Prague, September 17, 2023

1. Strong minimal model theorem

notes by M. Markl

Theorem 1 (Strong minimal model theorem). *Let* $(A,\partial,\cdot\,)$ *be a chain* $\Bbbk$*-algebra,* $\Bbbk$ *a field, and* $H := \mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ *its homology. Then* H *admits a* **canonical-up-to-isotopy** A_∞*-structure* $(H,0,\boldsymbol{\mu})$*, having the homotopy type of* $(A,\partial,\cdot\,)$.

The principal difference against the standard Minimal Model Theorem as formulated for instance in [2] and repeated at many places since (cf. e.g. [1, Theorem 1.1]), is the word "isotopy" instead of "isomorphism." This stronger version offers a "coordinate-free" approach to Massey products, my original motivation. The proof is shockingly simple.

Proof. Choose a chain map $p : (A,\partial) \to (H,0)$ and its left homotopy inverse $i : (H,0) \to (A,\partial)$ such that $pi = \mathrm{id}_H$. Suppose moreover that

(1) the map p restricted to $\mathrm{Ker}\,\partial \subset A$ is the canonical projection to H.

Notice there might be choices of p and its left homotopy inverse i that do not satisfy (1), see Example 2 below, but there is always a choice that does.

Let (p',i') resp. (p'',i'') be two such choices. Given $x \in H$, it follows from (i) and the obvious fact that $\mathrm{Im}\ i \subset \mathrm{Ker}\ \partial$ that $i'(x) - i''(x) = \partial b$, for some $b \in A$. Since p' is a chain map, $p'(\partial b) = \partial p'(b) = 0$, thus

$$p'i'' = p'i' = \mathrm{id}_H. \tag{2}$$

Using the celebrated Theorem 5 of [4], we extend p' and i' to A_∞-maps

$$(A,\partial,\cdot\,) \underset{i'_\infty}{\overset{p'_\infty}{\rightleftarrows}} (H,0,\boldsymbol{\mu}') \tag{3a}$$

and, likewise,

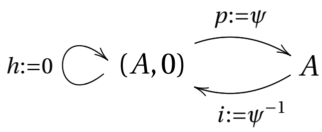
$$(A,\partial,\cdot\,) \underset{i''_\infty}{\overset{p''_\infty}{\rightleftarrows}} (H,0,\boldsymbol{\mu}''). \tag{3b}$$

Let us show that $(H,0,\boldsymbol{\mu}')$ and $(H,0,\boldsymbol{\mu}'')$ are isotopic. By (2), the linear part of $p'_\infty i''_\infty$ equals id_H, therefore the composite

$$(H,0,\boldsymbol{\mu}'') \xrightarrow{i''_\infty} (A,\partial,\cdot\,) \xrightarrow{p'_\infty} (H,0,\boldsymbol{\mu}') \tag{4}$$

is the required isotopy. □

Example 2. Let A be a $\Bbbk$-vector space regarded as a chain complex $(A,0)$ with trivial differential concentrated in degree 0, and $\psi : A \to A$ an automorphism. The homology $H(A,0)$ is canonically isomorphic to A, and the pair



with the trivial homotopy $h = 0 : A \to A$ between $ip = \mathrm{id}_A$ and id_A satisfies (1) if and only if $\psi = \mathrm{id}_A$. Notice that the infamous, confusing and superfluous side conditions

$$ih = hp = h^2 = 0$$

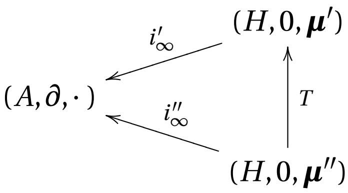
are satisfied for an arbitrary ψ.

Assume that A has a multiplication $* : A \otimes A \to A$ and see what Theorem 1 says in this situation. There is clearly a unique A_∞-structure on the "homology" $A = H(A,0)$ such that ψ extends to an A_∞-map $\psi_\infty : (A,0,*) \to (A,0,\boldsymbol{\mu})$, namely the associative multiplication $\star$ given by

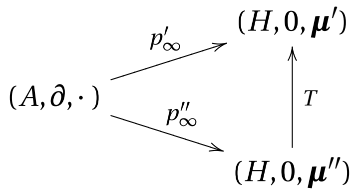
$$x \star y := \psi\big(\psi^{-1}(x) * \psi^{-1}(y)\big),\ x, y \in A. \tag{5}$$

If (1) is fulfilled, $\psi = \mathrm{id}_A$, and thus $\star = *$. Since the only isotopy $T : A \to A$ is the identity, there must be precisely one canonical A_∞-structure of Theorem 1. It is the one given by (5).

Remark 3. Let (p,i) be a pair as in (1) and (3a)–(3b) be two A_∞-extensions given by [4, Theorem 5]. Using [5, Proposition 14] one can show that the isotopy $T : (H,0,\boldsymbol{\mu}'') \to (H,0,\boldsymbol{\mu}')$ in (4) can be chosen such that the diagrams



and



commute up to an A_∞-homotopy.

2. Transfers as Grothendieck bifibrations
notes by M. Markl

In what follows, by $f' \sim f''$ we express that the chain map f' is chain homotopic to f''. The notation $F' \overset{\infty}{\sim} F''$ will mean that the A_∞-morphism F' and F'' are A_∞-homotopic.

Proposition 4. *Let $F' : (A,\partial,\boldsymbol{\mu}') \to (B,\partial,\boldsymbol{\nu}')$ and $F'' : (A,\partial,\boldsymbol{\mu}'') \to (B,\partial,\boldsymbol{\nu}'')$ be A_∞-morphisms with the underlying chain maps $f_1', f_1'' : (A,\partial) \to (B,\partial)$ which are chain homotopy equivalences. Then the following are equivalent:*

(i) $f_1' \sim f_1''$, and $(A,\partial,\boldsymbol{\mu}')$ and $(A,\partial,\boldsymbol{\mu}'')$ are isotopic,
(ii) $f_1' \sim f_1''$, and $(B,\partial,\boldsymbol{\nu}')$ and $(B,\partial,\boldsymbol{\nu}'')$ are isotopic,

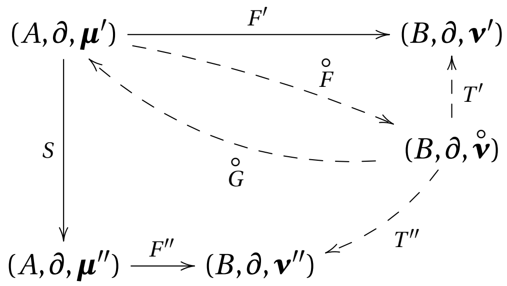
(iii) $f_1' \sim f_1''$, $(A,\partial,\boldsymbol{\mu}')$ *and* $(A,\partial,\boldsymbol{\mu}'')$ *are isotopic, and also* $(B,\partial,\boldsymbol{\nu}')$ *and* $(B,\partial,\boldsymbol{\nu}'')$ *are isotopic,*

(iv) There are isotopies $S : (A,\partial,\boldsymbol{\mu}') \to (A,\partial,\boldsymbol{\mu}'')$ *and* $T : (B,\partial,\boldsymbol{\nu}') \to (B,\partial,\boldsymbol{\nu}'')$ *such that the diagram*

$$\begin{array}{ccc} (A,\partial,\boldsymbol{\mu}') & \xrightarrow{F'} & (B,\partial,\boldsymbol{\nu}') \\ {\scriptstyle S}\big\downarrow & \overset{\infty}{\approx} & \big\downarrow{\scriptstyle T} \\ (A,\partial,\boldsymbol{\mu}'') & \xrightarrow{F''} & (B,\partial,\boldsymbol{\nu}'') \end{array} \tag{6}$$

commutes up to A_∞*-homotopy.*

Proof. Obviously (iv) implies all the remaining items. It would therefore suffice to prove e.g. that (i) $\Longrightarrow$ (iv). To this end, consider the diagram



in which $S : (A,\partial,\boldsymbol{\mu}') \to (A,\partial,\boldsymbol{\mu}'')$ is an isotopy. Theorem 5 of [4] guarantees the existence of A_∞-morphisms $\mathring{F}$ and $\mathring{G}$ such that $\mathring{F}$ extends f_1', the linear part $\mathring{g}_1$ of $\mathring{G}$ is a chain homotopy inverse of f_1', and $\mathring{G}\mathring{F} \overset{\infty}{\approx} \mathrm{id}_A$.

Now $F'\mathring{G} : (B,\partial,\mathring{\boldsymbol{\nu}}) \to (B,\partial,\boldsymbol{\nu}')$ is an A_∞-map with the linear part $f_1'\mathring{g}_1$ chain homotopic to id_B thus, by Corollary 6 below, there exists an isotopy $T' : (B,\partial,\mathring{\boldsymbol{\nu}}) \to (B,\partial,\boldsymbol{\nu}')$, A_∞-homotopic to $F'\mathring{G}$. Since S is an isotopy, it does not affect the linear parts of A_∞-maps, so the linear part of $F''S\mathring{G}$ is $f_1''\mathring{g}_1$ which is chain homotopic to $f_1'\mathring{g}_1 \sim \mathrm{id}_B$ since $f_1' \sim f_1''$ by assumption. Corollary 6 thus gives an isotopy $T'' : (B,\partial,\mathring{\boldsymbol{\nu}}) \to (B,\partial,\boldsymbol{\nu}'')$, A_∞-homotopic to $F''S\mathring{G}$. The above morphisms fulfill the hypotheses of the obvious implications

$$F'\mathring{G} \overset{\infty}{\approx} T' \ \&\ \mathring{G}\mathring{F} \overset{\infty}{\approx} \mathrm{id}_A \Longrightarrow F' \overset{\infty}{\approx} T'\mathring{F} \ \text{ and } \ F''S\mathring{G} \overset{\infty}{\approx} T'' \ \&\ \mathring{G}\mathring{F} \overset{\infty}{\approx} \mathrm{id}_A \Longrightarrow F''S \overset{\infty}{\approx} T''\mathring{F}. \tag{7}$$

We claim that item (iv) is satisfied with $T := T''T'^{-1}$. Indeed, the homotopy commutativity $F''S \overset{\infty}{\approx} T''T'^{-1}F' = TF'$ of (6) is equivalent to $T''\mathring{F} \overset{\infty}{\approx} T''T'^{-1}F'$ by the rightmost homotopy in (7). Applying T''^{-1} to both sides from the left, we see that the latter is equivalent to $\mathring{F} \overset{\infty}{\approx} T'^{-1}F'$, which is in turn equivalent to $T'\mathring{F} \overset{\infty}{\approx} F'$, established in (7). □

Lemma 5. *Let* $R = (r_1, r_2, r_3, \ldots) : (X,\partial,\boldsymbol{\mu}) \to (Y,\partial,\boldsymbol{\nu})$ *be an* A_∞*-map and* $\mathring{t}_1 : (X,\partial) \to (Y,\partial)$ *a chain map, chain homotopic to* r_1*. Then* $\mathring{t}_1$ *extends to an* A_∞*-map*

$$\mathring{T} = (\mathring{t}_1, \mathring{t}_2, \mathring{t}_3, \cdots) : (X,\partial,\boldsymbol{\mu}) \longrightarrow (Y,\partial,\boldsymbol{\nu})$$

which is A_∞*-homotopic to* R*.*

Corollary 6. *Let* $E = (e_1, e_2, e_3, \dots) : (X,\partial,\boldsymbol{\mu}') \to (X,\partial,\boldsymbol{\mu}'')$ *be an* A_∞*-endomorphism whose linear part* e_1 *is chain homotopic to the identity* $\mathrm{id}_X : (X,\partial) \to (X,\partial)$*. Then* E *is* A_∞*-homotopic to an isotopy* $\overset{\circ}{T} : (X,\partial,\boldsymbol{\mu}') \to (X,\partial,\boldsymbol{\mu}'')$.

Proof of Lemma 5. The existence of an extension $\overset{\circ}{T}$ is the content of [5, Proposition 14], an A_∞-homotopy between R and $\overset{\circ}{T}$ was in fact constructed in its proof.[1] □

Let $\mathtt{CHE}$ be the category whose objects are chain complexes and morphisms are homotopy classes of chain maps which have (unspecified) chain homotopy inverses. Next, let $\mathtt{Isot}_\infty$ be the category whose objects are isotopy classes of A_∞-algebras. The arrows (i.e. morphisms) in $\mathtt{Isot}_\infty$ are equivalence classes of A_∞-morphisms $F : (A,\partial,\boldsymbol{\mu}) \to (B,\partial,\boldsymbol{\nu})$ such that the underlying chain maps f_1 are chain homotopy equivalences, modulo the relation that identifies F' and F'' as in Proposition 4 if and only if one (and hence all) of (i)–(iv) of that proposition is satisfied. The composition is defined as follows.

Assume that $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are isotopy classes of A_∞-algebras and $\phi : \mathbb{A} \to \mathbb{B}$, resp. $\psi : \mathbb{B} \to \mathbb{C}$ two arrows in $\mathtt{Isot}_\infty$. Assume that ϕ is the equivalence class of some $F : (A,\partial,\boldsymbol{\mu}) \to (B,\partial,\boldsymbol{\nu}')$ and ψ the class of $Y : (B,\partial,\boldsymbol{\nu}'') \to (C,\partial,\boldsymbol{\omega})$. Since the objects of $\mathtt{Isot}_\infty$ are isotopy classes, there exists an isotopy $S : (B,\partial,\boldsymbol{\nu}') \to (B,\partial,\boldsymbol{\nu}'')$. We define the composite $\psi\phi : \mathbb{A} \to \mathbb{C}$ as the equivalence class of the composed A_∞-map

$$\begin{array}{ccccc}
(A,\partial,\boldsymbol{\mu}) & \xrightarrow{F} & (B,\partial,\boldsymbol{\nu}') & & \\
& & \downarrow{\scriptstyle S} & & \\
& & (B,\partial,\boldsymbol{\nu}'') & \xrightarrow{Y} & (C,\partial,\boldsymbol{\omega}).
\end{array}$$

Let us prove that the equivalence class of $YSF : (A,\partial,\boldsymbol{\mu}) \to (C,\partial,\boldsymbol{\omega})$ does not depend on S. Consider thus the diagram

$$\begin{array}{ccccc}
& & (B,\partial,\boldsymbol{\nu}'') & \xrightarrow{\quad Y \quad} & (C,\partial,\boldsymbol{\omega}) \\
& & \uparrow{\scriptstyle S''} & & \vdots \\
(A,\partial,\boldsymbol{\mu}) & \xrightarrow{F} & (B,\partial,\boldsymbol{\nu}') & \overset{\infty}{\sim} & \vdots\, {\scriptstyle T} \\
& & \downarrow{\scriptstyle S'} & & \downarrow \\
& & (B,\partial,\boldsymbol{\nu}'') & \xrightarrow{\quad Y \quad} & (C,\partial,\boldsymbol{\omega})
\end{array}$$

[1] Confirmed by Chris Rogers.

in which S' and S'' are two such isotopies. Then $S'S''^{-1} : (B,\partial,\boldsymbol{\nu}') \to (B,\partial,\boldsymbol{\nu}'')$ is an isotopy too, thus Proposition 4 gives an isotopy $T : (C,\partial,\boldsymbol{\omega}) \to (C,\partial,\boldsymbol{\omega})$ making the square in the above diagram A_∞-homotopy commutative. Thus also the square

$$\begin{array}{ccc} (A,\partial,\boldsymbol{\mu}) & \xrightarrow{YS''F} & (C,\partial,\omega) \\ \Vert & \overset{\infty}{\sim} & \big\downarrow T \\ (A,\partial,\boldsymbol{\mu}) & \xrightarrow{YS'F} & (B,\partial,\boldsymbol{\omega}) \end{array}$$

is homotopy commutative, therefore $YS'F$ and $YS''F$ belongs to the same equivalence class. We leave the verification that the composite $\psi\phi$ does not depend on the choices of the representatives of ϕ and ψ as an exercise.

Notice that isotopic A_∞-algebras have the same underling chain complex. Moreover, the underlying chain maps of all representatives of an arrow in $\mathtt{Isot}_\infty$ are chain homotopic to each other. One therefore has an obvious forgetful functor $\square : \mathtt{Isot}_\infty \to \mathtt{CHE}$. Here goes the main result of this section:

Theorem 7. *The forgetful functor* $\square : \mathtt{Isot}_\infty \to \mathtt{CHE}$ *is a discrete Grothendieck bifibration.*

Proof. The opfibration property by definition means that, for homotopy class $[f]$ of a chain homotopy equivalence $f : (A,\partial) \to (B,\partial)$ and an isotopy class $\mathbb{A} \in \mathtt{Isot}_\infty$ such that $\square(\mathbb{A}) = (A,\partial)$, there exists a unique arrow $\phi : \mathbb{A} \to \mathbb{B}$ in $\mathtt{Isot}_\infty$ such that $\square(\phi) = [f]$. Let us start with the uniqueness. Assume that ϕ', ϕ'' are two such arrows, represented by A_∞-maps

$$F' : (A,\partial,\boldsymbol{\mu}') \to (B,\partial,\boldsymbol{\nu}') \text{ resp. } F'' : (A,\partial,\boldsymbol{\mu}'') \to (B,\partial,\boldsymbol{\nu}'').$$

The linear parts f_1' resp. f_1'' of F' resp. F'' are chain homotopic to $f : (A,\partial) \to (B,\partial)$, therefore $f_1' \sim f_1''$. Moreover, $(A,\partial,\boldsymbol{\mu}')$ is isotopic to $(A,\partial,\boldsymbol{\mu}'')$, via an isotopy S in the diagram

$$\begin{array}{ccc} (A,\partial,\boldsymbol{\mu}') & \xrightarrow{F'} & (B,\partial,\boldsymbol{\nu}') \\ S\big\downarrow & & \\ (A,\partial,\boldsymbol{\mu}'') & \xrightarrow{F''} & (B,\partial,\boldsymbol{\nu}''). \end{array}$$

We are thus in the situation of item (i) of Proposition 4, so F' and F'' belong to the same equivalence class. The existence of a lift follows from Theorem 5 of [4]. The fibration property can be established similarly, only using **(M1')** on page 133 of [3] instead of Theorem 5 loc. cit. $\square$

Theorem 7 implies the homotopy functoriality of lifts up to isotopy, as formulated in

Corollary 8. *Consider a homotopy commutative diagram of chain maps*

$$\begin{array}{ccccc} & & (B,\partial) & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ (A,\partial) & & \xrightarrow{h:=gf} & & (C,\partial) \end{array}$$

along with the data consisting of

- *an* A_∞*-structure* $(A,\partial,\boldsymbol{\mu})$ *on* (A,∂),
- *an extension* $F=(f,f_2,f_3,\ldots):(A,\partial,\boldsymbol{\mu})\to(B,\partial,\boldsymbol{\nu})$ *of* f,
- *an extension* $G=(g,g_2,g_3,\ldots):(B,\partial,\boldsymbol{\nu})\to(C,\partial,\boldsymbol{\omega}')$ *of* g *and*
- *an extension* $H=(h,h_2,h_3,\ldots):(A,\partial,\boldsymbol{\mu})\to(C,\partial,\boldsymbol{\omega}'')$ *of the composite* $h:=gf$.

Then there exists an isotopy $T:(C,\partial,\boldsymbol{\omega}')\to(C,\partial,\boldsymbol{\omega}'')$ *such that the diagram of* A_∞*-maps*

$$
\begin{array}{ccccc}
 & & (B,\partial,\boldsymbol{\nu}) & \xrightarrow{G} & (C,\partial,\boldsymbol{\omega}') \\
 & {}^{F}\nearrow & \overset{\infty}{\sim} & & \big\downarrow T \\
(A,\partial,\boldsymbol{\mu}) & & \xrightarrow{\qquad H \qquad} & & (C,\partial,\boldsymbol{\omega}'')
\end{array}
$$

commutes up to A_∞*-homotopy.*

There classically exists an one-to-one correspondence between presheaves R (i.e. contravariant functors to the category $\mathtt{Set}$ of sets) on a small category $\mathtt{C}$ and discrete fibrations $\mathtt{E}\to\mathtt{C}$. The category $\mathtt{E}$ is the 'category of elements' of the functor R, usually denoted by $\int_{\mathtt{C}} R$. Likewise, there exits a one-to-one correspondence between covariant functors $L:\mathtt{C}\to\mathtt{Set}$ and discrete opfibrations $\mathtt{D}\to\mathtt{C}$, where the category $\mathtt{D}$ corresponding to L is usually denoted by $\int^{\mathtt{C}} L$.

In the situation of Theorem 7, the contravariant functor $R:\mathtt{CHE}\to\mathtt{Set}$ assigns to a chain complex $(A,\partial)\in\mathtt{CHE}$ the set $R(A,\partial)$ of isotopy classes of A_∞-algebras having (A,∂) as its underlying chain complex. Given a chain map $f:(A,\partial)\to(B,\partial)$, the set map $R([f]):E(B,\partial)\to E(A,\partial)$ sends the isotopy class of an A_∞-algebra $(B,\partial,\boldsymbol{\nu})$ to the isotopy class of an A_∞-algebra $(A,\partial,\boldsymbol{\mu})$ such that there exists an A_∞-map $G:(B,\partial,\boldsymbol{\nu})\to(A,\partial,\boldsymbol{\mu})$ whose linear part is a chain homotopy inverse of f.

The above rule indeed defines a contravariant functor $R:\mathtt{CHE}\to\mathtt{Set}$ by Proposition 4. The construction of the covariant functor $L:\mathtt{CHE}\to\mathtt{Set}$ is even simpler since it does not require homotopy inverses. The following proposition is easy to verify.

Proposition 9. *There are isomorphisms of categories*

$$\mathtt{Isot}_\infty\cong\int_{\mathtt{CHE}} R\cong\int^{\mathtt{CHE}} L.$$

References

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