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A cohomology theory for A(m)-algebras and applications

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Abstract

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For an A(m)-algebra A and an A-bimodule M we define the cohomology $H^*_{(m)}(A;M)$ of A with coefficients in M. If the algebra A is balanced, we define also the balanced cohomology $HB^*_{(m)}(A;M)$. Our main result says that, for such an algebra A, there exists a natural Hodge-type decomposition of $H^*_{(m)}(A;M)$ whose first component can be identified with $HB^*_{(m)}(A;M)$. Some applications are given, especially in rational homotopy theory.

Introduction and main results

A(m)-algebras were introduced in [22, p. 294] in connection with the study of homotopy associative H-spaces. An A(m)-algebra is a graded space A together with a set of multilinear operations $\mu_k : \bigotimes^k A \to A$, $1 \le k \le m$, satisfying certain associativity relations (see 1.4). The category of unitary augmented A(m)-algebras and their strong homomorphisms (see again 1.4 for the definitions) will be denoted by $\mathbf{A}(m)$. For an A(m)-algebra A, an A-bimodule is then an object of the category A-bi**Mod** $:= (\mathbf{A}(m)/A)_{\mathrm{ab}}$ of abelian group objects in the category $\mathbf{A}(m)/A$ of A(m)-algebras over A; an axiomatic characterization of A-bimodules is given in 1.10. Let **Vect** be the category of graded vector spaces and $\bigcirc$: A-bi**Mod** $\to$ **Vect** be the 'forgetful functor'. By a free A-bimodule is then meant an object of the category A-bi**Mod** having the form FV, where $V \in$ **Vect** and F is a left-adjoint to $\bigcirc$; an explicit description of free bimodules is given in 1.14. For simplicity we assume that all objects are defined over a field **k** of characteristic zero.

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Our first goal is to define a cohomology of an A(m)-algebra A with coefficients in an A-bimodule M, denoted by $H^*_{(m)}(A; M)$. The definition is given with the aid of certain free differential A-bimodule $(\mathscr{B}_{(m)}(A), \partial)$ (see 2.1) as

$$H^*_{(m)}(A; M) = H^*(C^*_{(m)}(A; M), \delta),$$

where $C^*_{(m)}(A; M) = \mathrm{Hom}^{*+1}_{A\text{-}\mathbf{biMod}}(\mathscr{B}_{(m)}(A), M)$ and δ is induced by ∂, see 3.6 for degree conventions.

Our second aim is to show that there exists a suitable concept of commutativity for A(m)-algebras. The rôle of commutative algebras will be played by A(m)-algebras (A, μ_k) such that the operations μ_k are, for $k \geq 2$, zero on decomposables of the shuffle product (see 1.4). Such algebras will be called balanced. They form a full subcategory of the category $\mathbf{A}(m)$, denoted by $\mathbf{A}(m)_{\mathrm{B}}$. The following two indications justify the definition of this property.

For an A(m)-algebra A, let $(\tilde{\mathscr{B}}(A), \tilde{\partial})$ denote the homotopy-bar (or tilde) construction (see [22, p. 295], [18, Définition 3.13] or Example 3.3). It is easy to show that the 'shuffle' product on $\tilde{\mathscr{B}}(A)$ is compatible with the differential $\tilde{\partial}$ provided A is balanced in the above sense, similarly as for a commutative algebra D this product induces the structure of a differential Hopf algebra on the bar construction $B(D)$ ([11, Chapter X, Section 12] or [24, 0.6.(1)]). The second indication is the following result of Kadeishvili. He constructed, for a chain algebra (C, ∂), a certain structure $(H(C, \partial), X_k)$ of an A(∞)-algebra on the graded vector space $H(C, \partial)$, called the homology A(∞)-algebra of (C, ∂) (see [9, Theorem 1] or 1.7). In fact, the operations X_k are constructed as obstructions to the existence of a homomorphism $\phi : (H(C, \partial), 0) \to (C, \partial)$ of differential algebras, inducing an isomorphism on the homology level. In Theorem 1.8 we prove that the algebra C is commutative if and only if the homology A(∞)-algebra can be constructed to be balanced.

Let $A \in \mathbf{A}(m)_{\mathrm{B}}$ be a balanced A(m)-algebra. By a balanced A-bimodule we mean an element of the category $A\text{-bi}\mathbf{Mod}_{\mathrm{B}} := (\mathbf{A}(m)_{\mathrm{B}}/A)_{\mathrm{ab}}$. This category consists of A-bimodules satisfying the additional condition (10). An alternative description of $A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$ is given in Proposition 1.13. For a balanced algebra A and an A-bimodule $M \in A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$ we define the balanced cohomology of A with coefficients in M, denoted by $HB^*_{(m)}(A; M)$. This objected appears together with a natural transformation

$$\Pi^*(A\ M) : HB^*_{(m)}(A; M) \to H^*_{(m)}(A; M).$$

The main result of this paper (Theorem 2.9) says that there is a natural decomposition $H^*_{(m)}(A; M) = \prod_{j \geq 0} H^{*,j}_{(m)}(A; M)$ with $H^{*,1}_{(m)}(A; M) = HB^*_{(m)}(A; M)$ such that the transformation $\Pi^*(A; M)$ corresponds to the map

$$HB^*_{(m)}(A; M) = H^{*,1}_{(m)}(A; M) \hookrightarrow \prod_{j \geq 0} H^{*,j}_{(m)}(A; M) = H^*_{(m)}(A; M).$$

As a consequence we get that the map $\Pi^*(A; M)$ is a naturally splitting monomorphism of graded vector spaces. The decomposition $H^*_{(m)}(A; M) = \prod_{j\geq 0} H^{*,j}_{(m)}(A; M)$ is an analog of the Hodge-type decomposition for the Hochschild cohomology of a commutative algebra as it was constructed for example in [5, pp. 231–234] or [6, pp. 7–8], but our situation is slightly different because the differential δ in the defining complex $C^*_{(m)}(A; M)$ is not homogeneous with respect to the 'simplicial degree'. Our theory covers the following situations.

1. Cohomology of algebras. (For details see 3.1.) Let A be a (graded) algebra, M an A-bimodule (in the usual sense) and let $\mathrm{Hoch}^{n,p}(A; M)$ denote the Hochschild cohomology of A with coefficients in M, where n is the simplicial and p the total degree, respectively. If we consider A as an A(∞)-algebra (Example 1.5) and M as an element of A-bi**Mod** (Example 1.11), then there exists an isomorphism $\omega^*(A; M): H^*_{(\infty)}(A; M) \xrightarrow{\cong} \prod_{n\geq 0} \mathrm{Hoch}^{n,-*}(A; M)$. If, moreover, the algebra A is commutative and M is a symmetric A-bimodule (i.e. $am = ma$ for all $a \in A$ and $m \in M$), then A can be considered as an element of $\mathbf{A}(\infty)_{\mathrm{B}}$ and M as an element of A-bi$\mathbf{Mod}_{\mathrm{B}}$. In this case we construct an isomorphism $\omega^*_B(A; M): HB^*_{(\infty)}(A; M) \xrightarrow{\cong} \prod_{n\geq 0} \mathrm{Harr}^{n,-*}(A; M)$, where $\mathrm{Harr}^{*,*}(A; M)$ denotes the Harrison cohomology of the commutative algebra A with coefficients in M (see [2, 25]). Moreover, if $\phi^{*,*}(A; M): \mathrm{Harr}^{*,*}(A; M) \to \mathrm{Hoch}^{*,*}(A; M)$ is the canonical map [2, p. 314], then $\prod_{n\geq 0} \phi^{n,-*}(A; M) \circ \omega^*_B(A; M) = \omega^*(A; M) \circ \Pi^*(A; M)$ and therefore, by Theorem 2.9, the map $\phi^{*,*}(A; M)$ is a monomorphism. This is a graded version of [2, Theorem 1.1]. Notice that this result has an immediate application in the rational homotopy theory [16, Theorem I.4.1].

2. Cohomology of algebras of derivations. (For details see 3.2.) Let V be a graded vector space of finite type and let (TV, ∂) be a free differential graded algebra, $\deg(\partial) = -1$. Let Der*(TV) be the Lie algebra of derivations of the algebra TV and define a differential Δ on $\mathrm{Der}^*(TV)$ by $\Delta(\theta) = [\partial; \theta]$. Then there exists an $A(\infty)$-algebra A and an isomorphism $\tilde{\Omega}^*(A): H^*_{(\infty)}(A; \bar{A}) \xrightarrow{\cong} H^{-*+1}(\mathrm{Der}^*(TV), \Delta)$, where $\bar{A}$ denotes the augmentation ideal of A (see 1.4). Let $\widetilde{\mathrm{Der}}^*(LV) = \{\theta \in \mathrm{Der}^*(TV) \mid \theta(V) \subset LV \oplus \mathbf{k}\}$, where LV is the free graded Lie algebra on V. If $\partial(V) \subset LV$, then the A(∞)-algebra A can be constructed to be balanced and there exists an isomorphism $\tilde{\mathscr{L}}^*(A): HB^*_{(\infty)}(A; \bar{A}) \xrightarrow{\cong} H^{-*+1}(\widetilde{\mathrm{Der}}^*(LV), \Delta)$. If we denote $I(A)^*: H^*(\widetilde{\mathrm{Der}}^*(LV), \Delta) \to H^*(\mathrm{Der}(TV), \Delta)$ the map induced by the inclusion $\widetilde{\mathrm{Der}}(LV) \hookrightarrow \mathrm{Der}(TV)$, then, moreover, $\tilde{\Omega}^*(A) \circ \Pi^*(A; \bar{A}) = I^{-*+1}(A) \circ \tilde{\mathscr{L}}^*(A)$ and Theorem 2.9 implies that the map $I^*(A)$ is a monomorphism. We remark that algebras of derivations of this kind play an important rôle in the rational homotopy theory, see [4], [13] and [15].

3. Homotopy associative *H*-spaces. Let X be a topological space admitting an A_m-form [22, p. 279]. Then the singular chain complex $C_*(X)$ has the structure of an A(m)-algebra (see [22, p. 295, Theorem 2.3] and Example 1.6) and it easily follows from [22, p. 296, Theorem 2.7] and the computation of 3.3 that

$$H^*_{(m)}(C_*(X); \mathbf{k}) \cong H^*(XP(m); \mathbf{k}) ,$$

where $XP(m)$ is the X-projective m-space [22, p. 280] and $\mathbf{k}$ carries the natural structure of a $C_*(X)$-bimodule as in 1.4.

4. Cohomology of a manifold. Let M be a simply connected smooth manifold having rational cohomology of finite type and let $\mathscr{E}(M)$ be the algebra of DeRham exterior forms on M. Let $\hat{\mathscr{E}}(M)$ be the 'opposite' algebra given by $\hat{\mathscr{E}}^p(M) = \mathscr{E}^{-p}(M)$. Since $\hat{\mathscr{E}}(M)$ is commutative, it can be considered as a balanced A(∞)-algebra, see Example 1.5. Then there are isomorphisms (see 3.4.) $\phi^*_H(M) : H^{-*}_{(\infty)}(\hat{\mathscr{E}}(M); \mathbb{R}) \xrightarrow{\cong} H_*(\Omega M; \mathbb{R})$ and $\phi^*_\pi(M) : HB^{-*}_{(\infty)}(\hat{\mathscr{E}}(M); \mathbb{R}) \xrightarrow{\cong} \pi_*(\Omega M) \otimes \mathbb{R}$, such that the diagram

$$\begin{array}{ccc} H^{-*}_{(\infty)}(\hat{\mathscr{E}}(M); \mathbb{R}) & \xrightarrow[\cong]{\phi^*_H(M)} & H_*(\Omega M; \mathbb{R}) \\ {\scriptstyle H^{-*}(\hat{\mathscr{E}}(M);\mathbb{R})} \Big\uparrow & & \Big\uparrow {\scriptstyle h_{\Omega M} \otimes \mathbb{R}} \\ HB^{-*}_{(\infty)}(\hat{\mathscr{E}}(M); \mathbb{R}) & \xrightarrow[\cong]{\phi^*_\pi(M)} & \pi_*(\Omega M) \otimes \mathbb{R} \end{array}$$

where $h_{\Omega M} : \pi_*(\Omega M) \to H_*(\Omega M)$ is the Hurewicz homomorphism, is commutative. Theorem 2.9 then says that $h_{\Omega M} \otimes \mathbb{R}$ is a monomorphism. This is, of course, a consequence of the Milnor–Moore theorem [17]. This computation remains valid if M is replaced by an arbitrary simply connected space having rational cohomology of finite type and $\mathscr{E}(M)$ by the algebra A_{PL} of Sullivan–DeRham polynomial forms [23, Section 7].

5. The canonical map $l : H_{\mathrm{Lie}}(L; L) \to \mathrm{Hoch}(\mathscr{U}L; \mathscr{U}L)$. (For details see 3.5.) Let L be a (graded) Lie algebra of finite type with $L_{\leq 0} = 0$ and let $\mathscr{U}L$ denote its universal enveloping algebra. In this situation there exists a map $l^{*,*} : H^{*,*}_{\mathrm{Lie}}(L; L) \to \mathrm{Hoch}^{*,*}(\mathscr{U}L; \mathscr{U}L)$ of the Lie algebra cohomology of L with coefficients in L considered in the clear way as an L-module, to the Hochschild cohomology of $\mathscr{U}L$ with coefficients in $\mathscr{U}L$ with the evident structure of a bimodule over itself. In 3.5 we, roughly speaking, prove the existence of a balanced A(∞)-algebra A such that $H_{\mathrm{Lie}}(L; L)$ can be identified with $HB_{(\infty)}(A; \bar{A})$ and that $\mathrm{Hoch}(\mathscr{U}L; \mathscr{U}L)$ can be identified with $H_{(\infty)}(A; \bar{A})$. Moreover, the map l is represented, under this identification, by the map $H(A; \bar{A})$ of Theorem 2.9. Especially, the map l is a monomorphism. This result has the following immediate application in rational homotopy theory.

Let S be a simply connected topological space having the rational cohomology of finite type. Let $L_* = \pi_*(\Omega S) \otimes \mathbb{Q}$, then, by the Milnor–Moore theorem [17],

$\mathcal{U}L_* \cong H_*(\Omega S; \mathbb{Q})$. The space S is said to be coformal [24, III.4.(6)], if there exists a free differential graded commutative algebra $(\wedge V, d)$ such that $d(V) \subset V \wedge V$, and a homomorphism $\phi : (\wedge V, d) \to A_{PL}(S)$, inducing an isomorphism in the cohomology; here $A_{PL}(S)$ denotes the algebra of Sullivan–DeRham polynomial forms on S [23, Section 7]. Similarly, S is said to be coquadratic [14, Definition 2.7], if there exists a free differential graded algebra (TW, δ) with $\delta(W) \subset W \otimes W$ and a homomorphism $\psi : (TW, \delta) \to C^*(S; \mathbb{Q})$, inducing an isomorphism on the cohomology level; here $C^*(S; \mathbb{Q})$ denotes the differential graded (associative but noncommutative) algebra of rational singular cochains of S. It is well known that the natural obstructions to coformality are elements $\tilde{\mathcal{P}}_n \in H_{\mathrm{Lie}}(L; L)$, $n \geq 3$, see [4, Annexe 2] and [15, Introduction]. Similarly, the obstructions to coquadraticity are elements $\mathcal{P}_n \in \mathrm{Hoch}(\mathcal{U}L; \mathcal{U}L)$, $n \geq 3$, and, moreover, $l(\tilde{\mathcal{P}}_n) = \mathcal{P}_n$ [14, Proposition 2.8 and the comments following Theorem 2.9]. We thus have proved the following theorem, as promised in [14].

Theorem. *A simply connected topological space is coformal if and only if it is coquadratic.* □

Indeed, it is easily seen that a coformal space is coquadratic. The opposite implication is an easy consequence of $l(\tilde{\mathcal{P}}_n) = \mathcal{P}_n$ and the fact that l is a monomorphism.

1. Shuffles, A(*m*)-algebras, A(*m*)-(bi)modules, etc.

1.1. All objects are assumed to be defined over a fixed ground field $\mathbf{k}$ of characteristic zero, although the assumption $\mathrm{char}(\mathbf{k}) = 0$ is not really necessary in all statements and proofs below.

For graded objects we will usually omit the $*$. If it is necessary to indicate the grading explicitly, the corresponding symbol (star or index) will be sometimes written as a superscript, sometimes as a subscript, in accord with the usual conventions.

Denote by **Vect** the category of graded $\mathbf{k}$-vector spaces, by $\mathrm{Hom}^p_{\mathbf{Vect}}(V, W)$ we denote the set of linear homogeneous maps $f : V \to W$ of degree p. For $V \in \mathbf{Vect}$ let $\uparrow V$ (resp. $\downarrow V$) be the suspension (resp. the desuspension) of V, i.e. the graded vector space defined by $(\uparrow V)_p = V_{p-1}$ (resp. $(\downarrow V)_p = V_{p+1}$). By $\#V$ we denote the dual of V, i.e. the graded vector space defined by $(\#V)_p = \mathrm{Hom}^{-p}_{\mathbf{Vect}}(V, \mathbf{k})$. Let $\hat{V}$ denote the graded vector space defined by $\hat{V}_p = V_{-p}$. Finally, let $\bigotimes^m V$ stand for $\mathbf{k}$, if $m = 0$, and for $V \otimes \cdots \otimes V$ (m-times), if $m > 0$; let $\bigotimes^{\leq m} V = \bigoplus_{0 \leq i \leq m} \bigotimes^i V$ and $T(V) = \bigoplus_{i \geq 0} \bigotimes^i V$.

As to graded objects in general, we will systematically use the following sign convention (called in [18, pp. 2–3] the Koszul sign convention): commuting two 'things' of degrees p and q, respectively, we multiply the sign by $(-1)^{pq}$. The fact that an object a is of degree p will be expressed as $\deg(a) = p$ or simply $|a| = p$. The degree and sign conventions used in our definitions of A(m)-algebras, shuffles, resolutions, etc. are commented in 3.6.

1.2. Let X be a graded **k**-module. For $x_1, \ldots, x_n \in X$ and a permutation $\sigma \in S_n$ define the number $\varepsilon(\sigma; x_1, \ldots, x_n)$ (the Koszul sign) by the equality

$$x_1 \wedge \cdots \wedge x_n = \varepsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma^{-1}(1)} \wedge \cdots \wedge x_{\sigma^{-1}(n)}$$

which has to be satisfied in the free graded commutative algebra $\wedge(x_1, \ldots, x_n)$ on graded indeterminates $x_1, \ldots, x_n$ [24, 0.2(11)]. Also, let

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) = \operatorname{sgn}(\sigma) \cdot \varepsilon(\sigma; x_1, \ldots, x_n) .$$

The *shuffle product* $s_{i,n-i}$ is, for $0 \le i \le n$, the linear map $s_{i,n-i} : \bigotimes^n X \to \bigotimes^n X$ defined by

$$s_{i,n-i}(x_1, \ldots, x_n) = \sum \chi(\sigma) x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} ,$$

where the summation is taken over all permutations $\sigma \in S_n$ with $\sigma(1) < \cdots < \sigma(i)$, $\sigma(i+1) < \cdots < \sigma(n)$. Sometimes we will write $s(x_1, \ldots, x_i \mid x_{i+1}, \ldots, x_n)$ instead of $s_{i,n-i}(x_1, \ldots, x_n)$; note that by definition

$$s(x_1, \ldots, x_n) = s_{0,n}(x_1, \ldots, x_n) = s_{n,0}(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n .$$

1.3. Let us sum up some properties of the shuffles. The first two of them are well known [20].

(i) *Associativity*:

$$\begin{aligned} &s(s(a_1, \ldots, a_k \mid b_1, \ldots, b_l) \mid c_1, \ldots, c_m) \\ &\quad = s(a_1, \ldots, a_k \mid s(b_1, \ldots, b_l \mid c_1, \ldots, c_m)) . \end{aligned}$$

Due to this property, we can write the multiple shuffle products without parentheses, i.e. use the notation like $s(a_1, \ldots, a_k \mid b_1, \ldots, b_l \mid c_1, \ldots, c_m)$, etc.

(ii) *Commutativity*:

$$\begin{aligned} &s(a_1, \ldots, a_k \mid b_1, \ldots, b_l) \\ &\quad = (-1)^{(|a_1| + \cdots + |a_k|)(|b_1| + \cdots + |b_l|) + kl} s(b_1, \ldots, b_l \mid a_1, \ldots, a_k) . \end{aligned}$$

The following equalities can also be easily verified.

(iii) Let $(a_1, \ldots, a_m)^{\leftarrow}$ denote $(a_m, \ldots, a_1)$. Then

$$\{s(a_1, \ldots, a_k \mid b_1, \ldots, b_l)\}^{\leftarrow} = s((b_1, \ldots, b_l)^{\leftarrow} \mid (a_1, \ldots, a_k)^{\leftarrow}) .$$

(iv) Let $\rho \in S_m$ denote, for each $m \geq 1$, the permutation which reverses the order, i.e. which sends $(1, \ldots, m)$ to $(m, \ldots, 1)$. Then

$$\begin{aligned} & s(x_1, \ldots, x_k \mid y_1, \ldots, y_l) \otimes w \\ & \quad = s(x_1, \ldots, x_k \mid y_1, \ldots, y_l, w) \\ & \quad\quad + \sum_{i=2}^{k} (-1)^{k+i+1} \chi_i \cdot s(x_1, \ldots, x_{i-1} \mid y_1, \ldots, y_l, w, x_k, \ldots, x_i) \\ & \quad\quad + (-1)^k \chi_1 \cdot y_1 \otimes \cdots \otimes y_l \otimes w \otimes x_k \otimes \cdots \otimes x_1 , \end{aligned}$$

where $\chi_i = (-1)^{(|x_i| + \cdots + |x_k|)(|y_1| + \cdots + |y_l| + |w|) + (k+i+1)(l+1)} \cdot \chi(\rho; x_i, \ldots, x_k)$.

Also the following equality will be useful in the sequel:

(v) $$\sum_{j=0}^{m} (-1)^{m+j} \chi(\rho; a_{j+1}, \ldots, a_m) \cdot s(a_{j+1}, \ldots, a_m \mid a_j, \ldots, a_1) = 0 .$$

1.4. Let m be a natural number or ∞. An *$A(m)$-algebra* is defined to be a graded **k**-module A together with a set $\{\mu_k \mid 1 \leq k \leq m, k < \infty\}$ of linear maps, $\mu_k : \bigotimes^k A \to A$, such that $\mu_k(\{\bigotimes^k A\}_p) \subset A_{p+k-2}$ for each p and

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} (-1)^{k+\lambda+k\lambda+k(|a_1|+\cdots+|a_\lambda|)} \cdot \mu_{n-k+1}(a_1, \ldots, a_\lambda, \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_n) = 0 \tag{1}$$

for all homogeneous $a_\lambda \in A$, $1 \leq \lambda \leq n$, and $n \leq m$ (see [22, p. 294], [18, Définition 3.2] and [9, p. 231]). By a *homomorphism* of two A(m)-algebras $A = (A, \mu_k)$ and $A' = (A', \mu'_k)$ we mean a sequence $\{f_r : \bigotimes^r A \to A' \mid r \geq 1\}$ of multilinear maps of degree $r - 1$ such that the following condition is satisfied for each $n \geq 1$:

$$\begin{aligned} & \sum_{k=1}^{\min(m,n)} \sum_{r_1+\cdots+r_k=n} (-1)^{\eta} \cdot \mu'_k(f_{r_1}(a_1, \ldots, a_{r_1}), \ldots, f_{r_k}(a_{n-r_k+1}, \ldots, a_n)) \\ & \quad = \sum_{k=1}^{\min(m,n)} \sum_{\lambda=0}^{n-k} (-1)^{\xi} \cdot f_{n-k+1}(a_1, \ldots, a_\lambda, \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}) , \\ & \qquad\qquad a_{\lambda+k+1}, \ldots, a_n) , \end{aligned} \tag{2}$$

where

$$\eta = \sum_{1 \leq \alpha < \beta \leq k} (|a_{r_1+\cdots+r_{\alpha-1}+1}| + \cdots + |a_{r_1+\cdots+r_\alpha}| + r_\alpha)(r_\beta + 1) ,$$

$$\xi = k(|a_1| + \cdots + |a_\lambda|) + n + k + k\lambda .$$

Such a homomorphism is called *strict* [18, Définition 3.6] if $f_2 = f_3 = \cdots = 0$. In this case the condition (2) for $f = f_1$ has the following simple form:

$$f(\mu_k(a_1, \ldots, a_k)) = \mu'_k(f(a_1), \ldots, f(a_k)) .$$

An A(m)-algebra A is called *unitary*, if there exists a linear homogeneous map $\eta = \eta_A : \mathbf{k} \to A$ of degree zero such that

$$\mu_2(\mathrm{id} \otimes \eta) = \mu_2(\eta \otimes \mathrm{id}) = \mathrm{id} ,$$
$$\mu_k(\mathrm{id}^{\lambda} \otimes \eta \otimes \mathrm{id}^{k-\lambda-1}) = 0 \quad \text{for } 1 \le k \le m,\ k \ne 2 \text{ and } 0 \le \lambda \le k-1 .$$

The element $1_A = \eta(1_{\mathbf{k}})$ is then called a *unit* of A. We will assume, similarly as in [18], that a unitary algebra A is automatically *augmented*, i.e. that there exists a strict homomorphism $\varepsilon = \varepsilon_A : A \to \mathbf{k}$ such that $\varepsilon \circ \eta = \mathrm{id}$; here we consider $\mathbf{k}$ as an A(m)-algebra with $\mu_2(m, n) = m \cdot n$, $m, n \in \mathbf{k}$, and $\mu_k = 0$ for $k \ne 2$. A morphism $\{f_k \mid k \ge 1\}$ of two unitary A(m)-algebras is then required to satisfy, besides (2), also

$$f_1 \circ \eta = \eta , \qquad \varepsilon \circ f_1 = \varepsilon ,$$
$$\varepsilon \circ f_k = 0 , \quad f_k(\mathrm{id}^{\lambda} \otimes \eta \otimes \mathrm{id}^{k-\lambda-1}) = 0 \qquad \text{for } k \ge 2 \text{ and } 0 \le \lambda \le k-1 .$$

For such a unitary algebra A let $\bar{A} = \mathrm{Ker}(\varepsilon)$. Then $\bar{A}$ can easily be shown to be an (nonunitary) A(m)-algebra which is called the *augmentation ideal* of A. On the other hand, for a nonunitary A(m)-algebra B there exists a natural structure of a unitary A(m)-algebra on $\tilde{B} = B \oplus \mathbf{k}$ such that $\eta(k) = 0 \oplus k$ and $\varepsilon(a \oplus k) = k$; the correspondence $A \mapsto \bar{A}$ and $B \mapsto \tilde{B}$ being one-to-one [18, Lemme 3.10].

Denote by $\mathbf{A}(m)$ the category of *unitary* A(m)-algebras and their *strict* homomorphisms. An algebra $(A, \mu_k) \in \mathbf{A}(m)$ will be said to be *balanced* if $\mu_k \circ s_{i,k-i} = 0$ for $2 \le k \le m$ and $1 \le i \le k-1$. We denote by $\mathbf{A}(m)_{\mathrm{B}}$ the full subcategory of $\mathbf{A}(m)$ consisting of balanced algebras.

1.5. Example. Let (C, ∂) be a differential graded algebra, $\deg(\partial) = -1$. Putting $\mu_1 = \partial$, $\mu_2(x, y) = x \cdot y$, $\mu_k = 0$ for $2 < k \le m$, then (C, μ_k) forms an A(m)-algebra for any $m \ge 2$ (compare also [22, p. 294, Proposition 2.2]). As $s(x \mid y) = x \otimes y - (-1)^{|x| \cdot |y|} y \otimes x$, the A($m$)-algebra (C, μ_k) is balanced if and only if C is commutative (in the graded sense). Clearly $(C, \mu_k) \in \mathbf{A}(m)$ if and only if the algebra C is unitary and augmented in the usual sense.

1.6. Example. In [22, p. 294], where A(m)-algebras were introduced for the first time, a slightly different sign convention than that used in formula (1) was introduced. Of course, both conventions are equivalent; if (A, m_k) is an A(m)-

algebra satisfying the sign convention of [22], then (A, μ_k) with $\mu_k = (-1)^{k(k-1)/2} m_k$, $1 \leq k \leq m$, is an A(m)-algebra satisfying our sign convention. Especially, if a topological space X admits an A_m-form [22, p. 279], then [22, p. 295, Theorem 2.3] shows that there exists a naturally induced structure of an A(m)-algebra (over $\mathbb{Z}$) on the singular complex $C_*(X)$, denoted by $(C_*(X), m_k)$; therefore $(C_*(X), \mu_k)$ with $\mu_k = (-1)^{k(k-1)/2} m_k$ is an A(m)-algebra in the sense used here.

1.7. Let $C = (C, \partial)$ be a chain algebra. We will consider it, as in Example 1.5, as an A(∞)-algebra with $\mu_1 = \partial$ and $\mu_2 =$ the product. Kadeishvili proved in [9, p. 232] the following theorem:

Theorem. *Let (C, ∂) be a chain algebra such that $H(C)$ is free (this is always the case over a field). Then there exists an A(∞)-structure $\{X_k \mid k \geq 1\}$ on the graded space $H(C)$ having $X_1 = 0$ and $X_2(a, b) = a \cdot b$, together with an A(∞)-homomorphism $f : (H(C), X_k) \to (C, \mu_1, \mu_2, 0, \ldots)$ such that $f_1 : H(C) \to C$ is a homology isomorphism.* □

Notice that the map f_1 need not be, in general, an algebra homomorphism. Nevertheless, the condition (2) gives, for $n = 1$, $\partial f_1 = 0$ which means that f_1 is a homomorphism of differential spaces $(H(C), 0)$ and (C, ∂). For $n = 2$ the condition (2) gives

$$f_1(a \cdot b) - f_1(a) \cdot f_1(b) = \partial f_2(a, b)$$

which means that f_1 is a homotopy multiplicative map of chain algebras $(H(C), 0)$ and (C, ∂), the homotopy being provided by f_2.

The A(∞)-algebra $(H(C), X_k)$ whose existence is guaranteed by the theorem above, is called the *cohomology A(∞)-algebra* of C. Note that in the original formulation of [9] $X_2(a, b) = (-1)^{|a|+1} a \cdot b$. This change is due to different sign conventions used here. We prove the following theorem:

1.8. Theorem. *Let C be a chain algebra as in 1.7. If it is commutative (in the graded sense), then the operations X_k, $k \geq 1$, can be constructed so that the cohomology A(∞)-algebra $(H(C), X_k)$ of C is balanced. Moreover, the homomorphism $f : (H(C), X_k) \to (C, \mu_1, \mu_2, 0, \ldots)$ can be constructed so that $f_k \circ s_{i,k-i} = 0$ for $k \geq 2$, $1 \leq i \leq k-1$.*

Proof. Recall briefly the proof of [9]. Put $X_1 = 0$ and let $f_1 : H(C) \to C$ be a cycle-selection homomorphism.

Suppose that X_i and f_i have been already constructed for $i < n$. Then we can define the function $U_n : \bigotimes^n H(C) \to C$ by the formula $U_n = U_n^1 + U_n^2$, where

$$
\begin{aligned}
U_n^1(a_1, \ldots, a_n) &= \sum_{t=1}^{n-1} (-1)^{(|a_1|+\cdots+|a_t|+t)(n+t+1)} \\
&\qquad \cdot \mu_2(f_t(a_1, \ldots, a_t), f_{n-t}(a_{t+1}, \ldots, a_n)) , \\
U_n^2(a_1, \ldots, a_n) &= -\sum_{k=2}^{n-1} \sum_{\lambda=0}^{n-k} (-1)^{k(|a_1|+\cdots+|a_\lambda|)+\lambda+n+k+k\lambda} \\
&\qquad \cdot f_{n-k+1}(a_1, \ldots, a_\lambda, X_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_n)
\end{aligned}
\tag{3}
$$

which involves X_i and f_i for $i < n$ only. Then $U_n(a_1, \ldots, a_n)$ can be shown to be a cycle and X_n is defined as $[U_n] \in H(C)$, the homology class of this cycle. Then $f_1 \circ X_n - U_n$ is homologous to zero and f_n is defined to satisfy $\partial f_n = f_1 \circ X_n - U_n$.

Let us come back to the proof of our theorem. Suppose that it has been proved that f_i and X_i are zero on decomposables of the shuffle product for all $i < n$ and prove it for $i = n$. From the construction of f_n and X_n as described above it is clear that it is enough to show that $U_n \circ s_{i,n-i} = 0$ for all $1 \le i \le n-1$. We choose such an i and prove this equality, i.e. prove that

$$
U_n(s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n)) = 0 , \quad a_1, \ldots, a_n \in H(C) . \tag{4}
$$

We prove first that the map U_n^1 of (3) vanishes on $s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n)$. To this end, notice that, for any $0 \le t \le n$,

$$
\begin{aligned}
&s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n) \\
&\quad = \sum_{\substack{0 \le \alpha \le i \\ 0 \le \beta \le n-i \\ \alpha+\beta=t}} (-1)^{|B_\alpha|\cdot|C_\beta|+\beta(\alpha+i)} \cdot s(A_\alpha \mid C_\beta) \otimes s(B_\alpha \mid D_\beta) ,
\end{aligned}
$$

where we have used the abbreviations $A_\alpha = (a_1, \ldots, a_\alpha)$, $B_\alpha = (a_{\alpha+1}, \ldots, a_i)$, $C_\beta = (a_{i+1}, \ldots, a_{i+\beta})$ and $D_\beta = (a_{i+\beta+1}, \ldots, a_n)$, the meaning of the abbreviations like $|B_\alpha|$ being clear. Then $U_n^1(s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n))$ is equal to

$$
\begin{aligned}
&\sum_{t=1}^{n-1} \sum_{\substack{0 \le \alpha \le i \\ 0 \le \beta \le n-i \\ \alpha+\beta=t}} (-1)^{\{(|A_\alpha|+|C_\beta|+t)(n+t+1)+|B_\alpha|\cdot|C_\beta|+\beta(\alpha+i)\}} \\
&\qquad \cdot \mu_2(f_t(s(A_\alpha \mid C_\beta)), f_{n-t}(s(B_\alpha \mid D_\beta))) .
\end{aligned}
\tag{5}
$$

By our induction assumption on the functions f_i, $i < n$, the expression $\mu_2(f_t(s(s(A_\alpha \mid C_\beta))), f_{n-t}(s(B_\alpha \mid D_\beta)))$ is zero for $(\alpha, \beta) \neq (0, 0), (i, 0), (0, n-i), (i, n-i)$, therefore (5) reduces to

$$(-1)^{(|A_i|+i)(n+i+1)} \cdot \mu_2(f_i(A_i), f_{n-i}(D_0))$$
$$+ (-1)^{\{(|C_{n-i}|+n+i)(i+1)+|B_0|\cdot|C_{n-i}|+(n+i)i\}} \cdot \mu_2(f_{n-i}(C_{n-i}), f_i(B_0)),$$

which is easily seen to be zero, observing that $B_0 = A_i$, $D_0 = C_{n-i}$ and using the commutativity of μ_2.

To finish, we must prove also that $U_n^2(s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n)) = 0$. First, let $\alpha, \beta, \gamma, \delta$ be natural numbers with $0 \le \alpha + \beta \le i$, $0 \le \gamma + \delta \le n-i$ and let $A_\alpha = (a_1, \ldots, a_\alpha)$, $B_\beta = (a_{\alpha+1}, \ldots, a_{\alpha+\beta})$, $C_{\alpha\beta} = (a_{\alpha+\beta+1}, \ldots, a_i)$, $D_\gamma = (a_{i+1}, \ldots, a_{i+\gamma})$, $E_\delta = (a_{i+\gamma+1}, \ldots, a_{i+\gamma+\delta})$ and $F_{\gamma\delta} = (a_{i+\gamma+\delta+1}, \ldots, a_n)$. Then, for fixed λ and k with $\lambda + k \le n$, we can easily prove that

$$s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n)$$
$$= \sum_{\substack{0 \le \alpha+\beta \le i \\ 0 \le \gamma+\delta \le n-i \\ \alpha+\gamma=\lambda, \beta+\delta=k}} (-1)^\phi \cdot s(A_\alpha \mid D_\gamma) \otimes s(B_\beta \mid E_\delta) \otimes s(C_{\alpha\beta} \mid F_{\gamma\delta})$$

with

$$\phi = |C_{\alpha\beta}| \cdot (|E_\delta| + |D_\gamma|) + |D_\gamma| \cdot |B_\beta| + (\delta + \gamma)(i + \alpha + \beta) + \gamma\beta .$$

Therefore $U_n^2(s(a_1, \ldots, a_i \mid a_{i+1}, \ldots, a_n))$ is equal to

$$\sum_{\substack{2 \le k \le n-1 \\ 0 \le \lambda \le n-k}} \sum_{\substack{0 \le \alpha+\beta \le i \\ 0 \le \gamma+\delta \le n-i \\ \alpha+\gamma=\lambda, \beta+\delta=k}} (-1)^\psi \cdot f_{n-k+1}(s(A_\alpha \mid D_\gamma), X_k(s(B_\beta \mid E_\delta)),$$
$$s(C_{\alpha\beta} \mid F_{\gamma\delta})),$$

where

$$\psi = k(|A_\alpha| + |D_\gamma|) + |C_{\alpha\beta}| \cdot (|E_\delta| + |D_\gamma|) + |D_\gamma| \cdot |B_\beta|$$
$$+ \lambda + n + k + k\lambda + (\delta + \gamma)(i + \alpha + \beta) + \gamma\beta .$$

By our induction assumption $X_k(s(B_\beta \mid E_\delta)) = 0$ for $\beta \ne 0 \ne \delta$, hence the sum above reduces to

$$\sum_{2 \le k \le n-1} \sum_{\substack{0 \le \alpha \le i \\ \gamma+k \le n-i}} (-1)^\theta \cdot f_{n-k+1}(s(A_\alpha \mid D_\gamma), X_k(E_k), s(C_{\alpha 0} \mid F_{\gamma k}))$$
$$+ \sum_{2 \le k \le n-1} \sum_{\substack{\alpha+k \le i \\ 0 \le \gamma \le n-i}} (-1)^\pi \cdot f_{n-k+1}(s(A_\alpha \mid D_\gamma), X_k(B_k), s(C_{\alpha k} \mid F_{\gamma 0})), \tag{6}$$

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where

$$\theta = k(|A_\alpha| + |D_\alpha|) + |C_{\alpha 0}| \cdot (|E_k| + |D_\gamma|) \\ + \alpha + \gamma + n + k + k(\alpha + \gamma) + (k + \gamma)(i + \alpha)$$

and

$$\pi = k(|A_\alpha| + |D_\gamma|) + |C_{\alpha k}| \cdot |D_\gamma| + |D_\gamma| \cdot |B_k| \\ + n + k + \alpha + \gamma + k(\alpha + \gamma) + \gamma(i + \alpha + k) + \gamma k\,.$$

On the other hand, for a fixed γ we have

$$s(a_1, \ldots, a_i \mid D_\gamma, X_k(E_k), F_{\gamma k}) \\ = \sum_{0 \le \alpha \le i} (-1)^{|C_{\alpha 0}| \cdot (|D_\gamma| + |E_k| + k) + (i+\alpha)(\gamma+1)} \\ \cdot s(A_\alpha | D_\gamma) \otimes X_k(E_k) \otimes s(C_{\alpha 0} \mid F_{\gamma k})\,. \tag{7}$$

The first summation in (6) is of the form

$$\sum_{\substack{2 \le k \le n-1 \\ \gamma \le n-k-i}} \sum_{0 \le \alpha \le i} C \cdot (-1)^{|C_{\alpha 0}| \cdot (|D_\gamma| + |E_k| + k) + (i+\alpha)(\gamma+1)} \\ \cdot f_{n-k+1}(s(A_\alpha \mid D_\gamma), X_k(E_k), s(C_{\alpha 0} \mid F_{\gamma k}))\,,$$

where the number $C = \pm 1$ does not depend on α; this shows, by (7) and the induction assumption that it is zero. A similar discussion applies also to the second summation in (6) and this completes our verification of formula (4). □

1.9. Example. All graded vector spaces in this example will be tacitly assumed of finite type. For the notation see 1.1 and 1.4. For a graded vector space V we have the natural map $\uparrow : V \to \uparrow V$; let $\uparrow^n$ denote $\bigotimes^n \uparrow : \bigotimes^n V \to \bigotimes^n \uparrow V$, the meaning of $\downarrow^n$ being analogous. Notice that $\uparrow^n \circ \downarrow^n = \downarrow^n \circ \uparrow^n = (-1)^{n(n-1)/2} \cdot \mathrm{id}$.

Let $A = (A, \mu_k)$ be an A(m)-algebra. Denote $\tilde{\Omega}(A) = (T(\downarrow \# \hat{A}), \partial)$, where $\partial = \partial_1 + \partial_2 + \cdots + \partial_m$ is the derivation defined by $\partial_k|_{\downarrow \# \hat{A}} = \downarrow^k \# \hat{\bar{\mu}}_k \uparrow$, where $\bar{\mu}_k = \mu_k|_{\bigotimes^k A}$. Then $\deg(\partial) = -1$ and (1) easily implies that $\pi_{\le m} \circ \partial^2 = 0$, where $\pi_{\le m} : T(\downarrow \# \hat{A}) \to \bigotimes^{\le m}(\downarrow \# \hat{A})$ is the natural projection. On the other hand, starting from $(T(V), \partial)$, where the derivation $\partial = \partial_1 + \cdots + \partial_m$ of degree -1 satisfies $\pi_{\le m} \circ \partial^2 = 0$, it is easy to see that the object $(B, \bar{\mu}_k)$, where $B = (\uparrow \# V)\hat{}$ and $\bar{\mu}_k$ is, for $1 \le k \le m$, defined by $\hat{\bar{\mu}}_k = (-1)^{k(k-1)/2} \uparrow \#(\partial_k|_V) \downarrow^k$, is an A($m$)-algebra. Denoting $B = \tilde{\Omega}^{-1}(TV, \partial)$, it is not hard to see that $\tilde{\Omega}$ and $\tilde{\Omega}^{-1}$ are inverse functors. We have obtained an equivalence of the category of unitary A(m)-algebras of finite type and the category whose objects have the form

$(T(V), \partial)$, where $\partial = \partial_1 + \cdots + \partial_m$ is a derivation of degree -1, $\pi_{\leq m} \circ \partial^2 = 0$, and whose morphisms are algebra homomorphisms, $f : TV \to TW$ with $f(V) \subset W$, which commute with the differentials. For $m = \infty$ this is the category of free differential graded (chain) algebras and their *linear* homomorphisms.

Suppose that the A(m)-algebra A is balanced and let $\tilde{\Omega}(A) = (TV, \partial)$. Then $\partial(LV) \subset LV$, where LV denotes the free graded Lie algebra on V [24, 0.4.(11)]. This is an easy consequence of the fact that the kernel of the natural map $\#J : \#TV \to \#LV$, where $J : LV \to TV$ is the natural inclusion, consists of decomposables of the shuffle product (an easy graded version of [20, Theorem 2.2]). Thus it makes sense to denote $\tilde{\mathscr{L}}(A) = (LV, \partial|_{LV})$. Similarly, it is easily seen that, if $\partial(LV) \subset LV$, then $\tilde{\Omega}^{-1}(TV, \partial) \in \mathbf{A}(m)_{\mathrm{B}}$. We have chosen the notation $\tilde{\Omega}$ and $\tilde{\mathscr{L}}$ to emphasize the analogy with the functors $\mathscr{L}_*$ and Ω defined in [24, 1.1.(7)] and [7, Appendix].

1.10. Let A be a unitary A(m)-algebra and M a graded vector space. Denote

$$J^k_{A,M} = \bigoplus_{j=1}^{k} \left[\bigotimes^{j-1} A \otimes M \otimes \bigotimes^{k-j} A \right] \subset T(A, M) .$$

Then the space M together with a set $\{ \mu_k : J^k_{A,M} \to M \mid 1 \leq k \leq m \}$ of linear maps, $\deg(\mu_k) = k - 2$, is said to be an *A-bimodule*, if the equality

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} (-1)^{k+\lambda+k\lambda+k(|a_1|+\cdots+|a_\lambda|)} \cdot \mu_{n-k+1}(a_1, \ldots, a_\lambda, \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_n) = 0 \tag{8}$$

(formally the same as (1)) is satisfied for all $n \leq m$ and $(a_1, \ldots, a_n) \in J^n_{A,M}$, and if

$$\begin{aligned} &\mu_2(\mathrm{id} \otimes \eta) = \mu_2(\eta \otimes \mathrm{id}) = \mathrm{id} , \\ &\mu_k(\mathrm{id}^\lambda \otimes \eta \otimes \mathrm{id}^{k-\lambda-1}) = 0 \quad \text{for } 3 \leq k \leq m \text{ and } 0 \leq \lambda \leq k-1 . \end{aligned} \tag{9}$$

Notice that for the homogeneity of our notation we denote by the same symbol both the 'multiplication' in A and the operations on M. Let $M = (M, \mu_k)$ and $M' = (M', \mu'_k)$ be two A-bimodules. By a *morphism* of M and M' of degree p we mean here a linear map $f : M \to M'$ of degree p satisfying for all homogeneous $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \in A$, $m \in M$, with $1 \leq j \leq n$, $n \leq m$:

$$\begin{aligned} &f(\mu_n(a_1, \ldots, a_{j-1}, m, a_{j+1}, \ldots, a_n)) \\ &\quad = (-1)^{pn+p(|a_1|+\cdots+|a_{j-1}|)} \cdot \mu'_n(a_1, \ldots, a_{j-1}, f(m), a_{j+1}, \ldots, a_n) . \end{aligned}$$

The set of all homomorphisms of degree p will be denoted $\mathrm{Hom}^p_{A\text{-}\mathbf{biMod}}(M, M')$.

The category of A-bimodules and their homomorphisms of *degree zero* will be denoted by A-bi**Mod**. Denote also by A-bi$\mathbf{Mod}_{\mathrm{B}}$ the full subcategory of A-bi**Mod** of all *balanced* bimodules, i.e. bimodules (M, μ_k) satisfying

$$\mu_n(s(a_1, \ldots, a_j \mid a_{j+1}, \ldots, a_n)) = 0 \tag{10}$$

for all $1 \le n \le m$, $1 \le j \le n-1$ and $(a_1, \ldots, a_n) \in J^n_{A,M}$.

1.11. Example. Let A be a (graded) algebra. We can relate with A the category $\mathbf{Mod}_{A-A}$ of A-bimodules in the traditional sense [11, V.3]. On the other hand, we can consider our algebra A as an A(∞)-algebra (Example 1.5) and take the category A-bi**Mod** of A-bimodules in the sense of 1.10. Any element $M \in \mathbf{Mod}_{A-A}$ can be considered as an element $(M, \mu_k) \in A$-bi**Mod** with $\mu_2(a, m) = am$, $\mu_2(m, b) = mb$ for $a, b \in A$ and $m \in M$, and $\mu_k = 0$ for $k \ne 2$. This correspondence identifies the category $\mathbf{Mod}_{A-A}$ with a full subcategory of A-bi**Mod**, the inclusion being strict in general. We hope that it will be always clear from the context whether, for an algebra A, by an A-bimodule we mean an element of $\mathbf{Mod}_{A-A}$ or an element of A-bi**Mod.**

For a category $\mathscr{D}$ and an object A of $\mathscr{D}$ denote by $\mathscr{D}/A$ the category of objects of $\mathscr{D}$ over A. The following lemma shows that A-bi**Mod** (resp. A-bi$\mathbf{Mod}_{\mathrm{B}}$) are appropriate coefficient categories for our cohomology theories.

1.12. Proposition. *Let A be an $A(m)$-algebra. Then the category A-bi**Mod** is equivalent with the category $(\mathbf{A}(m)/A)_{\mathrm{ab}}$ of Abelian group objects in the category $\mathbf{A}(m)/A$. Similarly, A-bi$\mathbf{Mod}_{\mathrm{B}}$ is equivalent with the category $(\mathbf{A}(m)_{\mathrm{B}}/A)_{\mathrm{ab}}$.*

Proof. Recall first some definitions. Let $\mathscr{D}$ be a category with finite products and a terminal object T (this is the case of $\mathbf{A}(m)/A$). Following the definitions of [12, p. 75], an *abelian group object* in $\mathscr{D}$ is an object $X \in \mathrm{Ob}(\mathscr{D})$ together with three maps $\mu : X \times X \to X$, $\zeta : T \to X$ and $\iota : X \to X$ such that:

(1) μ has the standard commutativity and associativity properties,

(2) the diagrams

$$\begin{array}{ccc} X \times X & \xrightarrow{\ \mu\ } & X \\ \uparrow{\scriptstyle \mathrm{id}_X \times \zeta} & & \| \\ X \times T & \xleftarrow{(\mathrm{id}_X, t_X)} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \times X & \xrightarrow{\ \mu\ } & X \\ \uparrow{\scriptstyle \zeta \times \mathrm{id}_X} & & \| \\ T \times X & \xleftarrow{(t_X, \mathrm{id}_X)} & X \end{array}$$

where $t_X : X \to T$ is the unique map (notice that (id_X, t_X) and (t_X, id_X) are actually isomorphisms), are commutative and also

(3) the diagrams

$$\begin{array}{ccc} X\times X & \xrightarrow{\mu} & X \\ \uparrow{\scriptstyle(\iota,\mathrm{id}_X)} & & \uparrow{\scriptstyle\zeta} \\ X & \xrightarrow{t_X} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X\times X & \xrightarrow{\mu} & X \\ \uparrow{\scriptstyle(\mathrm{id}_X,\iota)} & & \uparrow{\scriptstyle\zeta} \\ X & \xrightarrow{t_X} & X \end{array}$$

are commutative.

Let $\mathscr{D}_{\mathrm{ab}}$ be the category whose objects are abelian group objects (X, μ, ζ, ι) in $\mathscr{D}$ as above and the hom-set $\mathrm{Hom}_{\mathscr{D}_{\mathrm{ab}}}((X, \mu, \zeta, \iota), (X', \mu', \zeta', \iota'))$ is the set of all $f \in \mathrm{Hom}_{\mathscr{D}}(X, X')$ for which $\mu'(f \times f) = f \circ \mu$, $\zeta' = f \circ \zeta$ and $\iota' \circ f = f \circ \iota$. Notice that if $\mathscr{D}$ has a zero object (i.e. an object which is both terminal and initial), the definition of $\mathscr{D}_{\mathrm{ab}}$ can be reduced to the form as it is given for example in [8, p. 58]. This is, however, not the case of $\mathbf{A}(m)/A$, this category has both an initial object (given by the augmentation $\varepsilon : \mathbf{k} \to A$) and terminal object (given by the identity $\mathrm{id}_A : A \to A$) but these objects, for $\mathbf{k} \neq A$, do not coincide.

Let us begin our discussion of the case $\mathscr{D} = \mathbf{A}(m)/A$. As we have already observed, the identity map $\mathrm{id}_A : A \to A$ is the terminal object of this category. Let $X = (B \xrightarrow{\alpha} A)$ be an abelian group object in $\mathbf{A}(m)/A$ and let μ, ζ and ι be as above. The very existence of ζ immediately implies the existence of some $s : A \to B$ (morphism in $\mathbf{A}(m)$) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ \downarrow{\scriptstyle\mathrm{id}_A} & & \downarrow{\scriptstyle\alpha} \\ A & = & A \end{array}$$

is commutative, i.e. $\alpha \circ s = \mathrm{id}_A$. In other words, α is a *spitting epimorphism*, the splitting being a part of the structure.

We show that both μ and ι are entirely determined by the splitting s. First, notice that $X \times X$ is given by the pullback $E \xrightarrow{\bar{\alpha}} A$, with $E = \{(b, b') \in B \oplus B \mid \alpha(b) = \alpha(b')\}$ and $\bar{\alpha}(b, b') = \alpha(b)$. Suppose that $\mu : X \times X \to X$ is represented by a map $M : E \to B$ for which, of course, the diagram

$$\begin{array}{ccc} E & \xrightarrow{M} & B \\ \downarrow{\scriptstyle\bar{\alpha}} & & \downarrow{\scriptstyle\alpha} \\ A & \xrightarrow{\mathrm{id}_A} & A \end{array}$$

commutes, i.e. $\alpha M(b, b') = \alpha(b)$, for $(b, b') \in E$. In our setting, it is not hard to rewrite the second condition as

$$M(s\alpha(b), b) = M(b, s\alpha(b)) = b\ . \tag{11}$$

Let $(b, b') \in E$. Using the vector-space structure of E, we have the following equation in E:

$$(b, b') = (b - s\alpha(b) + s\alpha(b), b' - s\alpha(b') + s\alpha(b'))$$
$$= (s\alpha(b), s\alpha(b)) + (b - s\alpha(b), 0) + (0, b' - s\alpha(b)),$$

consequently

$$M(b, b') = M(s\alpha(b), s\alpha(b)) + M(0, b' - s\alpha(b)) + M(b - s\alpha(b), 0).$$

Writing in (11) $s\alpha(b)$ instead of b (and invoking $\alpha s = \mathrm{id}$) we get that $M(s\alpha(b), s\alpha(b)) = s\alpha(b)$. Similarly, noticing that $\alpha(b - s\alpha(b)) = 0$, (11) gives, for $b - s\alpha(b)$ in place of b, $M(b - s\alpha(b), 0) = b - s\alpha(b)$ and, similarly, $M(0, b' - s\alpha(b)) = b' - s\alpha(b)$. Thus the identities above give

$$M(b, b') = b + b' - s\alpha(b) \quad (= b + b' - s\alpha(b')). \tag{12}$$

By exactly the same method we can prove also that $\iota : X \to X$ is given by $Z : B \to B$ defined as

$$Z(b) = 2s\alpha(b) - b. \tag{13}$$

On the other hand, given a splitting s, it is easy to verify that (12) and (13) define an abelian group structure on $X = (B \xrightarrow{\alpha} A)$. Now it is clear that the category $(\mathbf{A}(m)/A)_{\mathrm{ab}}$ can be described as the category whose objects are triples (B, α, s), where $B \in \mathbf{A}(m)$, $\alpha \in \mathrm{Hom}_{\mathbf{A}(m)}(B, A)$ and $s \in \mathrm{Hom}_{\mathbf{A}(m)}(A, B)$ are maps with $\alpha \circ s = \mathrm{id}_A$. The morphism from (B, α, s) to (B', α', s') is then a map $f \in \mathrm{Hom}_{\mathbf{A}(m)}(B, B')$ such that $\alpha' f = \alpha$ and $fs = s'$.

Put $F(B, \alpha, s) = \mathrm{Ker}(\alpha)$ and define on $\mathrm{Ker}(\alpha)$ the structure of an A-bimodule by

$$\mu_k(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k)$$
$$= \nu_k(s(a_1), \ldots, s(a_{j-1}), x, s(a_{j+1}), \ldots, s(a_k)),$$

where ν_k are, for $1 \le k \le m$, the structure maps of B, $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \in A$ and $x \in \mathrm{Ker}(\alpha)$. This clearly defines a functor $F : (\mathbf{A}(m)/A)_{\mathrm{ab}} \to A$-bi**Mod.**

On the other hand, let $M \in A$-bi**Mod** and define on $A \oplus M$ the structure of an A(m)-algebra by

$$\nu_k(a_1 \oplus x_1, \ldots, a_k \oplus x_k)$$
$$= \mu_k(a_1, \ldots, a_k) \oplus \left\{ \sum_{1 \le j \le k} \mu_k(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_k) \right\}$$

for $1 \leq k \leq m$, $a_1, \ldots, a_k \in A$ and $x_1, \ldots, x_k \in M$. Let $G(M) = (A \oplus M, \pi_A, s_A)$, where $A \oplus M$ has the A(m)-structure as above, $\pi_A(a, x) = a$ and $s_A(a) = (a, 0)$ for $a \in A$ and $x \in M$. This clearly defines a functor $G : A\text{-bi}\mathbf{Mod} \to (\mathbf{A}(m)/A)_{\mathrm{ab}}$.

It is immediate to see that FG is the identity functor. On the other hand, for $(B, \alpha, s) \in (\mathbf{A}(m)/A)_{\mathrm{ab}}$ we have $GF(B, \alpha, s) = (A \oplus \mathrm{Ker}(\alpha), \pi_A, s_A)$ and the map $M \to A \oplus \mathrm{Ker}(\alpha)$ given by $b \mapsto (\alpha(b), b - s\alpha(b))$ clearly defines a natural equivalence of GF and the identity functor. This finishes the proof. The argument for balanced categories is the same. □

Now we give an alternative description of the category $A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$. By a *left A-module* we mean a graded vector space M together with a set $\{\mu_k : \bigotimes^{k-1} A \otimes M \to M \mid 1 \leq k \leq m\}$ of linear maps, $\deg(\mu_k) = k - 2$, satisfying (8) for all $(a_1, \ldots, a_n) \in \bigotimes^{n-1} A \otimes M$, $1 \leq n \leq m$, and satisfying also the evident version of (9). By a *morphism of degree p* of two left A-modules (M, μ_k) and (M', μ'_k) we mean a homogeneous linear map $f : M \to M'$ of degree p satisfying

$$f(\mu_n(a_1, \ldots, a_{n-1}, m)) = (-1)^{pn + p(|a_1| + \cdots + |a_{n-1}|)} \mu'_n(a_1, \ldots, a_{n-1}, f(m))$$

for all homogeneous $a_1 \otimes \cdots \otimes a_{n-1} \otimes m \in \bigotimes^{n-1} A \otimes M$, $1 \leq n \leq m$. The category of left A-modules and their homomorphisms of *degree zero* will be denoted A-left**Mod.**

1.13 Proposition. *Suppose that the $A(m)$-algebra A is balanced. Then the categories A-bi**Mod**$_{\mathrm{B}}$ and A-left**Mod** are equivalent.*

Proof. We will construct inverse functors $G : A\text{-bi}\mathbf{Mod}_{\mathrm{B}} \to A\text{-left}\mathbf{Mod}$ and $F : A\text{-left}\mathbf{Mod} \to A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$. For $(M, \mu_k) \in A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$, let $G(M, \mu_k) = (M, \nu_k)$, where ν_k is simply the restriction $\mu_k|_{\bigotimes^{k-1} A \otimes M}$, $1 \leq k \leq m$. This clearly defines a functor $G : A\text{-bi}\mathbf{Mod}_{\mathrm{B}} \to A\text{-left}\mathbf{Mod}$.

On the other hand, let $(M, \nu_k) \in A\text{-left}\mathbf{Mod}$. For $1 \leq k \leq m$, $1 \leq j \leq k$, $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \in A$ and $m \in M$ define

$$\begin{aligned} &\mu_k(a_1, \ldots, a_{j-1}, m, a_{j+1}, \ldots, a_k) \\ &\quad = \chi(\rho; a_{j+1}, \ldots, a_k) \cdot (-1)^{|m| \cdot (|a_{j+1}| + \cdots + |a_k|)} \\ &\quad \cdot \nu_k(s(a_1, \ldots, a_{j-1} \mid a_k, a_{k-1}, \ldots, a_{j+1}), m) \end{aligned} \tag{14}$$

(for the notation see 1.3). It can be easily verified that the above formula defines a functor $F : A\text{-left}\mathbf{Mod} \to A\text{-bi}\mathbf{Mod}_{\mathrm{B}}$. The fact that $G \circ F = \mathrm{id}$ is clear. Using 1.3(iv) and the commutativity 1.3(ii) of the shuffles, we see immediately that in the balanced bimodule (M, μ_k) the following condition is always satisfied:

$$\begin{aligned}
&\mu_k(a_1,\ldots,a_{j-1},m,a_{j+1},\ldots,a_k)\\
&\quad=\chi(\rho;a_{j+1},\ldots,a_k)\cdot(-1)^{(|a_{j+1}|+\cdots+|a_k|)(|a_1|+\cdots+|a_{j-1}|+|m|)+(j+k)(j+1)}\\
&\qquad\cdot\mu_k(s(a_k,\ldots,a_{j+1}\mid a_1,\ldots,a_{j-1}),m)\\
&\quad=\chi(\rho;a_{j+1},\ldots,a_k)\cdot(-1)^{|m|\cdot(|a_{j+1}|+\cdots+|a_k|)}\\
&\qquad\cdot\mu_k(s(a_1,\ldots,a_{j-1}\mid a_k,a_{k-1},\ldots,a_{j+1}),m)\,. \qquad (15)
\end{aligned}$$

Comparing it with the equation (14) defining the functor F we see also that $F\circ G=\mathrm{id}$. □

The above theorem clearly generalizes the following well-known simple observation. Let A be a graded commutative algebra. Then the category of left (graded) A-modules is equivalent with the full subcategory of the category of A-bimodules consisting of bimodules satisfying $a\cdot m=(-1)^{|a|\cdot|m|}m\cdot a$ for each $a\in A$ and $m\in M$.

1.14. Let now $\bigcirc : A\text{-bi}\mathbf{Mod}\to\mathbf{Vect}$ to the 'underlying' functor. It has, as does every algebraic functor by [10, p. 870], a left adjoint $F:\mathbf{Vect}\to A\text{-bi}\mathbf{Mod}$. For a graded vector space V it is then natural to call $F(V)$ the *free A-bimodule* on V. It comes together with an adjunction unit $\varepsilon: V\to FV$ and it is characterized by an obvious universal property. An explicit description of FV can be obtained as follows. Take $\tilde F_0V=V$ and let

$$\tilde F_{i+1}V=\bigoplus_{\substack{1\le k\le m\\ 1\le j\le k}}\bigotimes^{j-1}A\otimes\tilde F_iV\otimes\bigotimes^{k-j-1}A\,.$$

For $a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_k\in A$ and $m\in\tilde F_iV$ write $\mu_k(a_1,\ldots,a_{j-1},m,a_{j+1},\ldots,a_k)=a_1\otimes\cdots\otimes a_{j-1}\otimes m\otimes a_{j+1}\otimes\cdots\otimes a_k\in\tilde F_{i+1}V$. Then FV is $\tilde{FV}=\bigoplus_{i\ge0}\tilde F_iV$ factored subject to the relations (8) and (9) and ε is defined as the composition $V=\tilde F_0V\hookrightarrow\bigoplus_{i\ge0}\tilde F_iV\xrightarrow{\text{projection}}FV$. The map ε can be easily shown to be an injection and we will identify elements of V and their images under the map ε.

1.15. Lemma. *Let V be a graded space and let $M=(M,\mu_k)\in A$-bi***Mod**. *Then for any homogeneous linear map $f:V\to M$ of graded vector spaces of degree p there exists a unique $\phi\in\mathrm{Hom}^p_{A\text{-biMod}}(FV,M)$ with $\phi\circ\varepsilon=f$.*

Proof. Put $\tilde\phi|_V=f$ and suppose we have already defined $\tilde\phi$ on $\tilde F_iV$ for $i\le q$. The formula

$$\begin{aligned}
&\tilde\phi(\mu_k(a_1,\ldots,a_{j-1},m,a_{j+1},\ldots,a_k)\\
&\quad=(-1)^{p(|a_1|+\cdots+|a_{j-1}|)+pk}\cdot\mu_k(a_1,\ldots,a_{j-1},\tilde\phi(m),a_{j+1},\ldots,a_k)
\end{aligned}$$

then defines $\tilde{\phi}$ on $\tilde{F}_{q+1}V$ and this process gives rise to a map $\tilde{\phi} : \tilde{F}V \to M$. This map clearly factors through the projection $\tilde{F}V \to FV$ to give the requisite ϕ. The uniqueness is a consequence of the clear fact that FV is generated by V as an A(m)-bimodule. □

Similarly, we define the notion of the free *balanced* A-bimodule on a graded vector space V, denoted by $F_B V$. The adjunction unit $\varepsilon_B : V \to F_B V$ determines a canonical epimorphism $\pi : FV \to F_B V$. Clearly $F_B V$ can be thought of as FV factored by the relations (10). This has the following consequence:

1.16 Lemma. *Let $f : FV \to FV$ be a homomorphism of degree p. Then there exists a unique homomorphism $f_B : F_B V \to F_B V$ such that $f_B \circ \pi = \pi \circ f$.*

The next statement is merely an observation.

1.17. Observation. *For a graded space V and $M \in A\text{-bi}\mathbf{Mod}_B$ the map $\pi : FV \to F_B V$ induces, for any p, an isomorphism*

$$\pi^p : \mathrm{Hom}^p_{A\text{-bi}\mathbf{Mod}_B}(F_B V, M) \cong \mathrm{Hom}^p_{A\text{-bi}\mathbf{Mod}}(FV, M) . \qquad \square$$

2. Cohomology theory of A(m)-algebras

From now on, all A(m)-algebras will be tacitly assumed to be unitary and augmented (see 1.4). For such an algebra $A = (A, \mu_k)$, the augmentation ideal will be denoted $\bar{A} = (\bar{A}, \bar{\mu}_k)$.

2.1. For an A(m)-algebra A let $\mathcal{B}^n(A)$ be, for $n \geq 0$, the free A-bimodule (see 1.14) on the space $\bar{A} \otimes \cdots \otimes \bar{A}$ (n-times) graded by $\deg([a_1, \ldots, a_n]) = \sum_{i=1}^n \deg(a_i) + n - 1$, here we denote as usually in this context, $a_1 \otimes \cdots \otimes a_n$ by $[a_1, \ldots, a_n]$. For $1 \leq k \leq n+1$ define the linear map $\partial_E^k : \bigotimes^n \bar{A} \to \mathcal{B}^{n-k+1}(A)$ of degree -1 by

$$\begin{aligned} &\partial_E^k([a_1, \ldots, a_n]) \\ &\quad = -\sum_{\lambda=0}^{k-1} (-1)^{(n+k)(|a_1| + \cdots + |a_\lambda| + \lambda)} \\ &\qquad \cdot \mu_k(a_1, \ldots, a_\lambda, [a_{\lambda+1}, \ldots, a_{\lambda+n-k+1}], a_{\lambda+n-k+2}, \ldots, a_n) , \end{aligned}$$

for $k > n+1$ put $\partial_E^k = 0$. For $1 \leq k \leq n$ define also the linear map $\partial_I^k : \bigotimes^n \bar{A} \to \mathcal{B}^{n-k+1}(A)$ of degree -1 by

$$\partial_I^k([a_1,\ldots,a_n]) = \sum_{\lambda=0}^{n-k}(-1)^{k(|a_1|+\cdots+|a_\lambda|+\lambda)+\lambda+n+k} \cdot [a_1,\ldots,a_\lambda,\mu_k(a_{\lambda+1},\ldots,a_{\lambda+k}),a_{\lambda+k+1},\ldots,a_n],$$

for $k>n$ put $\partial_I^k=0$. Notice that $\partial_E^{n+1}([a_1,\ldots,a_n])$ may be nonzero while $\partial_I^{n+1}([a_1,\ldots,a_n])$ is always trivial be definition. By Lemma 1.15 these maps induce unique homomorphisms (denoted by the same symbol), $\partial_E^k,\partial_I^k:\mathcal{B}^n(A)\to\mathcal{B}^{n-k+1}(A)$ of A-bimodules of degree -1. Finally, let $\partial_E=\sum_{k=1}^m\partial_E^k$, $\partial_I=\sum_{k=1}^m\partial_I^k$ and $\partial=\partial_E+\partial_I$. Then ∂ is an endomorphism of the A-bimodule $\mathcal{B}_{(m)}(A):=\bigoplus_{0\le n\le m}\mathcal{B}^n(A)$ of degree -1. It can be easily verified, using the 'associativity' relation (1) that $\partial\circ\partial=0$.

2.2. For an A-bimodule M define

$$C^p_{(m)}(A;M)=\mathrm{Hom}^{-p+1}_{A\text{-}\mathbf{biMod}}(\mathcal{B}_{(m)}(A),M).$$

The formula $\delta=-\mathrm{Hom}_{A\text{-}\mathbf{biMod}}(\partial,M)$ then defines a differential of degree $+1$ on $C^*_{(m)}(A,M)$ (one has $\delta f=(-1)^p f\partial$ for $f\in C^p_{(m)}(A,M)$) and the *cohomology of A with coefficients in M* is defined as

$$H^*_{(m)}(A;M)=H(C^*_{(m)}(A;M),\delta)$$

(for our degree and sign conventions see 3.6). Properties of free bimodules give rise to the natural identification

$$C^*_{(m)}(A;M)=\mathrm{Hom}^{-*}_{\mathbf{Vect}}\left(\bigoplus_{0\le k\le m}\bigotimes^k\uparrow\bar{A},M\right).$$

Under this identification, the differential δ takes the form

$$\begin{aligned}\delta f(a_1,\ldots,a_n) &= \sum_{k=1}^{n}\sum_{\lambda=0}^{n-k}(-1)^{k(|a_1|+\cdots+|a_\lambda|+\lambda)+p+\lambda+n+k} \cdot f(a_1,\ldots,a_\lambda,\mu_k(a_{\lambda+1},\ldots,a_{\lambda+k}),a_{\lambda+k+1},\ldots,a_n)\\ &\quad-\sum_{k=1}^{n+1}\sum_{\lambda=0}^{k-1}(-1)^{(n-k+p)(|a_1|+\cdots+|a_\lambda|)+p(k+1)+(n+k)\lambda} \cdot\mu_k(a_1,\ldots,a_\lambda,f(a_{\lambda+1},\ldots,a_{\lambda+n-k+1}),\\ &\qquad a_{\lambda+n-k+2},\ldots,a_n)\end{aligned} \tag{16}$$

for $f\in C^p_{(m)}(A;M)$ and $a_1,\ldots,a_n\in\uparrow\bar{A}$, $1\le n\le m$.

2.3. Let $\mathscr{B}B^n(A)$ be, for $n \geq 0$, the *balanced* free A-bimodule on $\bigotimes^n \bar{A}$ with the same degree convention as in 2.1. By Lemma 1.16 the differential ∂ defines a differential ${}_B\partial$ on $\mathscr{B}B_{(m)}(A) = \bigoplus_{0 \leq n \leq m} \mathscr{B}B^n(A)$. Notice that, by Observation 1.17, for a *balanced* A-bimodule M,

$$C^*_{(m)}(A; M) \cong \mathrm{Hom}^{-*+1}_{A\text{-biMod}_B}(\mathscr{B}B_{(m)}(A), M)$$

and the differential $-\mathrm{Hom}_{A\text{-biMod}_B}({}_B\partial, M)$ clearly coincides with δ defined above. Therefore, for such a bimodule the cohomology $H_{(m)}(A; M)$ can be computed with the aid of the complex $(\mathscr{B}B_{(m)}(A), {}_B\partial)$. This complex will play an important rôle in our cohomology theory for balanced algebras.

The shuffles $s_{i,n-i} : \bigotimes^n \bar{A} \to \bigotimes^n \bar{A}$ induce, for $0 \leq i \leq n$, $n > 0$, the linear maps (denoted by the same symbol) $s_{i,n-i} : \mathscr{B}^n(A) \to \mathscr{B}^n(A)$ and $s_{i,n-i} : \mathscr{B}B^n(A) \to \mathscr{B}B^n(A)$.

2.4. Lemma. *Let* $s_n = \sum_{i=0}^n s_{i,n-i}$ *and* $s = \bigoplus_{1 \leq n \leq m} s_n$. *Then, for any* k, $1 \leq k \leq m$,

$$s \circ {}_B\partial^k_I = {}_B\partial^k_I \circ s \quad \textit{and} \quad s \circ \partial^k_I = \partial^k_I \circ s$$

provided A *is balanced.*

Proof. Clearly, it is enough to verify the equality only on 'generators' $[a_1, \dots, a_n]$, $1 \leq n \leq m$. As s maps a generator into a linear combination of generators and as ∂_I and ${}_B\partial_I$ agree on these generators, it is enough to verify our equality only for ∂_I. Using elementary combinatorial arguments we get

$$\begin{aligned}
&\partial^k_I s([a_1, \dots, a_n]) \\
&= \sum_{0 \leq i \leq n} \partial^k_I s_{i,n-i}([a_1, \dots, a_n]) \\
&= \sum_{\substack{0 \leq i \leq n, 0 \leq \alpha \leq \beta \leq i \\ i \leq \delta \leq \phi \leq n \\ \alpha - \beta + \phi - \delta = k}} (-1)^\theta [s(a_1, \dots, a_\alpha \mid a_{i+1}, \dots, a_\delta), \\
&\qquad\qquad \mu_k(s(a_{\alpha+1}, \dots, a_\beta \mid a_{\delta+1}, \dots, a_\phi)), \\
&\qquad\qquad s(a_{\beta+1}, \dots, a_i \mid a_{\phi+1}, \dots, a_n)],
\end{aligned}$$

where

$$\begin{aligned}
\theta = {}& (|a_{i+1}| + \cdots + |a_\delta|)(|a_{\alpha+1}| + \cdots + |a_i|) \\
&+ (|a_{\beta+1}| + \cdots + |a_i|)(|a_{\delta+1}| + \cdots + |a_\phi|) \\
&+ k(|a_1| + \cdots + |a_\alpha| + |a_{i+1}| + \cdots + |a_\delta| + \alpha + i + \delta) \\
&+ \alpha + i + \delta + n + k + (i + \delta)(\alpha + i).
\end{aligned}$$

As our algebra A is supposed to be balanced, the term $\mu_k(s(a_{\alpha+1}, \ldots, a_\beta \mid a_{\delta+1}, \ldots, a_\phi))$ and hence all the summation can be nonzero only if $\alpha = \beta$ or $\delta = \phi$. Therefore our sum can be reduced to

$$\sum_{\substack{0 \le i \le n-k, 0 \le \alpha \le i \\ i \le \delta \le n-k}} (-1)^{\xi}[s(a_1, \ldots, a_\alpha \mid a_{i+1}, \ldots, a_\delta), \mu_k(a_{\delta+1}, \ldots, a_{\delta+k}),$$

$$s(a_{\alpha+1}, \ldots, a_i \mid a_{\delta+k+1}, \ldots, a_n)]$$

$$+ \sum_{\substack{k \le i \le n, 0 \le \alpha \le i-k \\ i \le \delta \le n}} (-1)^{\eta}[s(a_1, \ldots, a_\alpha \mid a_{i+1}, \ldots, a_\delta),$$

$$\mu_k(a_{\alpha+1}, \ldots, a_{\alpha+k}),$$

$$s(a_{\alpha+k+1}, \ldots, a_i \mid a_{\delta+1}, \ldots, a_n)],$$

where

$$\begin{aligned}\xi = {} & (|a_{i+1}| + \cdots + |a_{\delta+k}|)(|a_{\alpha+1}| + \cdots + |a_i|) \\ & + k(|a_1| + \cdots + |a_\alpha| + |a_{i+1}| + \cdots + |a_\delta| + \alpha + i + \delta) \\ & + \alpha + i + \delta + n + k + (i + \delta + k)(\alpha + i)\end{aligned}$$

and

$$\begin{aligned}\eta = {} & (|a_{i+1}| + \cdots + |a_\delta|)(|a_{\alpha+1}| + \cdots + |a_i|) \\ & + k(|a_1| + \cdots + |a_\alpha| + |a_{i+1}| + \cdots + |a_\delta| + \alpha + i + \delta) \\ & + \alpha + i + \delta + n + k + (i + \delta)(a + i).\end{aligned}$$

On the other hand, it is easily seen that

$$\begin{aligned} & s\,\partial_l^k[a_1, \ldots, a_n] \\ & = \sum_{\substack{0 \le i \le n-k \\ 0 \le \lambda \le n-k}} (-1)^{k(|a_1| + \cdots + |a_\lambda| + \lambda) + \lambda + n + k} \\ & \qquad \cdot s_{i,n-k-i}[a_i, \ldots, a_\lambda, \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_n] \\ & = \sum_{\substack{0 \le \lambda \le n-k \\ 0 \le i \le \lambda \\ 0 \le j \le i}} (-1)^{\pi}[s(a_1, \ldots, a_j \mid a_{i+1}, \ldots, a_\lambda), \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), \\ & \qquad s(a_{j+1}, \ldots, a_i \mid a_{\lambda+k+1}, \ldots, a_n)] \\ & \quad + \sum_{\substack{0 \le \lambda \le n-k \\ \lambda+k \le i \le n \\ i \le j \le n}} (-1)^{\phi}[s(a_1, \ldots, a_\lambda \mid a_{i+1}, \ldots, a_j), \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), \\ & \qquad s(a_{\lambda+k+1}, \ldots, a_i \mid a_{j+1}, \ldots, a_n)],\end{aligned}$$

where

$$\pi = (|a_{j+1}| + \cdots + |a_i|)(|a_{i+1}| + \cdots + |a_{\lambda+k}| + k)$$
$$+ k(|a_1| + \cdots + |a_\lambda| + \lambda) + (j+i)(\lambda + i + 1) + \lambda + n + k$$

and

$$\phi = (|a_{i+1}| + \cdots + |a_j|)(|a_{\lambda+1}| + \cdots + |a_i| + k)$$
$$+ k(|a_1| + \cdots + |a_\lambda| + \lambda) + (j+i)(\lambda + k + i + 1) + \lambda + n + k .$$

Using the substitution $j = \alpha$, $\lambda = \delta$ in the first summand and the substitution $j = \delta$, $\lambda = \alpha$ in the second summand, one can easily verify that the last formula coincides with the formula for $\partial_I^k \circ s$ above. □

2.5. Lemma. *For any* k, $1 \leq k \leq m$,

$$_B\partial_E^k \circ s = s \circ {_B\partial_E^k} .$$

Proof. We have

$$\begin{aligned} &_B\partial_E^k s([a_1, \ldots, a_n]) \\ &= {_B\partial_E^k} \sum_{0 \leq i \leq n} s_{i,n-i}([a_1, \ldots, a_n]) \\ &= \sum_{\substack{0 \leq i \leq n \\ p+q+r+s=k-1}} (-1)^{\Gamma} \cdot \mu_k(s(C_p \mid E_{iq}), [s(F_{ipr} \mid G_{iqs})], s(H_{ir} \mid D_s)) , \end{aligned} \tag{17}$$

where

$$\Gamma = (n+k)(|C_p| + |E_{iq}| + p + q) + (|F_{ipr}| + |H_{ir}|) \cdot |E_{iq}|$$
$$+ |G_{iqs}| \cdot |H_{ir}| + (i+p)q + (i+q+n+s)r$$

and where we, as in the proof of Theorem 1.8, simplify our notation by putting $C_p = (a_1, \ldots, a_p)$, $E_{iq} = (a_{i+1}, \ldots, a_{i+q})$, $F_{ipr} = (a_{p+1}, \ldots, a_{i-r})$, $G_{iqs} = (a_{i+q+1}, \ldots, a_{n-s})$, $H_{ir} = (a_{i-r+1}, \ldots, a_i)$ and $D_s = (a_{n-s+1}, \ldots, a_n)$. The substitution $\alpha = p$, $\gamma = s$, $\delta + j = i$, $j = r$ and $\beta - j = q$ gives

$$\begin{aligned} &_B\partial_E^k s([a_1, \ldots, a_n]) \\ &= \sum_{\substack{\alpha+\beta+\gamma=k-1 \\ 0 \leq j \leq \beta, \alpha \leq \delta \leq n-\gamma-\beta}} (-1)^{\Delta} \\ &\qquad \cdot \mu_k(s(A_\alpha \mid W_{j\beta\delta}), [s(X_{\alpha\delta} \mid Y_{\alpha\beta\gamma})], s(Z_{j\delta} \mid B_\gamma)) , \end{aligned} \tag{18}$$

where

$$\Delta = (n+k)(|A_\alpha| + |W_{j\beta\delta}| + \alpha + \beta + j) + (|X_{\alpha\delta}| + |Z_{j\delta}|)\cdot |W_{j\beta\delta}| + |Y_{\alpha\beta\gamma}|\cdot|Z_{j\delta}| + (\delta + j + \alpha)(\beta + j) + (\delta + \beta + \gamma + n)j\,,$$

and where $A_\alpha = (a_1, \dots, a_\alpha)$, $W_{j\beta\delta} = (a_{\delta+j+1}, \dots, a_{\delta+\beta})$, $X_{\alpha\delta} = (a_{\alpha+1}, \dots, a_\delta)$, $Y_{\alpha\beta\gamma} = (a_{\delta+\beta+1}, \dots, a_{n-\gamma})$, $Z_{j\delta} = (a_{\delta+1}, \dots, a_{\delta+j})$ and $B_\gamma = (a_{n-\gamma+1}, \dots, a_n)$. We will show that the terms with $\beta > 0$ give no contribution to the sum in (18). Suppose for a moment that we have proved it. Then (18) gives

$$\begin{aligned}
{}_B\partial_E^k s([a_1, \dots, a_n]) &= \sum_{\substack{\alpha+\gamma=k-1\\ \alpha\leq\delta\leq n-\gamma}} (-1)^{(n+k)(|A_\alpha|+\alpha)}\cdot \mu_k(A_\alpha, [s(X_{\alpha\delta} \mid Y_{\alpha0\gamma})], B_\gamma)\\
&= \sum_{\substack{\alpha+\gamma=k-1\\ \alpha\leq\delta\leq n-\gamma}} (-1)^{(n+k)(|a_1|+\cdots+|a_\alpha|+\alpha)}\\
&\qquad \cdot \mu_k(a_1, \dots, a_\alpha, [s(a_{\alpha+1}, \dots, a_\delta \mid a_{\delta+1}, \dots, a_{n-\gamma})],\\
&\qquad\quad a_{n-\gamma+1}, \dots, a_n)\,, \qquad (19)
\end{aligned}$$

which is exactly the formula for $s \circ {}_B\partial_E^k([a_1, \dots, a_n])$. The lemma is thus proved.

It remains to show that, having fixed α, γ, δ and $\beta > 0$, the sum

$$\sum_{0\leq j\leq\beta} (-1)^\Delta \cdot \mu_k(s(A \mid W_j), [s(X \mid Y)], s(Z_j \mid B))\,, \qquad (20)$$

where we write $A = A_\alpha$, $W_j = W_{j\beta\delta}$, $X = X_{\alpha\delta}$, $Y = Y_{\alpha\beta\gamma}$, $Z_j = Z_{j\delta}$ and $B = B_\gamma$, is zero. Note that $\chi(\rho; x_1, \dots, x_n) = \chi(\rho; x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation σ. Notice also that $\chi(\rho; Z_j, B) = (-1)^{|Z_j|\cdot|B|+\gamma j}\cdot\chi(\rho; Z_j)\cdot\chi(\rho; B)$. Let $\overleftarrow{B} = (a_n, \dots, a_{n-\gamma+1})$ and $\overleftarrow{Z}_j = (a_{\delta+j}, \dots, a_{\delta+1})$. These remarks together with 1.3(i), 1.3(iii) and (15) give

$$\begin{aligned}
&\mu_k(s(A \mid W_j), [s(X \mid Y)], s(Z_j \mid B))\\
&\quad = (-1)^{(|B|+|Z_j|)\cdot(|X|+|Y|+\alpha+\beta+\gamma+n+1)+|Z_j|\cdot|B|+j\gamma}\\
&\qquad \cdot \chi(\rho; Z_j)\cdot\chi(\rho; B)\cdot \mu_k(s(A \mid W_j \mid \overleftarrow{B} \mid \overleftarrow{Z}_j), [s(X \mid Y)])\,,
\end{aligned}$$

while

$$\begin{aligned}
&\mu_k(s(A \mid W_j \mid \overleftarrow{B} \mid \overleftarrow{Z}_j), [s(X \mid Y)])\\
&\quad = (-1)^{|Z_j|\cdot|B|+j\gamma}\cdot \mu_k(s(A \mid W_j \mid \overleftarrow{Z}_j \mid \overleftarrow{B}), [s(X \mid Y)])
\end{aligned}$$

due to the commutativity of the shuffle 1.3(ii). Using these equalities, the sum in (20) is equal to

$$\sum_{0\leq j\leq \beta} C\cdot \chi(\rho; Z_j)\cdot(-1)^{|Z_j|\cdot|W_j|+j\beta}\cdot \mu_k(s(A \mid W_j \mid \overleftarrow{Z_j} \mid \overleftarrow{B}), [s(X \mid Y)]) ,$$

where $C = \pm 1$ and this value does not depend on j. On the other hand, the equality 1.3(v) says that, for $\beta > 0$,

$$\sum_{0\leq j\leq \beta} \chi(\rho; W_j)\cdot(-1)^{\beta+j}\cdot s(W_j \mid \overleftarrow{Z_j}) = 0 .$$

Clearly $\chi(\rho; W_j) = (-1)^{|W_j|\cdot|Z_j|+j+\beta j}\cdot \chi(\rho; W_j, Z_j)\cdot \chi(\rho; Z_j)$, where $\chi(\rho; W_j, Z_j)$ does not depend on j. The last equality then gives

$$\sum_{0\leq j\leq \beta} C'\cdot \chi(\rho; Z_j)\cdot(-1)^{|W_j|\cdot|Z_j|+\beta j}\cdot s(W_j \mid \overleftarrow{Z_j}) = 0 ,$$

where $C' = \pm 1$ again does not depend on j. We see that the sum in (20) is zero for $\beta > 0$ which completes our proof. □

For a graded **k**-module X the formula

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \chi(\sigma)x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)} , \quad \sigma \in S_n ,$$

where the sign $\chi(\sigma)$ was introduced in 1.2, defines an action of the symmetric group S_n on $\bigotimes^n X$. This observation enables us to consider $s_n := \sum_{i=0}^{n} s_{i,n-i}$ as an element of $\mathbb{Q}(S_n)$ and we may try to construct a Hodge-type decomposition similarly as it was done in [5, Section 1] for Hochschild cohomology of commutative algebras. We have, for $n \geq 1$, the elements $e_n(1), \ldots, e_n(n) \in \mathbb{Q}(S_n)$ which are polynomials in s_n (given by [5, formula (2), p. 233]) such that

$$e_n(1) + \cdots + e_n(n) = \mathrm{id} ,$$
$$e_n(j)^2 = e_n(j) , \quad 1 \leq j \leq n ,$$
$$e_n(i)e_n(j) = 0 , \quad i \neq j .$$

For an A(m)-algebra A we get (putting $X = A$) an action of $\mathbb{Q}(S_n)$ on $\bigotimes^n A$ which in turn gives the actions of $\mathbb{Q}(S_n)$ on $\mathscr{B}^n(A)$ and $\mathscr{B}B^n(A)$. For any $j \geq 1$ define the endomorphism $e(j)$ of $\mathscr{B}B_{(m)}(A) = \bigoplus_{0\leq n\leq m} \mathscr{B}B^n(A)$ by

$$e(j)(x) = \begin{cases} e_n(j)(x) & \text{for } x \in \mathscr{B}B^n(A) \text{ and } j \leq n , \\ 0 & \text{otherwise} , \end{cases}$$

and put

$$e(0)(x) = \begin{cases} x & \text{for } x \in \mathcal{B}B^0(A)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We clearly have $e(j)^2 = e(j)$, $e(i) \circ e(j) = 0$ for $i \neq j$ and $e(0)(x) + e(1)(x) + \cdots = x$ (finite sum) for any $x \in \mathcal{B}B_{(m)}(A)$. In the following corollary $s = \bigoplus_{1 \leq n \leq m} s_n$.

2.6. Corollary. *For a balanced $A(m)$-algebra A*

$$_B\partial \circ s = s \circ {_B\partial} \quad \textit{and} \quad {_B\partial} \circ e(j) = e(j) \circ {_B\partial}\,.$$

Proof. The first equation follows immediately from Lemmas 2.4 and 2.5. For $_B\partial^k = {_B\partial^k_E} + {_B\partial^k_I}$ we have (again from Lemmas 2.4 and 2.5) $_B\partial^k \circ s_n(x) = s_{n-k+1} \circ {_B\partial^k}(x)$ for $x \in \mathcal{B}B^m(A)$ which enables us to prove by exactly the same method as in the proof of [5, Theorem 1.3(ii)] that $_B\partial^k \circ e_n(j)(x) = e_{n-k+1}(j) \circ {_B\partial^k}(x)$. This easily gives the second equation of our corollary. □

2.7. Proposition. *Let M be a balanced bimodule over a balanced $A(m)$-algebra A. Define the endomorphisms $\mathfrak{s}$ and $\mathfrak{e}(j)$ of $C_{(m)}(A; M) = \mathrm{Hom}^{-*+1}_{A\text{-}\mathbf{biMod}}(\mathcal{B}B_{(m)}(A), M)$ to be the duals to the maps s and $e(j)$ defined above. Then $\delta \circ \mathfrak{s} = \mathfrak{s} \circ \delta$, $\delta \circ \mathfrak{e}(j) = \mathfrak{e}(j) \circ \delta$, $\mathfrak{e}(j)^2 = \mathfrak{e}(j)$ and $\mathfrak{e}(i) \circ \mathfrak{e}(j) = 0$ for $i \neq j$. Moreover, $C_{(m)}(A; M) = \varprojlim_n C_{(m)}(A; M)/G_n(A; M)$ with $G_n(A; M) := \{f \in C_{(m)}(A; M) \mid \mathfrak{e}(i)(f) = 0$ for $i \leq n\}$, and $\mathfrak{e}(0)(f) + \cdots + \mathfrak{e}(n)(f) = f$ modulo $G_n(A; M)$ for all $n \geq 0$ and $f \in C_{(m)}(A; M)$.*

Proof. All statements of the proposition easily follows from the definitions, properties of $e_n(j)$'s and Corollary 2.6. □

2.8. We can now introduce an analog of a Hodge-type decomposition for our cohomology of A(m)-algebras. Put

$$C^{*,j}_{(m)}(A; M) = \mathfrak{e}(j)(C^*_{(m)}(A; M))\,, \quad j \geq 0\,.$$

It immediately follows from Proposition 2.7 that $C^{*,j}_{(m)}(A; M)$ is, for each $j \geq 0$, a δ-stable subspace of $C^*_{(m)}(A; M)$ and that the maps $\mathfrak{e}(j) : C^*_{(m)}(A; M) \to C^{*,j}_{(m)}(A; M)$ induce an identification

$$C^*_{(m)}(A; M) \cong \prod_{j \geq 0} C^{*,j}_{(m)}(A; M)\,.$$

Here $C^{*,1}_{(m)}(A; M)$ will be of a special importance for us and we introduce the notation $CB^*_{(m)}(A; M) := C^{*,1}_{(m)}(A; M)$ (again B from balanced).

Having in mind future applications, we give the following alternative description of $CB^*_{(m)}(A;M)$. First, define, for each $q \geq 2$, the graded subspace $\mathrm{Sh}^q(A) \subset \mathcal{B}B^q(A)$ by $\mathrm{Sh}^q(A) = \bigoplus_{1\leq i\leq q-1} \mathrm{Im}(s_{i,q-i})$ and let $\mathrm{Sh}(A) = \bigoplus_{2\leq q\leq m} \mathrm{Sh}^q(A) \subset \mathcal{B}B_{(m)}(A)$. It is not hard to show that ${}_B\partial(\mathrm{Sh}(A)) \subset \mathrm{Sh}(A)$, which implies that ${}_B\partial$ induces on $\mathcal{B}B_{(m)}(A)/\mathrm{Sh}(A)$ a differential (denoted again by ${}_B\partial$). It easily follows from the general properties on the projections $e(j)$ (similarly as in the case of commutative algebras discussed in [5, 6]) that

$$CB^p_{(m)}(A;M) = \mathrm{Hom}^{-p+1}_{A\text{-biMod}}(\mathcal{B}B_{(m)}(A)/\mathrm{Sh}(A), M)$$

and that the restriction of δ on $CB^*_{(m)}(A;M) \subset C^*_{(m)}(A;M)$ coincide with the differential δ_B induced by ${}_B\partial$. Moreover, the inclusion $CB^*_{(m)}(A;M) \hookrightarrow C^*_{(m)}(A;M)$ is dual to the canonical projection $\mathcal{B}B_{(m)}(A) \to \mathcal{B}B_{(m)}(A)/\mathrm{Sh}(A)$. Put

$$H^{*,j}_{(m)}(A;M) = H^*(C^{*,j}_{(m)}(A;M), \delta^{(j)}), \quad j \geq 0,$$

where $\delta^{(j)}$ denotes the restriction of the differential δ on $C^{*,j}_{(m)}(A;M)$. Again, $H^{*,1}_{(m)}(A;M)$ will play an important rôle and we denote it by

$$HB^*_{(m)}(A;M) = H^*(CB^*_{(m)}(A;M), \delta_B)$$

and call it the *balanced cohomology* of A with coefficients in M. Denote also by $\Pi^*(A;M) : HB^*_{(m)}(A;M) \to H^*_{(m)}(A;M)$ the map induced by the inclusion $CB^*_{(m)}(A;M) \hookrightarrow C^*_{(m)}(A;M)$. Summing up the results above, we may formulate the central result of the paper.

2.9. Theorem. *Suppose that A is a balanced $A(m)$-algebra and M a balanced A-bimodule. Then there exists a natural decomposition*

$$H^*_{(m)}(A;M) = \prod_{j\geq 0} H^{*,j}_{(m)}(A;M)$$

such that the natural transformation $\Pi^*(A;M)$ coincides with the map

$$HB^*_{(m)}(A;M) = H^{*,1}_{(m)}(A;M) \hookrightarrow \prod_{j\geq 0} H^{*,j}_{(m)}(A;M) = H^*_{(m)}(A;M).$$

Consequently, the map $\Pi^*(A;M)$ is a monomorphism. □

2.10. Since the differential ∂ is not homogeneous with respect to the 'simplicial' degree q in $\mathcal{B}_{(m)}(A) = \bigoplus_{0\leq q\leq m} \mathcal{B}^q(A)$, the simplicial degree does not induce a grading of the cohomology. Nevertheless, we can use it to define a filtration.

Put $F_iC_{(m)}(A;M) = \{f \in C_{(m)}(A;M) \mid f|_{\mathcal{B}^q(A)} = 0 \text{ for } q \leq i\}$. Then plainly $\delta F_iC_{(m)}(A;M) \subset F_iC_{(m)}(A;M)$, $F_{-1}C_{(m)}(A;M) = C_{(m)}(A;M)$, $\bigcap_i F_iC_{(m)}(A;M)$

$=0$ and $F_iC_{(m)}(A;M)=0$ for $i \geq m$. In a similar way we can obtain also a filtration $F_iCB_{(m)}(A;M)$ of $CB_{(m)}(A;M)$ having the analogous properties. It is also easy to show, using (16), that the map $p: C_{(m)}(A;M) \to F_0C_{(m)}(A;M)$, given by $p(f)|_{\mathcal{B}^q(A)} = f|_{\mathcal{B}^q(A)}$ for $1 \leq q \leq m$ and $p(f)|_{\mathcal{B}^0(A)} = 0$, commutes with the differential and splits the inclusion $F_0C_{(m)}(A;M) \hookrightarrow C_{(m)}(A;M)$, provided M is balanced. The similar result holds also for $CB_{(m)}(A;M)$.

The filtrations above induce the filtrations on $H_{(m)}(A;M)$ and $HB_{(m)}(A;M)$, respectively. The splitting constructed above then shows that, for a balanced A-bimodule M, $F_0H_{(m)}(A;M) \cong H(F_0C_{(m)}(A;M), \delta)$, similarly for $HB_{(m)}(A;M)$. Moreover, the constructions of 2.8 are clearly compatible with our filtrations, therefore the inclusion $CB_{(m)}(A;M) \hookrightarrow C_{(m)}(A;M)$ induces, for each i, *a monomorphism*

$$HF_i: H(F_iCB_{(m)}(A;M), \delta_B) \to H(F_iC_{(m)}(A;M), \delta).$$

3. Examples and applications

3.1. Cohomology of algebras. Let A be an (augmented) algebra and M a (graded) bimodule over A. Then A can be considered as an A(∞)-algebra (Example 1.5) and also M can be converted into an element of A-bi**Mod** taking $\mu_k = 0$, $k \neq 2$, and $\mu_2(a,m) = a \cdot m$, $\mu_2(m,b) = m \cdot b$ for $a,b \in A$ and $m \in M$ (Example 1.11).

Denote $C^{n,p}_{(\infty)}(A;M) = \mathrm{Hom}^{-p+1}_{A\text{-}\mathbf{biMod}}(\mathcal{B}^n(A), M)$. Then $C^{n,p}_{(\infty)}(A;M)$ can be naturally identified with a subspace of $C^p_{(\infty)}(A;M)$; under this identification $C^p_{(\infty)}(A;M) = \prod_{n \geq 0} C^{n,p}_{(\infty)}(A;M)$ and $\delta C^{n,p}_{(\infty)}(A;M) \subset C^{n+1,p+1}_{(\infty)}(A;M)$, the last inclusion being a consequence of (16).

Let us recall briefly the notion of (two-sided, normalized) bar-resolution $\mathcal{B}(A,A)$ of the algebra A; we use the sign and degree conventions of [11, Chapter X, Section 10]. For $n \geq 0$, let $\mathcal{B}^n(A,A)$ be the free graded A-bimodule on $\bigotimes^n \bar{A}$ (in the usual sense, see the discussion in Example 1.11); the element $a_1 \otimes \cdots \otimes a_n \in \bigotimes^n \bar{A}$ being graded by $\deg(a_1 \otimes \cdots \otimes a_n) = |a_1| + \cdots + |a_n| + n$. The differential ∂_{A-A} is then a map of A-bimodules defined by

$$\partial_{A-A}[a_1, \ldots, a_n] = a_1[a_2, \ldots, a_n] + (-1)^{e_n}[a_1, \ldots, a_{n-1}]a_n + \sum_{i=1}^{n-1} (-1)^{e_i}[a_1, \ldots, a_i a_{i+1}, \ldots, a_n],$$

where $e_i = |a_1| + \cdots + |a_i| + i$ and $[a_1, \ldots, a_n]$ is an abbreviation for $a_1 \otimes \cdots \otimes a_n \in \bigotimes^n \bar{A}$. Let $C^{n,p}_{\mathrm{Hoch}}(A;M) = \mathrm{Hom}^p_{\mathbf{Mod}_{A-A}}(\mathcal{B}^n(A,A), M)$ and $\delta_{\mathrm{Hoch}} = \mathrm{Hom}_{\mathbf{Mod}_{A-A}}(\partial_{A-A}, M)$. The *Hochschild cohomology* of A with coefficients in M is then defined as $\mathrm{Hoch}^{n,p}(A;M) = H^{n,p}(C^{*,*}_{\mathrm{Hoch}}(A;M), \delta_{\mathrm{Hoch}})$. For any p and $n \geq 0$ define the linear map $\omega: C^{n,p}_{\mathrm{Hoch}}(A;M) \to C^{n,n+p}_{(\infty)}(A;M)$ by

$$(\omega f)([a_1,\ldots,a_n]) = (-1)^{n|a_1|+(n-1)|a_2|+\cdots+|a_n|+n(n-1)/2}\cdot f([a_1,\ldots,a_n]).$$

It is easy to verify that the map ω commutes with the differentials, hence it induces the isomorphisms

$$\omega^{*,*}(A;M): H^{*,*}(C^{*,*}_{(\infty)}(A;M),\delta)\xrightarrow{\cong}\mathrm{Hoch}^{*,-*}(A;M)$$

and

$$\omega^{*}(A;M): H^{*}_{(\infty)}(A;M)\xrightarrow{\cong}\prod_{n\geq 0}\mathrm{Hoch}^{n,-*}(A;M).$$

Suppose that the algebra A is commutative (i.e. balanced as an A(∞)-algebra; see Example 1.5) and let M be a *left* A-module. Then there exists on M a natural structure of an A-bimodule with $m\cdot a=(-1)^{|a|\cdot|m|}\cdot a\cdot m$ for $a\in A$ and $m\in M$ (compare Proposition 1.13 and the remark following it). It makes sense to denote $CB^{n,p}_{(\infty)}(A;M)=\mathrm{Hom}^{-p+1}_{A\text{-biMod}_B}(\mathscr{B}B^n(A)/\mathrm{Sh}^n(A),M)\subset CB^{p}_{(\infty)}(A;M)$. Again δ_B restricts to a differential (denoted by the same symbol) $\delta_B: CB^{n,p}_{(\infty)}(A;M)\to CB^{n+1,p+1}_{(\infty)}(A;M)$. If we denote by $\mathrm{Harr}^{i,j}(A;M)$ the Harrison cohomology of A with coefficients in M [2, 25], where i is the simplicial degree and j is the total degree, respectively, it can be easily verified that the map ω constructed above gives rise to the identifications

$$\omega^{*,*}_B(A;M): H^{*,*}(CB^{*,*}_{(\infty)}(A;M),\delta_B)\xrightarrow{\cong}\mathrm{Harr}^{*,-*}(A;M),$$

$$\omega^{*}_B(A;M): HB^{*}_{(\infty)}(A;M)\xrightarrow{\cong}\prod_{n\geq 0}\mathrm{Harr}^{n,-*}(A;M),$$

these identifications being compatible with the natural maps $\phi^{*,*}(A;M)$: $\mathrm{Harr}^{*,*}(A;M)\to\mathrm{Hoch}^{*,*}(A;M)$ [2, p. 314] and $\Pi^*(A;M): HB^*_{(\infty)}(A:M)\to H^*_{(\infty)}(A;M)$ (Theorem 2.9).

3.2. Cohomology of algebras of derivations. In this paragraph we refer to the notation introduced in Example 1.9. So, let $V=V_*$ be a (graded) vector space of finite type and $m\geq 1$ a natural number, the case $m=\infty$ being especially important. Consider an object (TV,∂), where $\partial=\partial_1+\partial_2+\cdots+\partial_m$ is a derivation of degree -1 and $\pi_{\leq m}\circ\partial^2=0$. Let $\mathrm{Der}^*(TV)$ denote the graded Lie algebra of derivations of the algebra TV (see [24, 0.2.(4)]) and let $\mathrm{Der}^*_{(m)}(TV)=\mathrm{Der}^*(T)/\sim$, where $\theta'\sim\theta''$ if and only if $\pi_{\leq m}\circ\theta'=\pi_{\leq m}\circ\theta''$. For $\theta\in\mathrm{Der}^*(TV)$ let $\{\theta\}$ denote the corresponding class in $\mathrm{Der}^*_{(m)}(TV)$. The formula $\Delta(\{\theta\})=\{[\partial;\theta]\}$ then defines on $\mathrm{Der}^*_{(m)}(TV)$ a differential of degree -1. Moreover, let $\widetilde{\mathrm{Der}}^*_{(m)}(LV)=\{\{\theta\}\in\mathrm{Der}^*_{(m)}(TV)\mid\theta(V)\subset L(V)\oplus\mathbf{k}\}$. If $\partial(V)\subset L(V)$, then Δ restricts to a differential (denoted again by Δ) on $\widetilde{\mathrm{Der}}^*_{(m)}(LV)$.

Let A be the A(m)-algebra $\tilde{\Omega}^{-1}(TV, \delta)$; recall that $V = \downarrow \# \hat{\bar{A}}$. Then $\bar{A}$ can be considered in the clear sense as a bimodule over itself and we define a linear isomorphism $\tilde{\Omega} : C^*_{(m)}(A; \bar{A}) \to \mathrm{Der}^{*+1}_{(m)}(TV)$ in the following manner.

Every $f \in C^p_{(m)}(A; \bar{A})$ can be expressed as $\coprod_{n=0}^m f_n$ with $f_n \in \mathrm{Hom}^{-p+1}_{A\text{-}\mathbf{biMod}}(\mathcal{B}^n(A), \bar{A})$, the last object being naturally isomorphic with $\mathrm{Hom}^{p+n}_{\mathbf{Vect}}(\bigotimes^n \bar{A}, \bar{A})$. Then we put $\tilde{\Omega}(f) = (-1)^p\{\theta_0 + \cdots + \theta_m\}$, where $\theta_m|_V = \downarrow^n \# \hat{f}_n \uparrow$ for $0 \le n \le m$. It can be verified immediately that $\tilde{\Omega}$ is an isomorphism of differential spaces, hence it induces an isomorphism $\tilde{\Omega}^*(A) : H^*_{(m)}(A; \bar{A}) \cong H^{*+1}(\mathrm{Der}^*_{(m)}(TV), \Delta)$. If $\partial(V) \subset L(V)$, the algebra A is balanced (see Example 1.9) and it is not hard to verify that $\tilde{\Omega}$ restricts to an isomorphism $\tilde{\mathcal{I}} : CB^*_{(m)}(A; \bar{A}) \to \widetilde{\mathrm{Der}}^{*+1}_{(m)}(LV)$. So, denoting by $I : \widetilde{\mathrm{Der}}^*_{(m)}(LV) \hookrightarrow \mathrm{Der}^*_{(m)}(TV)$ the canonical inclusion, we have the commutative diagram

$$\begin{array}{ccc} H^*_{(m)}(A; \bar{A}) & \xrightarrow[\cong]{\tilde{\Omega}^*(A)} & H^{*+1}(\mathrm{Der}^*_{(m)}(TV), \Delta) \\ {\scriptstyle H^*(A;A)} \big\uparrow & & \big\uparrow {\scriptstyle I^{*+1}(A)} \\ HB^*_{(m)}(A; \bar{A}) & \xrightarrow[\cong]{\tilde{\mathcal{I}}^*(A)} & H^{*+1}(\widetilde{\mathrm{Der}}_{(m)}(LV), \Delta) \end{array}$$

where $\tilde{\mathcal{I}}^*(A)$ is the map induced by $\tilde{\mathcal{I}}$ and $I^*(A)$ is induced by the inclusion I. This diagram, together with Theorem 2.9, imply that the map $I^*(A)$ is a monomorphism.

3.3. The homotopy-bar construction. Let A be an A(m)-algebra and let $\tilde{\mathcal{B}}(A) = \bigoplus_{0 \le n \le m} \bigotimes^n \uparrow \bar{A}$. Define on $\tilde{\mathcal{B}}(A)$ the differential $\tilde{\partial} = \tilde{\partial}_1 + \cdots + \tilde{\partial}_m$, where

$$\begin{aligned} &\tilde{\partial}_k([a_1, \ldots, a_n]) \\ &\quad = \sum_{\lambda=0}^{n-k} (-1)^{k(|a_1| + \cdots + |a_\lambda| + \lambda) + \lambda + n + k} \\ &\qquad \cdot [a_1, \ldots, a_\lambda, \mu_k(a_{\lambda+1}, \ldots, a_{\lambda+k}), a_{\lambda+k+1}, \ldots, a_n] . \end{aligned}$$

Then $(\tilde{\mathcal{B}}(A), \tilde{\partial})$ is the *homotopy-bar* (or *tilde*) *construction* on the A(m)-algebra A, introduced in [22, p. 295] (but we use the sign convention of [18, Définition 3.13]).

As in 1.4 consider the ground field $\mathbf{k}$ as an A-bimodule. Then the natural isomorphism

$$\mathrm{Hom}^*_{A\text{-}\mathbf{biMod}}(\mathcal{B}_{(m)}(A), \mathbf{k}) \cong \mathrm{Hom}^{*-1}_{\mathbf{Vect}}\Big(\bigoplus_{0 \le n \le m} \bigotimes^n \uparrow \bar{A}, \mathbf{k}\Big) ,$$

coming from the universal property of the free A-bimodule $\mathcal{B}_{(m)}(A)$, induces the identification $C^*_{(m)}(A; \mathbf{k}) \cong \mathrm{Hom}^*_{\mathbf{Vect}}(\tilde{\mathcal{B}}(A), \mathbf{k}) \cong (\# \tilde{\mathcal{B}}(A))^*$, compatible with the differentials δ and $\# \tilde{\partial}$, respectively, and we have an isomorphism

$$H^*_{(m)}(A; \mathbf{k}) \cong H^*(\# \tilde{\mathcal{B}}(A), \# \tilde{\partial}) .$$

3.4. Cohomology of the DeRham complex. Recall briefly the notion of the cobar construction over a coalgebra as it is given for example in [1] or [24, 0.6.(2)]. Let $C=(C, \Delta, d)$ be a differential graded (coaugmented) coalgebra, $\deg(d)=+1$ and let $\bar{C}=C/\mathbf{k}$ be the coaugmentation ideal. Denote $F(C)=T(\downarrow\bar{C})$ and equip $F(C)$ with the differential d_F by the formula

$$\begin{aligned} d_F([c_1,\dots,c_n]) &= -\sum_{i=1}^{n}(-1)^{e_{i-1}}\cdot[c_1,\dots,dc_i,\dots,c_n] \\ &\quad+\sum_{i=1}^{n-1}\sum_{\mu}(-1)^{e_{i-1}+|c'_{i\mu}|+1}\cdot[c_1,\dots,c_{i-1},c'_{i\mu},c''_{i\mu},c_{i+1},\dots,c_n]\,, \end{aligned}$$

where we abbreviate $\downarrow c_1\otimes\cdots\otimes\downarrow c_n$ by $[c_1,\dots,c_n]$, $e_i=|c_1|+\cdots+|c_i|+i$ and $\bar{\Delta}(c_i)=\sum_\mu c'_{i\mu}\otimes c''_{i\mu}$ for $1\le i\le n$. Then $(F(C), d_F)$ is called the *cobar construction* on the coalgebra C.

Let A be a graded differential chain algebra, augmented and of finite type. Then the dual $\#A$ is in the evident sense of coaugmented coalgebra, and we can apply the cobar construction to it. Consider the following composite of maps:

$$\begin{aligned} C^{-p}_{(\infty)}(\hat{A};\mathbf{k}) &= \mathrm{Hom}^{p+1}_{\hat{A}\text{-}\mathbf{biMod}}(\mathscr{B}_{(\infty)}(\hat{A}),\mathbf{k}) \xleftarrow[\cong]{\omega} \mathrm{Hom}^{p}_{\mathbf{Vect}}(T\uparrow\bar{\hat{A}},\mathbf{k}) \\ &= (\#T\uparrow\bar{\hat{A}})^{-p} \xleftarrow{\chi} (T\uparrow\#\bar{\hat{A}})^{-p} = T(\downarrow\#\bar{A})^{p}\,, \end{aligned}$$

where ω is an isomorphism defined by

$$\omega(f)([a_1,\dots,a_n]) = (-1)^{n|a_1|+\cdots+|a_n|+n(n-1)/n} f(\uparrow a_1\otimes\cdots\otimes\uparrow a_n)$$

(a similar map is used also in 3.1), χ is the canonical injection (as A is supposed to be of finite type, it is also an isomorphism), the other identifications being clear. It is not hard to verify that the isomorphism $\Xi=\omega\circ\chi: T(\downarrow\#\bar{A})^*\to C^{-*}_{(\infty)}(\hat{A};\mathbf{k})$ commutes with the differentials d_F and δ, respectively, therefore

$$H^{-*}_{(\infty)}(\hat{A};\mathbf{k})\cong H^*(F(\#A), d_F)\,.$$

Supposing that the algebra A is commutative, i.e. balanced as an A(∞)-algebra, it can be shown, using the characterization [20, Theorem 2.2], that Ξ restricts to an isomorphism $(\mathscr{L}_*(A), \partial_{\mathscr{L}})\cong CB^{-*}_{(\infty)}(\hat{A};\mathbf{k})$, where the $\mathscr{L}_*$-functor is defined in [24, I.1.(7)], therefore

$$HB^{-*}_{(\infty)}(\hat{A};\mathbf{k})\cong H^*(\mathscr{L}_*(A), \partial_{\mathscr{L}})\,.$$

Let now S be a simply connected topological space having the rational cohomology of finite type. Let $\mathscr{M}$ be Sullivan minimal model of S (see [24, III.2.(1)]). Then $\mathcal{M}$ is of finite type and it is well known that $H^*(F(\#\mathcal{M}), d_F) \cong H_*(\Omega S; \mathbf{k})$ and that $H^*(\mathcal{L}_*(\mathcal{M}), \partial_{\mathcal{L}}) \cong \pi_*(\Omega S) \otimes \mathbf{k}$ (see [1] and [24, III.3.(7)]). The isomorphisms constructed above then give

$$H_{(\infty)}^{*}(\hat{\mathcal{M}}; \mathbf{k}) \cong H_*(\Omega S; \mathbf{k}) \quad \text{and} \quad HB_{(\infty)}^{-*}(\hat{\mathcal{M}}; \mathbf{k}) \cong \pi_*(\Omega S) \otimes \mathbf{k}\,.$$

Finally, notice that the functors $H_{(\infty)}^{*}(*; \mathbf{k})$ and $HB_{(\infty)}^{*}(*; \mathbf{k})$ map weakly isomorphic algebras into isomorphic objects (use the spectral sequence induced by the filtration of 2.10). We see that in the last formula we can replace the minimal model $\mathcal{M}$ by the algebra $A_{PL}(S)$ of Sullivan–DeRham forms [23, Section 7] or even by the algebra $\mathcal{E}(S)$ of DeRham differential forms in the case when S is a smooth manifold and the field $\mathbf{k}$ contains the field of real numbers $\mathbb{R}$.

3.5. The canonical map $l : H_{\mathrm{Lie}}(L; L) \to \mathrm{Hoch}(\mathcal{U}L, \mathcal{U}L)$. In the classical (non-graded) case the existence of such a map l easily follows from the 'inverse process' as it is described in [3, X.6]. To be more precise, let ${}_E\mathcal{U}L$ denote the universal enveloping algebra $\mathcal{U}L$ considered as a left L-module with the action given by $(\lambda, u) \mapsto \lambda \cdot u - u \cdot \lambda$ for $\lambda \in L$ and $u \in \mathcal{U}L$. Then we have, by [3, p. 277, Theorem 5.1], the identification $H_{\mathrm{Lie}}(L; {}_E\mathcal{U}L) \cong \mathrm{Hoch}(\mathcal{U}L; \mathcal{U}L)$ and we can define l to be the composition $H_{\mathrm{Lie}}(L; L) \to H_{\mathrm{Lie}}(L; {}_E\mathcal{U}L) \cong \mathrm{Hoch}(\mathcal{U}L; \mathcal{U}L)$, where the first map is induced by the inclusion $L \hookrightarrow {}_E\mathcal{U}L$ of left L-modules. The same construction would perhaps work also in the graded case, but we prefer to give a more transparent explicit definition.

To this end, recall briefly the definition of the Lie algebra cohomology as it is used here. For a (graded) Lie algebra L let, for $n \geq 0$, $D^n(L)$ denote the free L-module on $\wedge^n \uparrow L$; the element $\uparrow\lambda_1 \wedge \cdots \wedge \uparrow\lambda_n \in \wedge^n \uparrow L \subset D^n(L)$ will be abbreviated by $\langle \lambda_1, \ldots, \lambda_n \rangle$. The differential ∂_{Lie} of degree -1 is then defined by the formula

$$\begin{aligned}
&\partial_{\mathrm{Lie}}(\langle \lambda_1, \ldots, \lambda_n \rangle) \\
&\quad = \sum_{i=1}^{n} (-1)^{f_{i-1}(|\lambda_i|+1)} \cdot \lambda_i \langle \lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_n \rangle \\
&\qquad + \sum_{1 \leq i < j \leq n} (-1)^{f_{i-1}(|\lambda_i|+1) + f_{j-1}(|\lambda_j|+1) + (|\lambda_i|+1)\cdot|\lambda_j|} \\
&\qquad\qquad \cdot \langle [\lambda_i; \lambda_j], \lambda_1, \ldots, \hat{\lambda}_i, \ldots, \hat{\lambda}_j, \ldots, \lambda_n \rangle\,,
\end{aligned}$$

where $f_i = |\lambda_1| + \cdots + |\lambda_i| = i$, $1 \leq i \leq n$, and $\hat{}$ denotes, as usual, omission. For an L-module $M \in L$-**Mod** let $C_{\mathrm{Lie}}^{n,p}(L; M) = \mathrm{Hom}^p_{L\text{-}\mathbf{Mod}}(D^n(L), M)$ and $\delta_{\mathrm{Lie}} = \mathrm{Hom}_{L\text{-}\mathbf{Mod}}(\partial_{\mathrm{Lie}}, M)$. The cohomology of L with coefficients in M is then defined as

$$H_{\mathrm{Lie}}^{n,p}(L; M) = H^{n,p}(C_{\mathrm{Lie}}^{*,*}(L; M), \delta_{\mathrm{Lie}})\,.$$

Let $U = \mathcal{U}L$ be the universal enveloping algebra of L. It comes together with a morphism $\iota : L \to \mathcal{U}L$ which is, in fact, a monomorphism by the Poincaré–Birkhoff–Witt theorem [19, p. 281]; we will identify L with a graded subspace of $\mathcal{U}L$ via ι.

Consider the Hochschild cochains $C^{*,*}_{\mathrm{Hoch}}(U; U)$ as they are introduced in 3.1 and let

$$\begin{aligned}&\tilde{C}^{n,p}_{\mathrm{Hoch}}(U; U)\\ &\quad = \Big\{ f \in C^{n,p}_{\mathrm{Hoch}}(U; U) \,\Big|\\ &\qquad \sum_{\sigma \in S_n} \varepsilon(\sigma) f([\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}]) \subset L,\ \lambda_1, \dots, \lambda_n \in L \Big\},\end{aligned}$$

where S_n denotes the nth permutation group and the Koszul sign $\varepsilon(\sigma)$ is defined by the equality

$$\langle \lambda_1, \dots, \lambda_n \rangle = \varepsilon(\sigma) \langle \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)} \rangle$$

which has to be satisfied in $\wedge^n \uparrow L$. It can be verified that $\tilde{C}^{*,*}_{\mathrm{Hoch}}(U; U)$ is a δ_{Hoch}-stable subspace of $C^{*,*}_{\mathrm{Hoch}}(U; U)$; denote by J the inclusion. Define also the map $\kappa : \tilde{C}^{*,*}_{\mathrm{Hoch}}(U; U) \to C^{*,*}_{\mathrm{Lie}}(L; L)$ by

$$\kappa(f)(\langle \lambda_1, \dots, \lambda_n \rangle) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \cdot f([\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}])$$

for $f \in \tilde{C}^{n,*}_{\mathrm{Hoch}}(U; U)$ and $\lambda_1, \dots, \lambda_n \in L$. It can be verified by a direct computational argument that κ commutes with the differentials.

Supposing that L is *of finite type and* $L_{\leq 0} = 0$, it is possible to prove (it was done, in fact, in [14, Lemma 2.1], using an equivalent description in terms of algebras of derivations) that κ induces an isomorphism in cohomology and we can define the canonical map $l^{*,*} : H^{*,*}_{\mathrm{Lie}}(L; L) \to \mathrm{Hoch}^{*,*}(U; U)$ as $l = H(J) \circ H(\kappa)^{-1}$.

We aim to show the existence of a balanced A(∞)-algebra A such that there are isomorphisms

$$K^*_B : H^{\geq 1,*}_{\mathrm{Lie}}(L; L) \xrightarrow{\cong} F_0 HB^{-*+2}_{(\infty)}(A; \bar{A})$$

and

$$K^* : \mathrm{Hoch}^{\geq 1,*}(U; U) \xrightarrow{\cong} F_0 H^{-*+2}_{(\infty)}(A; \bar{A}),$$

where F_0 refers to the filtration introduced in 2.10, $H^{\geq 1,*}_{\mathrm{Lie}}(L; L) = \prod_{n \geq 1} H^{n,*}_{\mathrm{Lie}}(L; L)$ and $\mathrm{Hoch}^{\geq 1,*}(U; U) = \prod_{n \geq 1} \mathrm{Hoch}^{n,*}(U; U)$. Moreover, these isomorphism satisfy $K^* \circ l^{\geq 1,*} = \Pi^*(A; \bar{A}) \circ K^*_B$. It is easy to see that Theorem 2.9 implies that $l^{*,*}$ is a monomorphism.

The construction of A is based on various properties of algebras of derivations as they are studied for example in [4], [13], [15] and [21]. Since this paper is not meant as an exposition of methods of the rational homotopy theory, our arguments will be merely sketchy.

First, let $(\wedge V, d_2) = \mathscr{C}^*(L, \partial = 0)$, where the functor $\mathscr{C}^*$ is introduced in [24, I.1.(1)]; recall that $V = \uparrow \# L$. Let $\mathrm{Der}^*(\wedge V)$ denote the Lie algebra of derivations of $\wedge V$, the differential $d_{\wedge V}$ of degree $+1$ is defined by $d_{\wedge V}(\theta) = [d_2; \theta]$. Then $\mathrm{Der}^*_{\geq 1}(\wedge V) = \{\theta \in \mathrm{Der}^*(\wedge V) \mid \theta(V) \subset \bigoplus_{i \geq 1} \wedge^i V\}$ is clearly a $d_{\wedge V}$-stable subspace of $\mathrm{Der}^*(\wedge V)$. It can be shown, using a similar method as in 3.2, that there exists a canonical isomorphisms of differential modules $(C^{\geq 1,*}_{\mathrm{Lie}}(L; L), \delta_{\mathrm{Lie}})$ and $(\mathrm{Der}^{-*+1}_{\geq 1}(\wedge V), d_{\wedge V})$, hence

$$H^{\geq 1,*}_{\mathrm{Lie}}(L; L) \cong H^{-*+1}(\mathrm{Der}^*_{\geq 1}(\wedge V, d_{\wedge V}) .$$

Notice that the existence of isomorphisms of this type is well known (see [15, Introduction]) and that $(\wedge V, d_2)$ is the Sullivan minimal model of the coformal space S with $\pi_*(\Omega S) \otimes \mathbf{k} \cong L_*$. Let $(L(W), \partial_1 + \partial_2) = \mathscr{L}_*(\wedge V, d_2)$, where $\mathscr{L}_*$ is the functor defined in [24, I.1.(7)]; recall that $W = \downarrow \# \overline{\wedge V}$. We can define a map $\chi^* : \mathrm{Der}^*_{\geq 1}(\wedge V) \to \mathrm{Der}^{-*}(LV)$ by $\chi(\theta)|_W = \downarrow \# \bar{\theta} \uparrow$. It is easy to verify that χ commutes with the differentials $d_{\wedge V}$ and $\partial_{LW} = [*; \partial_1 + \partial_2]$, respectively. It is also possible to show, using various properties of the functors $\mathscr{L}_*$ and $\mathscr{C}^*$ ([24, Chapitre I] and [21, Section 3]), that χ induces an isomorphism in cohomology. The differential $\partial_1 + \partial_2$ induces on TW a differential (denoted by the same symbol) and let $A = \tilde{\Omega}^{-1}(TW, \partial_1 + \partial_2)$, where the functor $\tilde{\Omega}^{-1}$ is introduced in Example 1.9. The constructions in 3.2 give rise to an isomorphism $\mathrm{Der}^*(LW), \partial_{LW}) \cong (F_0 CB^{-*+1}_{(\infty)}(A; \bar{A}), \delta_B)$. Note also that $H^*(F_0 CB^*_{(\infty)}(A; \bar{A}), \delta_B) \cong F_0 H^*(CB^*_{(\infty)}(A; \bar{A}), \delta_B)$, see 2.10. Composing all the identifications above, we obtain the desired isomorphism K^*_B. The isomorphism K^* is constructed by the same method, only replacing the functors $\mathscr{L}_*$ and $\mathscr{C}^*$ by the 'dual bar construction' Ω of [7, Appendix].

3.6. Note on the sign and degree conventions. Our definition of an A(m)-algebra relies upon the sign convention explained in 1.1. Having chosen the definition of an A(m)-algebra, we get automatically the signs in the formulas for the differentials on $\mathscr{B}_{(m)}(A)$ and $\mathscr{B}B_{(m)}(A)$, respectively, and, in the light of Theorem 1.8, also in the definition of the shuffle product.

As for the degree conventions of 2.1, the homogeneity of the differential ∂ requires that $\deg([a_1, \ldots, a_q]) = |a_1| + \cdots + |a_q| + q + \text{const.}$, the choice const. $\equiv$ 1 (mod 2) is necessary for the validity of the formulas in the proof of Lemma 2.5. The remaining conventions are chosen so that the gradings on $H^*_{(m)}(A; \mathbf{k})$ and $H^*(\# \tilde{\mathscr{B}}(A), \tilde{\partial})$ (see 3.3) agree.

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