Operadic categories

Hefei, 9.11.2018

Goal. Build a theory of operad-like (aka operadic) structures that will be

(1) general enough to cover all known examples, meaning ..

(2) offer unifying approach to bar constructions, koszulity, resolutions, &c.

(3) give systematic approach to s.h. versions

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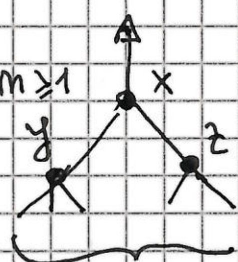
1 Aproach based on pasting schemes. I introduced in 2008, later taken up by Borisov, Manin, Kaufman ($\Rightarrow$ Feynman cat.)

Pasting schemes: graphs of certain types that are hereditary = by contracting subgraph get graph of same type.

Examples: • classical operads. PS = Directed rooted trees

$P = \{P(n)\}_{n \geq 1}$

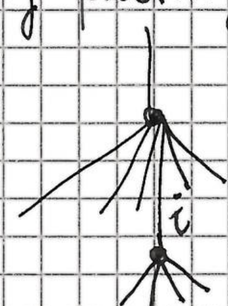


"paste" $\Longrightarrow$ element of $P(5)$

$(x \circ_2 z) \circ_1 y = (x \circ_1 y) \circ_4 z$

$x, z \in P(2)$
$y \in P(3)$

elementary pasting scheme: one edge

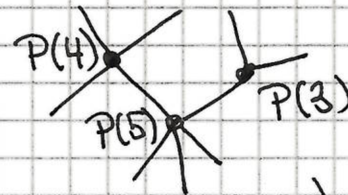


$\Rightarrow$ $\circ_i$ operation

Full contraction is iteration of

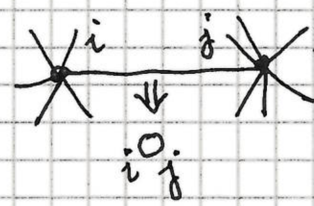
·) Cyclic operads: unrooted trees

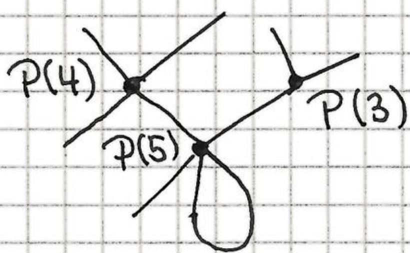
pasting



$\Longrightarrow \in P(8)$

·) Modular operads: PS are graphs

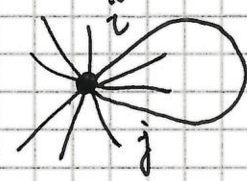




$\Longrightarrow P(6)$

elementary pasting: ${}_i\circ_j$

+

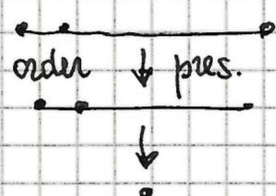


$\circ_{ij}$

Likewise
Directed graphs => PROPs
connected directed => properads
etc.

Examples that do not fit: Batanin's n-trees, $n \geq 1$.

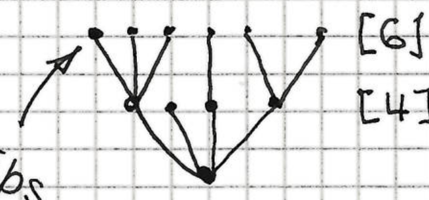
$\boxed{n=2}$ 2-tree



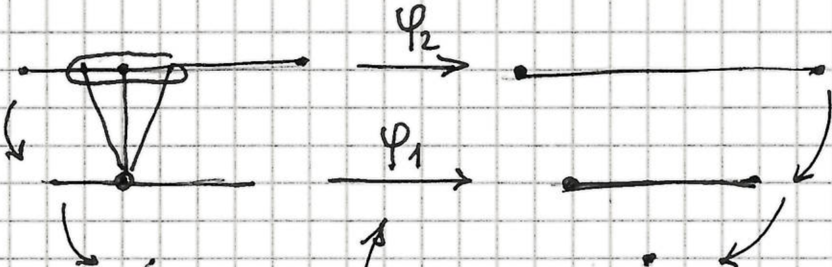
$[n] = \{1, \dots, n\}$

$[k]$

$[1]$

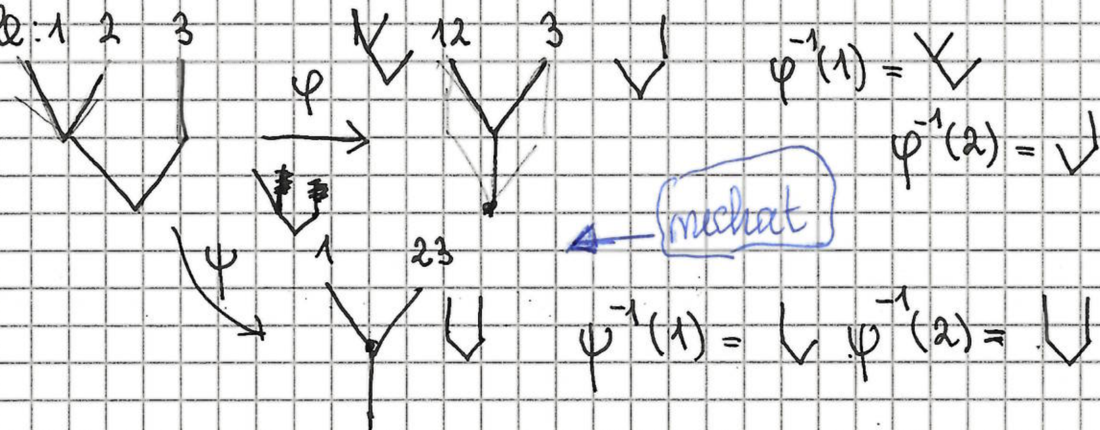


morphisms

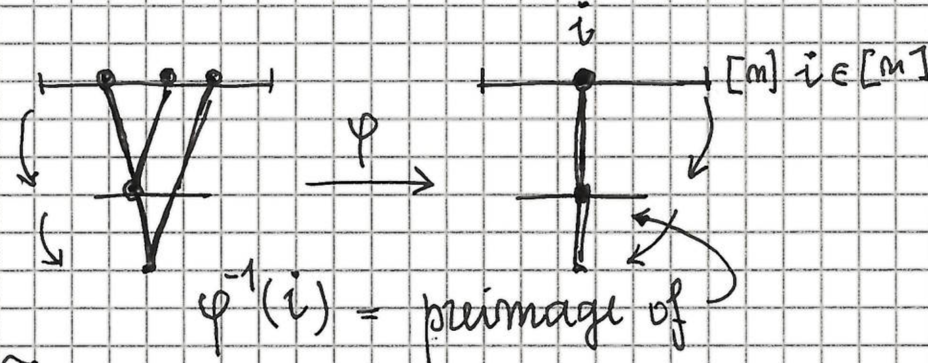


not necessarily order-preserving
but φ_2 ~″~ on preimages

Example:



Notion of fibres
examples are above



2 trees form category $\mathcal{T}_2$

Definition: 2-operad $A = \{A(T)\}_{T \in \mathcal{T}_2}$ + structure maps

$$\gamma_\varphi : A(T) \otimes A(F_1) \otimes \cdots \otimes A(F_m) \longrightarrow A(S)$$

Defined $\forall$ $F_1, \dots, F_m$ $\quad S \xrightarrow{\varphi} T$

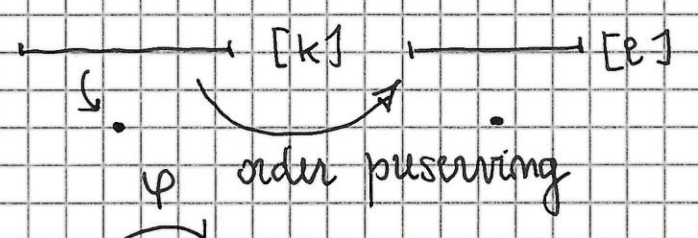
fibres of φ $\qquad$ m tips

+ axioms.

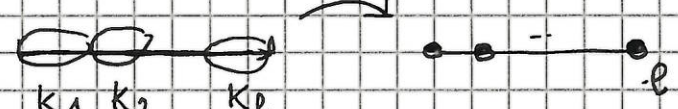
m-operads defined analogously.

Theorem (Batanin) X is an E_m-space iff acted by contractible m-operad.

$\boxed{m = 1}$ $\quad \mathcal{T}_1 = \Delta$:



fibers:



$k_1 + \cdots + k_l = k$

$$\gamma_\varphi : A(l) \otimes A(k_1) \otimes \cdots \otimes A(k_l) \longrightarrow A(k)$$

structure maps of non-Σ operad

Therefore 1-operads = non-Σ operads.
Other examples without obvious pasting schemes:
ones acting on dg-categories; properads, Leray's ..

Operadic category (BM + MM 2015): category $\mathcal{O}$ +
functor $|\text{-}| : \mathcal{O} \longrightarrow \mathrm{Fin}$, + — not inverse
$\cdot)$ $\forall\, f: S \longrightarrow T$ & $i \in |T|$ fiber $f^{-1}(i)$
$\cdot)$ choice of local terminal objects U_c; $c \in \pi_0(\mathcal{O})$
+ some axioms.

Examples: T_n w $|T|$ = tips, Δ with $|\text{-}| : \Delta \hookrightarrow \mathrm{Fin}$
Fin itself w $|\text{-}| = \mathrm{id}$ & $f^{-1}(i)$ = "real" preimage

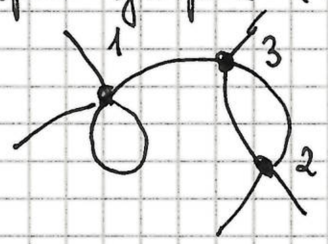
Definition: $\mathcal{O}$-operad $A = \{A(T)\}_{T \in \mathcal{O}}$ + structure operations

$$\gamma_f : A(T) \otimes \bigotimes_{i \in |T|} A(f^{-1}(i)) \longrightarrow A(S),\ \forall f: S \longrightarrow T$$

+ associativity + unitality involving U_c's
Examples: We saw T_n, Δ in particular. Fin-operads are ordinary ones. Local terminal = terminal ($\pi_0(\ ) = *$)
for T_n $\Big\}$ n levels; for Δ & Fin $[1]$. In all cases $\pi_0(\mathcal{O}) = *$

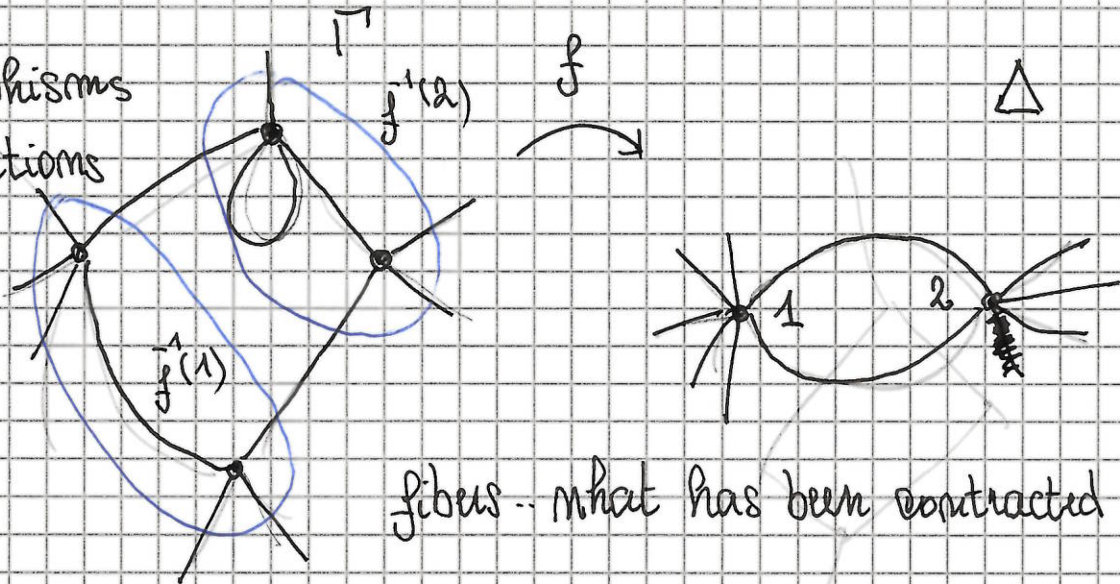
Example: graphs (modulo subtleties) $\mathcal{G}r$ ← cardinality

Γ



Γ has vertices $|\Gamma| = \{1,2,3\}$
Γ has legs
Γ has edges

morphisms
contractions



fibers .. what has been contracted

$f^{-1}(1) =$ , $f^{-1}(2) =$  Gr is operadic category

structure map corresponding to f; $\mathcal{M} = \{\mathcal{M}(\Gamma)\}_{\Gamma \in Gr}$

$$\mathcal{M}(\Delta) \otimes \mathcal{M}(f^{-1}(1)) \otimes \mathcal{M}(f^{-1}(2)) \longrightarrow \mathcal{M}(\Gamma)$$

Gr - operads = hyperoperads à la Getzler-Kapranov

On defining algebras over operads.

Definition. For $T \in O$ define

target $t(T) := [T] \in \pi_0(O)$, $s(T) := \{[1_T^{-1}(i)] \in \pi_0(O);\ i \in |T|\}$

Definition P be operad in O. P-algebra X is collection

$X = \{X_c;\ c \in \pi_0(O)\}$ + structure operations

$$\alpha_T : P(T) \longrightarrow \mathrm{Hom}\Big(\bigotimes_{c \in s(T)} X_c, X_{t(T)}\Big) =: End_X(T)$$

+ some axioms.

Example (dull) $O = \Delta$, remember Δ-operads = non-Σ ones

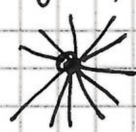
$\pi_0(\Delta) = \{*\}$, represented by $[1]$, $X = \{X_*\}$; just X

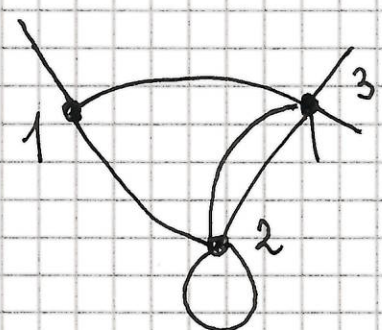
$s([m]) = \{\underbrace{*,\dots,*}_{m \text{ times}}\}$; $t([m]) = *$

$\alpha_{[m]} : P(m) \longrightarrow \mathrm{Hom}(\bigotimes^m X, X) = \mathrm{End}_X(m)$

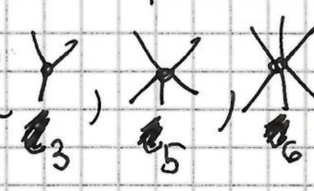
$\Rightarrow$ P-algebras just ordinary algebras.

Example $\mathcal{O} = \mathcal{G}r$, $\pi_0(\mathcal{G}r) = \{c_0, c_1, c_2, \dots\} \cong \mathbb{N}$

$c_m :=$  m corolla



$t(\Gamma) =$  $= c_3$

$s(\Gamma) = \{$  $\}$

$\alpha_\Gamma : \mathcal{M}(\Gamma) \longrightarrow \mathrm{Hom}(X(c_3) \otimes X(c_5) \otimes X(c_6); X(c_3))$

Df

Theorem. $\mathcal{M}$ is constant $\mathcal{G}r$-operad; $\mathcal{M}(\Gamma) = \mathbb{K}$ $\forall \Gamma$,

$\Rightarrow$ $\mathcal{M}$-algebras are modular operads.

Hint: α_Γ determined by $\alpha_\Gamma(1)$, which is precisely "pasting X along Γ". ∎

Basic feature of our theory (1) Operad-like structures appear as algebras over an (generalized) operad in an operadic category $\mathcal{O}$.

NB. No operadic category $\mathcal{O}_0$ s.t. modular operads are $\mathcal{O}$-operads.

Jacob ladder

(2) Opetopic property: O-operads are algebras over O^{+}-operads which are algebras over O^{++}-operads &c.

$\cdots \Rightarrow$ $\boxed{\text{$O^{+}$-operads = algebras for O^{++} operads}}$ $\Rightarrow$ $\boxed{\text{O-operads = algebras for O^{+}-operads}}$ $\Rightarrow$ $\boxed{\text{structures I want describe = algebras of O-operads}}$

Gain: Operad-like structures appear as algebras, thus I may resolve, define quadraticity, models, koszulity, etc.

Example: $\mathcal{M}$-constant Gr-operad so $\mathcal{M}$-algebras are modular operads, $\mathrm{Min} \longrightarrow \mathcal{M}$ minimal model $\Rightarrow$ Min-algebras are sh modular operads.

$\mathcal{M}^{!} = \mathrm{Odd}$; $\mathrm{Min} = \Omega(\mathrm{Odd})$; $\mathcal{M}$ quadratic Koszul.

Other tools available, as Grothendieck's construction, that creates new operadic categories from old ones.