RESEARCH STATEMENT

CHRIS LAMBIE-HANSON

To what extent does a mathematical structure's local behavior determine its global behavior? How much can one determine about a given structure by examining its small substructures? Such questions of compactness are among the most fundamental in modern mathematics, and my research is largely devoted to their investigation. My work lies primarily in logic and set theory, which are often particularly well suited to addressing questions of compactness, and in applications of set theoretic tools to other areas of mathematics, such as graph theory, algebra, and topology.

My set-theoretic work is largely combinatorial in nature and comes in two primary flavors: ZFC results and independence results. ZFC stands for *Zermelo-Fraenkel axioms with choice* and is the standard set of axioms used as the basis for set theory and for mathematics more broadly. Many interesting set-theoretic statements can be proven outright from the axioms of ZFC. However, it has been known since the 1930s [18] that, given any sufficiently strong, effectively axiomatizable formal system (such as ZFC), there are statements that can be neither proven nor disproven within the system. In 1963, Cohen [6] introduced the method of *forcing*, which, together with earlier work of Gödel [19], allowed him to show that the Continuum Hypothesis, the most famous open problem of set theory at the time, is undecidable by the axioms of ZFC. Since then, forcing, which allows one to construct new models of ZFC by introducing certain "generic" sets, has become a central technique in set theory, and I make extensive use of it in my work.

Another central concept in modern set theory, and in my work on independence results, is the notion of a *large cardinal*. Roughly speaking, a large cardinal is a type of infinite cardinal number whose consistent existence is not implied by ZFC. For example, a *weakly compact cardinal* is an uncountable cardinal κ for which the following generalization of Ramsey's theorem holds:

> Whenever the edges of the complete graph on κ vertices are colored with two colors, there is a complete subgraph of size κ, all of whose edges are the same color.

There is a great variety of large cardinal notions, and, rather strikingly, they form a largely linear hierarchy when ordered in terms of consistency strength. Assuming the existence of large cardinals can allow set theorists to prove certain consistency results that could not otherwise be obtained and, indeed, one can often show a natural infinitary combinatorial statement to be equiconsistent with the existence of a certain type of large cardinal. For example, the *tree property at* $\aleph_2$ ($\mathrm{TP}(\aleph_2)$), which is a generalization of König's infinity lemma, is equiconsistent over ZFC with the existence of a weakly compact cardinal [39]. In other words, if there is a model of ZFC in which $\mathrm{TP}(\aleph_2)$ holds, then there is a model in which there is a weakly compact cardinal, and vice versa.

My recent work applying set theoretic tools to other fields has focused in particular on graph theory and homological algebra. In graph theory, my work has been around questions about the extent to which global behavior of infinite graphs can be determined by looking at small subgraphs. In a recent result [29], for instance, I resolved an open question of Erdős, Hajnal, and Szemerédi about the growth rates of chromatic numbers of finite subgraphs of uncountably chromatic graphs. My work in homological algebra is joint with Jeffrey Bergfalk and has focused on the investigation of nontrivial coherent set theoretic objects which arise from considerations in algebraic topology. In one instance [5], we solved a long-standing open problem arising from investigations into the additivity of strong homology (and also recently independently from work of Clausen and Scholze on condensed mathematics).

Notation: We recall here some basic notions and notations. An *ordinal* is the order-type of a well-ordered set; in practice, an ordinal is identified with the set of all ordinals less than it. A *cardinal* is an equivalence class of sets under the equivalence relation of "having a bijection between." Cardinals are identified with the least ordinal of their cardinality. $\aleph_0$ is the smallest infinite cardinal number, $\aleph_1$ is the next smallest cardinal

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number, and so on. If κ is a cardinal, then κ^+ denotes the smallest cardinal greater than κ. If X is a set, then $|X|$ denotes the cardinality of X. If X is a set and κ is a cardinal, then $[X]^\kappa$ denotes the collection of all subsets of X of cardinality κ.

1. Compactness and incompactness in set theory

Questions about compactness (and, dually, reflection) are among the most fundamental that can be asked about classes of mathematical structures and concern the extent to which we can learn about the global properties of a structure simply by examining its local behavior. Compactness questions often take the following general form:

> Suppose a structure is such that all (or most) of its "small" substructures have a certain property. Must the entire structure have the same property?

The study of compactness principles has played a major role in modern mathematics. A number of seminal theorems of the twentieth century center around instances of compactness in a finitary setting. These include:

- König's infinity lemma, which asserts that every infinite tree with finite levels has an infinite branch.
- The compactness theorem for first-order logic.
- Tychonoff's theorem, which states that any product of compact topological spaces is compact.
- The de Bruijn-Erdős compactness theorem: if k is a natural number and $\mathcal{G}$ is a graph, all of whose finite subgraphs have chromatic number at most k, then $\mathcal{G}$ has chromatic number at most k.

Taken together, these theorems assert that the cardinal $\aleph_0$, the smallest infinite cardinal, exhibits a high degree of compactness. When the cardinal context of these theorems is shifted up one level, though, they typically become false:

- There is always a tree of height $\aleph_1$ with countable levels and no branch of size $\aleph_1$.
- The logic $L_{\omega_1\omega}$, which is obtained from first order logic by allowing countably infinite disjunctions and conjunctions, fails to be compact.
- There are two Lindelöf topological spaces whose product fails to be Lindelöf.
- Under rather mild hypotheses, there is a graph $\mathcal{G}$ of size $\aleph_2$ such that all subgraphs of size $\aleph_1$ have countable chromatic number but $\mathcal{G}$ has uncountable chromatic number.

These results, in turn, assert that the cardinal $\aleph_1$ exhibits a high degree of *incompactness*.

For cardinals greater than $\aleph_1$, questions of compactness become more complicated, and they are inextricably linked with large cardinals and independence over ZFC. For example, the assertion that every tree of height $\aleph_2$ with levels of size $\aleph_1$ has a branch of size $\aleph_2$ is equiconsistent over ZFC with the existence of a weakly compact cardinal. For another, if κ is an uncountable cardinal, then the compactness of the logic $L_{\kappa\kappa}$, which extends first-order logic by allowing conjunctions, disjunctions, and chains of quantifiers of any length less than κ, is equivalent to κ being a strongly compact cardinal.

Compactness and incompactness phenomena provide a useful way of organizing and classifying a great deal of set theoretic ideas. Canonical inner models such as L, for example, typically exhibit a great deal of incompactness, whereas large cardinals or forcing axioms such as the Proper Forcing Axiom (PFA) or Martin's Maximum (MM), which are strong generalizations of the Baire Category Theorem, typically imply instances of compactness. The tension between compactness and incompactness phenomena and the intricate web of implications and non-implications that exists among various compactness and incompactness statements has provided and continues to provide fertile ground for set theoretic research, with applications not just in set theory and logic but throughout mathematics.

1.1. **Constructions with small approximations.** A portion of my set theoretic work is devoted to the development of tools for the direct construction of incompact objects, i.e., structures whose behavior differs dramatically from the behavior of their small substructures. For an infinite cardinal κ, it is often helpful when constructing incompact objects of size κ^+ to be able to piece together these objects from small approximations, i.e., approximations of size $< \kappa$. Such constructions were done, making use of combinatorial objects known as *morasses*, by Shelah and Stanley [48, 49] and by Velleman [51]. Shortly thereafter, similar constructions were carried out using variants of the guessing principle known as *diamond* by Shelah et al., culminating in [47].

One prominent area in which these constructions are relevant is the study of generalizations of the Souslin Hypothesis (SH). SH asserts that every complete, dense linear order without endpoints and with the countable chain condition is isomorphic to the real numbers. SH has inspired a tremendous variety of research in set theory and was eventually found to be independent of ZFC. SH can be seen as $\mathrm{SH}_{\aleph_1}$ and has natural generalizations SH_κ for regular cardinals $\kappa > \aleph_1$. SH_κ can also be thought of as a weakening of the tree property at κ.

In joint work with Rinot [32], we developed a new technique for carrying out such constructions, using a combination of a square principle and a diamond principle. We encapsulate this technique in a new principle (of a sort known as a *forcing axiom*), which we call $\mathrm{SDFA}(\mathcal{P}_\kappa)$. We show that, for uncountable successor cardinals κ, the principle $\mathrm{SDFA}(\mathcal{P}_\kappa)$ is equivalent to an instance of the Generalized Continuum Hypothesis together with a particular square principle (cf. Subsection 1.2). In our main application, we prove that, for an infinite cardinal λ, $\mathrm{SDFA}(\mathcal{P}_{\lambda^+})$ implies that $\mathrm{SH}_{\lambda^{++}}$ fails. This leads to the following corollary regarding the large cardinal strength of the failure of the generalized Souslin Hypothesis, improving upon an almost forty-year-old result of Shelah and Stanley [48].

Theorem 1.1 (LH-Rinot [32]). *Suppose that λ is an uncountable cardinal, $2^\lambda = \lambda^+$, and $\mathrm{SH}_{\lambda^{++}}$ holds. Then λ^{++} is a Mahlo cardinal in L.*

We suspect that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ will have a number of other applications in a variety of fields of mathematics, and I plan to continue to investigate this. For example, we feel it is likely to entail the existence of exotic varieties of superatomic Boolean algebras or topological spaces, such as positive solutions to the Arhangel'skii Problem. As is the case with the generalized Souslin Hypothesis, this could provide better lower bounds for the large cardinal strength of the non-existence of these objects. We are also interested in removing the cardinal arithmetic assumptions from the statement of Theorem 1.1. The eventual goal would be an answer to the following question.

Question 1.2. *Is the existence of an infinite cardinal λ for which $\mathrm{SH}_{\lambda^{++}}$ holds equiconsistent with the existence of a weakly compact cardinal?*

1.2. **Square principles and stationary reflection.** Some of the most canonical incompactness and compactness principles in set theory are, respectively, square principles and principles of stationary reflection, and much of my work has touched directly or indirectly on these principles.

Very roughly speaking, a square principle at a cardinal λ asserts the existence of a certain coherent sequence of length λ that cannot be extended to have length $\lambda + 1$. It is thus manifestly an incompactness principle. A particularly useful family of square principles, introduced by Todorcevic, is denoted by $\square(\lambda, \kappa)$, where $1 \leq \kappa < \lambda$ are cardinals and λ is regular and uncountable. $\square(\lambda, \kappa)$ is an assertion of incompactness about the cardinal λ, and, if λ is held constant, then the strength of the principle $\square(\lambda, {<}\kappa)$ decreases as κ increases.

Stationary reflection principles lie on the other side of the compactness divide. Intuitively, a statonary subset of an an ordinal β of uncountable cofinality is a subset that is "large" in a particular sense. A stationary reflection principle at λ asserts that, for certain collections $\mathcal{S}$ of stationary subsets of λ, there are many ordinals $\beta < \lambda$ such that $\{S \cap \beta \mid S \in \mathcal{S}\}$ is a collection of stationary subsets of β. It is thus an assertion of compactness about λ. For a fixed nonzero cardinal $\kappa < \lambda$, $\mathrm{Refl}(\kappa, \lambda)$ denotes this reflection principle for all collections $\mathcal{S}$ of at most κ-many subsets of λ. The variant $\mathrm{Refl}({<}\kappa, \lambda)$ is about collections $\mathcal{S}$ of fewer than κ-many subsets. The strength of $\mathrm{Refl}(\kappa, \lambda)$ increases as κ increases.

It has long been known that square principles put a limit on the amount of stationary reflection that can hold. For example, it is a classical result that $\square(\lambda, 1)$ implies the failure of $\mathrm{Refl}(2, S)$ for every stationary $S \subseteq \lambda$. In joint work with Yair Hayut, I established a very strong link between the hierarchy of square principles and that of stationary reflection principles. (For simplicity, we state the following theorem slightly imprecisely.)

Theorem 1.3 (Hayut-LH, [20]). *Suppose that $\kappa < \lambda$ are infinite regular cardinals.*

1. *If $\square(\lambda, {<}\kappa)$ holds, then $\mathrm{Refl}({<}\kappa, \lambda)$ fails.*
2. *This is sharp in the sense that, modulo large cardinal assumptions, $\square(\lambda, \kappa)$ is compatible with $\mathrm{Refl}({<}\kappa, \lambda)$.*

The proofs of the two directions of Theorem 1.3 are very different in nature. The proof of (1) is a purely combinatorial argument establishing the result in ZFC. The proof of (2) is a consistency result in which we use forcing to construct a model of ZFC in which $\square(\lambda, \kappa)$ and $\mathrm{Refl}(<\kappa, \lambda)$ both hold.

In other papers, I have studied further aspects of the web of interconnections between various square principles ([23], [25], [26]) and between various principles of stationary reflection ([8], [24], [16]).

1.3. **Productivity of chain conditions.** The notion of chain conditions of partially ordered sets has played a major role in modern set theory, both in providing an impetus for set theoretic investigation and in being a key tool itself in the development of the technique of forcing.

Definition 1.4. Suppose that $\mathbb{P}$ is a partial order and κ is a cardinal.

1. If $p, q \in \mathbb{P}$, then p and q are *compatible* if there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.
2. An *antichain* in $\mathbb{P}$ is a subset $A \subseteq \mathbb{P}$ such that the elements of A are pairwise incompatible.
3. $\mathbb{P}$ satisfies the *κ-chain condition* (κ-c.c.) if, for every antichain A in $\mathbb{P}$, we have $|A| < \kappa$.
4. $\mathbb{P}$ is κ-Knaster if, whenever $A \subseteq \mathbb{P}$ and $|A| = \kappa$, there is a subset $B \subseteq A$ such that $|B| = \kappa$ and B consists of pairwise compatible elements.

Clearly, if a partial order $\mathbb{P}$ is κ-Knaster, then it satisfies the κ-c.c. One reason the κ-Knaster condition is useful is that it is *productive*: if $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster partial orders, then the product $\mathbb{P} \times \mathbb{Q}$ is also κ-Knaster. This is not necessarily true for the κ-chain condition, and the study of the productivity of chain conditions has led to much deep work in set theory.

Though the κ-Knaster condition is *finitely* productive, it is not necessarily *infinitely* productive, i.e., an infinite product of κ-Knaster partial orders may not be κ-Knaster. If κ is a weakly compact cardinal, then it is in fact the case that any product of fewer than κ κ-Knaster partial orders is again κ-Knaster. In joint work with Lücke, in conjunction with some work on variants of the tree property, we give a partial converse to this by showing that, if the κ-Knaster condition is even *countably* productive, then κ is weakly compact in L.

Theorem 1.5 (LH-Lücke, [30])**.** *Suppose that κ is an uncountable regular cardinal and $\square(\kappa)$ holds. Then there is a κ-Knaster partial order $\mathbb{P}$ such that $\mathbb{P}^{\aleph_0}$ does not satisfy the κ-chain condition. Consequently, if the κ-Knaster condition is countably productive, then κ is weakly compact in L.*

This work left open a question raised by Todorcevic in response to previous work of Cox and Lücke [7]: Is it consistent that the κ-Knaster condition is countably productive for some "small" cardinal κ, e.g., $\aleph_2$ or $\aleph_{\omega+1}$? In joint work with Rinot, we resolved this question negatively, in fact showing that the countable productivity of the κ-Knaster condition implies that κ is strongly inaccessible.

Theorem 1.6 (LH-Rinot, [31])**.** *Suppose that μ is an infinite cardinal and $\kappa = \mu^+$. Then there is a poset $\mathbb{P}$ such that $\mathbb{P}$ is κ-Knaster but $\mathbb{P}^{\aleph_0}$ does not satisfy the κ-c.c.*

Our proof of Theorem 1.6 led us to isolate a new combinatorial principle asserting the existence of certain "strongly unbounded" functions. Roughly speaking, these are functions on the space of two-element subsets of a cardinal κ into some smaller cardinal μ such that, for every unbounded subset of κ, the range of the function restricted to that subset is unbounded in μ (in fact, the principle gives somewhat more). This principle is a very strong negation of Ramsey's theorem at κ. We began a thorough analysis of this principle in [31], and this analysis continues today.

One of the most intriguing developments in this work with Rinot has been our isolation of a cardinal characteristic we call the *C-sequence number* [34]. This number provides a measure of the compactness of a cardinal and a concrete indication of how far away it is from being a weakly compact cardinal (weakly compact cardinals have a C-sequence number of 0, and the larger a cardinal's C-sequence number is relative to the cardinal itself, the less compact the cardinal is). We have proven a number of results, both ZFC results and consistency results, about the C-sequence number. For example, we have investigated the number at $\aleph_{\omega+1}$, the least successor of a singular cardinal.

Theorem 1.7 (LH-Rinot [34])**.** *Let $\chi(\aleph_{\omega+1})$ denote the C-sequence number of $\aleph_{\omega+1}$.*

1. $\aleph_0 \leq \chi(\aleph_{\omega+1}) \leq \aleph_\omega$.
2. *If $\square(\aleph_{\omega+1})$ holds, then $\chi(\aleph_{\omega+1}) = \aleph_\omega$. In particular, this is the case if $V = L$.*
3. *Modulo large cardinal assumptions, it is consistent that $\chi(\aleph_{\omega+1}) = \aleph_0$.*

1.4. **Singular cardinal combinatorics.** An infinite cardinal κ is *singular* if a set of size κ can be written as a union of fewer than κ sets, each of size less than κ. Many interesting and deep questions in set theory revolve around combinatorics and cardinal arithmetic at singular cardinals and their successors. For a variety of reasons, results at singular cardinals and their successors are typically more difficult to obtain than the analogous results at regular cardinals and their successors and often require stronger large cardinal assumptions and more intricate forcing arguments.

Some of the most fundamental tools of research into singular cardinal combinatorics are the so-called *Prikry-type* forcing notions. A typical Prikry-type forcing notion allows one to pass from a model of set theory with a suitably large cardinal, κ, to an outer model in which κ is a singular cardinal. We say that, in this situation, κ has become *singularized* in the outer model.

In joint work with Ben Neria and Unger [1], we introduce a new Prikry-type forcing notion, known as *diagonal supercompact Radin forcing*, and use it to prove the following global consistency result about the failure of the Singular Cardinals Hypothesis and the non-existence of special Aronszajn trees.

Theorem 1.8 (Ben Neria-LH-Unger, [1])**.** *If there are a supercompact cardinal κ and a weakly inaccessible cardinal $\theta > \kappa$, then there is a forcing extension in which κ is inaccessible and there is a club $E \subseteq \kappa$ of singular cardinals ν at which* SCH *and* AP *both fail.*

This is a step in the direction of a possible proof of the consistency of every regular cardinal $\kappa > \aleph_1$ satisfying the tree property, which would provide a striking example of global compactness and would answer a long-standing and central question of Magidor.

In other recent work [27], I have extended and generalized previous results [17, 10, 36] indicating that Prikry-type forcing notions are in some sense the *only* way to singularize cardinals. More precisely, I prove a quite general theorem stating that, if κ is a regular cardinal that becomes singular in some outer model of set theory, then, as long as a certain amount of cardinal structure is maintained, then there must be an object in the outer model which resembles a generic object for a Prikry-type forcing notion. Results such as this play a key role in providing limits on what kind of consistency results we can hope to prove.

Singular cardinal combinatorics is the setting for a suite of three deep connected problems that continue to guide much of my research on singular cardinal combinatorics. The questions concern relationships between regular cardinals $\kappa < \lambda$ such that λ is the successor of a singular cardinal μ and $\mathrm{cf}(\kappa) > \mu$. The simplest case is that in which $\lambda = \aleph_{\omega+1}$ and $\kappa = \aleph_2$.

Question 1.9. *Are there consistently models $V \subseteq W$ of ZFC with the same ordinals such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$?*

Question 1.10. *Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_1) \Rightarrow (\aleph_2, \aleph_1)$ consistent?*

Question 1.11. *Is it consistent that there is a PCF-theoretic scale of length $\aleph_{\omega+1}$ that has stationarily many bad points of cofinality $\aleph_2$?*

There is currently a rather peculiar situation regarding Questions 1.9–1.11. All three questions are known consistently to have positive answers if $\aleph_2$ is replaced by $\aleph_1$ (and $\aleph_1$ is replaced by $\aleph_0$ in Question 1.10). On the other hand, all three questions are known to have negative answers if $\aleph_2$ is replaced by $\aleph_n$ for some natural number $n \geq 4$ (and $\aleph_1$ is replaced by $\aleph_{n-1}$ in Question 1.10). All three questions, though, remain entirely open both as stated and with $\aleph_2$ replaced by $\aleph_3$. One promising angle of attack on them comes via combinatorial structures known as *covering matrices*, used by Viale to prove that the Singular Cardinals Hypothesis follows from the Proper Forcing Axiom [52]. My work in [23] and [25] is motivated in part by this line of questions, and by relationships between covering matrices and square principles. For example, in [25] I provide a new proof of a result of Sharon and Viale [44] stating that positive answers to either Questions 1.9 or 1.10 would entail nontrivial failures of stationary reflection at $\aleph_2$. I plan to continue pursuing this work.

Progress on these questions is also likely to shed light on the following major open problem in cardinal arithmetic.

Question 1.12. *Is it consistent that $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_{\omega_1}$?*

It follows from the work of Magidor [35] that $2^{\aleph_\omega}$ can be arbitrarily large below $\aleph_{\omega_1}$ in models in which $\aleph_\omega$ is strong limit. On the other hand, a celebrated theorem of Shelah [46] shows that, if $\aleph_\omega$ is strong limit, then we necessarily have $2^{\aleph_\omega} < \aleph_{\omega_4}$. The similarities between the current state of knowledge about Questions 1.9–1.11 and about Question 1.12 are not coincidental and arise from deep machinery employed in the proofs of the best current theorems. It seems likely that a breakthrough in one question will lead to a breakthrough in the others.

2. Graph theory

My work in graph theory is motivated by questions about the tension between local and global properties of uncountable graphs. In particular, I am interested in large graphs which display very complicated behavior but are locally rather simple. A vague paradigmatic example would be a graph with "very large" chromatic number such that all of its "small" subgraphs have "very small" chromatic number.

If G is a graph, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We will sometimes abuse notation and write $|G|$ instead of $|V(G)|$. Recall that, if G is a graph, the *chromatic number* of G is the least cardinal κ such that the vertices of G can be partitioned into κ-many independent sets. The *coloring number* is slightly more technical but often more amenable to set theoretic analysis: Given a well-ordering $\prec$ of the vertices $V(G)$ and a vertex v, let $N_G^{\prec}(v)$ denote the set $\{u \prec v \mid \{u,v\} \in E(G)\}$. Then the coloring number of G is the least cardinal κ such that there exists a well-ordering $\prec$ of $V(G)$ such that $|N_G^{\prec}(v)| < \kappa$ for all vertices v. A greedy coloring argument shows that the chromatic number of G is always at most the coloring number of G.

2.1. **Chromatic numbers of finite subgraphs.**

One of the earliest results indicating the extent to which a graph's global structure reflects its local structure is given by the De Bruijn-Erdős theorem. As a result of this theorem, if G is a graph with infinite chromatic number, then, for every natural number k, there is a finite subgraph of G with chromatic number k. We can therefore define a natural function $f_G : \mathbb{N} \to \mathbb{N}$ by letting $f_G(k)$ be the least number of vertices in a subgraph of G with chromatic number k. It is clear that f_G is an increasing function. The question then naturally arises: How fast can f_G grow? What can the behavior of f_G tell us about the global behavior of G; in particular, does it have an impact on the possible values for the chromatic number of G?

Using a result of Erdős from [11], it is not hard to show that if we only ask that G have countably infinite chromatic number, then f_G can grow arbitrarily quickly, so the question of the growth rate of f_G is most interesting for graphs of uncountable chromatic number. In 1982, Erdős, Hajnal, and Szemerédi [13] proved that, for every $n \in \mathbb{N}$, there is a graph G of uncountable chromatic number such that f_G grows faster than exp_n, where exp_n denotes the n-fold iterated exponential function. They then asked whether it is the case that, for every function $f : \mathbb{N} \to \mathbb{N}$, there is an uncountably chromatic graph G such that f_G grows faster than f (see also [14]). Recently, I resolved this question positively.

Theorem 2.1 (LH, [29])**.** *For every function $f : \mathbb{N} \to \mathbb{N}$, there is a graph G of size $2^{\aleph_1}$ and chromatic number $\aleph_1$ such that $f_G(k) > f(k)$ for all $3 \le k < \omega$.*

In subsequent work in progress, I have been able to reduce the size of G in the statement of the theorem to $2^{\aleph_0}$. A number of open question remain, which I plan to continue investigating. The most significant, I feel, is whether graphs G as in Theorem 2.1 can be found with arbitrarily large chromatic number.

Question 2.2. *Is it the case that, for every function $f : \mathbb{N} \to \mathbb{N}$ and every cardinal κ, there is a graph G of chromatic number at least κ such that f_G grows faster than f?*

The proof of Theorem 2.1 is very particular to the construction of graphs of chromatic number $\aleph_1$, so it seems that new ideas will be necessary to resolve Question 2.2. Even the consistency of a positive answer for Question 2.2 is open for $\kappa > \aleph_1$ and would be of significant interest.

2.2. **Compactness for chromatic and coloring numbers.**

In joint work with Rinot [33], we investigated the extent to which analogues of the De Bruijn-Erdős Compactness Theorem hold or fail at higher cardinals. Our main result indicates that rather mild assumptions imply a spectacular failure of a higher analogue of the De Bruijn-Erdős theorem and a maximal amount of incompactness for the chromatic number.

Theorem 2.3 (LH-Rinot, [33])**.** *Suppose that the Generalized Continuum Hypothesis holds and λ is an uncountable cardinal such that $\square(\lambda^+)$ holds. Then there is a graph G of size λ^+ such that*

(1) *every subgraph of G of size less than λ^+ has countable chromatic number;*
(2) *the chromatic number of G is λ^+.*

Since $\square(\lambda^+)$ is compatible with certain compactness principles, we obtain surprising corollaries stating that the maximal amount of incompactness for the chromatic number is compatible with large amounts of compactness in other areas. For example, it is compatible with compactness for the coloring number.

On the other hand, somewhat surprisingly, we show that, in contrast to the chromatic number, the coloring number cannot admit arbitrarily large compactness gaps.

Theorem 2.4 (LH-Rinot, [33]). *Suppose that μ is a cardinal, G is a graph, and every subgraph of G of strictly smaller cardinality than G has coloring number at most μ. Then G has coloring number at most μ^{++}.*

It is known that one-cardinal incompactness gaps in coloring number can be achieved. The question of whether two-cardinal gaps are possible remains open. Its solution is likely to involve deep combinatorial questions, and I plan to continue to investigate it. The most prominent special case of this question is the following:

Question 2.5. *Is it consistent that there is a graph G of size $\aleph_{\omega+1}$ and coloring number $\aleph_2$ such that every subgraph of G of size less than $\aleph_{\omega+1}$ has countable coloring number?*

2.3. **Highly connected Ramsey theory on the real numbers.** The obvious first attempts to generalize Ramsey's theorem to uncountable sets fails dramatically at the level of the real numbers. For example, if G is the complete graph with vertex set $\mathbb{R}$, then we have the following two strong failures to attempted generalizations of Ramsey's theorem:

- There is an edge-coloring of G using $\aleph_0$-many colors such that every infinite complete subgraph of G contains edges of infinitely-many different colors.
- There is an edge-coloring of G using only 2 colors such that every uncountable complete subgraph of G contains edges of both colors.

If one weakens the requirement that a witness to the Ramsey-like statement be a *complete* subgraph, though, one can obtain nontrivial statements that are at least consistent. One candidate for such a weakening was introduced by Bergfalk, Hrušák, and Shelah in [3], is the notion of a *highly connected graph.*

Definition 2.6. [3] A graph G is *highly connected* if it is connected and remains connected after the removal of any collection of fewer than $|G|$-many vertices.

In the realm of finite graphs, the highly connected graphs are precisely the complete graphs, so an infinitary version of Ramsey's theorem in which we ask for highly connected monochromatic graphs can be seen as a true analogue of Ramsey's theorem for finite graphs. With this notion, we were able to prove the consistency of a nontrivial instance of a Ramsey-type statement at the level of the real numbers. In fact, we were able to show that this statement is equiconsistent with a large cardinal notion.

Theorem 2.7. [28] *The following statements are equiconsistent over* ZFC*:*
(1) *There is a weakly compact cardinal.*
(2) *For every edge-coloring of the complete graph G with vertex set $\mathbb{R}$ using fewer than $|\mathbb{R}|$-many colors, there is a highly-connected subgraph H of G such that $|H| = |\mathbb{R}|$ and the edges of H have only 2 different colors.*

This theorem is optimal in the sense that the "2" in clause (2) provably cannot be reduced to a "1".

2.4. **Future work.** A number of questions asked by Erdős and Hajnal about uncountable graphs have since been addressed by consistency results but remain open in general. It seems plausible that they can be resolved using techniques similar to those we developed to prove Theorem 2.1, and I plan to investigate them. I mention a couple of them here concerning triangle-free graphs. Recall that K_n denotes the complete graph on n vertices.

Question 2.8 (Erdős-Hajnal, [12]). *Is there a K_4-free graph that cannot be written as the union of countably many triangle-free graphs?*

Shelah [45] proved that, assuming the consistency of certain large cardinals, Question 2.8 consistently has a positive answer. We plan to investigate whether a positive answer follows simply from ZFC.

Question 2.9 (Erdős-Hajnal)**.** *Let κ be an uncountable cardinal. If G is a graph with chromatic number κ, must G have a triangle-free subgraph of chromatic number κ?*

Rödl [42] proved that Question 2.9 has a positive answer if $\kappa = \aleph_0$. Komjáth and Shelah [22] proved that it consistently has a negative answer if $\kappa = \aleph_1$ and conjectured that this negative answer follows from ZFC.

3. Homological algebra

Much of my recent work has centered on questions arising from homological algebra. As questions of the tension between local and global behavior are prominent in both set theory and homological algebra, it is natural that there are a number of connections between the two fields, and though there has been some work done exploring these connections over the last fifty years, we feel that there remains a great deal to be done and that the common area inhabited by the two fields currently provides particularly fertile ground for new research.

3.1. **Strong homology and the vanishing of higher derived limits.** In recent work [5], Jeffrey Bergfalk and I solved a problem arising from the study of the additivity of strong homology.

Definition 3.1 ([38])**.** If $\mathcal{C}$ is a class of topological spaces, we say a homology theory is *additive on $\mathcal{C}$* if for every natural number p and every family $\{X_i \mid i \in I\}$ with each X_i and $\coprod_I X_i$ in $\mathcal{C}$, we have the isomorphism

$$\bigoplus_I \mathrm{H}_p(X_i) \cong \mathrm{H}_p(\coprod_I X_i)$$

via the obvious map induced by the inclusions $X_i \hookrightarrow \coprod_I X_i$.

Additivity is satisfied, for example, by singular homology. In [37], Mardešić and Prasolov investigated the additivity of strong homology and, by computing the strong homology of the k-dimensional Hawaiian earring and countably infinite topological sums thereof, were able to show that, if strong homology is additive on closed subspaces of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n \geq 1$, where $\mathbf{A}$ is a certain inverse system of abelian groups indexed by ${}^\omega\omega$. They then translated the statement "$\lim^1 \mathbf{A} = 0$" into a purely set theoretic statement which they were able to use to prove that the Continuum Hypothesis implies that $\lim^1 \mathbf{A} \neq 0$ and, therefore, that strong homology is not additive.

We briefly remark that the question of the vanishing of $\lim^n \mathbf{A}$ has also arisen recently, independently of its connection with the additivity of strong homology, in work of Dustin Clausen and Peter Scholze on condensed mathematics (cf. [43]). They introduce the category of *condensed abelian groups* as a setting in which to do algebra when the algebraic objects carry additional topological information, and a foundational question about this category reduces, in its simplest case, to the vanishing of $\lim^n \mathbf{A}$ for $n \geq 1$.

The vanishing of $\lim^n \mathbf{A}$ can be translated into a purely set theoretic statement about the nonexistence of certain nontrivial coherent families of functions which are clear instances of incompactness. Mardešić and Prasolov showed that the statement "$\lim^1 \mathbf{A} = 0$" is equivalent to the assertion that every coherent family is trivial. In [2], Bergfalk generalized the definitions of "trivial" and "coherent" to higher-dimensional families of functions, defining the notions of n-coherence and n-triviality for positive integers n. Again, the existence of non-n-trivial n-coherent families provides a clear instance of set theoretic incompactness. Bergfalk showed that the statement "$\lim^n \mathbf{A} = 0$" is equivalent to the assertion that every n-coherent family is n-trivial.

$\lim^1 \mathbf{A}$ and nontrivial coherent families have been the subject of a great amount of set theoretic research (cf. [9], [21], [15], [50]), but $\lim^n \mathbf{A}$ and non-n-trivial n-coherent families have remained much more mysterious. In fact, the question of the consistency of the statement "$\lim^1 \mathbf{A} = 0 = \lim^2 \mathbf{A}$" was open until very recently and was explicitly asked by Moore in his 2010 ICM survey on the Proper Forcing Axiom [40]. With Jeffrey Bergfalk, we answered this question and in fact proved the consistency of the statement "$\lim^n \mathbf{A} = 0$ for all $n \geq 1$".

Theorem 3.2 (Bergfalk-LH, [5])**.** *Suppose that κ is a weakly compact cardinal and $\mathbb{P}$ is a finite support iteration of Hechler forcings of length κ. Then, in $V^{\mathbb{P}}$, $\lim^n \mathbf{A} = 0$ for every $n \geq 1$.*

A number of questions are left open by this result, which we intend to pursue. These include questions about the optimality of the above theorem. The first asks whether the assumption of the existence of a weakly compact cardinal is really necessary.

Question 3.3. *What is the consistency strength of the statement* "$\lim^n \mathbf{A} = 0$ *for all* $n \geq 1$"*?*

The second asks whether the continuum must be as large as in the statement of the theorem.

Question 3.4. *What is the smallest value of the continuum that is compatible with the statement* "$\lim^n \mathbf{A} = 0$ *for all* $n \geq 1$"*?*

A natural model in which to try to investigate Question 3.4 is the model obtained by adding $\aleph_{\omega+1}$-many Cohen reals to a model of CH.

A more ambitious line of inquiry reconnects these investigations to the original question of the additivity of strong homology. In [41], Prasolov proved in ZFC that strong homology is not additive on the class of all topological spaces. His counterexample, though, is not paracompact and hence not metrizable, leaving open the possibility that strong homology could be additive on some smaller but still substantial class of spaces.

Question 3.5. *Is it consistent that strong homology is additive on some nice class of spaces properly extending the class of spaces homotopy equivalent to a CW-complex? For example, is it consistent that strong homology is additive on Polish spaces? On locally compact metric spaces? On metric spaces?*

The model $V^{\mathbb{P}}$ of Theorem 3.2 provides a natural model in which to begin investigating Question 3.5.

3.2. **Cohomology of the ordinals.** In other work with Jeffrey Bergfalk [4], we investigate Čech cohomology groups of ordinals (with the topology inherited from the natural ordering of the ordinals). We show that, for a fixed ordinal δ and natural number $n \geq 1$, the statement "$\check{\mathrm{H}}^n(\delta, \mathbb{Z}) \neq 0$" is equivalent to the existence of an n-dimensional coherent, nontrivial family of functions indexed by the n-element subsets of δ. This is a purely combinatorial object, and such coherent, nontrivial families provide examples of incompactness at δ.

We prove in ZFC that, if $k < n$ are natural numbers and δ is an ordinal of cofinality $\aleph_k$, then $\check{\mathrm{H}}^n(\delta, \mathbb{Z}) = 0$. This is sharp in the sense that forthcoming work of Bergfalk implies that, for such δ, $\check{\mathrm{H}}^k(\delta, \mathbb{Z}) \neq 0$. Things become much less clear when $\mathrm{cf}(\delta) > \aleph_n$, and here large cardinals and consistency results come into play. For example, we show that, if κ is a weakly compact cardinal, then $\check{\mathrm{H}}^n(\kappa, \mathbb{Z}) = 0$ for all n, and, if κ is strongly compact, then in fact $\check{\mathrm{H}}^n(\lambda, \mathbb{Z}) = 0$ for all n and every regular cardinal $\lambda \geq \kappa$. We also prove that, consistently, $\check{\mathrm{H}}^1(\kappa, \mathbb{Z}) = 0$ for all κ with $\mathrm{cf}(\kappa) > \aleph_1$. On the other hand, we show that, in Gödel's constructible universe L, the Čech cohomology groups of ordinals are nonzero wherever possible:

Theorem 3.6 (Bergfalk-LH, [4])**.** *Suppose that $V = L$. Then, for every positive integer n and every regular cardinal κ such that $\mathrm{cf}(\kappa) \geq \aleph_n$ and κ is not weakly compact, we have $\check{\mathrm{H}}^n(\kappa, \mathbb{Z}) \neq 0$.*

It remains a question of interest whether we can arrange nontrivial instances of $\check{\mathrm{H}}^n(\kappa, \mathbb{Z}) = 0$ for $n > 1$ and "small" κ. The simplest such open question, which we plan to continue working on, is the following.

Question 3.7. *Is $\check{\mathrm{H}}^2(\aleph_3, \mathbb{Z}) = 0$ consistent?*

We feel that Question 3.7 should have a positive answer relative to large cardinal assumptions, but it is likely that such an answer will require the development of new ideas in forcing.

In the case of $n = 1$, the nontrivial coherent families witnessing $\check{\mathrm{H}}^n(\kappa, \mathbb{Z}) \neq 0$ are very familiar combinatorial objects to set theorists: they are precisely the coherent κ-Aronszajn trees. Such trees have been studied extensively and have proven very useful in the study of 2-dimensional combinatorics (particularly Ramsey theory) on uncountable cardinals. For $n \geq 2$, however, the nontrivial coherent families seem to be genuinely new objects, and we feel that they provide fertile ground for study. In particular, we feel that their study could yield dividends in higher-dimensional Ramsey theory at uncountable cardinals, which until now has proven much more difficult than the 2-dimensional case.

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