central notions of HA

Higher structures & operads: Why?

What are higher structures depends on context, e.g. spectrum in topology vs. in functional analysis. I will mean in homotopy theory or homological algebra or category theory. Both enter mathematical physics.

HS - quite hot topics, I founded (w Batanin & Kaufmann) journal "Higher structures" already 3rd year, indexed by MathSciNet & Zentralblatt

<u>Example</u> in place of "<u>definition</u>." of higher struct

(nonunital) associative monoid topological

$$\mu : X \times X \longrightarrow X$$

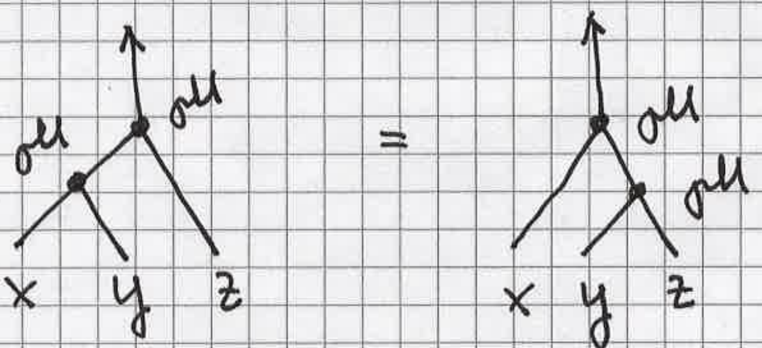
X topological space, μ <u>associative</u>:

$$\mu(\mu(x,y),z) = \mu(x, \mu(y,z)), \quad x,y,z \in X$$

or

$$\mu(\mu \times 1\!\!1_X) = \mu(1\!\!1_X \times \mu)$$

or

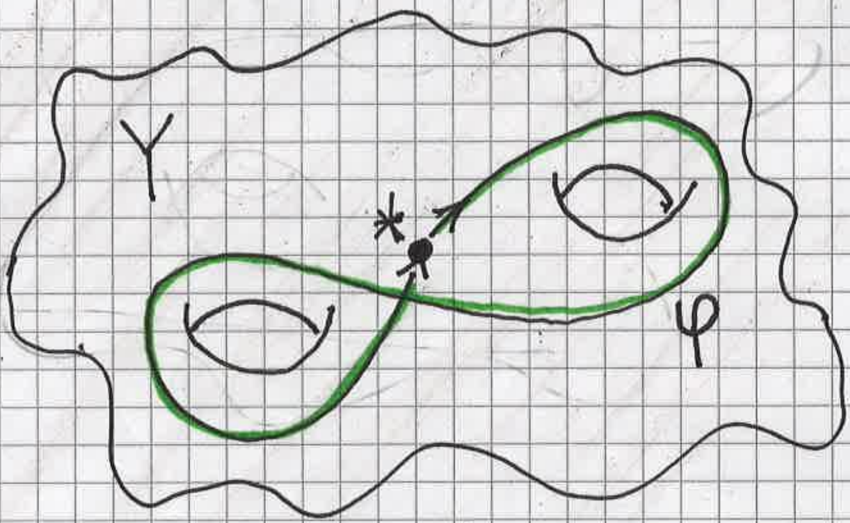


<u>Examples</u>: topological or Lie groups, matrix algebras as $M_2(\mathbb{R}) \in \mathbb{R}^4$, etc.

Non-example: loop space.

Y topological space w/ base point $*$

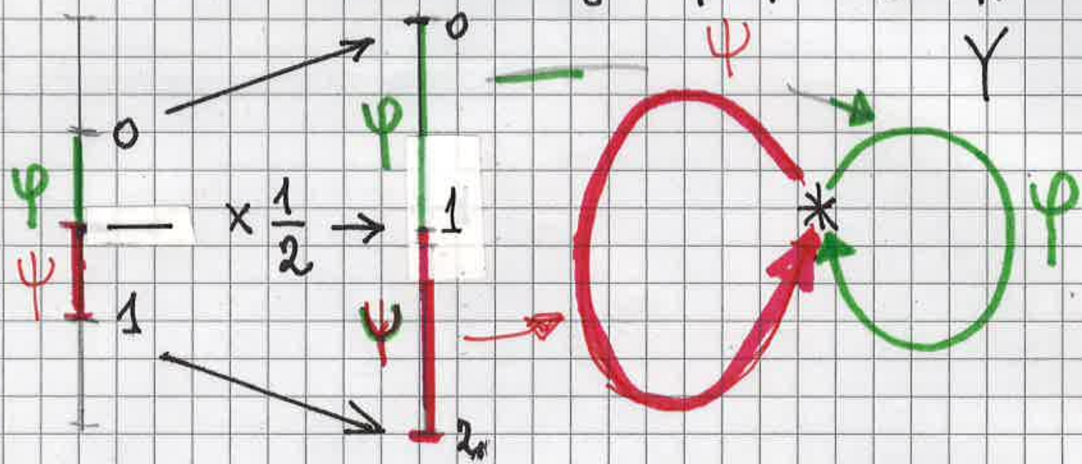
$$\Omega Y := \{\varphi : [0,1] \longrightarrow Y;\ \varphi(0) = \varphi(1) = *\} \subset Y^{[0,1]}$$



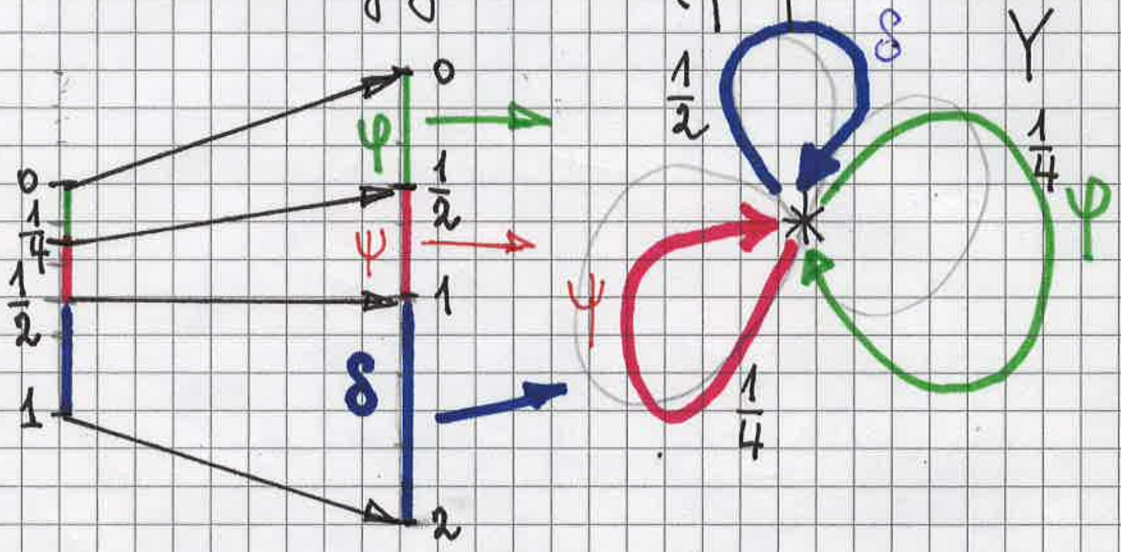
$\mu : \Omega Y \times \Omega Y \longrightarrow \Omega Y$ concatination & reparametrization

---

Construction of $\varphi \cdot \psi$ $(= \mu(\varphi, \psi))$: (4)

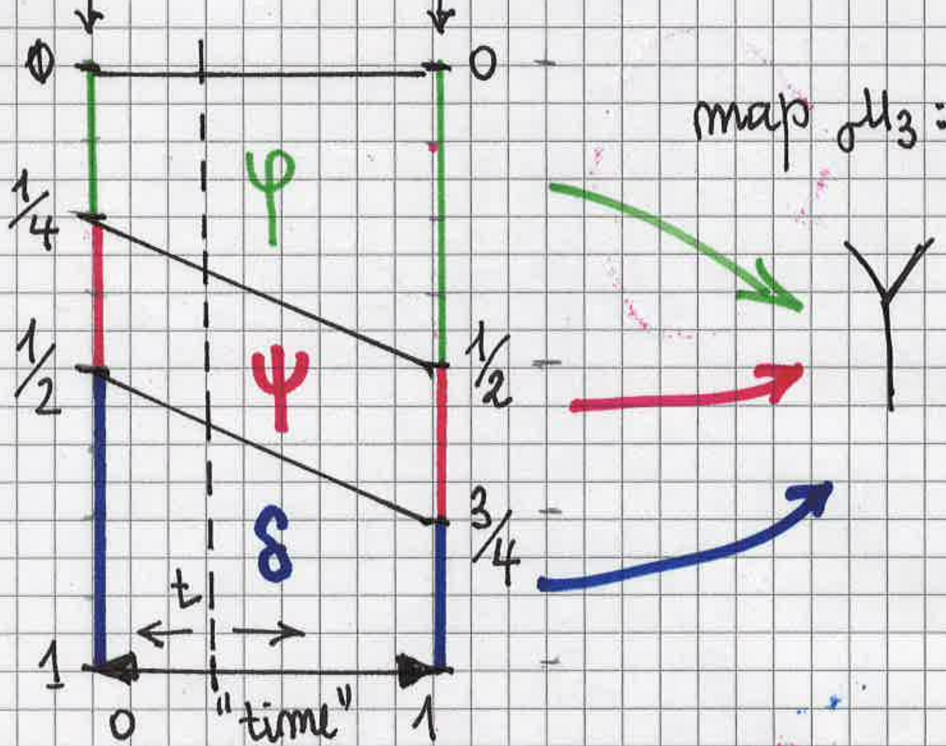


Associativity fails: $(\varphi \cdot \psi) \cdot \delta$



Above is $(\varphi \cdot \psi) \cdot \delta : [0,1] \longrightarrow Y.$

$(\varphi\cdot\psi)\cdot\delta \neq \varphi\cdot(\psi\cdot\delta)$ thus non-associative



map $\mu_3 : \Omega Y^{\times 3} \times [0,1] \xrightarrow{\ \varphi,\psi,\delta \ \downarrow\ t\ } \Omega Y$

Yet, one has a map (above) μ_3:

$$\begin{tikzcd}
\Omega Y^{\times 3} \times \{0\} \arrow[d, hook] \arrow[dr, "(\varphi\cdot\psi)\cdot\delta \,=\, \mu(\mu(\varphi,\psi),\delta)"] & \\
\Omega Y^{\times 3} \times [0,1] \arrow[r, "\mu_3"] & \Omega Y \\
\Omega Y^{\times 3} \times \{1\} \arrow[u, hook] \arrow[ur, "\varphi\cdot(\psi\cdot\delta) \,=\, \mu(\varphi,\mu(\psi,\delta))"'] &
\end{tikzcd}$$

μ_3 is a <u>homotopy</u> (="deformation") between $\mu(\mu \times \mathbb{1}_{\Omega Y})$ & $\mu(\mathbb{1}_{\Omega Y} \times \mu)$.

<u>Definition</u>: Topological monoid X is homotopy associative if its structure operation $\mu : X \times X \longrightarrow X$ satisfies that $\mu(\mu \times 1\!\!1_X)$ is homotopic, as a map $X^{\times 3} \longrightarrow X$, to $\mu(1\!\!1_X \times \mu)$.

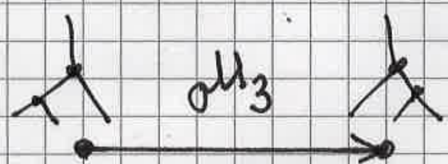
Meaning $\exists\ \mu_3 : [0,1] \times X^{\times 3} \longrightarrow X$

s.t.
$$\mu_3 \big|_{\{0\} \times X^{\times 3}} = \mu(\mu \times 1\!\!1_X)$$
$$\mu_3 \big|_{\{1\} \times X^{\times 3}} = \mu(1\!\!1_X \times \mu)$$

Expressed by Diagrams

$$\mu(\mu\times 1\!\!1) \xrightarrow{\ \mu_3\ } \mu(1\!\!1\times\mu)$$

or

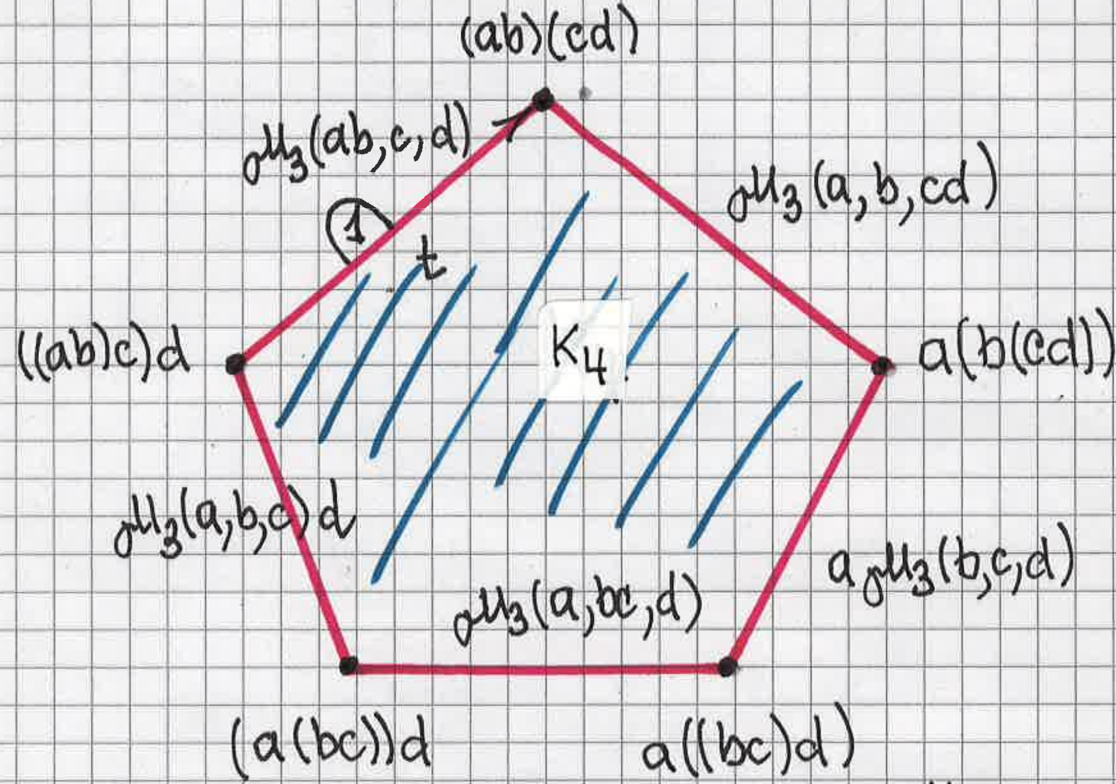


or

$$(ab)c \xrightarrow[\ \mu_3(a,b,c)\ ]{} a(bc)$$

where t in μ_3 omited

Assume X is a homotopy associative space. Then one may draw



One reads off a map  $\times X^{\times 4} \longrightarrow X$

For instance, on ① given by $a,b,c,d;t \to \mu_3(ab,c,d;$

For $t=0$ I get by definition $\mu(\mu(ab,c),d) = ((ab)c)d$

for $t=1$ I get likewise $(ab)(cd) = \mu(\mu\times 1\!\!1)(ab,c,d)$

Assume there exists an extension to the interior μ_4:  $\times X^4 \longrightarrow X$

μ_4 example of higher homotopy

Assume one has

•) $\mu : X^{\times 2} \longrightarrow X$

(Denoted $\mu_2 : K_2 \times X^{\times 2} \longrightarrow X$, $K_2 = \{*\}$)

•) $\mu_3 : [0,1] \times X^{\times 3} \longrightarrow X$

(Denoted $\mu_3 : K_3 \times X^{\times 3} \longrightarrow X$, $K_3 = [0,1]$)

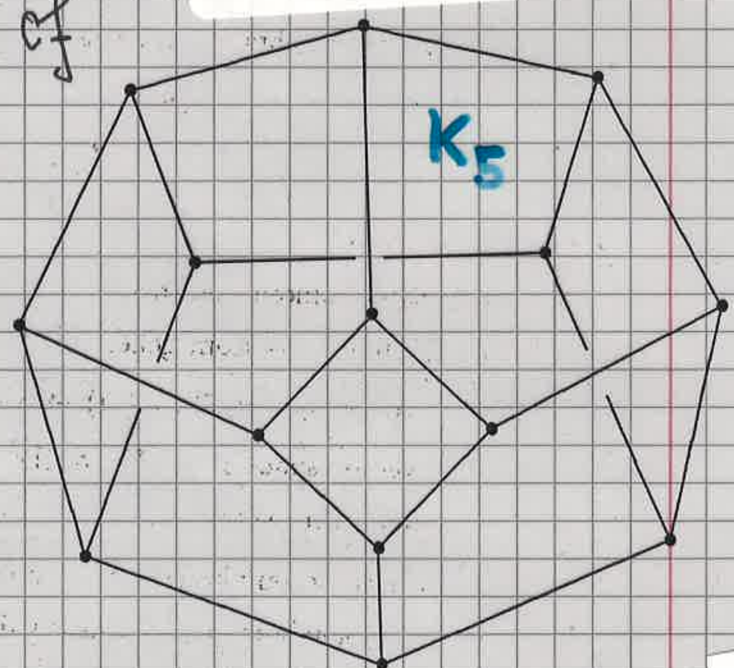
& $\mu_4 : K_4 \times X^{\times 4} \longrightarrow X$, $K_4 =$ 

Out of it, one likewise constructs a map

$$\partial K_5 \times X^{\times 5} \longrightarrow X,$$

∂K_5 is boundary of

K_5:



May look for extension

$\mu_5 : K_5 \times X^{\times 5} \longrightarrow X$

etc.

glimpse of a scheme:

<u>Definition</u>. Strongly homotopy associative aka A_∞-space is a topological space X & system $\mu_n : K_n \times X^{\times n} \longrightarrow X$, $n \geq 2$, where K_n is a specific $(n-2)$ dimensional convex polyhedron & μ_n <u>extends</u> map $\partial K_n \times X^n \longrightarrow X$ pasted from $\mu_2, \mu_3, \ldots, \mu_{n-1}$.

Historically first example of a $\boxed{\text{higher structure:}}$

Stasheff 1963 ← I wrote a couple of papers & book w/ him.

<u>Salient feature</u>: homotopy invariant concept: X & Y have the same homotopy type ("one is deformation of other") then X is A_∞ iff Y is A_∞.

<u>Example</u>: ΩY is A_∞

Aside: <u>Recognition principle</u> - Boardman - Vogt 70's

X has homotopy type of a loop space iff it is A_∞.

Where <u>operads</u> chime in?

In this context they organize higher homotopies: $\{K_n\}_{n\geq 2}$ is a topological operad; i.e. $\exists$ maps

$$\circ_i : K_m \times K_n \longrightarrow K_{m+n-1}$$

$m, n \geq 2$, $1 \leq i \leq m$, satisfying suitable associativity.

A_∞ space is then an algebra (in an appropriate sense) over $\{K_n\}_{n\geq 2}$.

Stasheff's operad of associahedra

Origin of higher structures in topology or homological algebra: One starts w/ a structure (e.g. monoid) having some operations & axioms. Then one requires these axioms to hold up to homotopy only. This calls for a bunch (usually infinite) of "higher operations." Operads organize them.

More precisely:

resolution

$$P \longleftarrow P_\infty$$

operad for original version

operad for higher version

Describes the "weakening" entirely in terms of operads

Random closing remarks

•) HS in <u>category theory</u>: relaxing associativity of composition $(fg)h = f(gh)$:

$$\begin{tikzcd}
 & \bullet \arrow[r, "g)"] & \bullet \arrow[dr, "f"] & \\
\bullet \arrow[ur, "(h"] \arrow[dr, "h"'] \arrow[rrr, phantom, "\Updownarrow"] & & & \bullet \\
 & \bullet \arrow[r, "(g"'] & \bullet \arrow[ur, "f)"'] &
\end{tikzcd}$$

invertible 2-cell ("2-map", "higher morphism") calls for still higher cells ("higer morphisms")

•) HS in matematical (quantum) physics

Very superficially: quantum world is deformation of classical one.

Thus classical structures (such as Poisson bracket of classical fields) appear in their higher versions thanks to their homotopy invariance (invariance w.r. to "deformations"). Operads sneak in accordingly.

·) Jacob's ladder (bit of kabbalah)

$\vdots$

$\downarrow$

hyper-hyper operads

$\downarrow$

hyperoperads

$\downarrow$

operads

$\downarrow$

algebras

• level $n+1$

$\downarrow$ ← means "is governed by"

• level n

conceptual understanding level n requires working in level $n+1$

That is all, folks!