# ROBUST REFLECTION PRINCIPLES

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Abstract. A cardinal $\lambda$ satisfies a property P *robustly* if, whenever $\mathbb{Q}$ is a forcing poset and $|\mathbb{Q}|^+ < \lambda$, $\lambda$ satisfies P in $V^{\mathbb{Q}}$. We study the extent to which certain reflection properties of large cardinals can be satisfied robustly by small cardinals. We focus in particular on stationary reflection and the tree property, both of which can consistently hold but fail to be robust at small cardinals. We introduce natural strengthenings of these principles which are always robust and which hold at sufficiently large cardinals, consider the extent to which these strengthenings are in fact stronger than the original principles, and investigate the possibility of these strengthenings holding at small cardinals, particularly at successors of singular cardinals.

## 1. Introduction

Large cardinal properties have, among others, the following two appealing attributes: they imply certain strong reflection properties, and they are robust under small forcing. The study of the extent to which reflection properties of large cardinals can hold at small cardinals, and in particular at successors of singular cardinals, has been a fruitful line of research in set theory. We continue this line here, adding the requirement of robustness under small forcing to these reflection properties and focusing in particular on stationary reflection and the tree property at successors of singular cardinals.

**Definition 1.1.** Let $P$ be a property that can hold of a cardinal $\lambda$. We say that $\lambda$ *satisfies $P$ robustly* or that $\lambda$ *has the robust property $P$* if, whenever $\mathbb{Q}$ is a forcing poset and $|\mathbb{Q}|^+ < \lambda$, $\lambda$ satisfies $P$ in $V^{\mathbb{Q}}$.

**Remark 1.2.** The requirement $|\mathbb{Q}|^+ < \lambda$, rather than the seemingly more natural $|\mathbb{Q}| < \lambda$, is necessary for our purposes in order to obtain consistent principles. If $\lambda = \mu^+$ and $\mathbb{Q} = \mathrm{Coll}(\omega, \mu)$, then $|\mathbb{Q}| < \lambda$ and, in $V^{\mathbb{Q}}$, $\lambda = \omega_1$ and therefore cannot satisfy, for example, stationary reflection or the tree property.

Most large cardinal properties are always robust. For example, if $\lambda$ satisfies the property "is inaccessible," "is weakly compact," "is measurable," "is strongly compact," "is supercompact," etc., then, by an argument of Levy and Solovay (see [9]), $\lambda$ satisfies the property robustly. Therefore, reflection principles, when they hold due to large cardinal properties, are themselves robust. Of particular interest to us are the following.

**Fact 1.3.** *Suppose $\lambda$ is weakly compact. Then $\lambda$ satisfies robust stationary reflection and the robust tree property.*

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**Fact 1.4.** *Suppose $\mu$ is a singular limit of strongly compact cardinals and $\lambda = \mu^+$. Then $\lambda$ satisfies robust stationary reflection and the robust tree property.*

As we will see, though, these reflection principles need not be robust when they hold at small cardinals. We consider here natural strengthenings of reflection principles, in particular stationary reflection and the tree property, that are always robust, and investigate the extent to which they can hold at small cardinals and the extent to which they are true strengthenings of the more classical principles.

The general outline of the paper is as follows. In Section 2, we consider robust stationary reflection. We show that this is equivalent to a natural condition studied by Cummings and the author in [3] and that it is not in general equivalent to stationary reflection at inaccessible cardinals. The rest of the paper is devoted to the tree property and the strong system property. In Section 3, we introduce the strong system property, a robust strengthening of the tree property that is equivalent to the tree property at inaccessible cardinals. In Section 4, we show that fairly weak square principles imply the failure of the strong system property, and we provide a characterization of the robustness of having no special $\mu^+$-Aronszajn trees for infinite $\mu$. In Section 5, we present some branch preservation lemmas for systems and a technical lemma about systems in a generic extension by a product of Levy collapses. In Section 6, we adapt arguments of Fontanella and Magidor from [4] to show that we have some control over the extent of the failure of the strong system property at $\aleph_{\omega^2+1}$ and that the strong system property can consistently hold at $\aleph_{\omega^2+1}$. We conclude with some open questions.

Our notation is, for the most part, standard. Our reference for all undefined terms and notations is [6]. If $\kappa < \lambda$ are infinite cardinals, with $\kappa$ regular, then $S^\lambda_\kappa = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$. $S^\lambda_{<\kappa}$, $S^\lambda_{\leq\kappa}$, etc. are given the natural meanings. By 'inaccessible,' we always mean 'strongly inaccessible.' If $R$ is a binary relation on a set $X$, we will often write $x <_R y$ in place of $(x, y) \in R$. If $A$ is a set of ordinals, then $A'$ denotes $\{\alpha < \sup(A) \mid \alpha = \sup(A \cap \alpha)\}$.

## 2. Stationary reflection

Recall the following definitions.

**Definition 2.1.** Let $\lambda$ be a regular, uncountable cardinal, and let $S \subseteq \lambda$ be stationary.

1. If $\alpha < \lambda$ and $\mathrm{cf}(\alpha) > \omega$, then $S$ *reflects* at $\alpha$ if $S \cap \alpha$ is stationary in $\alpha$.
2. $S$ *reflects* if there is $\alpha < \lambda$ such that $S$ reflects at $\alpha$.
3. $S$ *reflects at arbitrarily high cofinalities* if, for every regular $\kappa < \lambda$, there is $\alpha \in S^\lambda_{\geq\kappa}$ such that $S$ reflects at $\alpha$.
4. $\mathrm{Refl}(\lambda)$ is the assertion that every stationary subset of $\lambda$ reflects.

The following proposition is easily proven (see [3], for example).

**Proposition 2.2.** *Suppose* $\mathrm{Refl}(\aleph_{\omega+1})$ *holds. Then every stationary subset of* $\aleph_{\omega+1}$ *reflects at arbitrarily high cofinalities.*

Also, standard arguments yield that, if $\lambda$ is weakly compact, then every stationary subset of $\lambda$ reflects at arbitrarily high cofinalities. However, the situation is different in general. In [3], Cummings and the author show that, assuming the existence of sufficiently large cardinals, it is consistent that there is a singular cardinal $\mu > \aleph_\omega$ such that $\mathrm{Refl}(\mu^+)$ holds and there is a stationary subset of $\mu^+$ that does not reflect at arbitrarily high cofinalities. In [7], the author extends this result to show that, assuming the existence of a proper class of supercompact cardinals, there is a class forcing extension in which, whenever $\mu > \aleph_\omega$ is a singular cardinal, $\mathrm{Refl}(\mu^+)$ holds and there is a stationary subset of $\mu^+$ that does not reflect at arbitrarily high cofinalities.

It turns out that this notion is closely related to robust stationary reflection and that the models constructed in [3] and [7] provide instances in which stationary reflection holds non-robustly at successors of singular cardinals.

**Theorem 2.3.** *Suppose $\lambda$ is a regular, uncountable cardinal. The following are equivalent.*

(1) *$\lambda$ satisfies robust stationary reflection.*
(2) *Every stationary subset of $\lambda$ reflects at arbitrarily high cofinalities.*

*Proof.* First note that, if $\lambda = \kappa^+$ and $\kappa$ is regular, then $S^\lambda_\kappa$ is a non-reflecting stationary subset of $\lambda$. Hence, both (1) and (2) imply that $\lambda$ is either weakly inaccessible or the successor of a singular cardinal. In particular, if $\kappa < \lambda$ is regular, then $\kappa^+ < \lambda$.

(1) $\Rightarrow$ (2): Assume (1) holds. Suppose for sake of contradiction that $S \subseteq \lambda$ is stationary, $\kappa < \lambda$ is regular, and $S$ does not reflect at any ordinal in $S^\lambda_{\geq\kappa}$. Let $\mathbb{P} = \mathrm{Coll}(\omega, \kappa)$. $|\mathbb{P}| = \kappa$, so $|\mathbb{P}|^+ < \lambda$. In particular, $\mathbb{P}$ has the $\lambda$-c.c., so $S$ remains stationary in $V^{\mathbb{P}}$. Also, if $\alpha < \lambda$ and $\mathrm{cf}^{V^{\mathbb{P}}}(\alpha) > \omega$, then $\mathrm{cf}^V(\alpha) > \kappa$. Since $S$ does not reflect at any ordinal in $S^\lambda_{\geq\kappa}$ in $V$, there is a club $C_\alpha$ in $\alpha$ such that $C_\alpha \cap S = \emptyset$. $C_\alpha$ still witnesses that $S$ does not reflect at $\alpha$ in $V^{\mathbb{P}}$, so $S$ is a non-reflecting stationary subset of $\lambda$ in $V^{\mathbb{P}}$, contradicting (1).

(2) $\Rightarrow$ (1): Assume (2) holds. Suppose for sake of contradiction that $|\mathbb{P}|$ is a forcing poset, $|\mathbb{P}|^+ < \lambda$, $p \in \mathbb{P}$, and $\dot{S}$ is a $\mathbb{P}$-name such that $p \Vdash$ "$\dot{S}$ is a non-reflecting stationary subset of $\lambda$." For all $q \leq p$, let $S_q = \{\eta < \lambda \mid q \Vdash \text{“}\eta \in \dot{S}\text{”}\}$. Since $p \Vdash$ "$\dot{S} \subseteq \bigcup_{q \leq p} S_q$" and $|\mathbb{P}| < \lambda$, there must be $q \leq p$ such that $S_q$ is stationary in $\lambda$. Fix such a $q$. By (2), we may find $\alpha \in S^\lambda_{\geq|\mathbb{P}|^+}$ such that $S_q$ reflects at $\alpha$. Since $\mathbb{P}$ trivially has the $|\mathbb{P}|^+$-c.c., $S_q \cap \alpha$ remains stationary in $V^{\mathbb{P}}$. But then, since $q \Vdash$ "$S_q \subseteq \dot{S}$", we have $q \Vdash$ "$\dot{S}$ reflects at $\alpha$," which is a contradiction. $\square$

The next result shows that robust stationary reflection is not necessarily equivalent to stationary reflection, even for inaccessible cardinals.

**Theorem 2.4.** *Suppose there is an inaccessible limit of supercompact cardinals. Then there is a forcing extension with an inaccessible cardinal $\lambda$ such that* $\mathrm{Refl}(\lambda)$ *holds but there is a stationary $S \subseteq S^\lambda_\omega$ that does not reflect at any ordinal in $S^\lambda_{>\aleph_\omega}$.*

*Proof.* The proof largely follows the proof of Theorem 3.1 in [3], so we omit many of the details. Let $\lambda$ be the least inaccessible limit of supercompact cardinals. In particular, $\lambda$ is not Mahlo. Let $\langle \kappa_i \mid i < \lambda \rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$;
- if $i = 0$ or $i$ is a limit ordinal, $\kappa_{i+1} = \kappa_i^+$;
- if $i$ is a successor ordinal, $\kappa_{i+1}$ is supercompact;
- $\sup(\{\kappa_i \mid i < \lambda\}) = \lambda$.

We first define a forcing iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \lambda, j < \lambda \rangle$, taken with full supports, as follows. If $i = 0$ or $i$ is a limit ordinal, then $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is trivial." If $i$ is a successor ordinal, then $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i = \mathrm{Coll}(\kappa_i, < \kappa_{i+1})$." Let $\mathbb{P} = \mathbb{P}_\lambda$. In $V^{\mathbb{P}}$, $\lambda$ is the least inaccessible cardinal and, for all $i < \lambda$, $\kappa_i = \aleph_i$. In $V^{\mathbb{P}}$, let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of all bounded subsets of $\lambda$, and let $\mathbb{A}$ be the forcing to shoot a club through the set of ordinals below $\lambda$ that are approachable with respect to $\vec{a}$. In $V^{\mathbb{P}*\dot{\mathbb{A}}}$, let $\mathbb{S}$ be the poset whose conditions are of the form $s = (\gamma^s, x^s)$ such that:

- $\gamma^s < \lambda$;
- $x^s \subseteq (\gamma^s + 1) \cap \mathrm{cof}(\omega)$;
- for all $\beta \in S^\lambda_{>\aleph_\omega}$, $x^s \cap \beta$ is not stationary in $\beta$.

If $s, t \in \mathbb{S}$, then $t \leq s$ iff $\gamma^t \geq \gamma^s$ and $x^t \cap (\gamma^s + 1) = x^s$.

If $G * H * I$ is generic for $\mathbb{P} * \dot{\mathbb{A}} * \dot{\mathbb{S}}$, then $V[G * H * I]$ is the desired model, with $S = \bigcup_{s \in I} x^s$ being the witnessing stationary subset of $S^\lambda_\omega$ not reflecting at any ordinal in $S^\lambda_{>\aleph_\omega}$. The verification is as in [3] and is thus omitted. $\square$

## 3. Systems

**Definition 3.1.** Let $R$ be a binary relation on a set $X$.

- If $a, b \in X$, then $a$ and $b$ are *$R$-comparable* if $a = b$, $a <_R b$, or $b <_R a$. Otherwise, $a$ and $b$ are *$R$-incomparable*, which is denoted $a \perp_R b$.
- $R$ is *tree-like* if, for all $a, b, c \in X$, if $a <_R c$ and $b <_R c$, then $a$ and $b$ are $R$-comparable.

**Definition 3.2.** Let $\lambda$ be an infinite, regular cardinal. $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a *$\lambda$-system* if:

(1) $I \subseteq \lambda$ is unbounded and, for all $\alpha \in I$, $0 < \kappa_\alpha < \lambda$. We sometimes identify $S$ with $\{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}$. For each $\alpha \in I$, we say that $S_\alpha := \{\alpha\} \times \kappa_\alpha$ is the $\alpha^{\text{th}}$ level of $S$;
(2) $\mathcal{R}$ is a set of binary, transitive, tree-like relations on $S$ and $|\mathcal{R}| < \lambda$;
(3) for all $R \in \mathcal{R}$, $\alpha_0, \alpha_1 \in I$, $\beta_0 < \kappa_{\alpha_0}$, and $\beta_1 < \kappa_{\alpha_1}$, if $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$, then $\alpha_0 < \alpha_1$;
(4) for all $\alpha_0 < \alpha_1$, both in $I$, there are $\beta_0 < \kappa_{\alpha_0}$, $\beta_1 < \kappa_{\alpha_1}$, and $R \in \mathcal{R}$ such that $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$.

If $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a $\lambda$-system, then we define $\mathrm{width}(S) = \max(\sup(\{\kappa_\alpha \mid \alpha \in I\}), |\mathcal{R}|)$ and $\mathrm{height}(S) = \lambda$. $S$ is a *narrow $\lambda$-system* if $\mathrm{width}(S)^+ < \lambda$.

$S$ is a *strong $\lambda$-system* if it satisfies the following strengthening of (4):

(4′) for all $\alpha_0 < \alpha_1$, both in $I$, and for every $\beta_1 < \kappa_{\alpha_1}$, there are $\beta_0 < \kappa_{\alpha_0}$ and $R \in \mathcal{R}$ such that $(\alpha_0, \beta_0) <_R (\alpha_1, \beta_1)$.

If $R \in \mathcal{R}$, a *branch of $S$ through $R$* is a set $b \subset S$ such that for all $a_0, a_1 \in b$, $a_0$ and $a_1$ are $R$-comparable. $b$ is a *cofinal branch* if, for unboundedly many $\alpha \in I$, $b \cap S_\alpha \neq \emptyset$.

**Notation 3.3.** If $S$ is a $\lambda$ system, we will sometimes write $\mathcal{R}(S)$ to denote the set of relations of $S$.

**Remark 3.4.** In previous presentations of systems (e.g. [11] and [15]), $\lambda$-systems were typically considered only for successor cardinals $\lambda$, and it was assumed that all $\lambda$-systems were of the form $\langle I \times \kappa, \mathcal{R} \rangle$, i.e. that all levels of the system were of the same width. If $\lambda$ is a successor cardinal and $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ is a $\lambda$-system, or if $\lambda$ is weakly inaccessible and $S$ is a narrow $\lambda$-system, then there is an unbounded $J \subseteq I$ and a $\kappa < \lambda$ such that, for all $\alpha \in J$, $\kappa_\alpha = \kappa$. It will then be sufficient for us to work with subsystems of the form $\langle J \times \kappa, \mathcal{R} \rangle$, so, in the case that $\lambda$ is a successor cardinal, we will assume our systems are of this form (and typically, we will in fact have $\lambda = \kappa^+$). If $\lambda$ is weakly inaccessible and we do not want to assume narrowness, though, our more general notion of system seems to us to be the correct notion to work with.

**Proposition 3.5.** *Suppose $\mu$ is an infinite cardinal. Then there is a strong $\mu^+$-system $S$ such that $|\mathcal{R}(S)| = \mu$ and $S$ has no cofinal branch.*

We provide two simple and quite different proofs of this proposition.

*Proof 1.* We define a system $S = \langle \mu^+ \times 2, \mathcal{R} \rangle$, where $\mathcal{R} = \{R^i_\eta \mid i < 2, \eta < \mu\}$. For each $\alpha < \mu^+$, let $f_\alpha : \alpha \to \mu$ be injective. Fix $i < 2$ and $\eta < \mu$, and suppose $\alpha < \beta < \mu^+$ and $k_\alpha, k_\beta < 2$. Then $(\alpha, k_\alpha) <_{R^i_\eta} (\beta, k_\beta)$ if and only if:

- $k_\beta = i$;
- $k_\alpha = 1 - i$;
- $f_\beta(\alpha) = \eta$.

The statement that each $R^i_\eta$ is transitive and tree-like is vacuously true, and it is easily verified that this defines a strong system. $S$ does not even have a branch of length 3, so it certainly does not have a cofinal branch. □

*Proof 2.* If $\mu = \omega$, then there is a $\mu^+$-Aronszajn tree, which is a strong $\mu^+$-system with 1 relation and no cofinal branch. If $\mu > \omega$, let $\mathbb{P} = \mathrm{Coll}(\omega, \mu)$. In $V^{\mathbb{P}}$, $\mu^+ = \omega_1$, so there is a $\mu^+$-Aronszajn tree. Let $\dot{T}$ be a $\mathbb{P}$-name for a $\mu^+$-Aronszajn tree. Without loss of generality, the underlying set of $\dot{T}$ is forced to be $\mu^+ \times \omega$.

In $V$, we define a system $S = \langle \mu^+ \times \omega, \mathcal{R} \rangle$, where $\mathcal{R} = \{R_p \mid p \in \mathbb{P}\}$. If $p \in \mathbb{P}$, $\alpha < \beta < \mu^+$, and $n_\alpha, n_\beta < \omega$, then let $(\alpha, n_\alpha) <_{R_p} (\beta, n_\beta)$ if and only if $p \Vdash$ "$(\alpha, n_\alpha) <_{\dot{T}} (\beta, n_\beta)$." It is easily verified that $S$ is a strong $\mu^+$-system. If $S$ had a cofinal branch, there would be an unbounded set $I \subseteq \mu^+$ and a condition $p \in \mathbb{P}$ such that, for every $\alpha \in I$, there is $n_\alpha < \omega$ such that, whenever $\alpha < \beta$ are both in $I$, $p \Vdash$ "$(\alpha, n_\alpha) <_{\dot{T}} (\beta, n_\beta)$." But then $p$ forces that the downward closure of $\{(\alpha, n_\alpha) \mid \alpha \in I\}$ is a cofinal branch in $\dot{T}$, contradicting the fact that $\dot{T}$ is a name for an Aronszajn tree. □

**Definition 3.6.** Let $\lambda$ be a regular cardinal. $\lambda$ satisfies the *strong system property* if, whenever $S$ is a strong $\lambda$-system and $|\mathcal{R}(S)|^+ < \lambda$, $S$ has a cofinal branch.

**Remark 3.7.** Note that, if $\lambda$ is a regular cardinal, then a $\lambda$-tree $(T, <_T)$ can be viewed as a strong $\lambda$-system with 1 relation. Thus, if $\lambda$ satisfies the strong system property, then $\lambda$ also satisfies the tree property.

**Proposition 3.8.** *The strong system property is robust.*

*Proof.* Suppose $\lambda$ is a regular, uncountable cardinal, $\lambda$ satisfies the strong system property, and $\mathbb{P}$ is a forcing poset such that $|\mathbb{P}|^+ < \lambda$. We must show that $\lambda$ satisfies the strong system property in $V^{\mathbb{P}}$. Suppose for sake of contradiction that there is $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{S}$ such that $p$ forces $\dot{S}$ to be a strong $\lambda$-system with no cofinal branch. Without loss of generality, by extending $p$ if necessary, we may assume that there is a cardinal $\nu$ such that $\nu^+ < \lambda$ and $p$ forces $\dot{S}$ to be of the form

$$\langle \{\{\alpha\} \times \dot{\kappa}_\alpha \mid \alpha \in \dot{I}\}, \{\dot{R}_\eta \mid \eta < \nu\} \rangle.$$

For all $\alpha < \lambda$ such that $p \not\Vdash$ "$\alpha \notin I$," find $q_\alpha \leq p$ and $\kappa^*_\alpha < \lambda$ such that $q_\alpha \Vdash$ "$\alpha \in \dot{I}$ and $\dot{\kappa}_\alpha = \kappa^*_\alpha$." As $|\mathbb{P}| < \lambda$, we can find an unbounded $J \subseteq \lambda$ and a $q \leq p$ such that, for all $\alpha \in J$, $q_\alpha = q$. Define a system

$$T = \langle \{\{\alpha\} \times \kappa^*_\alpha \mid \alpha \in J\}, \{R_{\eta,s} \mid \eta < \nu, s \leq q\} \rangle$$

in $V$ as follows: for all $\alpha_0 < \alpha_1$, both in $J$, for all $\beta_0 < \kappa^*_{\alpha_0}$ and $\beta_1 < \kappa^*_{\alpha_1}$, for all $\eta < \nu$, and for all $s \leq q$, let $(\alpha_0, \beta_0) <_{R_{\eta,s}} (\alpha_1, \beta_1)$ iff $s \Vdash$ "$(\alpha_0, \beta_0) <_{\dot{R}_\eta} (\alpha_1, \beta_1)$." Since $p$ forces $\dot{S}$ to be a strong $\lambda$-system, it is easily verified that $T$ is a strong $\lambda$-system with $\max(|\mathbb{P}|, \nu)$ relations. By the strong system property, there are $b \subseteq T$, $\eta < \nu$, and $s \leq q$ such that $b$ is a cofinal branch in $T$ through $R_{\eta,s}$. But then $s \Vdash$ "$b$ is a cofinal branch in $\dot{S}$ through $\dot{R}_\eta$," contradicting the assumption that $p$ forces $\dot{S}$ to have no cofinal branches. □

**Remark 3.9.** Note that, by the proof of Proposition 3.8, if $\lambda$ is regular, $\kappa$ and $\mu$ are such that $\kappa^+, \mu^+ < \lambda$, $\mathbb{P}$ is a forcing poset with $|\mathbb{P}| = \kappa$, and, in $V^{\mathbb{P}}$, there is a strong $\lambda$-system with $\mu$ relations and no cofinal branch, then, in $V$, there is a strong $\lambda$-system with $\max(\mu, \kappa)$ relations and no cofinal branch.

In Section 2, we saw that robust stationary reflection is equivalent to the property that every stationary set reflects at arbitrarily high cofinalities. It is not clear that we have an exactly analogous situation here with the robust tree property and the strong system property. We do, however, have the following.

**Proposition 3.10.** *Suppose $\lambda$ is a regular, uncountable cardinal. The following are equivalent.*

1. *$\lambda$ satisfies the strong system property.*
2. *Every strong $\lambda$-system with only countably many relations has a cofinal branch, and this property is robust under small forcing.*

*Proof.* (1) $\Rightarrow$ (2) follows immediately from Proposition 3.8. To prove (2) $\Rightarrow$ (1), suppose (2) holds, and let $S$ be a strong $\lambda$-system with $|\mathcal{R}(S)|^+ < \lambda$. Let $\mathbb{P} = \mathrm{Coll}(\omega, |\mathcal{R}(S)|)$, and let $G$ be $\mathbb{P}$-generic. $|\mathbb{P}|^+ < \lambda$ and, in $V[G]$, $S$ is a strong $\lambda$-system with countably many relations. Thus, by (2), there is a cofinal branch $b \subseteq S$ in $V[G]$. Let $\dot{b} \in V$ be a $\mathbb{P}$-name for a cofinal branch through $S$. For $p \in \mathbb{P}$, let $b_p = \{u \in S \mid p \Vdash \text{"}u \in \dot{b}\text{"}\}$. Each $b_p$ is a branch through $S$. Since $|\mathbb{P}| < \lambda$, there is $p \in G$ such that $b_p$ is cofinal. □

We now show that the strong system property holds at large cardinals. Since the strong system property is a robust generalization of the tree property and large cardinals are themselves robust, it is not surprising that the proofs are straightforward generalizations of the proofs of the corresponding facts about the tree property. In particular, the proof of Proposition 3.12 is extremely similar to the proof of Theorem 3.1 from [11].

**Proposition 3.11.** *Suppose $\lambda$ is weakly compact. Then $\lambda$ satisfies the strong system property.*

*Proof.* Let $S = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R} \rangle$ be a strong $\lambda$-system. $S$ can be coded in a natural way by a set $A \subseteq V_\lambda$. By the weak compactness of $\lambda$, find a transitive set $X \neq V_\lambda$ and $B \subseteq X$ such that $(V_\lambda, \in, A) \prec (X, \in, B)$. By elementarity and the fact that $|\mathcal{R}| < \lambda$, $B$ codes a strong system $T = \langle \{\{\alpha\} \times \kappa_\alpha \mid \alpha \in J\}, \mathcal{R} \rangle$ such that $J$ is unbounded in the ordinals of $X$ and $T$ extends $S$. Choose $\gamma \in J \setminus \kappa$ and, for all

$\alpha \in I$, find $\beta_\alpha < \kappa_\alpha$ and $R_\alpha \in \mathcal{R}$ such that $(\alpha, \beta_\alpha) <_{R_\alpha} (\gamma, 0)$ in $T$. Since $|\mathcal{R}| < \lambda$, there is an unbounded $I^* \subseteq I$ and a fixed $R \in \mathcal{R}$ such that, for all $\alpha \in I^*$, $R_\alpha = R$. Then $b = \{(\alpha, \beta_\alpha) \mid \alpha \in I^*\}$ is a cofinal branch in $S$ through $R$. □

Since $\lambda$ is weakly compact iff $\lambda$ is inaccessible and has the tree property, Proposition 3.11 implies that, for inaccessible $\lambda$, the tree property is equivalent to the strong system property. As we will see later, this equivalence does not necessarily hold for accessible cardinals. Also, note that this is in contrast to the situation with stationary reflection, as we saw in the previous section that, for inaccessible $\lambda$, stationary reflection is not necessarily equivalent to robust stationary reflection.

**Proposition 3.12.** *Suppose $\mu$ is a singular limit of strongly compact cardinals and $\lambda > \mu$ is a regular cardinal. If $S = \langle I \times \kappa, \mathcal{R}\rangle$ is a strong $\lambda$-system, $\kappa \leq \mu$, and $|\mathcal{R}| < \mu$, then $S$ has a cofinal branch. In particular, $\mu^+$ satisfies the strong system property.*

*Proof.* Fix a regular $\lambda > \mu$, and let $S = \langle I \times \kappa, \mathcal{R}\rangle$ be a strong $\lambda$-system with $\kappa \leq \mu$ and $|\mathcal{R}| < \mu$. We assume for this proof that $\kappa = \mu$, as the case $\kappa < \mu$ is easier. Let $\langle \mu_i \mid i < \mathrm{cf}(\mu)\rangle$ be an increasing sequence of strongly compact cardinals, cofinal in $\mu$, such that $\mathrm{cf}(\mu), |\mathcal{R}| < \mu_0$.

Let $F$ be the filter of co-bounded subsets of $S$, i.e. the set of $X \subseteq I \times \kappa$ such that $|(I \times \kappa) \setminus X| < \lambda$, and let $U$ be a $\mu_0$-complete ultrafilter on $S$ extending $F$. For each $\alpha \in I$ and $u \in S_{>\alpha}$, pick $(i^\alpha_u, R^\alpha_u)$, with $i^\alpha_u < \mathrm{cf}(\mu)$ and $R^\alpha_u \in \mathcal{R}$, such that, for some $\beta < \mu_{i^\alpha_u}$, $(\alpha, \beta) <_{R^\alpha_u} u$. Since $S_{>\alpha} \in U$ and $U$ is $\mu_0$-complete, there is $(i^\alpha, R^\alpha)$ such that the set $X_\alpha := \{u \in S_{>\alpha} \mid (i^\alpha_u, R^\alpha_u) = (i^\alpha, R^\alpha)\} \in U$. There is then an unbounded $J \subseteq I$ and $(i^*, R^*)$ such that, for all $\alpha \in J$, $(i^\alpha, R^\alpha) = (i^*, R^*)$. Now, if $\alpha_0 < \alpha_1$ are both in $J$, we can find $u \in X_{\alpha_0} \cap X_{\alpha_1}$. There are then $\beta_0, \beta_1 < \mu_{i^*}$ such that $(\alpha_0, \beta_0), (\alpha_1, \beta_1) <_{R^*} u$. Since $R^*$ is tree-like, we have $(\alpha_0, \beta_0) <_{R^*} (\alpha_1, \beta_1)$. This shows that $S' := \langle J \times \mu_{i^*}, \{R^*\}\rangle$ is a $\lambda$-system.

Next, fix $k > i^*$ with $k < \mathrm{cf}(\mu)$, and let $U'$ be a $\mu_k$-complete ultrafilter over $\lambda$ extending the co-bounded filter and such that $J \in U'$. Fix $\alpha \in J$. For all $\beta \in J \setminus (\alpha+1)$, fix $\gamma^\alpha_\beta, \delta^\alpha_\beta < \mu_{i^*}$ such that $(\alpha, \gamma^\alpha_\beta) <_{R^*} (\beta, \delta^\alpha_\beta)$. Since $J \setminus (\alpha+1) \in U'$ and $U'$ is $\mu_k$-complete, we can fix $\gamma^\alpha, \delta^\alpha < \mu_{i^*}$ such that $Y_\alpha := \{\beta \in J \setminus (\alpha+1) \mid (\gamma^\alpha_\beta, \delta^\alpha_\beta) = (\gamma^\alpha, \delta^\alpha)\} \in U'$. Next, fix an unbounded $J' \subseteq J$ and $\gamma^*, \delta^* < \mu_{i^*}$ such that, for all $\alpha \in J'$, $(\gamma^\alpha, \delta^\alpha) = (\gamma^*, \delta^*)$. Suppose $\alpha_0 < \alpha_1$ are both in $J'$. Fix $\beta \in X_{\alpha_0} \cap X_{\alpha_1}$. Then $(\alpha_0, \gamma^*), (\alpha_1, \gamma^*) <_{R^*} (\beta, \delta^*)$, so, since $R^*$ is tree-like, $(\alpha_0, \gamma^*) <_{R^*} (\alpha_1, \gamma^*)$. Hence, $\langle (\alpha, \gamma^*) \mid \alpha \in J'\rangle$ is a cofinal branch through $R^*$ in $S$. □

A similar argument shows that all systems with finite width have a cofinal branch.

**Proposition 3.13.** *Suppose $\lambda$ is a regular cardinal and $S = \langle I \times n, \mathcal{R}\rangle$ is a $\lambda$-system with $n, |\mathcal{R}| < \omega$. Then $S$ has a cofinal branch.*

*Proof.* Let $U$ be an ultrafilter over $\lambda$, extending the co-bounded filter, such that $I \in U$. Fix $\alpha \in I$. For all $\beta \in I \setminus (\alpha+1)$, choose $i^\alpha_\beta, j^\alpha_\beta < n$ and $R^\alpha_\beta \in \mathcal{R}$ such that $(\alpha, i^\alpha_\beta) <_{R^\alpha_\beta} (\beta, j^\alpha_\beta)$. Fix $i^\alpha, j^\alpha < n$ and $R^\alpha \in \mathcal{R}$ such that $X_\alpha := \{\beta \in I \setminus (\alpha+1) \mid (i^\alpha_\beta, j^\alpha_\beta, R^\alpha_\beta) = (i^\alpha, j^\alpha, R^\alpha)\} \in U$. Fix an unbounded $J \subseteq I$ and $(i^*, j^*, R^*)$ such that, for all $\alpha \in J$, $(i^\alpha, j^\alpha, R^\alpha) = (i^*, j^*, R^*)$. Now, suppose $\alpha_0 < \alpha_1$ are both in $J$, and find $\beta \in X_{\alpha_0} \cap X_{\alpha_1}$. $(\alpha_0, i^*), (\alpha_1, i^*) <_{R^*} (\beta, j^*)$, so $(\alpha_0, i^*) <_{R^*} (\alpha_1, j^*)$. Thus, $\{(\alpha, i^*) \mid \alpha \in J\}$ is a cofinal branch through $R^*$ in $S$. □

However, the existence of certain subadditive, unbounded functions implies the existence of strong systems of possibly small width with no cofinal branch.

**Definition 3.14.** Suppose $\kappa < \lambda$ are infinite, regular cardinals and $d : [\lambda]^2 \to \kappa$.

(1) $d$ is *subadditive* if, for all $\alpha < \beta < \gamma < \lambda$:
  (a) $d(\alpha, \gamma) \leq \max(\{d(\alpha, \beta), d(\beta, \gamma)\})$;
  (b) $d(\alpha, \beta) \leq \max(\{d(\alpha, \gamma), d(\beta, \gamma)\})$.
(2) $d$ is *unbounded* if, whenever $I \subseteq \lambda$ is unbounded, $d``[I]^2$ is unbounded in $\kappa$.

**Remark 3.15.** Note that there are different definitions of subadditivity in the literature. In particular, functions are sometimes (e.g., in [14]) called subadditive if they simply satisfy the inequality in (1a) of Definition 3.14. The stronger definition we give here is more appropriate for the study of systems and matches that in, e.g., Section 9 of [16]. For more on the consistency of subadditive, unbounded functions, see [8].

**Proposition 3.16.** *Suppose $\kappa < \lambda$ are infinite, regular cardinals and $d : [\lambda]^2 \to \kappa$ is subadditive and unbounded. Then there is a strong $\lambda$-system $S = \langle \lambda \times 1, \mathcal{R} \rangle$ such that $|\mathcal{R}| = \kappa$ and $S$ has no cofinal branch.*

*Proof.* We define $S = \langle \lambda \times 1, \mathcal{R} \rangle$, with $\mathcal{R} = \{R_\eta \mid \eta < \kappa\}$. Given $\alpha < \beta < \lambda$ and $\eta < \kappa$, let $(\alpha, 0) <_{R_\eta} (\beta, 0)$ if and only if $\eta \geq d(\alpha, \beta)$. The fact that each $R_\eta$ is transitive and tree-like follows from (a) and (b) of the definition of subadditivity, respectively. It is then easy to verify that $S$ is a strong $\lambda$-system. Suppose for sake of contradiction that $S$ has a cofinal branch. Then there is an unbounded $I \subseteq \lambda$ and an $\eta < \kappa$ such that, for all $\alpha < \beta$, both in $I$, we have $(\alpha, 0) <_{R_\eta} (\beta, 0)$. Since $d$ is unbounded, we can find $\alpha < \beta$ in $I$ such that $d(\alpha, \beta) > \eta$. But then $(\alpha, 0) \not<_{R_\eta} (\beta, 0)$. Contradiction. $\square$

## 4. Weak squares

Recall the following definition.

**Definition 4.1.** Let $\lambda$ and $\mu$ be cardinals, with $\mu$ infinite and $\lambda > 1$. A $\square_{\mu,<\lambda}$-sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \mu^+ \rangle$ such that:

(1) for all limit $\alpha < \mu^+$, if $C \in \mathcal{C}_\alpha$, then $C$ is a club in $\alpha$ and $\mathrm{otp}(C) \leq \mu$;
(2) for all limit $\alpha < \mu^+$, $1 \leq |\mathcal{C}_\alpha| < \lambda$;
(3) for all limit $\alpha < \beta < \mu^+$ and all $C \in \mathcal{C}_\beta$, if $\alpha \in C'$, then $C \cap \alpha \in \mathcal{C}_\alpha$.

$\square_{\mu,<\lambda}$ holds if there is a $\square_{\mu,<\lambda}$-sequence.

**Remark 4.2.** $\square_{\mu,<\lambda^+}$ is usually denoted $\square_{\mu,\lambda}$. It is immediate that, if $\lambda_0 < \lambda_1$, then $\square_{\mu,<\lambda_0}$ implies $\square_{\mu,<\lambda_1}$. $\square_{\mu,1}$ is Jensen's classical principle $\square_\mu$. $\square_{\mu,\mu}$ is also called *weak square* and denoted $\square^*_\mu$. $\square^*_\mu$ is equivalent to the existence of a special $\mu^+$-Aronszajn tree. $\square_{\mu,\mu^+}$ is also called *silly square* and holds in all models of ZFC.

We will be interested in the following variation on the classical square principles.

**Definition 4.3.** Let $\kappa$, $\lambda$, and $\mu$ be cardinals, with $\kappa \leq \mu$, $\kappa$ regular, and $\lambda > 1$. A $\square^{\geq\kappa}_{\mu,<\lambda}$-sequence is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in S \rangle$ such that:

(1) $S^{\mu^+}_{\geq\kappa} \subseteq S \subseteq \lim(\mu^+)$;
(2) for all $\alpha \in S$ and all $C \in \mathcal{C}_\alpha$, $C$ is a club in $\alpha$ and $\mathrm{otp}(C) \leq \mu$;
(3) for all $\alpha \in S$, $1 \leq |\mathcal{C}_\alpha| < \lambda$;

(4) for all $\beta \in S$, for all $C \in \mathcal{C}_\beta$, and for all $\alpha \in C'$, we have $\alpha \in S$ and $C \cap \alpha \in \mathcal{C}_\alpha$.

$\square^{\geq\kappa}_{\mu,<\lambda}$ holds if there is a $\square^{\geq\kappa}_{\mu,<\lambda}$-sequence. As usual, we shall write $\square^{\geq\kappa}_{\mu,\lambda}$ instead of $\square^{\geq\kappa}_{\mu,<\lambda^+}$ and $\square^{\geq\kappa}_{\mu}$ instead of $\square^{\geq\kappa}_{\mu,1}$.

Baumgartner, in unpublished work, was the first to study such square sequences, particularly square sequences, using our notation, of the form $\square^{\geq\kappa}_{\mu}$. For more on this, see, e.g. Section 2.2 of [1] or Section 8 of [2].

For later use, we introduce a forcing poset designed to add such a square sequence. Given $\kappa$, $\lambda$, and $\mu$ as in the above definition, define a poset $\mathbb{B}(\kappa,\lambda,\mu)$ as follows. Conditions are of the form $p = \langle \mathcal{C}^p_\alpha \mid \alpha \in s^p \rangle$ such that:

- $s^p$ is a bounded subset of $\mu^+$ with a maximal element, which we denote $\gamma^p$;
- $(\gamma^p + 1) \cap \mathrm{cof}(\geq \kappa) \subseteq s^p$;
- for all $\alpha \in s^p$ and all $C \in \mathcal{C}^p_\alpha$, $C$ is a club in $\alpha$ and $\mathrm{otp}(C) \leq \mu$;
- for all $\alpha \in s^p$, $1 \leq |\mathcal{C}^p_\alpha| < \lambda$;
- for all $\beta \in s^p$, for all $C \in \mathcal{C}^p_\beta$, and for all $\alpha \in C'$, we have $\alpha \in s^p$ and $C \cap \alpha \in \mathcal{C}_\alpha$.

If $p, q \in \mathbb{B}(\kappa,\lambda,\mu)$, then $q \leq p$ iff $s^q$ end-extends $s^p$ and, for all $\alpha \in s^p$, $\mathcal{C}^q_\alpha = \mathcal{C}^p_\alpha$.

The following is easily verified. See, for example, [1] for the details in the case $\lambda = 2$. The proof is essentially the same for other values of $\lambda$.

**Proposition 4.4.** *Let $\kappa$, $\lambda$, and $\mu$ be cardinals as above.*

(1) *$\mathbb{B}(\kappa,\lambda,\mu)$ is $\kappa$-directed closed.*
(2) *$\mathbb{B}(\kappa,\lambda,\mu)$ is $\mu+1$-strategically closed.*
(3) *$\Vdash_{\mathbb{B}(\kappa,\lambda,\mu)}$ "$\square^{\geq\kappa}_{\mu,<\lambda}$ holds."*

**Proposition 4.5.** *Suppose $\kappa < \mu$ are infinite cardinals. Then the following are equivalent.*

(1) *$\square^{\geq\kappa^+}_{\mu,\mu}$ holds.*
(2) *There is a poset $\mathbb{P}$ such that $|\mathbb{P}| \leq \kappa$ and $\Vdash_{\mathbb{P}}$ "$\square^*_\mu$ holds."*

*Proof.* (1) $\Rightarrow$ (2): Suppose $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in S \rangle$ is a $\square^{\geq\kappa^+}_{\mu,\mu}$-sequence. Let $\mathbb{P} = \mathrm{Coll}(\omega,\kappa)$. Then $|\mathbb{P}| = \kappa$ and, in $V^{\mathbb{P}}$, $\kappa^+ = \omega_1$. Define a $\square^*_\mu$-sequence $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \mu^+ \rangle$ in $V^{\mathbb{P}}$ as follows. If $\alpha \in S$, then let $\mathcal{D}_\alpha = \mathcal{C}_\alpha$. If $\alpha \in \lim(\mu^+) \setminus S$, then $(\mathrm{cf}(\alpha))^{V^{\mathbb{P}}} = \omega$. Let $D$ be an arbitrary $\omega$-sequence cofinal in $\alpha$, and let $\mathcal{D}_\alpha = \{D\}$. It is easily verified that this defines a $\square^*_\mu$-sequence.

(2) $\Rightarrow$ (1): Suppose $\mathbb{P}$ is a forcing poset such that $|\mathbb{P}| \leq \kappa$ and $\Vdash_{\mathbb{P}}$ "$\square^*_\mu$ holds." Let $\dot{\vec{\mathcal{C}}} = \langle \dot{\mathcal{C}}_\alpha \mid \alpha < \mu^+ \rangle$ be a $\mathbb{P}$-name for a $\square^*_\mu$-sequence. For each $\mathbb{P}$-name $\dot{X}$ for a subset of $\mu^+$ and each $p \in \mathbb{P}$, let $\dot{X}_p = \{\alpha < \mu^+ \mid p \Vdash \text{“}\alpha \in \dot{X}\}$. For each $\beta < \mu^+$, let

$$A_\beta = \{p \in \mathbb{P} \mid \text{for some } \mathbb{P}\text{-name } \dot{C}, p \Vdash \text{“}\dot{C} \in \dot{\mathcal{C}}_\beta\text{” and } \dot{C}_p \text{ is club in } \beta\}.$$

Let $S = \{\beta < \mu^+ \mid A_\beta \neq \emptyset\}$. Easily, as $|\mathbb{P}| = \kappa$, $S^{\mu^+}_{\geq\kappa^+} \subseteq S$. Define a $\square^{\geq\kappa^+}_{\mu,\mu}$ sequence $\vec{\mathcal{D}} = \langle \mathcal{D}_\alpha \mid \alpha \in S \rangle$ by letting

$$\mathcal{D}_\alpha = \{\dot{C}_p \mid p \in A_\alpha, p \Vdash \text{“}\dot{C} \in \dot{\mathcal{C}}_\alpha,\text{” and } \dot{C}_p \text{ is club in } \alpha\}.$$

It is easily verified that this defines a $\square^{\geq\kappa^+}_{\mu,\mu}$-sequence. $\square$

Since $\square^*_\mu$ is equivalent to the existence of a special $\mu^+$-Aronszajn tree, if $\square^{\geq\kappa^+}_{\mu,\mu}$ holds, then there is a strong $\mu^+$-system with $\kappa$ relations and no cofinal branch. We also get the following characterization for robustness of having no special trees.

**Corollary 4.6.** *Let $\mu$ be an infinite cardinal. The following are equivalent.*

(1) *There are no special $\mu^+$-Aronszajn trees, and this property holds robustly.*
(2) $\square^{\geq\kappa^+}_{\mu,\mu}$ *fails for all $\kappa < \mu$.*

**Remark 4.7.** In Section 8 of [2], Cummings, Foreman, and Magidor use similar ideas to show that, consistently, square sequences can be added by small forcing. In particular, they construct a model in which $\square^*_{\aleph_\omega}$ fails but in which $\square^{\geq\aleph_2}_{\aleph_\omega}$ holds and, therefore, $\square_{\aleph_\omega}$ holds after forcing with $\mathrm{Coll}(\omega, \omega_1)$. Proposition 4.5 shows that this is, in essence, the only scenario in which square sequences can be added by small forcing.

We would like to bring the reader's attention here to a related matter. In situations in which a special $\mu^+$-Aronszajn tree can be added by small forcing, it can always be added by $\mathrm{Coll}(\omega, \kappa)$ for a suitable $\kappa < \mu$. The question remains, though, whether it can be the case that there is no special $\mu^+$-Aronszajn tree in $V$ but there is one in an extension by a small forcing poset that preserves all cofinalities. Rinot, in [13], addresses this question in the case in which $\mu$ has uncountable cofinality, proving the following theorem.

**Theorem 4.8** (Rinot, [13])**.** *It is consistent relative to the existence of two supercompact cardinals that there is no special $\aleph_{\omega_1+1}$-Aronszajn tree but there is a special $\aleph_{\omega_1+1}$-Aronszajn tree in a forcing extension by a forcing of size $\aleph_3$ that preserves all cofinalities.*

As far as we are aware, the analogous question for successors of singular cardinals of countable cofinality remains open.

## 5. Preservation lemmas and the narrow system property

In this section, we present some results that will be useful in Section 6. We first make the following definition.

**Definition 5.1.** Let $\lambda$ be an uncountable regular cardinal, and let $S = \langle I \times \kappa, \mathcal{R}\rangle$ be a narrow $\lambda$-system. $\bar{b} = \{b_{\gamma,R} \mid \gamma < \kappa, R \in \mathcal{R}\}$ is a *full set of branches through* $S$ if:

(1) for all $\gamma < \kappa$ and $R \in \mathcal{R}$, $b_{\gamma,R}$ is a branch of $S$ through $R$;
(2) for all $\alpha \in I$, there are $\gamma < \kappa$ and $R \in \mathcal{R}$ such that $b_{\gamma,R} \cap S_\alpha \neq \emptyset$.

**Remark 5.2.** Note that, since $\lambda$ is regular and $\mathrm{width}(S) < \lambda$, condition (2) in Definition 5.1 implies that, for some $\gamma < \kappa$ and $R \in \mathcal{R}$, $b_{\gamma,R}$ is a cofinal branch.

The following result is due to Neeman and improves a similar result of Sinapova from [15].

**Lemma 5.3** (Neeman, [12])**.** *Suppose that $\lambda$ is a regular, uncountable cardinal, $S = \langle I \times \kappa, \mathcal{R}\rangle$ is a narrow $\lambda$-system, and* $\mathrm{width}(S) = \theta$*. Suppose $\mathbb{P}$ is a forcing poset, and let $\mathbb{P}^{\theta^+}$ denote the full-support product of $\theta^+$ copies of $\mathbb{P}$. Suppose moreover that $\mathbb{P}^{\theta^+}$ is $\theta^{++}$-distributive, $G$ is $\mathbb{P}$-generic over $V$, and, in $V[G]$, there is a full set of branches through $S$. Then there is a cofinal branch through $S$ in $V$.*

Next, we show that cofinal branches cannot be added to systems by forcing posets satisfying appropriate approximation properties.

**Definition 5.4.** Let $\lambda$ be a regular cardinal, and let $\mathbb{P}$ be a forcing poset. $\mathbb{P}$ has the $\lambda$-*approximation property* if, for every $y \in V$ and every $\mathbb{P}$-name $\dot{x}$ for a subset of $y$ such that, for all $z \in (\mathcal{P}_\lambda(y))^V$, $\Vdash_{\mathbb{P}}$ "$\dot{x} \cap z \in V$", we have $\Vdash_{\mathbb{P}} \dot{x} \in V$.

**Lemma 5.5.** *Suppose $\lambda$ is a regular cardinal, $S = \langle\{\{\alpha\} \times \kappa_\alpha \mid \alpha \in I\}, \mathcal{R}\rangle$ is a $\lambda$-system, and $\mathbb{P}$ has the $\lambda$-approximation property. If $G$ is $\mathbb{P}$-generic over $V$ and $S$ has a cofinal branch in $V[G]$, then $S$ has a cofinal branch in $V$.*

*Proof.* In $V[G]$, suppose $b \subseteq S$ is a cofinal branch through $R \in \mathcal{R}$. By closing $b$ downward, we may assume that $b$ is a maximal branch, i.e., if $\alpha \in I$ and $v \in b \cap S_\alpha$, then $b \cap S_{<\alpha} = \{u \in S \mid u <_R v\}$. It suffices to show that $b \cap z \in V$ for all $z \in (\mathcal{P}_\lambda(S))^V$, as then the $\lambda$-approximation property will imply that $b \in V$.

To this end, let $z \in (\mathcal{P}_\lambda(S))^V$. As $\lambda$ is regular, there is $\alpha < \lambda$ such that $z \subseteq S_{<\alpha}$. Find $v \in b \cap S_{\geq\alpha}$. Then $b \cap z = \{u \in S \mid u <_R v\} \cap z \in V$. □

The following lemma is due to Unger.

**Lemma 5.6** (Unger, [17])**.** *Suppose $\lambda$ is a regular cardinal, $\mathbb{P}$ is a forcing poset, and $\mathbb{P} \times \mathbb{P}$ has the $\lambda$-c.c. Then $\mathbb{P}$ has the $\lambda$-approximation property.*

**Definition 5.7.** Suppose $\lambda$ is a regular cardinal. $\lambda$ has the *narrow system property* if every narrow $\lambda$-system has a cofinal branch.

The narrow system property is a useful tool for analyzing trees and strong systems. It was implicitly introduced by Magidor and Shelah in [11] in order to establish the consistency of the tree property at the successor of a singular cardinal. Their general framework for establishing the tree property at the successor of a singular cardinal $\mu$ consists of two main steps. First, given a $\mu^+$-tree, one argues that it must have a narrow subsystem, $S$. Second, one argues that $\mu^+$ has the narrow system property. This yields a cofinal branch through $S$ and, in turn, a cofinal branch through $T$. For more on the narrow system property, see [8].

The following lemma is essentially due to Neeman (see the proof of Lemma 3.6 in [12]). For that reason, and because it will also follow from Lemma 6.7, which we prove in Section 6, we omit the proof here.

**Lemma 5.8.** *Suppose $\langle\kappa_n \mid n < \omega\rangle$ is an increasing sequence of indestructibly supercompact cardinals. Let $\mu = \sup(\{\kappa_n \mid n < \omega\})$, and let $\lambda = \mu^+$. For $m < \omega$, let $\mathbb{S}_m$ be the full-support product $\prod_{n \geq m} \mathrm{Coll}(\kappa_n^{+2}, < \kappa_{n+1})$. Then, in $V^{\mathbb{S}_m}$, $\lambda$ has the narrow system property.*

**Corollary 5.9.** *Assume the same hypotheses as in Lemma 5.8. In $V^{\mathbb{S}_m}$, suppose $S = \langle\lambda \times \mu, \mathcal{R}\rangle$ is a strong $\lambda$-system and $|\mathcal{R}| < \kappa_m$. Then $S$ has a cofinal branch.*

*Proof.* Let $G$ be $\mathbb{S}_m$-generic over $V$. In $V[G]$, $\kappa_m$ remains supercompact. Let $j : V[G] \to M$ witness that $\kappa_m$ is $\lambda$-supercompact. In $M$, $j(S) = \langle j(\lambda) \times j(\mu), \{j(R) \mid R \in \mathcal{R}\}\rangle$ is a strong $j(\lambda)$-system. Let $\delta = \sup(j``\lambda)$. For $\alpha < \lambda$, find $\beta_\alpha < j(\mu)$ and $R_\alpha \in \mathcal{R}$ such that $(j(\alpha), \beta_\alpha) <_{j(R_\alpha)} (\delta, 0)$. Let $n_\alpha < \omega$ be such that $\beta_\alpha < j(\kappa_{n_\alpha})$. Since $\lambda$ is regular, we can find $R^* \in \mathcal{R}$, $n^* < \omega$, and an unbounded $I \subseteq \lambda$ such that, for all $\alpha \in I$, $R_\alpha = R^*$ and $n_\alpha = n^*$. Now, if $\alpha_0 < \alpha_1$ are both in $I$, then $(j(\alpha_0), \beta_{\alpha_0}), (j(\alpha_1), \beta_{\alpha_1}) <_{j(R^*)} (\delta, 0)$, so $(j(\alpha_0), \beta_{\alpha_0}) <_{j(R^*)} (j(\alpha_1), \beta_{\alpha_1})$. Thus, $M \models$ "there are $\beta_0, \beta_1 < j(\kappa_{n^*})$ such that $(j(\alpha_0), \beta_0) <_{j(R^*)} (j(\alpha_1), \beta_1)$," so, by elementarity, $V[G] \models$ "there are $\beta_0, \beta_1 < \kappa_{n^*}$ such that $(\alpha_0, \beta_0) <_{R^*} (\alpha_1, \beta_1)$." Therefore, in $V[G]$, $S' = \langle I \times \kappa_{n^*}, \{R^*\} \rangle$ is a narrow system. By Lemma 5.8, $S'$ has a cofinal branch, $b$, which is then also a cofinal branch of $S$. $\square$

## 6. Strong systems at $\aleph_{\omega^2+1}$

In this section, we will prove the following two results.

**Theorem 6.1.** *Suppose there are infinitely many supercompact cardinals, and fix $\alpha < \omega^2$. Then there is a forcing extension in which every strong $\aleph_{\omega^2+1}$-system with $\aleph_\alpha$ relations has a cofinal branch but in which $\aleph_{\omega^2+1}$ fails to have the robust tree property.*

**Theorem 6.2.** *Suppose there are infinitely many supercompact cardinals. Then there is a forcing extension in which $\aleph_{\omega^2+1}$ satisfies the strong system property.*

The proofs of these theorems are modifications of the proof of Theorem 1.2 in [4]. We will provide the details for the proof of Theorem 6.1. Because the proof of Theorem 6.2 is very similar to that of Theorem 1.2 in [4] and because it also follows from a straightforward modification of the proof of Theorem 6.1, we will include only a brief paragraph indicating how Theorem 6.2 can be obtained from the proof of Theorem 6.1.

**Remark 6.3.** Similar results, due to Hayut and Magidor, appear in [5]. In particular, they construct a model in which $\aleph_{\omega+1}$ has the tree property but an $\aleph_{\omega+1}$-Aronszajn tree is added by forcing with $\mathrm{Coll}(\omega, \omega_1)$. This is close to our Theorem 6.1 with $\aleph_{\omega^2+1}$ replaced by $\aleph_{\omega+1}$ and $\aleph_\alpha$ replaced by 1 and, as in the proof of Theorem 6.1, the non-robustness of the tree property in the final model in their proof is witnessed by the existence of a partial square sequence of the type considered in Section 4. It is not clear whether their result can be strengthened to be a true analogue of our Theorem 6.1 at $\aleph_{\omega+1}$. Also in [5], Hayut and Magidor, using an argument different from our proof of Theorem 6.2, show the consistency of the robust tree property at $\aleph_{\omega^2+1}$ by showing that it holds in Sinapova's model from [15] for the tree property at $\aleph_{\omega^2+1}$.

The results will be obtained by using a version of diagonal Prikry forcing with interleaved collapses introduced in [10]. The reason that $\aleph_{\omega^2+1}$ is being considered rather than smaller cardinals is that, for technical reasons that will become clear in the proof of Lemma 6.12, it is necessary for our methods to preserve the first $\omega + 1$ cardinal successors of each of the Prikry points. It is an open question of much interest whether these and similar results can be pushed down to smaller cardinals, and in particular to $\aleph_{\omega+1}$. We note that the original application of the diagonal Prikry forcing in [10] provides an instance in which this cannot be done: the forcing is used there to produce a model in which the reflection principle $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ holds. In particular, this principle implies that every almost free Abelian group of cardinality $\aleph_{\omega^2+1}$ is free. On the other hand, in the same paper, it is shown that, for every regular, uncountable $\kappa < \aleph_{\omega^2}$, there is an almost free Abelian group of cardinality $\kappa$ that is not free.

We now begin working towards Theorem 6.1, first reviewing some facts about forcing and square sequences.

Recall that, if $\kappa, \lambda$, and $\mu$ are cardinals, with $\kappa \leq \mu$, $\kappa$ regular, and $\lambda > 1$, then $\mathbb{B}(\kappa, \lambda, \mu)$ is the forcing to add a $\square^{\geq\kappa}_{\mu,<\lambda}$-sequence. Temporarily fix values for $\kappa$, $\lambda$, and $\mu$, and let $\mathbb{B} = \mathbb{B}(\kappa, \lambda, \mu)$.

In $V^{\mathbb{B}}$, let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in S \rangle$ be the $\square^{\geq\kappa}_{\mu,<\lambda}$-sequence added by $\mathbb{B}$. Let $\nu < \mu$ be regular, and let $\mathbb{T}_\nu$ be the forcing poset whose conditions are closed, bounded subsets $t$ of $\mu^+$ such that:

- $|t| < \nu$;
- for all $\alpha \in t'$, we have $\alpha \in S$ and $t \cap \alpha \in \mathcal{C}_\alpha$.

If $s, t \in \mathbb{T}_\nu$, then $t \leq s$ iff $t$ end-extends $s$. For a cardinal $\epsilon$, let $\mathbb{T}^\epsilon_\nu$ denote the full-support product of $\epsilon$ copies of $\mathbb{T}_\nu$. Let $\dot{\mathbb{T}}_\nu$ be a $\mathbb{B}$-name for $\mathbb{T}_\nu$.

**Proposition 6.4.** *In $V$, for every $\epsilon < \lambda$, $\mathbb{B} * \dot{\mathbb{T}}^\epsilon_\nu$ has a dense $\nu$-directed closed subset.*

*Proof.* Fix $\epsilon < \lambda$. Let $\mathbb{U}$ be the set of $(p, \langle \dot{t}_\eta \mid \eta < \epsilon \rangle) \in \mathbb{B} * \dot{\mathbb{T}}$ such that, for all $\eta < \epsilon$:

- there is $t_\eta \in V$ such that $p \Vdash$ "$\dot{t}_\eta = t_\eta$";
- $\gamma^p = \max(t_\eta)$.

The verification that $\mathbb{U}$ is dense and $\nu$-directed closed is straightforward. See, for example, [2] for similar arguments. $\square$

**Corollary 6.5.** *Forcing with $\mathbb{T}_\nu$ over $V^{\mathbb{B}}$ adds a club $D$ in $\mu^+$ such that:*

- $\mathrm{otp}(D) = \nu$;
- *for all $\alpha \in D'$, we have $\alpha \in S$ and $D \cap \alpha \in \mathcal{C}_\alpha$.*

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of indestructibly supercompact cardinals such that, for all $n < \omega$, $2^{\kappa_n} = \kappa_n^+$. Let $\mu = \sup(\{\kappa_n \mid n < \omega\})$, and let $\lambda = \mu^+$. For $m < k < \omega$, let $\mathbb{S}_m$ be the full-support product $\prod_{m \leq n < \omega} \mathrm{Coll}(\kappa_n^{+2}, < \kappa_{n+1})$, and let $\mathbb{S}_{m,k} = \prod_{m \leq n < k} \mathrm{Coll}(\kappa_n^{+2}, < \kappa_{n+1})$. Note that $\mathbb{S}_m = \mathbb{S}_{m,k} \times \mathbb{S}_k$. For notational ease, let $\mathbb{S}_{m,m} = \{\emptyset\}$. Introduce an equivalence relation on $\mathbb{S}_0$ as follows. Given $s = \langle s_n \mid n < \omega \rangle$ and $t = \langle t_n \mid n < \omega \rangle$ in $\mathbb{S}_0$, let $s \sim t$ iff there is $m < \omega$ such that, for all $m \leq n < \omega$, $s_n = t_n$. Given $s \in \mathbb{S}_0$, let $[s]$ denote the equivalence class of $s$. If $m > 0$ and $s \in \mathbb{S}_m$, we abuse notation and let $[s]$ denote $[s^*]$, where $s^* \in \mathbb{S}_0$ is such that $s^*(n) = \emptyset$ if $n < m$ and $s^*(n) = s(n)$ if $n \geq m$. Let $\mathbb{S}^*$ be the poset whose conditions are equivalence classes $[s]$, where $s \in \mathbb{S}_0$. If $[s], [t] \in \mathbb{S}^*$, we let $[t] \leq [s]$ iff there is $m < \omega$ such that, for all $m \leq n < \omega$, $t_n \leq s_n$. It is clear that this is well-defined. It is also easily verified that, if $m < \omega$, the map $\pi_m : \mathbb{S}_m \to \mathbb{S}^*$ defined by $\pi_m(s) = [s]$ is a projection.

Fix an $m < \omega$, and, in $V^{\mathbb{S}^*}$, let $\mathbb{B} = \mathbb{B}(\kappa_m, \mu, \mu)$. Let $\dot{\mathbb{B}}$ be an $\mathbb{S}^*$-name for $\mathbb{B}$, and note that $\mathbb{S}_m * \dot{\mathbb{B}}$ is $\kappa_m$-directed closed.

**Lemma 6.6.** *In $V^{\mathbb{S}_m}$, $\mathbb{B}$ is $\lambda$-distributive.*

*Proof.* Since $\lambda = \mu^+$ and $\mu$ is singular, it suffices to show that $\mathbb{B}$ is $\nu$-distributive for all regular $\nu < \mu$. In particular, it suffices to show that $\mathbb{B}$ is $\kappa_k$-distributive for all $k < \omega$. To this end, fix $k < \omega$. Without loss of generality, $m \leq k$. In $V^{\mathbb{S}^* * \dot{\mathbb{B}}}$, let $\mathbb{T} = \mathbb{T}_{\kappa_k}$, as defined above. Recall that, in $V^{\mathbb{S}^*}$, $\mathbb{B} * \dot{\mathbb{T}}$ has a $\kappa_k$-directed closed dense subset. Note that, in $V$, $\mathbb{S}_{m,k} = \prod_{m \leq n < k} \mathrm{Coll}(\kappa_n^{+2}, < \kappa_{n+1})$ has the $\kappa_k$-c.c. Since $\mathbb{S}_k$ is $\kappa_k^{+2}$-directed closed, we still have that, in $V^{\mathbb{S}_k}$, $\mathbb{S}_{m,k}$ has the $\kappa_k$-c.c. and

$\mathbb{B} * \dot{\mathbb{T}}$ has a $\kappa_k$-directed closed dense subset. Thus, by Easton's Lemma, $\mathbb{B} * \dot{\mathbb{T}}$ is $\kappa_k$-distributive in $V^{\mathbb{S}_k \times \mathbb{S}_{m,k}} = V^{\mathbb{S}_m}$. In particular, $\mathbb{B}$ is $\kappa_k$-distributive in $V^{\mathbb{S}_m}$. □

**Lemma 6.7.** *In $V^{\mathbb{S}_m * \dot{\mathbb{B}}}$, $\lambda$ satisfies the narrow system property.*

*Proof.* First note that, by Lemma 6.6, $\lambda = \kappa_m^{+\omega+1}$ in $V^{\mathbb{S}_m * \dot{\mathbb{B}}}$. Let $\dot{S}$ be an $\mathbb{S}_m * \dot{\mathbb{B}}$-name for a narrow $\lambda$-system. Without loss of generality, we may assume there are $\kappa, \nu < \mu$ such that $\dot{S}$ is forced to be of the form $\langle \lambda \times \kappa, \{\dot{R}_\eta \mid \eta < \nu\}\rangle$. Fix $m < n^* < \omega$ such that $\kappa, \nu < \kappa_{n^*}$. In $V^{\mathbb{S}^* * \dot{\mathbb{B}}}$, let $\mathbb{T} = \mathbb{T}_{\kappa_{n^*+1}}$. Let $G_{n^*+1}$ be $\mathbb{S}_{n^*+1}$-generic over $V$ and let $H * I$ be $\mathbb{B} * \dot{\mathbb{T}}$-generic over $V[G_{n^*+1}]$. Since $\mathbb{S}_{n^*+1} * \dot{\mathbb{B}} * \dot{\mathbb{T}}$ has a $\kappa_{n^*+1}$-directed closed dense subset, $\kappa_{n^*+1}$ remains supercompact in $V[G_{n^*+1} * H * I]$. Move to $V[G_{n^*+1} * H * I]$, which we denote $V_1$, reinterpreting $\dot{S}$ as an $\mathbb{S}_{m,n^*+1}$-name.

Let $j : V_1 \to M$ witness that $\kappa_{n^*+1}$ is $\lambda$-supercompact. Note that $j(\mathbb{S}_{m,n^*}) = \mathbb{S}_{m,n^*}$ and

$$j(\mathbb{S}_{n^*,n^*+1}) = \mathrm{Coll}(\kappa_{n^*}^{+2}, < j(\kappa_{n^*+1})) \cong \mathbb{S}_{n^*,n^*+1} * \dot{\mathbb{R}},$$

where $\dot{\mathbb{R}}$ is $\kappa_{n^*}^{+2}$-closed in $V_1^{\mathbb{S}_{n^*,n^*+1}}$. Let $G_{n^*,n^*+1}$ be $\mathbb{S}_{n^*,n^*+1}$-generic over $V_1$, and let $\mathbb{R}$ be the interpretation of $\dot{\mathbb{R}}$ in $V_1[G_{n^*,n^*+1}]$. Then, letting $G_{m,n^*}$ be $\mathbb{S}_{m,n^*}$ over $V_1[G_{n^*,n^*+1}]$ and $J$ be $\mathbb{R}$-generic over $V_1[G_{n^*,n^*+1}][G_{m^*,n}]$, we can lift $j$ to

$$j : V_1[G_{n^*,n^*+1}][G_{m,n^*}] \to M[G_{n^*,n^*+1}][G_{m,n^*}][J].$$

Let $S = \langle \lambda \times \kappa, \{R_\eta \mid \eta < \nu\}\rangle$ be the realization of $\dot{S}$ in $V_1[G_{n^*,n^*+1}][G_{m,n^*}]$. In $M[G_{n^*,n^*+1}][G_{m,n^*}][J]$,

$$j(S) = \langle j(\lambda) \times \kappa, \{j(R_\eta) \mid \eta < \nu\}\rangle$$

is a $j(\lambda)$-system. Let $\delta = \sup(j``\lambda)$. For all $\gamma < \kappa$ and $\eta < \nu$, let

$$b_{\gamma, R_\eta} = \{(\alpha, \beta) \in \lambda \times \kappa \mid (j(\alpha), \beta) <_{j(R_\eta)} (\delta, \gamma)\}.$$

$\bar{b} := \{b_{\gamma,R_\eta} \mid \gamma < \kappa, \eta < \nu\}$ is easily seen to be a full set of branches through $S$, and $\bar{b} \in V[G_{n^*+1} * H * I][G_{n^*,n^*+1}][G_{m,n^*}][J]$.

In $V[G_{n^*+1}]$, $\mathbb{B} * \dot{\mathbb{T}}^{\kappa_{n^*}}$ has a dense $\kappa_{n^*+1}$-closed subset. Moreover, $\mathbb{S}_{n^*,n^*+1}$ is $\kappa_{n^*}^{+2}$-closed in $V[G_{n^*+1}]$ and, in $V[G_{n^*+1} * H * I][G_{n^*,n^*+1}]$, $\mathbb{R}$ is $\kappa_{n^*}^{+2}$-closed. Thus, in $V[G_{n^*+1}][G_{n^*,n^*+1}]$, $\mathbb{B} * \dot{\mathbb{T}}^{\kappa_{n^*}} * \dot{\mathbb{R}}^{\kappa_{n^*}} \cong \mathbb{B} * (\dot{\mathbb{T}} * \dot{\mathbb{R}})^{\kappa_{n^*}}$ has a dense $\kappa_{n^*}^{+2}$-closed subset. $\mathbb{S}_{m,n^*}$ has the $\kappa_{n^*}$-c.c. in $V[G_{n^*+1}][G_{n^*,n^*+1}]$, so, by Easton's Lemma, $\mathbb{B} * (\dot{\mathbb{T}} * \dot{\mathbb{R}})^{\kappa_{n^*}}$ is $\kappa_{n^*}^{+2}$-distributive in $V[G_{n^*+1}][G_{n^*,n^*+1}][G_{m,n^*}]$, so $(\mathbb{T} * \dot{\mathbb{R}})^{\kappa_{n^*}}$ is $\kappa_{n^*}^{+2}$-distributive in $V[G_{n^*+1}][G_{n^*,n^*+1}][G_{m,n^*}][H]$. Thus, since $\mathrm{width}(S) < \kappa_{n^*}$, we can apply Lemma 5.3 in $V[G_{n^*+1}][G_{n^*,n^*+1}][G_{m,n^*}][H]$ to $\mathbb{T} * \dot{\mathbb{R}}$ to conclude that $S$ has a cofinal branch in $V[G_{n^*+1}][G_{n^*,n^*+1}][G_{m,n^*}][H]$, thereby completing the proof. □

For $k < \omega$, let $U_k$ be a normal ultrafilter on $\kappa_k$. For $k \neq m$, the choice is arbitrary. For $k = m$, note that $\kappa_m$ remains supercompact in $V^{\mathbb{S}_m * \dot{\mathbb{B}}}$, so we may fix, in $V^{\mathbb{S}_m * \dot{\mathbb{B}}}$, a normal, fine ultrafilter $F_m$ on $\mathcal{P}_{\kappa_m}(\lambda)$. Let $U_m$ be the projection of $F_m$ on $\kappa_m$, i.e., if $X \subseteq \kappa_m$, then $X \in U_m$ iff $\{y \in \mathcal{P}_{\kappa_m}(\lambda) \mid y \cap \kappa_m \in X\} \in F_m$. By the distributivity of $\mathbb{S}_m * \dot{\mathbb{B}}$ and the fact that $2^{\kappa_m} = \kappa_m^+$, we have $U_m \in V$. Choose a condition $(s^*, \dot{t}^*) \in \mathbb{S}_m * \dot{\mathbb{B}}$ such that $(s^*, \dot{t}^*)$ forces $U_m$ to be the projection of a normal ultrafilter on $\mathcal{P}_{\kappa_m}(\lambda)$ in $V^{\mathbb{S}_m * \dot{\mathbb{B}}}$.

For $k < \omega$, let $M_k$ denote the transitive collapse of $\mathrm{Ult}(V, U_k)$, and let $j_k : V \to M_k$ be the associated embedding. Let $\mathbb{C}_k$ denote $\mathrm{Coll}(\kappa_k^{+\omega+2}, < j_k(\kappa_k))$ as defined in $M_k$. $M_k \models$ "there are $j_k(\kappa_k)$ maximal antichains of $\mathbb{C}_k$," $|j_k(\kappa_k)| = \kappa_k^+$, and $\mathbb{C}_k$ is $\kappa_k^+$-closed, so we can build in $V$ a filter $G_k$ that is $\mathbb{C}_k$-generic over $M_k$.

We now recall the diagonal Prikry forcing, which we denote $\mathbb{P}$, from [10]. For convenience, we let $\kappa_{-1} = \omega$. Conditions of $\mathbb{P}$ are of the form $p = \langle \alpha_0^p, \ldots, \alpha_{n-1}^p, \langle A_k^p \mid n \le k < \omega \rangle, g_0^p, \ldots, g_n^p, f_0^p, \ldots, f_{n-1}^p, \langle F_k^p \mid n \le k < \omega \rangle, \langle g_k^p \mid n < k < \omega \rangle \rangle$ such that:

- for all $i < n$, $\alpha_i^p$ is inaccessible and $\kappa_{i-1} < \alpha_i^p < \kappa_i$;
- for all $n \le k < \omega$, $A_k^p \in U_k$ and, for all $\alpha \in A_k^p$, $\alpha$ is inaccessible;
- for all $i < n$, $g_i^p \in \mathrm{Coll}(\kappa_{i-1}^{+2}, < \alpha_i^p)$ and $f_i^p \in \mathrm{Coll}((\alpha_i^p)^{+\omega+2}, < \kappa_i)$;
- for all $n \le k < \omega$, $g_k^p \in \mathrm{Coll}(\kappa_{k-1}^{+2}, < \kappa_k)$ and, for all $\alpha \in A_k^p$, $g_k^p \in \mathrm{Coll}(\kappa_{k+1}^{+2}, < \alpha)$;
- for all $n \le k < \omega$, $F_k^p$ is a function with domain $A_k^p$ such that, for all $\alpha \in A_k^p$, $F_k^p(\alpha) \in \mathrm{Coll}(\alpha^{+\omega+2}, < \kappa_k)$ and $j_k(F_k^p)(\kappa_k) \in G_k$.

The number $n$ as above is referred to as the length of $p$, denoted $\ell(p)$. If $q, p \in \mathbb{P}$, then $q \le p$ iff:

- $\ell(q) \ge \ell(p)$;
- for all $i < \ell(p)$, $\alpha_i^q = \alpha_i^p$ and $f_i^q \le f_i^p$;
- for all $i < \omega$, $g_i^q \le g_i^p$;
- for all $\ell(q) \le k < \omega$, $A_k^q \subseteq A_k^p$ and, for all $\alpha \in A_k^q$, $F_k^q(\alpha) \le F_k^p(\alpha)$;
- for all $\ell(p) \le k < \ell(q)$, $\alpha_k^q \in A_k^p$ and $f_k^q \le F_k^p(\alpha_k^q)$.

Following [10], given $p \in \mathbb{P}$ as above, we call $\langle \alpha_k^p \mid k < \ell(p) \rangle$ its $\alpha$-part, $\langle A_k^p \mid \ell(p) \le k < \omega \rangle$ its $A$-part, $\langle f_k^p \mid k < \ell(p) \rangle$ its $f$-part, $\langle g_k^p \mid k \le \ell(p) \rangle$ its $g$-part, $\langle F_k^p \mid \ell(p) \le k < \omega \rangle$ its $F$-part, and $\langle g_k^p \mid \ell(p) < k < \omega \rangle$ its $S$-part. The $\alpha$-part, $g$-part, and $f$-part together comprise the lower part of $p$, denoted $a(p)$. If $k \le \ell(p)$, let $p \restriction k$ denote $\langle \langle \alpha_i^p \mid i < k \rangle, \langle g_i^p \mid i \le k \rangle, \langle f_i^p \mid i < k \rangle \rangle$. If $k > \ell(p)$, let $p \restriction k = a(p)^\frown \langle A_i^p, F_i^p, g_{i+1}^p \mid \ell(p) \le i < k \rangle$. Note that $p \restriction \ell(p) = a(p)$. We say that $q$ is a length-preserving extension of $p$ if $q \le p$ and $\ell(q) = \ell(p)$. If $k \le \ell(p)$, we say $q$ is a $k$-length-preserving extension of $p$ if $q$ is a length-preserving extension of $p$ and $q \restriction k = p \restriction k$. Finally, we say $q$ is a trivial extension of $p$ if it is an $\ell(p)$-length preserving extension of $p$.

The following facts hold about $\mathbb{P}$. Proofs can be found in [10].

- (*Prikry property*) If $p \in \mathbb{P}$, $k \le \ell(p)$, and $D$ is a dense open subset of $\mathbb{P}$, then there is a $k$-length preserving extension $q$ of $p$ such that, if $q^* \le q$ and $q^* \in D$, then, if $q^{**} \le q$, $\ell(q^{**}) = \ell(q^*)$, and $q^{**} \restriction k = q^* \restriction k$, then $q^{**} \in D$.
- $\mathbb{P}$ preserves all cardinals $\ge \mu$.
- The only cardinals below $\mu$ that are collapsed by forcing with $\mathbb{P}$ are those explicitly in the scope of the interleaved Levy collapses. In particular, if, in $V^{\mathbb{P}}$, $\langle \alpha_n \mid n < \omega \rangle$ is the generic Prikry sequence, then the infinite cardinals below $\mu$ in $V^{\mathbb{P}}$ are precisely those in the intervals $\{[\kappa_{n-1}, \kappa_{n-1}^{+2}], [\alpha_n, \alpha_n^{+\omega+2}] \mid n < \omega\}$. It follows that, in $V^{\mathbb{P}}$, $\mu = \aleph_{\omega^2}$, $\lambda = \aleph_{\omega^2+1}$, and, for all $n < \omega$, $\kappa_n = \aleph_{\omega \cdot (n+1)+3}$.
- The map $\pi : \mathbb{P} \to \mathbb{S}^*$ defined by $\pi(p) = [S(p)]$ is a projection.

The models witnessing Theorem 6.1 will be of the form $V^{\mathbb{P} * \dot{\mathbb{B}}}$. The result will follow easily from the following theorem.

**Theorem 6.8.** *There is a generic extension by $\mathbb{P} * \dot{\mathbb{B}}$ in which every strong $\lambda$-system with $\kappa_{m-1}^{+2}$ relations has a cofinal branch.*

*Proof.* The proof is similar to the arguments of Section 5 in [4]. The main differences are that we must deal with the forcing $\mathbb{B}$, that we are working with strong systems instead of just trees and therefore potentially have more relations (this will only come up in the proof of Lemma 6.12), and that our use of the narrow system property will simplify certain arguments.

Suppose the theorem fails, and let $\nu = \kappa_{m-1}^{+2}$. Then there is a $\mathbb{P} * \dot{\mathbb{B}}$-name $\dot{S} = \langle \lambda \times \mu, \{\dot{R}_\eta \mid \eta < \nu\}\rangle$ such that $\Vdash_{\mathbb{P}*\dot{\mathbb{B}}}$ "$\dot{S}$ is a strong $\lambda$-system with no cofinal branch."

Let $G * B$ be $\mathbb{S}_m * \dot{\mathbb{B}}$-generic over $V$ with $(s^*, \dot{t}^*) \in G * B$. Let $p \in \mathbb{P}$ be such that $\ell(p) = m$ and $S(p) = s^*$. Let $G^*$ be the $\mathbb{S}^*$-generic filter induced by $G$, and let $\mathbb{P}^* = \{q \leq p \mid [S(q)] \in G^*\}$.

**Lemma 6.9.** *In $V[G^* * B]$, $\mathbb{P}^* \times \mathbb{S}_m/G^* \times \mathbb{S}_m/G^*$ has the $\lambda$-c.c.*

*Proof.* The proof is essentially the same as the proof of Proposition 4.5 from [4]. The difference is that we must deal with the forcing $\mathbb{B}$ in addition to $\mathbb{S}^*$. We provide the argument for completeness.

For all $m \leq k < \omega$ and all $q \in \mathbb{P}^*$ such that $n := \mathrm{lh}(q) \leq k$, let

$$h_k(q) = \langle \langle \alpha_i^q \mid i < n\rangle, \langle g_i^q \mid i < k\rangle, \langle f_i^q \mid i < n\rangle\rangle,$$

and let $H_k = \{h_k(q) \mid q \in \mathbb{P}^* \text{ and } \mathrm{lh}(q) \leq k\}$. Note that $|H_k| = \kappa_k$. Enumerate $H_m \times \mathbb{S}_{m,m} \times \mathbb{S}_{m,m}$ as $\langle (h_\beta, s_\beta^0, s_\beta^1) \mid \beta < \kappa_m\rangle$. For $k \geq m$, enumerate $H_{k+1} \times \mathbb{S}_{m,k} \times \mathbb{S}_{m,k}$ as $\langle (h_\beta, s_\beta^0, s_\beta^1) \mid \kappa_k \leq \beta < \kappa_{k+1}\rangle$.

Work in $V$. Suppose that $\dot{A}$ is an $\mathbb{S}^* * \dot{\mathbb{B}}$-name, $(r, \dot{b}) \in \mathbb{S}_0 * \dot{\mathbb{B}}$, and $([r], \dot{b})$ forces that $\dot{A}$ is a maximal antichain in $\mathbb{P}^* \times \mathbb{S}_m/G^* \times \mathbb{S}_m/G^*$. We will recursively construct a decreasing sequence $\langle (r_\beta, \dot{b}_\beta) \mid \beta < \mu\rangle$ of conditions in $\mathbb{S}_0 * \dot{\mathbb{B}}$ such that, for all $\beta < \gamma < \mu$, if $k$ is the least element of the interval $[m, \omega)$ such that $\beta < \kappa_k$, then $r_\beta \restriction k = r_\gamma \restriction k$. This will ensure that there is a lower bound for the sequence $\langle r_\beta \mid \beta < \mu\rangle$. Recall that $\mathbb{B}$ is $\mu + 1$-strategically closed, so, by playing according to a name $\dot{\sigma}$ for our winning strategy in $G_{\mu+1}(\mathbb{B})$ at limit stages and in between $\dot{b}_\beta$ and $\dot{b}_{\beta+1}$ for all $\beta < \mu$, we will also ensure that $\langle (r_\beta, \dot{b}_\beta) \mid \beta < \mu\rangle$ has a lower bound.

The construction is as follows. Let $(r_0, \dot{b}_0) = (r, \dot{b})$. For limit $\beta < \mu$, $r_\beta$ is defined by letting $r_\beta(i) = \bigcup_{\alpha<\beta} r_\alpha(i)$ for all $i < \omega$, and $\dot{b}_\beta$ is determined according to $\dot{\sigma}$. Suppose $\beta < \mu$ and $(r_\beta, \dot{b}_\beta)$ has been defined. First, find $\dot{b}_\beta^*$ such that $[r_\beta] \Vdash$ "$\dot{b}_\beta^* \leq \dot{b}_\beta$" by appealing to $\dot{\sigma}$. Let $k \in [m, \omega)$ be least such that $\beta < \kappa_k$.

Let $\Phi_\beta$ be the statement in the forcing language for $\mathbb{S}^* * \dot{\mathbb{B}}$ asserting that there is $(q, t^0, t^1) \in \dot{A}$ such that:

1. $\mathrm{lh}(q) \leq k$ and $h_k(q) = h_\beta$;
2. $t^0 \restriction [m, k) = s_\beta^0$ and $t^1 \restriction [m, k) = s_\beta^1$;
3. for all $i \geq k$, the conditions $g_j^q$, $t^0(j)$, $t^1(j)$, and $r_\beta(j)$ are pairwise compatible.

Find $r_{\beta+1} \leq r_\beta$ and $\dot{b}_{\beta+1}$ such that $[r_{\beta+1}] \Vdash$ "$\dot{b}_{\beta+1} \leq \dot{b}_\beta^*$" and $([r_{\beta+1}], \dot{b}_{\beta+1})$ decides $\Phi_\beta$. Since $[r_{\beta+1}]$ is unchanged by finite modifications to $r_{\beta+1}$, we may assume that $r_{\beta+1} \restriction k = r_\beta \restriction k$.

If $([r_{\beta+1}], \dot{b}_{\beta+1}) \Vdash \neg\Phi_\beta$, then move on to the next step in the construction. If $([r_{\beta+1}], \dot{b}_{\beta+1}) \Vdash \Phi_\beta$, then fix a witness, $(q_\beta, t_\beta^0, t_\beta^1)$. Since $[r_{\beta+1}]$ forces that $[S(q_\beta)]$,

$[t^0_\beta]$, and $[t^1_\beta]$ are in $G^*$, $[r_{\beta+1}]$ must extend all three. Therefore, by this fact and clause (3) in $\Phi_\beta$, by adjusting $r_{\beta+1}$ on finitely many coordinates if necessary, we may assume that, for all $k \leq i < \omega$, $r_{\beta+1}(i)$ extends $g^{q_\beta}_i \cup t^0_\beta(i) \cup t^1_\beta(i)$.

At the end of the construction, let $(r_\infty, \dot{b}_\infty)$ be a lower bound for $\langle (r_\beta, \dot{b}_\beta) \mid \beta < \mu\rangle$. Let $E = \{(q_\beta, t^0_\beta, t^1_\beta) \mid \beta < \mu$ and $(r_{\beta+1}, \dot{b}_{\beta+1}) \Vdash \Phi_\beta\}$.

**Claim 6.10.** $([r_\infty], \dot{b}_\infty) \Vdash$ "$\dot{A} = E$."

*Proof.* Clearly, $([r_\infty], \dot{b}_\infty) \Vdash$ "$E \subseteq \dot{A}$." We show the reverse inclusion. Thus, suppose $(r^*, \dot{b}^*) \leq (r_\infty, \dot{b}_\infty)$ and, for some $(q, t^0, t^1)$, $([r^*], \dot{b}^*) \Vdash$ "$(q, t^0, t^1) \in \dot{A}$." As before, $[r^*]$ must extend $[S(q)], [t^0]$, and $[t^1]$, so there is $m \leq k < \omega$ such that, for all $k \leq i < \omega$, $g^q_i, t^0(i)$, and $t^1(i)$ are pairwise compatible and $r^*(i)$ extends $g^q_i \cup t^0(i) \cup t^1(i)$. Fix $\beta < \kappa_k$ such that $(h_k(q), t^0 \restriction [m, k), t^1 \restriction [m, k)) = (h_\beta, s^0_\beta, s^1_\beta)$.

$(q, t^0, t^1)$ is a witness to $\Phi_\beta$, so we must have $([r_{\beta+1}], \dot{b}_{\beta+1}) \Vdash \Phi_\beta$. We claim that $(q, t^0, t^1)$ is compatible with $(q_\beta, t^0_\beta, t^1_\beta)$. To see this, first note that, as $h_k(q) = h_k(q_\beta) = h_\beta$, we know that $\langle (\alpha^q_i, f^q_i) \mid i < \mathrm{lh}(q)\rangle = \langle (\alpha^{q_\beta}_i, f^{q_\beta}_i)\rangle$ and $\langle g^q_i \mid i < k\rangle = \langle g^{q_\beta}_i \mid i < k\rangle$. Moreover, for all $k \leq i < \omega$, we have $s^*(i) \leq g^q_i, g^{q_\beta}_i$, so, in particular, $g^q_i$ and $g^{q_\beta}_i$ are compatible. Therefore, it is easily verified that $q$ and $q^\beta$ are compatible. The argument that $(t^0, t^1)$ and $(t^0_\beta, t^1_\beta)$ are compatible is similar.

Therefore, since $\dot{A}$ is forced to be an antichain, we must have $(q, t^0, t^1) = (q_\beta, t^0_\beta, t^1_\beta)$, and the claim is proved. $\square$

Since $|E| \leq \mu$, this finishes the proof of the lemma. $\square$

Let $\mathbb{P}' = \{q \leq p \mid S(q) \in G\}$. We are abusing notation here in the sense that, if $q \in \mathbb{P}$ and $\ell(q) > m$, then $S(q)$ is not in $\mathbb{S}_m$. However, in this situation, $G$ naturally projects to a generic filter $G_{\ell(q)}$ for $\mathbb{S}_{\ell(q)}$, so $S(q) \in G$ should be interpreted as $S(q) \in G_{\ell(q)}$. The following is proven in [10]

**Lemma 6.11.** *Forcing with $\mathbb{P}'$ over $V[G]$ adds a $\mathbb{P}$-generic filter over $V$.*

Note also that, if $q_0, q_1 \in \mathbb{P}'$ and $a(q_0) = a(q_1)$, then $q_0$ and $q_1$ are compatible. In particular, $\mathbb{P}'$ has the $\lambda$-c.c. in $V[G * B]$. We now work in $V[G * B]$ and use the name $\dot{S}$, reinterpreted as a $\mathbb{P}'$-name, to extract a narrow system.

**Lemma 6.12.** *There are $n, k < \omega$, $\eta < \nu$, and a cofinal set $I \subseteq \lambda$ such that, for all $\beta_0 < \beta_1$, both in $I$, there are $\gamma_0, \gamma_1 < \kappa_n$ and a condition $q \in \mathbb{P}'$ with $\ell(q) = k$ such that $q \Vdash$ "$(\beta_0, \gamma_0) <_{\dot{R}_\eta} (\beta_1, \gamma_1)$."*

*Proof.* This Lemma is analogous to Lemma 5.1 in [4]. Recall that, in $V[G*B]$, since $(s^*, \dot{t}^*) \in G * B$, there is a normal measure, $F_m$, on $\mathcal{P}_{\kappa_m}(\lambda)$ such that $F_m$ projects to $U_m$. Let $M \cong \mathrm{Ult}(V[G * B], F_m)$ be the transitive collapse of the ultrapower, and let $j : V[G*B] \to M$ be the associated embedding. Find $r \leq j(p)$ in $j(\mathbb{P}')$ such that $\alpha^r_m = \kappa_m$. This is possible because, since $A^p_m \in U_m$, we have $\kappa_m \in j(A^p_m)$. Let $H$ be $j(\mathbb{P}')$-generic with $r \in H$. Note that, in $M[H]$, all cardinals in the interval $[\kappa_m, \kappa_m^{+\omega+2}]^{V[G*B]}$ are preserved. In particular, since $\lambda = (\kappa_m^{+\omega+1})^{V[G*B]}$, $\lambda$ is preserved in $M[H]$. Let $S^*$ be the realization of $j(\dot{S})$ in $M[H]$. $S^*$ is of the form $\langle j(\lambda) \times j(\mu), \{R^*_\eta \mid \eta < \nu\}\rangle$, where $R^*_\eta$ denotes the realization of $j(\dot{R}_\eta)$ for all $\eta < \nu$, and $S^*$ is a strong $j(\lambda)$-system in $M[H]$.

Let $\delta = \sup(j``\lambda)$. For each $\beta < \lambda$, find $q_\beta \in H$, $\gamma^*_\beta < j(\mu)$, and $\eta_\beta < \nu$ such that $q_\beta \Vdash$ "$(j(\beta), \gamma^*_\beta) <_{j(\dot{R}_{\eta_\beta})} (\delta, 0)$." Without loss of generality, assume that, for all $\beta < \lambda$, $q_\beta \leq r$. There is then an unbounded $I^* \subseteq \lambda$ together with $n, k < \omega$ and $\eta < \nu$ such that, for all $\beta \in I^*$, we have $\ell(q_\beta) = k$, $\gamma^*_\beta < j(\kappa_n)$, and $\eta_\beta = \eta$.

For $\beta \in I^*$, let $q_\beta = \langle \alpha^\beta_0, \ldots, \alpha^\beta_{k-1}, \langle A^\beta_i \mid k \leq i < \omega \rangle, g^\beta_0, \ldots, g^\beta_k, f^\beta_0, \ldots, f^\beta_{k-1}, \langle F^\beta_i \mid k \leq i < \omega \rangle, \langle g^\beta_i \mid k < i < \omega \rangle \rangle$. Since the $q_\beta$'s are pairwise compatible, there is a sequence $\langle \alpha_i \mid i < k \rangle$ such that, for all $\beta \in I^*$ and all $i < k$, $\alpha^\beta_i = \alpha_i$. Moreover, we know that $k > m$, $\alpha_i = \alpha^p_i$ for all $i < m$, and $\alpha_m = \kappa_m$. There are thus fewer than $\lambda$ choices for the sequence $\langle g^\beta_0, \ldots, g^\beta_m, f^\beta_0, \ldots, f^\beta_{m-1} \rangle$, so we can find a cofinal $I \subseteq I^*$ and a sequence $\langle g_0, \ldots, g_m, f_0, \ldots, f_{m-1} \rangle$ such that, for all $\beta \in I$, $\langle g^\beta_0, \ldots, g^\beta_m, f^\beta_0, \ldots, f^\beta_{m-1} \rangle = \langle g_0, \ldots, g_m, f_0, \ldots, f_{m-1} \rangle$. Finally, if $m < i \leq k$ and $\beta \in I$, then $g^\beta_i$ comes from a forcing that is $\lambda^+$-directed closed. Similarly, if $m \leq i < k$ and $\beta \in I$, then $f^\beta_i$ comes from a forcing that is $\lambda^+$-directed closed. Thus, since $M$ is closed under $\lambda$-sequences, we may assume, by taking lower bounds on the relevant coordinates, that there is a lower part $a^* = \langle \alpha_0, \ldots, \alpha_{k-1}, g_0, \ldots, g_k, f_0, \ldots, f_{k-1} \rangle$ such that, for all $\beta \in I$, $a(q_\beta) = a^*$.

We claim that $I$, $n$, $k$, and $\eta$ are as desired. Work in $V[G * B]$. Fix $\beta_0 < \beta_1$, both in $I$. In $M$, we have $q_{\beta_0}, q_{\beta_1} \in j(\mathbb{P}')$ with $a(q_{\beta_0}) = a(q_{\beta_1}) = a^*$ and $\gamma^*_{\beta_0}, \gamma^*_{\beta_1} < j(\kappa_n)$ such that, for $\epsilon < 2$, $q_\epsilon \Vdash$ "$(j(\beta_\epsilon), \gamma^*_{\beta_\epsilon}) <_{j(\dot{R}_\eta)} (\delta, 0)$." Since $a(q_{\beta_0}) = a(q_{\beta_1}) = a^*$, we can find $q^* \leq q_{\beta_0}, q_{\beta_1}$ with $a(q^*) = a^*$. Then, since $j(\dot{R}_\eta)$ is forced to be tree-like, we have $q^* \Vdash$ "$(j(\beta_0), \gamma^*_{\beta_0}) <_{j(\dot{R}_\eta)} (j(\beta_1), \gamma^*_{\beta_1})$." By elementarity, there are $q \in \mathbb{P}'$ with $\ell(q) = k$ and $\gamma_0, \gamma_1 < \kappa_n$ such that $q \Vdash$ "$(\beta_0, \gamma_0) <_{\dot{R}_\eta} (\beta_1, \gamma_1)$." □

Fix $n$, $k$, $\eta$, and $I$ as in Lemma 6.12. Define a system

$$S_0 = \langle I \times \kappa_n, \{R_a \mid a \text{ is a lower part of length } k\} \rangle$$

by letting $(\beta_0, \gamma_0) <_{R_a} (\beta_1, \gamma_1)$ if and only if there is $q \in \mathbb{P}'$ such that $a(q) = a$ and $q \Vdash$ "$(\beta_0, \gamma_0) <_{\dot{R}_\eta} (\beta_1, \gamma_1)$." By Lemma 6.12, $S_0$ is a narrow $\lambda$-system. By Lemma 6.7, $\lambda$ satisfies the narrow system property in $V[G * B]$, so there is a cofinal branch through $S_0$. Namely, there is a cofinal $J \subseteq I$ and a lower part $a$ of length $k$ such that, for every $\beta \in J$, there is $\gamma_\beta < \kappa_n$ such that, whenever $\beta_0 < \beta_1$ are both in $J$, there is $q \in \mathbb{P}'$ with $a(q) = a$ such that $q \Vdash$ "$(\beta_0, \gamma_{\beta_0}) <_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$". Fix such a $J$ and $a$ and an assignment $\beta \mapsto \gamma_\beta$ for $\beta \in J$.

For $k \leq i < \omega$, let $H_i$ be the set of $(A, F, g)$ such that $A \in U_i$, $F$ is a function with domain $A$ such that, for all $\alpha \in A$, $F(\alpha) \in \mathrm{Coll}(\alpha^{+\omega+2}, < \kappa_i)$ and $j_i(F)(\kappa_i) \in G_i$, and $g \in \mathrm{Coll}(\kappa_i^{+2}, < \kappa_{i+1})$ is in the generic filter induced by $G$. Suppose that, for $\epsilon < 2$, $(A^\epsilon_i, F^\epsilon_i, g^\epsilon_{i+1}) \in H_i$. Then we define $(A^0_i, F^0_i, g^0_{i+1}) \wedge (A^1_i, F^1_i, g^1_{i+1}) = (A_i, F_i, g_{i+1})$ by letting

$$A_i = \{\alpha \in A^0_i \cap A^1_i \mid F^0_i(\alpha) \text{ and } F^1_i(\alpha) \text{ are compatible}\},$$

defining $F_i$ on $A_i$ by $F_i(\alpha) = F^0_i(\alpha) \cup F^1_i(\alpha)$, and letting $g_{i+1} = g^0_{i+1} \cup g^1_{i+1}$. Suppose that, for $\epsilon < 2$, $q_\epsilon \in \mathbb{P}'$ and $a(q_\epsilon) = a$, where $q_\epsilon = a^\frown \langle A^\epsilon_i, F^\epsilon_i, g^\epsilon_{i+1} \mid k \leq i < \omega \rangle$. Then the greatest lower bound for $q_0$ and $q_1$ is $q = a^\frown \langle A_i, F_i, g_{i+1} \mid k \leq i < \omega \rangle$ where, for all $k \leq i < \omega$, $(A_i, F_i, g_{i+1}) = (A^0_i, F^0_i, g^0_{i+1}) \wedge (A^1_i, F^1_i, g^1_{i+1})$.

By recursion on $i$, we now define $\langle (A^\beta_i, F^\beta_i, g^\beta_{i+1}) \mid \beta \in J, k \leq i < \omega \rangle$, maintaining the inductive hypothesis that, for all $\beta_0 < \beta_1$, both in $J$, and all $\ell$ with $k \leq \ell < \omega$, there is $q \in \mathbb{P}'$ such that

$$q \restriction \ell = a^\frown \langle (A^{\beta_0}_i, F^{\beta_0}_i, g^{\beta_0}_{i+1}) \wedge (A^{\beta_1}_i, F^{\beta_1}_i, g^{\beta_1}_{i+1}) \mid k \leq i < \ell \rangle$$

and $q \Vdash$ "$(\beta_0, \gamma_{\beta_0}) <_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$."

Suppose $k \leq \ell < \omega$ and $(A^\beta_i, F^\beta_i, g^\beta_{i+1})$ has been defined for all $\beta \in J$ and all $k \leq i < \ell$. Define a system

$$S_\ell = \langle J \times \{0\}, \{R_{A,F,g} \mid (A,F,g) \in H_\ell\}\rangle$$

by letting $(\beta_0, 0) <_{R_{A,F,g}} (\beta_1, 0)$ iff there is $q \in \mathbb{P}'$ such that

$$q \restriction (\ell+1) = a^\frown \langle (A^{\beta_0}_i, F^{\beta_0}_i, g^{\beta_0}_{i+1}) \wedge (A^{\beta_1}_i, F^{\beta_1}_i, g^{\beta_1}_{i+1}) \mid k \leq i < \ell\rangle^\frown \langle (A,F,g)\rangle$$

and $q \Vdash$ "$(\beta_0, \gamma_{\beta_0}) <_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$." By the inductive hypothesis, this defines a narrow $\lambda$-system, so, again by Lemma 6.7, it has a cofinal branch, namely a cofinal set $J_\ell \subseteq J$ and a fixed $(A,F,g) \in H_j$ such that, for all $\beta_0 < \beta_1$, both in $J_\ell$, $(\beta_0, 0) <_{R_{A,F,g}} (\beta_1, 0)$. We now define $(A^\beta_\ell, F^\beta_\ell, g^\beta_{\ell+1})$ for $\beta \in J$ as follows. If $\beta \in J_\ell$, then $(A^\beta_\ell, F^\beta_\ell, g^\beta_{\ell+1}) = (A,F,g)$. If $\beta \notin J_\ell$, let $\beta^* = \min(J_\ell \setminus \beta)$. Find $q \in \mathbb{P}'$ such that

$$q \restriction \ell = a^\frown \langle (A^\beta_i, F^\beta_i, g^\beta_{i+1}) \wedge (A^{\beta^*}_i, F^{\beta^*}_i, g^{\beta^*}_{i+1}) \mid k \leq i < \ell\rangle$$

and $q \Vdash$ "$(\beta, \gamma_\beta) <_{\dot{R}_\eta} (\beta^*, \gamma_{\beta^*})$", and let $(A^\beta_\ell, F^\beta_\ell, g^\beta_{\ell+1}) = (A^q_\ell, F^q_\ell, g^q_{\ell+1}) \wedge (A,F,g)$. It is tedious but straightforward to verify that this definition maintains the inductive hypothesis.

For $\beta \in J$, let $q_\beta = a^\frown \langle (A^\beta_i, F^\beta_i, g^\beta_{i+1}) \mid k \leq i < \omega\rangle$ and note that $q_\beta \in \mathbb{P}'$. For $\beta_0 < \beta_1$, both in $J$, let $q_{\beta_0,\beta_1} := q_{\beta_0} \wedge q_{\beta_1}$ denote the greatest lower bound of $q_{\beta_0}$ and $q_{\beta_1}$.

**Claim 6.13.** *Suppose $\beta_0 < \beta_1$, both in $J$. Then $q_{\beta_0,\beta_1} \Vdash$ "$(\beta_0, \gamma_{\beta_0}) <_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$."*

*Proof.* Suppose not, and let $r \in \mathbb{P}'$, $r \leq q_{\beta_0,\beta_1}$ be such that $r \Vdash$ "$(\beta_0, \gamma_{\beta_0}) \not<_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$." Let $i = \ell(r)$. By the inductive hypothesis in the previous construction, we can find $q \in \mathbb{P}'$ such that $q \restriction i = q_{\beta_0,\beta_1} \restriction i$ and $q \Vdash$ "$(\beta_0, \gamma_{\beta_0}) <_{\dot{R}_\eta} (\beta_1, \gamma_{\beta_1})$". But it is easily seen that $r$ and $q$ are compatible in $\mathbb{P}'$, which is a contradiction. $\square$

Since $\mathbb{P}'$ has the $\lambda$-c.c., we may find $q \in \mathbb{P}'$ such that $q \Vdash$ "for unboundedly many $\beta \in J, q_\beta \in \dot{H}$", where $\dot{H}$ is the canonical name for the generic filter. Let $H$ be $\mathbb{P}'$-generic with $q \in H$. Let $S$ be the realization of $\dot{S}$ in $V[G * B * H]$. Then, in $V[G * B * H]$, $\{(\beta, \gamma_\beta) \mid \beta \in J \text{ and } q_\beta \in H\}$ is a cofinal branch of $S$ through $R_\eta$. Thus, $S$ has a cofinal branch in $V[G * B * H]$. Note that, as $H$ is $\mathbb{P}'$-generic over $V[G*B]$, it is also $\mathbb{P}^*$-generic over $V[G^* * B]$, so $V[G*B*H] = V[G^* * B * H][G/G^*] = V[H * B][G/G^*]$, where $G/G^*$ is $\mathbb{S}_m/G^*$-generic over $V[G^* * B * H] = V[H * B]$. By Lemma 6.9, $\mathbb{S}_m/G^* \times \mathbb{S}_m/G^*$ has the $\lambda$-c.c. in $V[H * B]$, so, by Lemma 5.6, $\mathbb{S}_m/G^*$ has the $\lambda$-approximation property in $V[H * B]$. Therefore, by Lemma 5.5, $S$ has a cofinal branch in $V[H * B]$. But $p \in H$ and $p$ forces that $\dot{S}$ had no cofinal branch. This is a contradiction. $\square$

It is clear that, in any extension by $\mathbb{P} * \dot{\mathbb{B}}$, $\square^{\geq \kappa_m}_{\mu, <\mu}$ holds. In particular, there is a strong $\lambda$-system with $\aleph_{\omega \cdot (m+1)+2}$ relations that has no cofinal branch, so $\lambda$ fails to satisfy the robust tree property. By Theorem 6.8, there is an extension by $\mathbb{P} * \dot{\mathbb{B}}$ in which every strong $\aleph_{\omega^2+1}$-system with $\aleph_{\omega \cdot m + 5}$ relations has a cofinal branch. Since our choice of $m < \omega$ was arbitrary, this completes the proof of Theorem 6.1.

We now indicate how to obtain Theorem 6.2. In $V$, by the homogeneity of the forcings $\mathbb{S}_k$ for $k < \omega$, we may find ultrafilters $U_k$ on $\kappa_k$ for $k < \omega$ such that $U_k$ is forced by the empty condition to be the projection in $V^{\mathbb{S}_k}$ of a normal, fine ultrafilter $F_k$ on $\mathcal{P}_{\kappa_k}(\lambda)$. If we then use these ultrafilters to define the diagonal Prikry forcing

$\mathbb{P}$ as above, then there is a forcing extension by $\mathbb{P}$ in which the strong system property holds at $\aleph_{\omega^2+1}$. To see this, suppose for sake of contradiction that $\dot{S}$ is a $\mathbb{P}$-name forced to be a strong $\lambda$-system with no cofinal branch. Find $p \in \mathbb{P}$ deciding the number of relations in $\dot{S}$ to be equal to some $\nu < \mu$. We may assume without loss of generality that $\nu < \kappa_{\ell(p)}$. Now, letting $\ell(p)$ play the role that $m$ plays in the proof of Theorem 6.8 and working below $p$, the proof proceeds as in the proof of Theorem 6.8 but without the poset $\mathbb{B}$. As this argument is very similar to that of the proof of Theorem 1.2 of [4] and because all differences from that proof are exhibited in the proof of Theorem 6.8 of this paper, we omit the details.

We end with three open questions.

**Question 6.14.** *Can $\aleph_{\omega+1}$ consistently satisfy the strong system property?*

**Question 6.15.** *Suppose $\lambda$ is a regular uncountable cardinal and $\lambda$ satisfies the robust tree property. Must $\lambda$ satisfy the strong system property?*

**Question 6.16.** *Suppose $\lambda$ is a regular uncountable cardinal and $\lambda$ satisfies the tree property. Must it be the case that every strong $\lambda$-system with only countably many relations has a cofinal branch?*

Note that a 'Yes' answer to Question 6.16 would also entail a 'Yes' answer to Question 6.15.

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