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Deformations of varieties of structure constants

by M. Markl

Introduction.

The aim of this paper is to describe a deformation theory for varieties of structure constants which is well applicable namely to the study of the set of rational homotopy types with given homotopy or cohomology. Our theory, based on the approach of A. Nijenhuis and J. Richardson [NR], is motivated by the notion of "filtered cohomology" introduced by Y. Félix in [F] or [F1].

Similarly as in NR we consider the variety of structure constants as the set of solutions of a deformation equation in suitably defined graded Lie algebra. We show that our theory has all the features of other theories of deformations [NR],[G]. We prove also some results on morphisms of such objects. Our results are valid over an arbitrary field of characteristic zero.

We apply our results to the set of all rational homotopy types with given cohomology or homotopy. We shall see that the related cohomology theories are the Harrison cohomology, studied in the connection with the rational homotopy theory by D. Tanré [T], or the Félix filtered cohomology [F], or the analogical theories in the dual situation. From the general theory we get immediately the famous results on intrinsical formality of Y. Félix and D. Tanré, and also some results on the number of rational homotopy types.

As the application of our mapping theorems we shall study the behaviour of the number of rational homotopy types under products, wedges, attaching of cells and pullbacks.

Our paper is divided into three sections:

I. Bigraded Lie algebras

II. Derivations of a bigraded algebra

III. Applications: Rational homotopy types

Appendix: Homotopy types with given homotopy

I. Bigraded Lie algebras.

I.1. Basic definitions. All objects are considered over a field k of characteristic zero.

Definition I.1. By a *bigraded Lie algebra* we mean a (positively) graded Lie algebra $E^* = \bigoplus_{i \geq 0} E^i$ such that for each $i \geq 0$ a second grading $E^i_* = \bigoplus_{j \geq 0} E^i_j$ is given and $[E^i_j, E^p_q] \subset E^{i+p}_{j+q}$ for each $i,j,p,q \geq 0$. The symbol π_j will allways denote the *projection* from E^*_* onto E^*_j.

Definition I.2. Let G be a group. A *G-structure* (or a *G-action*) on a bigraded Lie algebra E is an object $\mathbb{E}$ of the form $\mathbb{E} = (G, \rho, E, m)$ where:

(i) $\rho : G \to \mathrm{Aut}(E)$ is a left action satisfying $\rho(g)(E^i_j) \subset E^i_{\geq j}$ for each $i,j \geq 0$, we shall write for simplicity $g(e)$ or ge instead of $\rho(g)(e)$,

(ii) $m \in E^1_0$ is a point satisfying $[m,m] = 0$ and $\pi_0(g(m)) = m$ for each $g \in G$,

(iii) there exists a map $\mathcal{K} : E^0_{\geq 1} \to G$ such that the element $\mathcal{K}(e)$ for each $x \in E^*_j$, $e \in E^0_k$, $j \geq 0$, $k \geq 1$, satisfies:

$$\pi_j(\mathcal{K}(e)(x)) = x,\ \pi_i(\mathcal{K}(e)(x)) = 0 \text{ for } j < i < j+k \text{ and } \pi_{j+k}(\mathcal{K}(e)(x)) = [e,x],$$

(iv) the action of the group G is complete in the following sense: For each sequence $0 < n_1 < n_2 < \dots$ (finite or infinite) of integers and for each choice of elements $e_i \in E^0_{n_i}$, $i = 1,2,\dots$ the sequence $\{h^j\} = \{\mathcal{K}(e_j)\mathcal{K}(e_{j-1}) \dots \mathcal{K}(e_1)\}$ converges to a point of G, more precisely, there exists $g \in G$ with $\pi_{p+k}(g(e)) = \pi_{p+k}(h^j(e))$ for each $e \in E^*_p$, $k < n_{j+1}$, $j \geq 1$ and $p \geq 0$.

Note that if the sequence $n_1, n_2, \dots$ is finite then (iv) is allways satisfied. Indeed, if n_N is the last element of this sequence then $g = h^N$ has the requisite properties by (iii). The object $\mathbb{E} = (G, \rho, E, m)$ of the previous definition will be called sometimes also simply the bigraded Lie algebra. The algebras of derivations of a bigraded algebra are natural examples of such objects, see the following part.

Definition I.3. Let $\mathbb{E} = (G, \varrho, E, m)$ be a bigraded Lie algebra. Denote by M_m the set

$$M_m = \{m+m_1+m_2+\ldots \in \prod_{j\geq 0} E^1_j;\ m_j \in E^1_j,\ [m+m_1+m_2+\ldots, m+m_1+m_2+\ldots] = 0\}.$$

Because our Lie algebra is supposed to satisfy I.2(ii), the action of the group G defines the (left) action of G on the set M_m. We denote by $\#(M_m/G)$ the number (possibly infinite) of elements of the orbit space M_m/G. The equation $[m+m_1+m_2+\ldots, m+m_1+m_2+\ldots] = 0$ plays here the role of the deformation equation in the sense of [NR].

Writting, for $x \in \prod_{j\geq 0} E^1_j$, $x = x_0+x_1+\ldots$ we allways mean that $x_i \in E^1_i$, $i \geq 0$, are the homogeneous components of the point x.

I.4. Let again $\mathbb{E} = (G, \varrho, E, m)$. The map $D_m : E \to E$ defined by $D_m(e) = [m,e]$ is clearly a homogeneous differential of bidegree (1,0). Thus the cohomology of the complex (E, D_m) is a bigraded space and we denote it by $H^*_*(E, D_m)$.

Definition I.5. Let $\overline{\mathbb{E}} = (\overline{G}, \overline{\varrho}, \overline{E}, \overline{m})$ be another bigraded Lie algebra. A morphism from $\overline{\mathbb{E}} = (\overline{G}, \overline{\varrho}, \overline{E}, \overline{m})$ to $\mathbb{E} = (G, \varrho, E, m)$ is a couple $\mathbb{P} = (\phi, P)$ where $P : \overline{E} \to E$ is a morphism of bigraded Lie algebras, homogeneous of bidegree (0,0), $\phi : \overline{G} \to G$ is a homomorphism of groups, $P(\overline{m}) = m$ and $P(\overline{g}(\overline{e})) = \phi(\overline{g})P(\overline{e})$ for $\overline{g} \in \overline{G}$ and $\overline{e} \in \overline{E}$. The objects, related by the algebra $\overline{\mathbb{E}}$, we indicate by $\overline{\ }$.

Note that the map P clearly induces the map from $\prod_{j\geq 0} \overline{E}_j$ to $\prod_{j\geq 0} E_j$, we abbreviate this map again by P. Clearly $P(\overline{M}_{\overline{m}}) \subset M_m$.

The map P also induces the morphisms of the cohomology. The kernel of P is a bigraded subspace, stable under the differential $\overline{D}_{\overline{m}}$.

Theorem I.6. Let $\mathbb{P} = (\phi, P)$ be a morphism of bigraded Lie algebras, $\mathbb{P} : \overline{\mathbb{E}} \longrightarrow \mathbb{E}$, such that the map $P : \overline{E} \to E$ is epic. Let $K^*_* = \mathrm{Ker}(P : \overline{E}^*_* \longrightarrow E^*_*)$ and denote by ∂ the differential induced on K by $\overline{D}_{\overline{m}}$.

If $H^2_{\geq 1}(K, \partial) = 0$, then $P(\overline{M}_{\overline{m}}) = M_m$, hence $\#(\overline{M}_{\overline{m}}/\overline{G}) \geq \#(M_m/G)$.

Proof. Let $x = m+m_1+m_2+\ldots$ be a point in M_m. Because P is epic, there exists a point $\overline{x} = \overline{m}+\overline{m}_1+\overline{m}_2+\ldots \in \prod_{j\geq 0} \overline{E}^1_j$ such that $P(\overline{x}) = x$. Computing explicitely the projection $\pi_k([\overline{x}, \overline{x}])$ we see that $\overline{x} \in \overline{M}_{\overline{m}}$ if and only if the equation

$$2[\bar{m},\bar{m}_k] = -(1/2)\sum_{i=1}^{k-1}[\bar{m}_i,\bar{m}_{k-i}] \qquad (\mathrm{def}_k)$$

is satisfied for each $k \geqslant 1$.

Suppose that there exists $p \geqslant 1$ such that (def_k) holds for all $k \leqslant p-1$. We claim that the point $\pi_p([\bar{x},\bar{x}])$ belongs to $\mathrm{Ker}(\vartheta)$. By the Jacobi identity $[\bar{x},[\bar{x},\bar{x}]] = 0$, hence also $\pi_p([\bar{x},[\bar{x},\bar{x}]]) = 0$. But $\pi_k([\bar{x},\bar{x}]) = 0$ for $k \leqslant p-1$, hence $\pi_p([\bar{x},[\bar{x},\bar{x}]]) = [\bar{m}, \pi_p([\bar{x},\bar{x}])] = \vartheta(\pi_p([\bar{x},\bar{x}]))$. By our assumption, $H^2_p(K,\vartheta) = 0$, hence there exists $y \in K^1_p$ with $-\vartheta(y) = -[\bar{m},y] = \pi_p([x,x]) = 2[m,m_p] + \sum_{i=1}^{k-1}[m_i,m_{k-i}]$. Then the point $\bar{x}' = \bar{m}+\bar{m}_1+\ldots+\bar{m}_{p-1}+(\bar{m}_p+y)+\ldots$ satisfies clearly (def_k) for all $k \leqslant p$. Because $y \in \mathrm{Ker}(P)$, $P(\bar{x}) = P(\bar{x}') = x$. Using this argument inductively, we modify the point $\bar{x}$ to some $\bar{x}^+ \in \bar{M}_m$ with $P(\bar{x}^+) = x$.

<u>Definition I.7</u>. A G-structure $\mathbb{E} = (G,\varrho,E,m)$ is said to be <u>regular</u>, if for each $k \geqslant 1$ and $g \in G$ with $\pi_i(gm) = 0$ for $0 < i < k$, there exists some $e \in E^0_k$ such that $\pi_i(gm) = \pi_i(\kappa(e)m)$ for $0 \leqslant i \leqslant k$.

<u>Definition I.8</u>. A bigraded Lie algebra $\bar{\mathbb{E}} = (\bar{G},\bar{\varrho},\bar{E},\bar{m})$ is a <u>subalgebra of</u> $\mathbb{E} = (G,\varrho,E,m)$ if $\bar{G}$ is a subgroup of the group G, $\bar{E}$ is a bigraded subalgebra of the algebra E, $\bar{\varrho} = \varrho|\bar{G}$ and $\bar{m} = m$.

Note that the differential D_m induces on $J^*_* = E^*_*/\bar{E}^*_*$ a differential of bidegree (1,0). The inclusion $\bar{M}_m \hookrightarrow M_m$ clearly induces the natural map $\iota : \bar{M}_m/\bar{G} \longrightarrow M_m/G$.

<u>Theorem I.9</u>. Suppose that $\bar{\mathbb{E}} = (\bar{G},\bar{\varrho},\bar{E},\bar{m})$ is a bigraded subalgebra of $\mathbb{E} = (G,\varrho,E,m)$. Denote by ω the differential induced by D_m on the bigraded space $J^*_* = E^*_*/\bar{E}^*_*$.

a) If $H^1_{\geqslant 1}(J,\omega) = 0$, then the natural map $\iota : \bar{M}_m/\bar{G} \to M_m/G$ is epic, consequently $\#(\bar{M}_m/\bar{G}) \geqslant \#(M_m/G)$.

b) Suppose that the G-structure on $\mathbb{E}$ is regular. If there exists $k > 0$ with $H^1_{\leqslant k}(\bar{E},\bar{D}_m) = 0$, $H^1_k(E,D_m) \neq 0$ and $H^2_{>k}(E,D_m) = 0$, then the map ι is not an epimorphism.

Applying the previous theorem to the case $\bar{\mathbb{E}}$ trivial, we obtain the following "rigidity" theorem.

Theorem I.10. If $H^1_{\geq 1}(E,D_m) = 0$, then $\#(M_m/G) = 1$. If the G-structure on E is regular, $H^2_{\geq 2}(E,D_m) = 0$ and $\#(M_m/G) = 1$, then $H^1_{\geq 1}(E,D_m) = 0$.

Proof of Theorem I.9.a): Let $x = m+m_1+m_2+\ldots \in M_m$ and suppose that $x \notin \prod_{j\geq 0} \bar{E}^1_j$. Then there exists $k_1 > 1$ such that $m_1, m_2, \ldots, m_{k_1-1} \in \bar{E}$ but $m_{k_1} \notin \bar{E}$. We claim that $D_m(m_{k_1}) \in \bar{E}$, hence m_{k_1} represents a cycle in $J^1_{k_1}$. Indeed, since $x \in M_m$ we have $[x,x] = 0$ and also $\pi_{k_1}([x,x]) = 0$. But $\pi_{k_1}([x,x]) = 2[m,m_{k_1}] + \sum_{i=1}^{k_1-1}[m_i,m_{k_1-i}]$, hence $D_m(m_{k_1}) = -(1/2)\sum_{i=1}^{k_1-1}[m_i,m_{k_1-i}] \in \bar{E}$.

Because $H^1_{k_1}(J,\omega) = 0$, there exists a point $e_1 \in E^0_{k_1}$ with $-[e_1,m] = m_{k_1}+\bar{z}$ for some $\bar{z} \in \bar{E}^1_{k_1}$. Then $\mathcal{K}(e_1)(m+m_1+\ldots) = m+m_1+\ldots+m_{k_1-1}+ +(m_{k_1}+[e_1,m])+\ldots = m+m_1+\ldots+m_{k_1-1}+\bar{z}+\ldots$. If we write $\mathcal{K}(e_1)(x) = m+m_1'+m_2'+\ldots$ then $m_i' \in \bar{E}$ for $i \leq k_1$.

If $\mathcal{K}(e_1)(x) \notin \prod_{j\geq 0} \bar{E}^1_j$ then there exists $k_2 > k_1$ with $m_1', \ldots, m_{k_2-1}' \in \bar{E}$ but $m_{k_2}' \notin \bar{E}$ ....

Repeating the process above sufficiently many times we get a (possibly finite) sequence $k_1 < k_2 < \ldots$ of integers and a sequence $e_i \in E^1_{k_i}$ of elements such that $\pi_j((\mathcal{K}(e_n)\mathcal{K}(e_{n-1})\ldots\mathcal{K}(e_1)(x)) \in \bar{E}$ whenever $j < k_{n+1}$. Then I.2(iv) gives $g \in G$ with $g(x) \in \bar{E}$.

Proof of Theorem I.9.b): Let $m_k \in \mathrm{Ker}(D_m : E^1_k \to E^2_k)$. At first, we show that there are $m_{k+1}, m_{k+2}, \ldots$ such that $m+m_k+m_{k+1}+m_{k+2}+\ldots \in M_m$. Suppose that we have finden $m_{k+1}, \ldots, m_p$ such that $\pi_i([m+m_k+\ldots+m_p, m+m_k+\ldots+m_p]) = 0$ for $0 \leq i \leq p$. The Jacobi identity gives, for each $a \in E^1$, $[a,[a,a]] = 0$, and also $\pi_{p+1}([a,[a,a]]) = 0$. Writting $a = m+m_k+\ldots+m_p$ and using the assumption $\pi_i([a,a]) = 0$ for $0 \leq i \leq p$ we get that $\pi_{p+1}([m+m_k+\ldots+m_p, m+m_k+\ldots+m_p]) \in \mathrm{Ker}(D_m : E^2_{p+1} \to E^3_{p+1})$ exactly as in the proof of Theorem I.6. Because $H^2_{p+1}(E,D_m) = 0$, there exists $m_{p+1} \in E^1_{p+1}$ with $-2D_m(m_{k+1}) = -2[m,m_{k+1}] = \pi_{p+1}([m+m_k+\ldots+m_p, m+m_k+\ldots+m_p])$ and we see that the point $x = m+m_k+\ldots \ldots m_p+m_{p+1}$ satisfies $\pi_i([x,x]) = 0$ for all i, $0 \leq i \leq p+1$. We proceed using the clear induction.

Because $H^1_k(E,D_m) \neq 0$ there exists $x = m+m_k+m_{k+1}+\ldots \in M_m$ such that m_k represents a nontrivial element $[m_k]$ of the group $H^1_k(E,D_m)$. Using the similar arguments as in the proof of Theorem I.6 we see that the condition $H^1_{\leqslant k}(\bar{E},\bar{D}_{\bar{m}})=0$ implies that each point $\bar{x} \in \bar{M}_{\bar{m}}$ is $\bar{G}$-equivalent with a point $\bar{y}$ of the form $\bar{y} = \bar{m}+\bar{m}_{k+1}+\bar{m}_{k+2}+\ldots$. We conclude that if the point x is G-equivalent with some $\bar{x} \in \bar{M}_{\bar{m}}$, it must be equivalent also with some $\bar{y}$ of the form $\bar{y} = \bar{m}+\bar{m}_{k+1}+\bar{m}_{k+2}+\ldots$ This means that there exists $g \in G$ with $g(\bar{m}+\bar{m}_{k+1}+\bar{m}_{k+2}+\ldots) = m+m_k+\ldots$, hence $\pi_i(gm) = \pi_i(m+m_k)$ for $0 \leqslant i \leqslant k$. By the regularity there exists $e \in E^0_k$ with $[e,m] = m_k = D_m(e)$ and $[m_k] = 0$ in $H^1_k(E,D_m)$-a contradiction.

<u>Theorem I.11</u>. Let $\bar{\mathbb{E}} = (\bar{G},\bar{\varrho},\bar{E},\bar{m})$ be a subalgebra of $\mathbb{E} = (G,\varrho,E,m)$. Suppose that the G-structure on E is regular and that the inclusion $\bar{E} \subsetneq E$ induces a monomorphism $H^1_{\geqslant 1}(\bar{E},\bar{D}_{\bar{m}}) \longrightarrow H^1_{\geqslant 1}(E,D_m)$. Then $\#(M_m/G) = 1$ implies that $\#(\bar{M}_{\bar{m}}/\bar{G}) = 1$.

<u>Proof</u>. Suppose that $\#(\bar{M}_{\bar{m}}/\bar{G}) > 1$. By the arguments using in the proofs of the previous theorems we see that there exists $\bar{x} \in \bar{M}_{\bar{m}}$, $\bar{x} = \bar{m}+\bar{m}_k+\bar{m}_{k+1}+\ldots$ with $[\bar{m}_k] \neq 0$ in $H^1_k(\bar{E},\bar{D}_{\bar{m}})$. By our assumption, $[\bar{m}_k]$ considering as an element of $H^1_k(E,D_m)$ is nonzero, too. Because $\#(M_m/G) = 1$ there exists $g \in G$ with $gm = m+m_k+m_{k+1}+\ldots$. The regularity now guarantees the existence of $e \in E^0_k$ with $[e,m] = m_k$, hence $[m_k] = 0$ in $H^1_k(E,D_m)$-a contradiction.

II. Derivations of a bigraded algebra.

II.1. Let $A^*_* = \bigoplus_{i,j\in \mathbb{Z}} A^i_j$ be a bigraded algebra (i.e. $A^i_j . A^p_q \subset A^{i+p}_{j+q}$ for $i,j,p,q \in \mathbb{Z}$) which is either graded commutative or graded Lie algebra with respect to the upper grading. Suppose that the following condition is satisfied:

(bound) The set $\{i \in \mathbb{Z};\ A^n_i \neq \{0\}\}$ is finite for each $n \in \mathbb{Z}$.

II.2. Denote by $Der^i(A)$ the vector space of all derivations of the graded algebra A^* of degree i and let $Der^i_j(A) = \{\Theta \in Der^i(A);\ \Theta(A_k) \subset \subset A_{k+j}$ for each $k \in \mathbb{Z}\}$. Let $d \in Der^1_1(A)$ be a derivation satisfying $[d,d] = = d^2 = 0$. Define the bigraded vector space $E^*_* = E^*_*(A,d)$ by:

$$E^k_j = Der^k_{j+k}(A) \text{ for } k>0, j\geq 0 \text{ or } k=0 \text{ and } j>0,$$
$$E^0_0 = \{\Theta \in Der^0_0(A);\ [d,\Theta] = 0\},$$
$$E^i_j = \{0\} \text{ otherwise.}$$

This vector space with the product defined by the commutator of derivations clearly forms a bigraded Lie algebra in the sense of Definition I.1.

II.3. For $g \in Aut(A)$ and $k \in \mathbb{Z}$ denote by $F_k(g)$ the linear endomorphism of A defined by $F_k(g)(y) = P_{k+i}(g(y))$ for $y \in A_i$. Finally, define $G = G(A,d) = \{g \in Aut(A);\ g(A^i_j) \subset A^i_{\geq j}$ and $F_0(g)\circ d = d\circ F_0(g)\}$. It is possible to show that G is really a subgroup of the group Aut(A), this need not be true if the condition (bound) is not satisfied.

<u>Remark</u>. If we denote $Der^i_+(A) = \{\Theta \in Der^i(A);\ \Theta(A^i_j) \subset A^i_{\geq j}\}$ thenthere are the natural inclusions $\bigoplus_{j\geq 0} Der^i_j(A) \hookrightarrow Der^i_+(A) \hookrightarrow \prod_{j\geq 0} Der^i_j(A)$ (i arbitrary fixed). On the other hand, if the condition (bound) is satisfied then each element of $\prod_{j\geq 0} Der^i_j(A)$ defines a derivation, therefore also the points of the set M_d defined by the bigraded algebra $E^*_*(A,d)$ as in Definition I.3 can be considered as derivations of the algebra A.

II.4. Let $i \geq 1$ and let $\Theta \in E^0_i = Der^0_i(A)$. The condition (bound) guarrantees that, for each $x \in A$, $\Theta^n(x) = 0$ for n sufficiently large. Hence the sum

$$\exp(\Theta)(x) = \sum_{n\geq 0} (1/n!)\, \Theta^n(x)$$

is finite and clearly defines an element of the group G which we denote by $\exp(\Theta)$. Finally, define the left action ϱ of the group G on E by $g(\Delta) = g\circ\Delta\circ g^{-1}$. It can be verified directly that, if we put $\kappa(e) = \exp(e)$, the object $\mathbb{E} = \mathbb{E}(A,d) = (G,\varrho,E,d)$ satisfies (i)-(iii) of Definition I.2.

II.5. We show that the action just defined is complete. If $e_1,e_2,\dots$ are elements of $E^0_{n_1},E^0_{n_2},\dots$ as in I.2(iv) and $h^i = \kappa(e_i)\kappa(e_{i-1})\dots\kappa(e_1)$, then clearly for each $x\in A$, $h^m(x) = h^n(x)$ for m and n sufficiently large because of (bound) and it makes sense to speak about $\lim_{n\to\infty} h^n(x)$. It is not hard to verify that the equation $g(x) = \lim_{n\to\infty} h^n(x)$ defines an element of G which satisfies the condition (comp).

Before proving that the action is also regular we prove the following

<u>Sublemma II.6</u>. Let $g = F_0(g)+F_1(g)+F_2(g)+\dots\in G$ (because of (bound) this sum makes sense althrough it may be infinite). Then

a) $F_0(g)\in G$,

b) if $F_0(g) = \mathrm{id}$, $F_1(g) = F_2(g) = \dots = F_{k-1}(g) = 0$, then $F_k(g)\in \mathrm{Der}^0_k(A)$.

<u>The proof</u> is trivial.

Recall that, under the abbreviation of I.2, $\pi_j(g(d))\in E^1_j$ denotes the "j-th homogeneous part" of the derivation $g(d) = g\circ d\circ g^{-1}$.

<u>Sublemma II.7</u>. If $j>0$ and $\Theta\in E^0_j = \mathrm{Der}^0_j(A)$, then

$$\pi_{nj}(\exp(-\Theta)\circ d\circ\exp(\Theta)) = (1/n!)[[\dots[d,\underbrace{\Theta],\dots,\Theta],\Theta}_{n\ \text{times}}],\qquad n\geq 1,$$

$$\pi_k(\exp(-\Theta)\circ d\circ\exp(\Theta)) = 0 \text{ for } k\not\equiv 0 \bmod j.$$

Especially, if $[d,\Theta] = D_d(\Theta) = 0$, then $\exp(-\Theta)\circ d\circ\exp(\Theta) = d$.

<u>The proof</u> is again a direct computation and we omit it.

II.8. Let us prove the regularity. Let $g\in G$ be such that, for some k, $\pi_i(g(d)) = 0$, $0<i<k$. We write again $g = F_0(g)+F_1(g)+\dots$. By Sublemma II.6, $F_0(g)\in G$ and $F_0^{-1}(g)(d) = d$ by the definition of the group G. If we denote $g' = gF_0^{-1}(g)$ then clearly $g'(d) = g(d)$ and $F_0(g') = \mathrm{id}$. Hence we may suppose that $F_0(g) = \mathrm{id}$.

In this case $g = \mathrm{id}+F_1(g)+\dots$ and $F_1(g)\in E^0_1$ by Sublemma II.6. Clearly

$\pi_1(g(d)) = [d,F_1(g)]$, hence $[d,F_1(g)] = 0$. If we write $g' = g\cdot\exp(-F_1(g))$ then $g(d) = g'(d)$ by sublemma II.7 and clearly $F_1(g') = 0$. Repeating this process sufficiently many times we find $h\in G$ with $h(d) = g(d)$ and $F_o(h) = = id$, $F_1(h) = \dots = F_{k-1}(h) = 0$. By Sublemma II.6, $F_k(h)\in E_k^0$ and clearly $\pi_k(h(d)) = [F_k(h),d] = \pi_k(\mathcal{K}(F_k(h))(d))$. The regularity is proved.

II.9. The free-algebra case. For a $\mathbb{Z}$-graded vector space Z^* we denote by $F^*(Z)$ either the free commutative graded algebra or the free graded Lie algebra on Z. If Z^* has another "lower" grading $Z^i_* = \bigoplus_{j\in\mathbb{Z}} Z^i_j$ for $i\in\mathbb{Z}$, then there is the natural "lower" grading on F(Z); suppose that the grading on Z is such that $F^*_*(Z)$ satisfies the condition (bound) of II.1. We denote by $^+F(Z)$ the linear subspace of F(Z) spanned by all elements of positive lenght. Note that there is one-to-one correspondence between elements of Der(F(Z)) and the linear maps from Z to F(Z).

II.10. Suppose that $Z^*_* = X^*_* \oplus Y^*_*$ as bigraded spaces. Then there are the canonical injections $F(X)\hookrightarrow F(Z)$ and $F(Y)\hookrightarrow F(Z)$ and we shall consider F(X) and F(Y) as subspaces of F(Z). Define the map $\oplus : Der^*_*(F(X)) \oplus Der^*_*(F(Y)) \longrightarrow Der^*_*(F(Z))$ by $(\Omega,\Theta)\mapsto\Omega\oplus\Theta$, where $\Omega\oplus\Theta$ is the derivation of F(Z) defined by $(\Omega\oplus\Theta)(x) = \Omega(x)$ for $x\in F(X)$ and $(\Omega\oplus\Theta)(y) = \Theta(y)$ for $y\in F(Y)$. The map $\times : Aut(F(X))\times Aut(F(Y))\to Aut(F(Z))$ can be defined similarly.

II.11. If $d'\in Der^1_1(F(X))$ and $d''\in Der^1_1(F(Y))$ satisfy $d'^2 = 0$ and $d''^2 = 0$, then clearly $d = d'\oplus d''\in Der^1_1(F(Z))$ satisfies $d^2 = 0$. In the algebra $\mathbb{E} = \mathbb{E}(F(Z)),d) = (G,\rho,E,d)$ the subspace of all derivations Θ with $\Theta(X)\subset F(X)$ and $\Theta(Y)\subset F(Y)$ can be shown to form a bigraded subalgebra $\bar{E}$ of the Lie algebra E. We also denote $\bar{G} = \{g\in G;\ g(X)\subset F(X)$ and $g(Y)\subset F(Y)\}$. Then $d\in\bar{E}^1_1$ and the object $\bar{\mathbb{E}} = (\bar{G},\bar{\rho},\bar{E},\bar{d})$, where we write $\bar{\rho} = \rho|\bar{G}$ and $\bar{d} = d$ forms a bigraded subalgebra of the algebra $\mathbb{E}$ in the sense of Definition I.8.

Proposition II.12. There exists a linear map $\Pi:E^*_*\longrightarrow\bar{E}^*_*$ which is a homogeneous morphism of differential spaces (E,D_d) and $(\bar{E},D_{\bar{d}})$ of bidegree (0,0) such that $\Pi\circ\iota = id$, where $\iota:\bar{E}\hookrightarrow E$ is the inclusion.

Corrolary II.13. The inclusion induces a monomorphism $H^*_*(\overline{E},\overline{D}_{\overline{d}}) \rightarrowtail H^*_*(E,D_d)$.

Proof of the proposition. Let $P_1:F(Z)\to F(X)$ and $P_2:F(Z)\to F(Y)$ be the canonical projections. For $\theta \in Der(F(Z))$ define the linear endomorphism $\Pi_1(\theta)$ of $F(X)$ by $\Pi_1(\theta)(x) = P_1(\theta(x))$ for $x\in F(X)$; the endomorphism $\Pi_2(\theta)$ of $F(Y)$ is defined similarly.

We show that $\Pi_1(\theta)$ is a derivation of $F(X)$. For each $a\in F(Z)$ we can decompose $\theta(a)$ uniquely in the form $P_1(\theta(a))+\theta_+(a)$ where $\theta_+(a)$ belongs to the ideal $\mathcal{J}$ generated by $^+F(X)$ in $F(Z)$. Let $a,b\in F(X)$ and compute $\Pi_1(\theta)(a.b)$. By the definition, $\Pi_1(\theta)(a.b) = P_1(\theta(a.b)) =$
$= P_1(\theta(a).b \pm a.\theta(b)) = P_1((P_1(\theta(a))+\theta_+(a))b \pm a(P_1(\theta(b))+\theta_+(b))) =$
$= P_1(\Pi_1(\theta)(a).b\pm a.\Pi_1(\theta)(b)+$an element of $\mathcal{J}) = \Pi_1(\theta)(a).b \pm a.\Pi_1(\theta)(b)$. The endomorphism $\Pi_2(\theta)$ is, of course, a derivation by the same argument.

We can now define the map $\Pi:E\to\overline{E}$ by $\Pi(\theta) = \Pi_1(\theta)\oplus\Pi_2(\theta)$; the map is clearly a linear endomorphism of bidegree (0,0). Note that it need not be a homomorphism.

We show that our map Π commutes with the differentials, i.e. that, for each $\theta\in Der(F(Z))$, $[\overline{d},\Pi(\theta)]= \Pi[d,\theta]$. For $x\in F(X)$ we have
$\Pi[d,\theta](x) = P_1(d\circ\theta(x) \pm \theta\circ d(x)) = P_1((d'\oplus d'')(P_1(\theta(x))+\theta_+(x)) \pm (P_1(\theta(d'(x)) \oplus \theta_+(d'(x))) = P_1(d'(P_1(\theta(x))+$an element of $\mathcal{J}) \pm (P_1(\theta(d'(x)) \oplus \theta_+(d'(x))) = [\overline{d},\Pi(\theta)](x)$. The same equation is clearly valid also on $F(Y)$ and therefore it is valid on $F(Z)$ as well.

II.14. In I.8 we related to a subalgebra $\overline{\mathbb{E}}$ of a bigraded Lie algebra $\mathbb{E}$ a differential space (J,ω). We give an explicit description of this space in the situation of the previous paragraphs.

For two bigraded spaces P^*_* and Q^*_* we denote by $Hom^p_q(P,Q)$ the space of all homogeneous linear maps from P to Q of bidegree (p,q).

The map Π constructed in Proposition II.12 splits the sequence $0\to\overline{E}\xrightarrow{\iota}E\to J\to 0$, hence J can be identified with $Im((id - \iota\circ\Pi):E\to E)$. Under this identification, the differential ω is induced by the differential D_d. Denote by $\mathcal{J}$ the ideal generated by $^+F(X)$ in $F(Z)$ and by $\mathcal{K}$

the ideal generated by ${}^{+}F(Y)$ in $F(Z)$. The last space clearly consists of all derivations $\theta \in E$ with $\theta(X) \subset \mathcal{K}$ and $\theta(Y) \subset \mathcal{J}$. Hence J^i_j can be naturally identified with the space $\mathrm{Hom}^i_j(X,\mathcal{K}) + \mathrm{Hom}^i_j(Y,\mathcal{J})$.

II.15. Let us denote $\mathbb{E}' = \mathbb{E}(F(X),d') = (G',\rho',E',d')$ and

$$\mathbb{E}'' = \mathbb{E}(F(Y),d'') = (G'',\rho'',E'',d'')$$

(see II.4 for the notation). We can define by the evident way the direct sum $\mathbb{E}' \oplus \mathbb{E}''$ of the algebras $\mathbb{E}'$ and $\mathbb{E}''$,

$$\mathbb{E}' \oplus \mathbb{E}'' = (G' \times G'', \rho' + \rho'', E' \oplus E'', d' \oplus d'')$$

and the map $\mathbb{P} = (\times, \oplus)$, where $\times$ and $\oplus$ are the maps defined in II.10, is clearly an isomorphism of $\mathbb{E}' \oplus \mathbb{E}''$ and $\bar{\mathbb{E}}$. It is not hard to verify that $M_{d'}/G' \times M_{d''}/G'' \cong \bar{M}_{\bar{d}}/\bar{G}$, consequently $\#(M_{d'}/G').\#(M_{d''}/G'') = \#(\bar{M}_{\bar{d}}/\bar{G})$. Theorem I.9.a) gives:

<u>Theorem II.16</u>. If $H^1_{\geq 1}(J,\omega) = 0$, then $\#(M_{d'}/G').\#(M_{d''}/G'') \geq \#(M_d/G)$.

Combining Theorem I.11, Corrolary II.13 and the notes above we obtain the following:

<u>Theorem II.17</u>. If $\#(M_d/G) = 1$, then $\#(M_{d'}/G') = 1$ and $\#(M_{d''}/G'') = 1$.

III. Application: Rational homotopy types.

III.1. From now on all objects are considered over the rationals (of course, that algebraic part of our statements remains valid over an arbitrary field of characteristic zero). The symbols H^*, H'^*... allways denote (positively) graded commutative algebras of finitie type with $H^0 \cong \mathbb{Q}$ and $H^1 = \{0\}$. We denote by $H^*(X)$ the singular cohomology of X with rational coefficients. By $\#(H)$ we denote the number of all rational homotopy types with the cohomology isomorphic to H. For a topological space X let $\#(X) = \#(H(X))$.

III.2. Deformations of the Quillen model. Being H^* an algebra as above, denote $\mathcal{L}_*((H,d=0)) = (\mathbb{L}(W), \partial_2)$ (hence W is the dual to $s^{-1}(H)$). The bigraded space Z_*^* is defined by

$$Z_1^i = W^{-i},\ Z_j^i = \{0\} \text{ for } i,j \in Z,\ j \neq 1.$$

Then clearly $\mathbb{L}(Z)$ satisfy the condition (bound) of II.1. and ∂_2 determines on $\mathbb{L}(Z)$ a differential $d \in Der_1^1(\ \ (Z))$. The object $\mathbb{E} = \mathbb{E}(\mathbb{L}(Z),d) = (G, \mathfrak{g}, E, d)$ constructed as in II.4 we call the Quillen deformation algebra of H^*. In this case, the set M_d is the set of all Quillen minimal algebras with the quadratic part $(\mathbb{L}(W), \partial_2)$ and $\#(M_d/G) = \#(H)$ (see [SL]).

Proposition III.3. There is the natural isomorphism

$$H_j^i(E,D_d) \cong Harr^{i+j+1,i}(H,H),\ i,j \geq 0,$$

where $Harr^{*,*}$ denotes the Harrison cohomology as in [Ta2, p.363] or [Ta3] (the notation in [Ta1] is slightly different).

The fact stated in the previous theorem is not surprising. The Harrison cohomology appear in the connection with the study of Felix-Halperin--Stasheff model [Ta1] and there is one-to-one correspondence between deformations of FHS-models and deformations of Quillen models.

The isomorphism above can be proved directly by some long, but straightforward computation and we omit it. The only nontrivial fact used in the proof is the description of the kernel of the canonical map $(\bigotimes(W))^* \longrightarrow (\mathbb{L}(W))^*$ as the subspace of reducible elements of the mixed product. From Proposition III.3 and Theorem I.10 we obtain the following famous result [Ta2, p.368].

Theorem III.4.(D. Tanré).

a) If $\mathrm{Harr}^{>2,1}(H,H) = 0$ then H is intrinsically formal.

b) If $\mathrm{Harr}^{>4,2}(H,H) = 0$ then H is intrinsically formal if and only if $\mathrm{Harr}^{>2,1}(H,H) = 0$.

We shall write simply $Ha^i_j(H)$ instead of $H^i_j(E(\mathbb{L}(Z),d),D_d) \cong$ $\cong \mathrm{Harr}^{i+j+1,i}(H,H)$.

III.5. Deformations of the Halperin-Stasheff filtered model. Being H^* as above, let $(\wedge X,d_{-1})$ be the Halperin-Stasheff bigraded model of the algebra H, $X = \bigoplus_{i,j\geq 0} X^i_j$. Define the bigraded space Z^*_* by

$$Z^i_j = X^i_{-j} \text{ for } i,j \geq 0,\ Z^i_j = \{0\} \text{ otherwise},$$

and denote by d the differential defined by d_{-1} on $\wedge Z$. Then the object $\mathbb{E} = \mathbb{E}(\wedge Z,d) = (G,\varrho,E,d)$ constructed as in II.4 we call the Halperin-Stasheff (HS) deformation algebra of H . Then the set M_d is the set of all deformations of the bifiltered model $(\wedge X,d_{-1})$ and again $\#(M_d/G) =$ $= \#(H)$. The cohomology of the complex (E^*_*,D_d) is naturally isomorphic with the "filtered cohomology" introduced vy Y. Félix [F1,F2] and we denote ${}_FH^i_j(H) = H^i_j(E(\wedge Z,d),D_d)$. Theorem I.10 gives the following result [F1,F2]:

Theorem III.6.(Y. Félix).

a) If ${}_FH^1_{\geq 1}(H) = 0$ then H is intrinsically formal.

b) If ${}_FH^2_{\geq 2}(H) = 0$ then H is intrinsically formal if and only if ${}_FH^1_{\geq 1}(H) = 0$.

Example III.7. Consider the space $X = S^5_c \vee S^5_d \vee S^{14} \vee S^{23}_a \vee S^{23}_b \cup_\omega e^{28}$, $\omega = [S^{14},S^{14}] + [S^{23}_a,S^5_c] + [S^{23}_b,S^5_d]$ studied by Y. Félix in [F1]. For this space

$${}_FH^1_{\geq 1}(H^*_*(X)) \neq 0 \text{ but } Ha^1_{\geq 1}(H^*_*(X)) = 0.$$

The fact that the filtered cohomology is nonzero is proved in [F1], there is also shown that this space is intrinsically formal. Compute the Harrison cohomology.

Clearly $\mathcal{L}_*((H^*(X),d=0)) = (\mathbb{L}(c,d,x,a,b,z),\partial)$, $\deg(c)=\deg(d)=4$, $\deg(x)=13$, $\deg(a)=\deg(b)=22$, $\deg(z)=27$, $\partial(c)=\partial(d)=\partial(x)=\partial(a)=\partial(b)=0$ and $\partial(z)=[x,x]+[c,a]+[b,d]$. The space $E^1_{\geqslant 1}$ consists of all derivations θ of the form $\theta(c)=\theta(d)=\theta(x)=\theta(z)=0$,

$$\theta(a) = \alpha[x,[c,d]]+\beta[x,[x,c]]+\gamma[d,[x,d]]+\delta[c,[x,d]],$$

$$\theta(b) = \phi[x,[c,d]]+\psi[x,[x,c]]+\mu[d,[x,d]]+\nu[c,[x,d]],$$

where $\alpha,\beta,\gamma,\delta,\phi,\psi,\mu,\nu\in\mathbb{Q}$, hence $E^1_{\geqslant 1}\cong\mathbb{Q}^8$. On the other hand, computing explicitely the basis of the Lie algebra, it is possible to verify that the differential $D_d: E^1_{\geqslant 1}\longrightarrow E^2{}_1$ is a monomorphism, hence $Ha^1_{\geqslant 1}(H^*(X)) = 0$.

Example III.8. Let $n_1,\dots,n_k$ be natural numbers $\geqslant 2$ and denote $X = S^{n_1}\vee\dots\vee S^{n_k}$. If $u_1,\dots,u_k$ are indeterminates, $\deg(u_j) = n_j-1$, $1\leqslant j\leqslant k$ then X is intrinsically formal if and only if $Der^{-1}_{\geqslant 3}(\mathbb{L}(u_1,\dots,u_k)) = \{0\}$. The last condition can be rewrited in the following form (we suppose $n_1\leqslant\dots\leqslant n_k$):

There is no m, $1\leqslant m\leqslant k$, such that $n_m-2 = \sum_1^{m-1} a_j(n_j-1)$ where a_j are integers $\geqslant 0$, $\sum_1^{m-1} a_j\geqslant 3$ and $a_j\neq 0$ for at least two indices j, $1\leqslant j\leqslant m-1$. For k=3 we obtain the result of [F1, Exemple 9 b)].

Let us prove our statement. The algebra $(\mathbb{L}(u_1,\dots,u_k),\partial=0)$ is the Quillen model for the space X, hence $Der^{-1}_{\geqslant 3}(\mathbb{L}(u_1,\dots,u_k)) = \{0\}$ implies $Ha^1_{\geqslant 1}(H^*(X)) = 0$ and X is intrinsically formal by Theorem I.10.

On the other hand, if there exists a nonzero derivation $\theta\in Der^{-1}_{\geqslant 3}(\mathbb{L}(u_1,\dots,u_k))$ with $\theta^2 = 0$, then X is not be intrinsically cally formal (because $\partial = 0$). Suppose $Der^{-1}_{\geqslant 3}(\mathbb{L}(u_1,\dots,u_k)) \neq 0$ and let u_m be an element of the minimal degree in the set $\{u_j;$ there exists $\Omega\in Der^{-1}_{\geqslant 3}(\mathbb{L}(u_1,\dots,u_k))$ with $\Omega(u_j)\neq 0\}$, denote $p = \deg(u_m)$. Then there exists $z\in\mathbb{L}^{p-1}_{\geqslant 3}(u_1,\dots,u_k)$, $z\neq 0$, and define θ by $\theta(u_m) = z$ and $\theta(u_j) = 0$, $j\neq m$. Since $\deg(z) = p-1$, $\theta(z) = 0$ and $\theta^2 = 0$.

Example III.9. The graded space V^* of finite type has the property that each algebra H^* having V^* as its underlying vector space is intrinsically formal if and only if $Der^{-1}_{\geqslant 3}(\mathbb{L}(s^{-1}(V))) = \{0\}$.

The rest of the paper is devoted to the study of the behaviour of the number $\#(*)$ under some canonical operations.

III.10. Wedges. Suppose that the algebra H^* is the "connected" direct sum of H'^* and H''^*, $H^* = H'^* \vee H''^*$ (i.e. $H^0 \cong \mathbb{Q}$ and $H^{\geq 1} = H'^{\geq 1} \oplus H''^{\geq 1}$ with the product defined by the clear way). Denote by $\mathbb{E} = \mathbb{E}(\mathbb{L}(Z),d)$, $\mathbb{E}' = \mathbb{E}(\mathbb{L}(X),d')$ and $\mathbb{E}'' = \mathbb{E}(\mathbb{L}(Y),d'')$ the Quillen deformation algebras of H^*, H'^* and H''^* respectively (see III.2). Because the Quillen model of the algebra $(H,d=0)$ is of the form $(\mathbb{L}(s^{-1}(H' \oplus H'')), \partial' \oplus \partial'')$, we can suppose that $Z_*^* = X_*^* \oplus Y_*^*$ and that $d = d' \oplus d''$. This situation was studied in II.10-II.17 and Theorem II.17 gives:

Theorem III.11. If the algebra $H^* = H'^* \vee H''^*$ is intrinsically formal, then both H'^* and H''^* are intrinsically formal, too.

The previous theorem clearly generalizes to the case of n factors. Because $H^*(X \vee Y) \cong H^*(X) \vee H^*(Y)$ we get immediately

Theorem III.12. Let $X_1,\dots,X_n$ be simply connected spaces having the cohomology of finite type and suppose that $X_1 \vee \dots \vee X_n$ is intrinsically formal. Then each $X_1,\dots,X_n$ is intrinsically formal, too.

The cohomology of the differential space (J_*^*,ω) introduced in I.8 and studied in II.14 clearly depends only on the algebras H'^* and H''^* and we denote it by $Ha_*^*(H^*;H'^*,H''^*)$. The explicit calculation related with this object can be done using II.14. Theorem II.16 gives immediately:

Theorem III.13. If $Ha^1_{\geq 1}(H^*;H'^*,H''^*) = 0$, then $\#(H) \leq \#(H').\#(H\ )$.

Example III.14. It can be shown that there are precisely two rational homotopy types with the cohomology isomorphic to $H^*(S^2 \vee S^2 \vee S^5)$, $\#(S^2 \vee S^2 \vee S^5) = 2$. For $k \geq 2$, $\#(S^2 \vee S^2 \vee S^5 \vee S^k) \geq 2$ and the equality holds if and only if k=3 or 4.

Indeed, writting $H' = H(S^2 \vee S^2 \vee S^5)$ and $H'' = H(S^k)$ we obtain the following description of the space $J^1_{\geq 1}$:

Let x,y,z,a be indeterminates, deg(x)=deg(y)=1, deg(z)=4 and deg(a)=k-1. Then $J^1_{\geq 1}$ consists of all derivations $\theta \in Der^{-1}_{\geq 3}(\mathbb{L}(x,y,z,a))$

with $\Theta(x), \Theta(y) \in \mathbb{L}^0_{\geqslant 3}(a)$, $\Theta(z) \in \mathcal{J}^3_{\geqslant 3}$ and $\Theta(a) \in \mathcal{K}^{k-2}_{\geqslant 3}$, where $\mathcal{J}$ is the ideal generated by ${}^+\mathbb{L}(a)$ in $\mathbb{L}(x,y,z,a)$ and $\mathcal{K}$ is the ideal generated by ${}^+\mathbb{L}(x,y,z)$ in $\mathbb{L}(x,y,z,a)$. We see that for k=3 or 4 (and only in this case) such derivations are zero, hence $J^1_{\geqslant 1} = 0$ and $\#(S^2 \vee S^2 \vee S^5 \vee S^k) = 2$ by Theorem III.13 (our space is not, by Example III.8, intrinsically formal). The fact that $\# > 2$ in other cases can be proved similarly as in Example III.8.

Example III.15. $(S^9 \vee S^9 \vee S^{26} \vee (S^7 \times S^7)) = 2.$

Put $H' = H(S^9 \vee S^9 \vee S^{26})$ and $H'' = H(S^7 \times S^7)$. Then $\mathcal{L}_*(H',d=0) = (\mathbb{L}(x,y,z), \partial'=0)$, $\mathcal{L}_*(H'',d=0) = (\mathbb{L}(a,b,c), \partial''(a)=\partial''(b)=0, \partial''(c) = [a,b])$, $\deg(x)=\deg(y)=8$, $\deg(z)=25$, $\deg(a)=\deg(b)=6$ and $\deg(c)=13$. Then the space $J^1_{\geqslant 1}$ consists of all derivations Θ of the form $\Theta(a)=\Theta(b)=\Theta(x)=\Theta(y)=\Theta(c)=0$, $\Theta(z) = \alpha[a,[a,[a,b]]] + \beta[b,[a,[a,b]]] + \gamma[b,[b,[a,b]]]$, $\alpha, \beta, \gamma \in \mathbb{Q}$. If we define Ω by $\Omega(a)=\Omega(b)=\Omega(x)=\Omega(y)=\Omega(c)=0$, $\Omega(z) = \alpha[a,[a,c]] + \beta[b,[a,c]] + \gamma[b,[b,c]]$, then clearly $[\Omega, \partial' \oplus \partial''] = \Theta$, hence $H^1_{\geqslant 1}(H;H',H'') = 0$ and our statement follows from Theorem III.13.

III.16. Attaching of cells. Let H^* be again 1-connected algebra of finite type and suppose that there exists $n > 0$ with $H^i = \{0\}$ for $i \geqslant n+1$. Let $\bar{H}^*$ be another graded algebra and suppose that there exists a graded vector space N^* with $N^i = \{0\}$ for $i \leqslant n$ such that

- $\bar{H}^* \cong H^* \oplus N^*$ as graded vector spaces,
- whenever $a,b \in H \subset \bar{H}$ are such that $\deg(a)+\deg(b) \leqslant n$, then the product of a and b in $\bar{H}$ is the same as this product in H.

The Quillen models $(\mathbb{L}(W), \partial_2)$ and $(\mathbb{L}(\bar{W}), \bar{\partial}_2)$ of $(H^*,d=0)$ and $(\bar{H}^*,d=0)$ can be choosen such that $\bar{W} = W \oplus V$ ($V \cong (s^{-1}N)^*$) and $\bar{\partial}_2 | W = \partial_2$. If X is a formal homotopy type with $H^*(X) \cong H^*$ then the formal homotopy type corresponding to $\bar{H}^*$ can be obtained attaching to X cells by suitably choosen attaching maps.

Let $\mathbb{E} = \mathbb{E}(\mathbb{L}(Z),d) = (G, \varrho, E, d)$ and $\bar{\mathbb{E}} = \mathbb{E}(\mathbb{L}(\bar{Z}), \bar{d}) = (\bar{G}, \bar{\varrho}, \bar{E}, \bar{d})$ be the Quillen deformation algebras for H^* and $\bar{H}^*$ respectively.

By definition, $Z^i = \{0\}$ for $i \leqslant -n$, $\overline{Z}^* = Z^* \oplus Y^*$ where $Y^i = \{0\}$ for $i \geqslant -n+1$ and $\bar{d}|\mathbb{L}(Z) = d$. We define the maps $P:\overline{E} \to E$ and $\phi:\overline{G} \to G$ as follows: If $\theta \in \overline{E}$ is a derivation, then $P(\theta) = \theta|\mathbb{L}(Z)$. Similarly, let $\phi(g) = g|\mathbb{L}(Z)$. It follows from our assertions that $\theta(\mathbb{L}(Z)) \subset \mathbb{L}(Z)$ for each derivation $\theta \in \overline{E}$. This fact guarrantees that P is a morphism of bigraded Lie algebras. The map ϕ is a homomorphism of groups by the similar argument. Note that the map P is epic. We can easily verify that the couple $\mathbb{P} = (\phi, P)$ is a morphism of bigraded Lie algebras in the sense of Definition I.5.

III.17. The bigraded space $Ker(P:\overline{E} \to E)$ consists of all derivations $\theta \in \overline{E}$ with $\theta(\mathbb{L}(Z)) = 0$. The cohomology of the complex (K^*_*, ∂) introduced in Theorem I.6 clearly depends only on the algebras H^* and $\overline{H}^*$ and we denote it by $Ha^*_*(\overline{H}^*, H^*)$. Applying Theorem I.6 we get the following:

Theorem III.18. If $Ha^2_{\geqslant 1}(\overline{H}^*, H^*) = 0$ then $\#(\overline{H}) \geqslant \#(H)$.

Example III.19. There are infinitely many rational homotopy types with the cohomology isomorphic to $H^*(S^3 \vee S^3 \vee S^{10})$, $\#(S^3 \vee S^3 \vee S^{10}) = \infty$ [SL]. We show that the space $X = (S^3_a \vee S^3_b \vee S^{10}) \cup_\omega e^{13}$, $\omega = [S^3_a, S^{13}]$, is intrinsically formal. Let $\mathbb{E} = (G, \rho, E, d)$ be the Quillen deformation algebra for $H^*(X)$. The Quillen model for X is $(\mathbb{L}(x,y,z,a), \partial_1 a = [x,z])$, $deg(x)= deg(y)=2$, $deg(z)=9$ and $deg(a) = 12$; we see that $E^1_{\geqslant 1}$ consists of all derivations θ with $\theta(x)= \theta(y)= \theta(a)=0$, $\theta(z) = \alpha[x,[x,[x,y]]] + \beta[x,[y,[x,y]]] + \gamma[y,[y,[x,y]]]$, $\alpha, \beta, \gamma \in \mathbb{Q}$. For such a derivation $[\theta, \partial_2](a) = 0$ if and only if $\alpha, \beta, \gamma = 0$, hence $Ha^1_{\geqslant 1}(H^*(X)) = 0$ and $\#(X) = 1$ by Theorem I.10. Note that $dim(Ha^2_{\geqslant 1}(H^*(X), H^*(S^3 \vee S^3 \vee S^{10})) = 6$.

Example III.20. It can be shown that $\#(S^5 \vee S^5 \vee S^{18}) = \infty$. Consider the space $X = (S^5_a \vee S^5_b \vee S^{18}) \cup_\omega e^{23} \vee S^{35}$, $\omega = [S^5_a, S^{18}]$. The dimension of the space $J^1_{\geqslant 1}$ is in this case equal to 8. But the differential on $J^1_{\geqslant 1}$ can be shown to be monic, hence $H^2_{\geqslant 1}(H^*(X), H^*(S^5 \vee S^5 \vee S^{18})) = 0$ and $\#(X) = \infty$ by Theorem III.18.

III.21. Products. Suppose that the algebra H^* is the tensor product of H'^* and H''^*, $H^* \cong H'^* \otimes H''^*$. If $\mathcal{U}':(\wedge U, d'_{-1}) \to (H', d=0)$ and $\mathcal{U}'':(\wedge V, d''_{-1}) \to (H'', d=0)$ are Halperin-Stasheff bigraded models then clearly

$\mathcal{N}'\otimes\mathcal{N}'' : (\Lambda(U\oplus V), d'_{-1}\oplus d''_{-1}) \to (H'\otimes H'', d=0) \cong (H, d=0)$ is a bigraded model. If now $\mathbb{E} = \mathbb{E}(\Lambda Z, d)$ is the Halperin-Stasheff deformation algebra then we can see similarly as in III.10 that $(\Lambda Z, d)$ is of the form $(\Lambda(X\oplus Y), d'\oplus d'')$ and we obtain analogically:

Theorem III.22. If $H^* \cong H'^* \otimes H''^*$ is intrinsically formal, then both H'^* and H''^* are intrinsically formal, too.

Because $H(X\times Y) \cong H(X)\otimes H(Y)$ we get

Theorem III.23. Let $X_1,\dots,X_n$ be simply connected spaces with the cohomology of finite type. If the space $X = X_1\times\dots\times X_n$ is intrinsically formal, then each $X_1,\dots,X_n$ is intrinsically formal, too.

We can define, similarly as in III.12, the cohomology group ${}_F H^*_*(H^*;H'^*,H''^*)$ and prove

Theorem III.24. If ${}_F H^1_{\geq 1}(H^*;H'^*,H''^*) = 0$ then $\#(H) \leq \#(H')\cdot\#(H'')$.

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We have listed some typical applications of our method to the study of the number of rational homotopy types with a given cohomology group. The behaviour of this number under wedges and attaching of cells was studied using the Quillen deformation algebra, because the Quillen model behaves well under these operations. Analogically, the behaviour under products was studied using the Halperin-Stasheff deformation algebra. It is possible to deduce the statement analogical to Theorem III.18 for pull-backs of the path fibrations over Eilenberg-MacLane spaces.

Appendix: Homotopy types with given homotopy.

Here we give an outline of application of our method to the set of rational homotopy types with a given homotopy Lie algebra. The symbol Π_* will denote a (positively) graded Lie algebra of finite type.

A.1. We call the bigraded algebra $\mathbb{E} = \mathbb{E}(\wedge Z,d)$, where $(\wedge Z,d) = \mathscr{L}_*(\Pi_*,\partial=0)$, the Sullivan deformation algebra of Π_* and we denote the associated cohomology by $Ha^*_*(\Pi_*)$. This cohomology was introduced by Y. Félix in [F1, Annexe 2] and denoted there by ${}_FH$. Our notation which differs from that used by Y. Felix, is based on the duality with the objects studied in the previous part.

A.2. Let $(\mathbb{L}(X),\partial_{-1})$ be the Lie Halperin-Stasheff bigraded model of Π_*, denote by Z^*_* the bigraded space defined by $Z^i_j = X^{-i}_{-j}$ and by d the differential induced by ∂_{-1} on $\mathbb{L}(Z)$. Then $\mathbb{E} = \mathbb{E}(\mathbb{L}(Z),d)$ can be called the Lie Halperin-Stasheff deformation algebra for Π_* and we denote the associated cohomology by ${}_FH^*_*(\Pi_*)$. Proposition III.3 and Theorem I.10 then gives:

Theorem A.3. (Y. Félix)

a) If $Ha^1_{\geq 1}(\Pi_*) = 0$ then Π_* is intrinsically coformal.

b) If $Ha^2_{\geq 2}(\Pi_*) = 0$ then Π_* is intrinsically coformal if and only if $Ha^1_{\geq 1}(\Pi_*) = 0$.

Theorem A.4. The previous theorem is valid also with ${}_FH^*_*(\Pi_*)$ instead of $Ha^*_*(\Pi_*)$.

The cohomology just defined can be used for the study of the behaviour of the number of rational homotopy types with fixed homotopy similarly as in Part III. For example, we can prove

Theorem A.5. Let $X_1,\dots,X_n$ be simply connected spaces of finite H-type. If $X_1 \times \dots \times X_n$ is intrinsically coformal, then each $X_1,\dots,X_n$ is intrinsically coformal, too.

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Matematický Ústav ČSAV,
Žitná 25,
115 67 Praha 1,
Czechoslovakia.