{A}, B \in M$, a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $j, M \in N$. We will lift j to $M[G]$. Let $p = \langle \kappa, \alpha \rangle$ be a condition in $j(\mathbb{F}_\kappa)$. Below p , $j(\mathbb{F}_\kappa)$ factors as $j(\mathbb{F}_\kappa) \upharpoonright p \cong \mathbb{F}_\kappa \times \mathbb{F}_{[\kappa, j(\kappa)]} \upharpoonright p$, where the second factor is $\leq \kappa$ -closed in N . In V , we can build an N -generic function f' for $\mathbb{F}_{[\kappa, j(\kappa)]}$ containing p , and so $f \times f'$ is N -generic for $j(\mathbb{F}_\kappa)$. Thus, we can lift j to $j : M[f] \rightarrow N[f][f']$, and clearly $M[f]$ and j are in $N[f][f']$.

It remains to verify that $M[f]$ is a κ -model and $N[f][f']$ is a Π_{n-1}^1 -correct κ -model. The argument to show that $M[f]$ is a κ -model in $V[f]$ will be more involved than usual because, as \mathbb{F}_κ is not κ -c.c., we cannot apply the generic closure criterion. Fixing $\beta < \kappa$, we will show that $M[f]^\beta \subseteq M[f]$ in $V[f]$. By density, there is an inaccessible cardinal $\alpha > \beta$ and a condition $p = \langle \gamma, \delta \rangle \in G$ such that $\gamma < \alpha < \delta$. Below p , \mathbb{F}_κ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta, \kappa)}$ and f factors as $f_\gamma \times f_{(\delta, \kappa)}$. Since \mathbb{F}_γ clearly has the α -c.c., by the generic closure criterion, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f_\gamma]$. Also, since $\mathbb{F}_{(\delta, \kappa)}$ is $\leq \alpha$ -closed, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f]$. Finally, by the ground closure criterion, $M[f_\gamma][f_{(\delta, \kappa)}]^\beta \subseteq M[f_\gamma][f_{(\delta, \kappa)}]$ in $V[f]$. The same argument shows that $N[f]$ is a κ -model in $V[f]$, and therefore, $N[f][f']$ is a κ -model as well. To show that $N[f]$ is Π_{n-1}^1 -correct, we argue essentially as in the proof of Theorem 3.7. Arguments from [Fri00] (section ‘‘Reverse Easton Forcing’’ in Chapter 2) show that since \mathbb{F}_κ can be factored as a forcing with an arbitrarily highly closed tail, it is pretame over V_κ , giving the definability of the forcing relation. Note also that we have $V_\kappa^{V[f]} = V_\kappa^{N[f]} = V_\kappa[f]$. Finally, the model $N[f][f']$ must also be Π_{n-1}^1 -correct because the tail forcing $\mathbb{F}_{[\kappa, j(\kappa)]}$ does not add any subsets to $V_\kappa[f]$ by closure. \square

It is not difficult to see that once we have a fast function, we also get a weak Laver function [Ham02].

Lemma 6.2. *Suppose κ is Π_n^1 -inaccessible. In the generic extension $V[f]$ by fast function forcing, there is a function $\ell : \kappa \rightarrow V_\kappa$ satisfying the following property. For all $A, B \in H(\kappa^+)^{V[f]}$, there are a κ -model M with $\ell, A, B \in M$, a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $j(\ell)(\kappa) = B$ and $j, M \in N$.*

Proof. Fix any bijection $b : \kappa \rightarrow V_\kappa$ in V . In $V[f]$, define $\ell : \kappa \rightarrow V_\kappa$ by letting $\ell(\gamma) = b(f(\gamma))_{f \upharpoonright \gamma}$ provided that $f \upharpoonright \gamma$ is \mathbb{F}_γ -generic over V and $b(f(\gamma))$ is an \mathbb{F}_γ -name. By adapting the proof of Lemma 6.1, we verify that ℓ has the desired properties as follows. Working in $V[f]$, fix $A, B \in H(\kappa^+)^{V[f]}$ and let $\dot{\ell}, \dot{A}, \dot{B}$ be nice \mathbb{F}_κ -names for ℓ, A and B respectively. By Theorem 2.6 (4), there are a κ -model M with $\dot{\ell}, \dot{A}, \dot{B}, \mathbb{F}_\kappa, b \in M$, a Π_{n-1}^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $j, M \in N$. Since \mathbb{F}_κ is κ^+ -c.c., we can assume without loss of generality that $\dot{B} \in j(V_\kappa)$. By elementarity $j(b) : j(\kappa) \rightarrow j(V_\kappa)$ is a bijection, and thus there is some ordinal $\alpha < j(\kappa)$ such that $j(b)(\alpha) = \dot{B}$. As in the proof of Lemma 6.1, we may lift j to $j : M[f] \rightarrow N[f][f']$ such that $j(f)(\kappa) = \alpha$. Now we have $j(\ell)(\kappa) = j(b)(j(f)(\kappa))_{j(f) \upharpoonright \kappa} = j(b)(\alpha)_f = \dot{B}_f = B$.

Now one may prove that $M[f]$ is a κ -model and $N[f][f']$ is a Π_{n-1}^1 -correct κ -model exactly as in the proof of Lemma 6.1. \square

Theorem 1.2. *Suppose κ is Π_2^1 -indescribable and GCH holds. Then there is a cofinality-preserving forcing extension $V[G]$ in which*

- (1) $\square_1(\kappa)$ holds,
- (2) $\text{Ref}_1(\kappa)$ holds and
- (3) κ is κ^+ -weakly compact.

Proof. By passing to an extension with a fast function, we can assume without loss of generality that there is a function $\ell: \kappa \rightarrow V_\kappa$ such that for any $A, B \in H(\kappa^+)$ there are a κ -model M with $\ell, A, B \in M$, a Π_1^1 -correct κ -model N and an elementary embedding $j: M \rightarrow N$ with critical point κ such that $j(\ell)(\kappa) = B$.

Let $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the Easton-support iteration defined as follows.

- If $\alpha < \kappa$ is inaccessible and $\ell(\alpha)$ is a \mathbb{P}_α -name for an α -strategically closed, α^+ -c.c. forcing notion, then $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$.
- Otherwise, $\dot{\mathbb{Q}}_\alpha$ is a \mathbb{P}_α -name for trivial forcing.

Let G be generic for \mathbb{P}_κ over V . In $V[G]$, we define a 2-step iteration

$$\mathbb{Q}_\kappa = \mathbb{Q}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$$

as follows.

- $\mathbb{Q}_{\kappa,0}$ is the forcing to add a $\square_1(\kappa)$ -sequence from Definition 5.1.
- $\dot{\mathbb{Q}}_{\kappa,2}$ is a $\mathbb{Q}_{\kappa,0}$ -name for the forcing $\mathbb{T}(\vec{C}(\kappa))$ to thread the generic $\square_1(\kappa)$ -sequence.
- $\dot{\mathbb{Q}}_{\kappa,1}$ is a $\mathbb{Q}_{\kappa,0}$ -name for an iteration $\langle \mathbb{R}_\eta, \dot{S}_\xi : \eta \leq \kappa^+, \xi < \kappa^+ \rangle$ with supports of size $< \kappa$ defined as follows. For each $\eta < \kappa^+$, a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$ -name \dot{S}_η is chosen for a stationary subset of κ such that

$$\Vdash_{\mathbb{Q}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})} \text{“there is a 1-club in } \kappa \text{ disjoint from } \dot{S}_\eta \text{”},$$

and then \dot{S}_η is a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$ -name for the forcing $T^1(\kappa \setminus \dot{S}_\eta)$ to shoot a 1-club through the complement of \dot{S}_η .

Notice that \mathbb{P}_κ is κ -c.c. and preserves GCH and, in $V[G]$, the forcing \mathbb{Q}_κ is κ^+ -c.c.. We also claim that \mathbb{Q}_κ is κ -strategically closed in $V[G]$. To see this, first note that, in $V[G]$, $\mathbb{Q}_{\kappa,0}$ is κ -strategically closed by Proposition 5.2. Next, $\dot{\mathbb{Q}}_{\kappa,2}$ is a $\mathbb{Q}_{\kappa,0}$ -name for a κ -strategically closed forcing, by Lemma 5.4. Finally, $\dot{\mathbb{Q}}_{\kappa,1}$ is a $\mathbb{Q}_{\kappa,0}$ -name for a $(< \kappa)$ -support iteration, each iterand of which is, by Lemma 4.1, κ -strategically closed. A standard argument, in which Player II plays according to the respective winning strategy on each coordinate, shows that such an iteration must itself be κ -strategically closed. It then similarly follows that $\mathbb{Q}_\kappa = \mathbb{Q}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$ is κ -strategically closed in $V[G]$.

By standard chain condition arguments and bookkeeping, we can ensure that in $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * \dot{\mathbb{Q}}_{\kappa,1}}$, if $S \subseteq \kappa$ is stationary and

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is a 1-club in } \kappa \text{ disjoint from } \check{S} \text{”},$$

then there is already a 1-club in κ disjoint from S .

Let $H = h_0 * (h_1 \times h_2)$ be generic for \mathbb{Q}_κ over $V[G]$. Our desired model will be $V[G * h_0 * h_1]$. We must show that in $V[G * h_0 * h_1]$, κ is κ^+ -weakly compact, $\text{Refl}_1(\kappa)$ holds and $\square_1(\kappa)$ holds.

In order to show that κ is κ^+ -weakly compact in $V[G * h_0 * h_1]$, we will first prove the following.

Claim 6.3. κ is Π_2^1 -indescribable in $V[G * H]$.

Proof. Fix $A \in P(\kappa)^{V[G * H]}$. We must find a κ -model M with $A \in M$, a Π_1^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ .

Let $\dot{A} \in V$ be a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ -name for A . Since $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$ has the κ^+ -c.c., we can fix $\eta < \kappa^+$ such that \dot{A} is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ -name. Moreover, we can assume that \dot{A} , \mathbb{P}_κ and $\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ are in $H(\kappa^+)$. For $\eta < \kappa^+$, let $h_1 \upharpoonright \eta$ be the generic for \mathbb{R}_η induced by h_1 .

By Proposition 6.2, there are a κ -model M with $\ell, \mathbb{P}_\kappa, \dot{A}, \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2}) \in M$, a Π_1^1 -correct κ -model N and an elementary embedding $j : M \rightarrow N$ with critical point κ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ and $j, M \in N$. Without loss of generality we may additionally assume that $M \models |\eta| = \kappa$ since a bijection witnessing this can easily be placed into such a κ -model.

Notice that $j(\mathbb{P}_\kappa)$ is an Easton-support iteration in N of length $j(\kappa)$ and

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})) * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$$

by our choice of $j(l)(\kappa)$. By Theorem 3.7, $N[G]$ is Π_1^1 -correct in $V[G]$, and by Corollary 3.8, $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is Π_1^1 -correct in $V[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Hence $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is Π_1^1 -correct in $V[G * h_0 * (h_1 \times h_2)]$ by Corollary 3.4.

Since $(\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G * h_0 * (h_1 \upharpoonright \eta \times h_2)} = \dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is κ -strategically closed in $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ and since $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$ is a κ -model in $V[G * H]$, we can build a filter $G'_{\kappa,j(\kappa)}$ which is generic for $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ over $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Since $j \upharpoonright G$ is the identity function, it follows that

$$j \upharpoonright G \subseteq \hat{G} =_{\text{def}} G * h_0 * (h_1 \upharpoonright \eta \times h_2) * G_{\kappa,j(\kappa)},$$

and thus j lifts to $j : M[G] \rightarrow N[\hat{G}]$.

Let $\vec{C} = \langle C_\alpha : \alpha \in \text{inacc}(\kappa) \rangle$ be the generic $\square_1(\kappa)$ -sequence added by h_0 , and let T be the thread added by $h_2 \subseteq \mathbb{Q}_{\kappa,2} = \mathbb{T}(\vec{C}(\kappa))$. By Lemma 5.5, T is a 1-club in $N[G * h_0 * (h_1 \upharpoonright \eta \times h_2)]$. Let $p_0 = \vec{C} \cup \{(\kappa, T)\}$. Then $p_0 \in N[\hat{G}]$ and $p_0 \in j(\mathbb{Q}_{\kappa,0})$. Moreover, $p_0 \leq_{j(\mathbb{Q}_{\kappa,0})} j(q)$ for all $q \in h_0$. By the strategic closure of $j(\mathbb{Q}_{\kappa,0})$ and the fact that $N[\hat{G}]$ is a κ -model in $V[G * H]$, we can build a filter $p_0 \in \hat{h}_0 \subseteq j(\mathbb{Q}_{\kappa,0})$ which is generic over $N[\hat{G}]$. Thus, j extends to $j : M[G * h_0] \rightarrow N[\hat{G} * \hat{h}_0]$.

Similarly, in $N[\hat{G} * \hat{h}_0]$, the set $p_2 = T \cup \{\kappa\}$ is a condition in $j(\mathbb{Q}_{\kappa,2})$ and $p_2 \leq_{j(\mathbb{Q}_{\kappa,2})} j(q)$ for all $q \in h_2$. Again, since $j(\mathbb{Q}_{\kappa,2})$ is κ -strategically closed in $N[\hat{G} * \hat{h}_0]$, which is a κ -model in $V[G * H]$, we can build a filter $p_2 \in \hat{h}_2 \subseteq j(\mathbb{Q}_{\kappa,2})$ which is generic for $j(\mathbb{Q}_{\kappa,2})$ over $N[\hat{G} * \hat{h}_0]$, and lift j to

$$j : M[G * h_0 * h_2] \rightarrow N[\hat{G} * \hat{h}_0 * \hat{h}_2].$$

Now we lift the embedding through $h_1 \upharpoonright \eta$. Let $\mathbb{R}_\eta = (\dot{\mathbb{R}}_\eta)_{G * h_0}$. By elementarity, $j(\mathbb{R}_\eta)$ is an iteration of length $j(\eta)$ with supports of size less than $j(\kappa)$. For each $\xi < \eta$, $\dot{\mathbb{S}}_{j(\xi)} = j(\dot{\mathbb{S}}_\xi)$ is, in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, a $j(\mathbb{R}_\xi) = \mathbb{R}_{j(\xi)}$ -name for the forcing to

shoot a 1-club disjoint from $j(\dot{S}_\xi)$. For all $\xi < \eta$, let

$$D_\xi = \bigcup \{(p(\xi))_{h_1 \upharpoonright \xi} : p \in h_1 \text{ and } \xi \in \text{dom}(p)\}$$

and note that D_ξ is a 1-club subset of κ in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ because the forcing after $\mathbb{R}_{\xi+1}$ is κ -strategically closed and therefore cannot affect Π_1^1 -truths by Lemma 3.3. Since $h_1 \upharpoonright \eta, j \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, we can define a function $p^* \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ such that $\text{dom}(p^*) = j \upharpoonright \eta$ by letting $p^*(j(\xi))$ be a $j(\mathbb{R}_\xi)$ -name for $D_\xi \cup \{\kappa\}$ for all $\xi < \eta$. In order to verify that $p^* \in j(\mathbb{R}_\eta)$, we must show that for all $\xi < \eta$, $p^* \upharpoonright j(\xi) \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \emptyset$.

Suppose this is not the case, and let $\xi < \eta$ be the minimal counterexample. It follows that $p^* \upharpoonright j(\xi) \in j(\mathbb{R}_\xi)$ and, for all $p \in h_1 \upharpoonright \xi$ we have $p^* \upharpoonright j(\xi) \leq j(p)$. By assumption,

$$p^* \upharpoonright j(\xi) \not\Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \emptyset$$

and thus we may let $p^{**} \leq_{j(\mathbb{R}_\xi)} p^* \upharpoonright j(\xi)$ be such that

$$p^{**} \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) \neq \emptyset.$$

Since $j(\mathbb{R}_\xi)$ is sufficiently strategically closed, we can build a filter $\hat{h}_1 \subseteq j(\mathbb{R}_\xi)$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ with $p^{**} \in \hat{h}_1$ and lift to

$$j : M[G * h_0 * h_2 * (h_1 \upharpoonright \xi)] \rightarrow N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

It follows that in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ we have $(D_\xi \cup \{\kappa\}) \cap j(S_\xi) \neq \emptyset$, where $S_\xi = (\dot{S}_\xi)_{h_0 \upharpoonright \xi}$. Since $j(S_\xi) \cap \kappa = S_\xi$, we know that $D_\xi \cap j(S_\xi) = \emptyset$, so it must be the case that $\kappa \in j(S_\xi)$. However, in $M[G * h_0 * (h_1 \upharpoonright \xi)]$, we have

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is a 1-club in } \kappa \text{ disjoint from } \check{S}_\xi \text{”}.$$

Therefore, we can fix such a 1-club E in $M[G * h_0 * h_2 * (h_1 \upharpoonright \xi)]$. Note that E is actually stationary because $M[G * h_0 * h_2 * (h_1 \upharpoonright \xi)]$ is Π_1^1 -correct by Theorem 3.7 and Corollary 3.8. But then $\kappa \in j(E)$ since in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$, $j(E)$ is 1-club in $j(\kappa)$ and $j(E) \cap \kappa = E$ is stationary in κ . Thus $\kappa \in j(E) \cap j(S_\xi) = \emptyset$, a contradiction.

Thus, $p^* \in j(\mathbb{R}_\eta)$ and we can build a filter $p^* \in \hat{h}_1$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$. This implies that the embedding lifts to

$$j : M[G * h_0 * h_2 * (h_1 \upharpoonright \eta)] \rightarrow N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

As we argued above, $N[G * h_0 * h_2 * (h_1 \upharpoonright \eta)]$ is Π_1^1 -correct in $V[G * H]$, and since the forcing

$$\mathbb{P}_{(\kappa, j(\kappa))} * j(\dot{\mathbb{Q}}_{\kappa,0}) * (j(\dot{\mathbb{Q}}_{\kappa,2}) \times j(\dot{\mathbb{R}}_\eta))$$

is $\leq \kappa$ -distributive, it follows that $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ is Π_1^1 -correct in $V[G * H]$. Since $A = \dot{A}_{G * h_0 * (h_1 \upharpoonright \eta * h_2)} \in M[G * h_0 * h_2 * (h_1 \upharpoonright \eta)]$, this shows that κ is Π_2^1 -indescribable in $V[G * H]$. \square

Now let us argue that κ is κ^+ -weakly compact in $V[G * h_0 * h_1]$. Fix $\zeta < \kappa^+$. We must argue that $\text{Tr}_1^\zeta([\kappa])^{V[G * h_0 * h_1]} \neq \emptyset$. Since κ is Π_2^1 -indescribable in $V[G * h_0 * (h_1 \times h_2)]$ by Claim 6.3, and since $\mathbb{Q}_{\kappa,2}$ is κ -strategically closed, it follows that $\text{Tr}_1^\zeta([\kappa])^{V[G * h_0 * (h_1 \times h_2)]} = [S]$, where $S \in V[G * h_0 * h_1]$ is Π_2^1 -indescribable in $V[G * h_0 * (h_1 \times h_2)]$. It follows that S is weakly compact in $V[G * h_0 * h_1]$ by

Proposition 3.5, and clearly $\text{Tr}_1^\zeta([\kappa])^{V[G * h_0 * h_1]} = [S]$. Thus, κ is κ^+ -weakly compact in $V[G * h_0 * h_1]$.

We next argue that $\text{Refl}_1(\kappa)$ holds in $V[G * h_0 * h_1]$. Fix a weakly compact set $S \subseteq \kappa$ in $V[G * h_0 * h_1]$. Since S intersects every 1-club in κ , our construction of $\mathbb{Q}_{\kappa,1}$ implies that there is $p \in \mathbb{Q}_{\kappa,2}$ such that

$$p \Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is no 1-club in } \kappa \text{ disjoint from } \check{S}\text{”}.$$

Let $g_2 \subseteq \mathbb{Q}_{\kappa,2}$ be generic over $V[G * h_0 * h_1]$ with $p \in g_2$. By the proof of Claim 6.3, κ is Π_2^1 -indescribable in $V[G * h_0 * h_1 * g_2]$. Therefore, in $V[G * h_0 * h_1 * g_2]$, $\text{Refl}_1(\kappa)$ holds and S is a weakly compact subset of κ , and thus there is some $\alpha < \kappa$ such that $S \cap \alpha$ is a weakly compact subset of α . But $V[G * h_0 * h_1 * g_2]$ and $V[G * h_0 * h_1]$ have the same V_κ , so $S \cap \alpha$ is a weakly compact subset of α in $V[G * h_0 * h_1]$. Thus, $\text{Refl}_1(\kappa)$ holds in $V[G * h_0 * h_1]$.

Finally, we argue that $\square_1(\kappa)$ holds in $V[G * h_0 * h_1]$. The sequence

$$\bigcup h_0 = \vec{C} = \langle C_\alpha : \alpha \in \text{inacc}(\kappa) \rangle$$

is a $\square_1(\kappa)$ -sequence in $V[G * h_0]$ by Theorem 5.11 because we can show that $\text{Refl}_0(\kappa)$ holds by essentially the same argument as for $\text{Refl}_1(\kappa)$ above. Suppose that \vec{C} is no longer a $\square_1(\kappa)$ -sequence in $V[G * h_0 * h_1]$. This implies that there is a condition $p \in h_1$ such that in $V[G * h_0]$,

$$p \Vdash_{\mathbb{Q}_{\kappa,1}} \text{“there is a 1-club } \dot{E} \subseteq \check{\kappa} \text{ that threads } \vec{C}\text{”}.$$

Let g_1 be generic for $\mathbb{Q}_{\kappa,1}$ over $V[G * h_0 * h_1]$ with $p \in g_1$. In $V[G * h_0 * (h_1 \times g_1)]$, let $E = \dot{E}_{h_1}$ and $E^* = \dot{E}_{g_1}$. By mutual genericity, we may fix $\alpha \in E \setminus E^*$. A proof almost identical to that of Claim 6.3 shows that κ is Π_2^1 -indescribable in $V[G * h_0 * (h_1 \times g_1 \times h_2)]$ and hence weakly compact in $V[G * h_0 * (h_1 \times g_1)]$. Now, in $V[G * h_0 * (h_1 \times g_1)]$, fix any $j : M \rightarrow N$ with critical point κ and $E, E^* \in M$. Since both are 1-clubs, $\kappa \in j(E) \cap j(E^*)$, and so by elementarity there is an inaccessible $\beta \in \kappa \setminus (\alpha + 1)$ such that $E \cap \beta$ and $E^* \cap \beta$ are both stationary in β . But then, as they both thread \vec{C} , it must be the case that $E \cap \beta = C_\beta = \hat{E} \cap \beta$. This contradicts the fact that $\alpha \in E \setminus E^*$ and finishes the proof of the theorem. \square

Remark 6.4. Observe that κ cannot be Π_2^1 -indescribable in $V[G * h_0 * h_1]$ because $\square_1(\kappa)$ holds there. Thus, the set S , where $\text{Tr}_1^\zeta([\kappa])^{V[G * h_0 * h_1]} = [S]$, cannot be Π_2^1 -indescribable in $V[G * h_0 * h_1]$, which shows that Proposition 3.5 can fail for Π_2^1 -indescribable sets.

7. AN APPLICATION TO SIMULTANEOUS REFLECTION

In this section we will show that the simultaneous reflection principle $\text{Refl}_n(\kappa, 2)$ is incompatible with $\square_n(\kappa)$.

Theorem 7.1. *Suppose that $1 \leq n < \omega$, κ is Π_n^1 -indescribable and $\square_n(\kappa)$ holds. Then there are two Π_n^1 -indescribable subsets $S_0, S_1 \subseteq \kappa$ that do not reflect simultaneously, i.e., there is no $\beta < \kappa$ such that $S_0 \cap \beta$ and $S_1 \cap \beta$ are both Π_n^1 -indescribable subsets of β .*

Proof. Suppose for the sake of contradiction that every pair of Π_n^1 -indescribable subsets of κ reflects simultaneously. Already, $\text{Refl}_n(\kappa)$ implies that the set $E = \{\alpha < \kappa : \text{Refl}_{n-1}(\alpha) \text{ holds}\}$ is a Π_n^1 -indescribable subset of κ because the set of

Π_n^1 -indescribable cardinals below κ is Π_n^1 -indescribable and $(n-1)$ -reflection holds at each of them (see [Cod19]).

Let $\vec{C} = \langle C_\alpha : \alpha \in \text{Tr}_{n-1}(\kappa) \rangle$ be a $\square_n(\kappa)$ -sequence. For all $\alpha \in \text{Tr}_{n-1}(\kappa)$, let

$$S_\alpha^0 = \{\beta \in \text{Tr}_{n-1}(\kappa) \setminus (\alpha + 1) : C_\beta \cap \alpha \in \Pi_{n-1}^1(\alpha)^+\} \text{ and}$$

$$S_\alpha^1 = \text{Tr}_{n-1}(\kappa) \setminus ((\alpha + 1) \cup S_\alpha^0).$$

Let $A = \{\alpha \in \text{Tr}_{n-1}(\kappa) : S_\alpha^0 \in \Pi_n^1(\kappa)^+\}$.

Claim 7.2. *A is Π_n^1 -indescribable in κ .*

Proof. Fix an n -club $C \subseteq \kappa$. Since $\text{Tr}_{n-1}(C)$ is an n -club in κ , it follows that $E \cap \text{Tr}_{n-1}(C)$ is Π_n^1 -indescribable in κ . For each $\beta \in E \cap \text{Tr}_{n-1}(C)$, $\text{Refl}_{n-1}(\beta)$ holds and $C_\beta \cap C$ is a Π_{n-1}^1 -indescribable subset of β . Thus, for $\beta \in E \cap \text{Tr}_{n-1}(C)$, we may let α_β be the least Π_{n-1}^1 -indescribable cardinal such that $C_\beta \cap C \cap \alpha_\beta$ is Π_{n-1}^1 -indescribable in α_β . Notice that $\alpha_\beta \in C$ for all $\beta \in E \cap \text{Tr}_{n-1}(C)$ because C is an n -club. Since the map $\beta \mapsto \alpha_\beta$ is regressive on $E \cap \text{Tr}_{n-1}(C)$, it follows by the normality of $\Pi_n^1(\kappa)$ that there is a fixed $\alpha \in C$ and a Π_n^1 -indescribable set $T \subseteq E \cap \text{Tr}_{n-1}(C)$ such that $\alpha_\beta = \alpha$ for all $\beta \in T$. This implies that $T \subseteq S_\alpha^0$, and thus $\alpha \in A \cap C$. \square

Claim 7.3. *There is $\alpha \in A$ such that S_α^1 is a Π_n^1 -indescribable subset of κ .*

Proof. Suppose not, and let $\alpha_0 < \alpha_1$ be elements of A . Since E is Π_n^1 -indescribable in κ and $S_{\alpha_0}^1$ and $S_{\alpha_1}^1$ are both in the Π_n^1 -indescribability ideal on κ , we can find $\beta \in E \setminus ((\alpha_1 + 1) \cup S_{\alpha_0}^1 \cup S_{\alpha_1}^1)$. It follows that $\beta \in S_{\alpha_0}^0 \cap S_{\alpha_1}^0$, so, by the coherence properties of the $\square_n(\kappa)$ -sequence, we have $C_\beta \cap \alpha_0 = C_{\alpha_0}$ and $C_\beta \cap \alpha_1 = C_{\alpha_1}$, and hence $C_{\alpha_1} \cap \alpha_0 = C_{\alpha_0}$. But then by Lemma 2.7 and Claim 7.2, we see that $\bigcup_{\alpha \in A} C_\alpha$ is a Π_n^1 -indescribable subset of κ . Thus, $\bigcup_{\alpha \in A} C_\alpha$ is a thread through \vec{C} , which is a contradiction. \square

We can therefore fix $\alpha \in \text{Tr}_{n-1}(\kappa)$ such that both S_α^0 and S_α^1 are Π_n^1 -indescribable subsets of κ . Let $S_0 = S_\alpha^0$ and $S_1 = S_\alpha^1$. We claim that S_0 and S_1 cannot reflect simultaneously. Otherwise, there is γ such that $S_0 \cap \gamma$ and $S_1 \cap \gamma$ are both Π_n^1 -indescribable subsets of γ . Consider the n -club C_γ . Since γ is Π_n^1 -indescribable, $\text{Tr}_{n-1}(C_\gamma)$ is also an n -club in γ . We can therefore find $\beta_0 < \beta_1$ in $\text{Tr}_{n-1}(C_\gamma)$ such that $\beta_0 \in S_0$ and $\beta_1 \in S_1$. But note that $C_{\beta_0} = C_\gamma \cap \beta_0$ and $C_{\beta_1} = C_\gamma \cap \beta_1$, so $C_{\beta_0} = C_{\beta_1} \cap \beta_0$, contradicting the fact that $C_{\beta_0} \cap \alpha$ is Π_{n-1}^1 -indescribable in α whereas $C_{\beta_1} \cap \alpha$ is not Π_{n-1}^1 -indescribable in α . \square

As a direct consequence of Theorem 1.2 and Theorem 7.1 we obtain the following.

Corollary 7.4. *Suppose κ is Π_2^1 -indescribable. Then there is a forcing extension in which $\text{Refl}_1(\kappa)$ and $\neg \text{Refl}_1(\kappa, 2)$ both hold.*

8. QUESTIONS

The theorems proved in this article about the principle $\square_1(\kappa)$ do not easily generalize to $\square_n(\kappa)$ because several key technical results about Π_1^1 -indescribability which we used crucially in the proofs no longer hold for higher orders of indescribability. For example, given an embedding $j : M \rightarrow N$, where N is Π_n^1 -correct, we cannot necessarily use a generic G for a poset $\mathbb{P} \in N$ from the ground model to lift j because $N[G]$ may no longer be Π_n^1 -correct. An illustration of this is given in

Remark 5.7. Also, while κ -strategically closed forcing cannot make a subset of κ Π_1^1 -indescribable if it was not so already in the ground model by Proposition 3.5, a set can become Π_2^1 -indescribable after κ -strategically closed forcing by Remark 6.4.

Question 8.1. For $n > 1$, can we force from a strong enough large cardinal that κ is Π_n^1 -indescribable and $\square_n(\kappa)$ holds nontrivially?

Question 8.2. Relative to large cardinals, for $n > 1$, is it consistent that $\text{Refl}_n(\kappa)$ and $\square_n(\kappa)$ both hold?

Question 8.3. Relative to large cardinals, is it consistent that $\text{Refl}_1(\kappa, 2)$ holds but $\text{Refl}_1(\kappa, 3)$ fails?

Question 8.4. Can we force any indestructibility of $\text{Refl}_1(\kappa)$?

Question 8.5. For $1 \leq n < \omega$, if κ is Π_n^1 -indescribable, does our principle $\square_n(\kappa)$ imply the Brickhill-Welch principle $\square^n(\kappa)$? See Remark 2.3 for a discussion of $\square^n(\kappa)$.

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