

## WHICH HOMOTOPY ALGEBRAS COME FROM TRANSFER?

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(Communicated by Julie Bergner)

**ABSTRACT.** We characterize  $A_\infty$ -structures that are equivalent to a given transferred structure over a chain homotopy equivalence or a quasi-isomorphism, answering a question posed by D. Sullivan. Along the way, we present an obstruction theory for weak  $A_\infty$ -morphisms over an arbitrary commutative ring. We then generalize our results to  $\mathcal{P}_\infty$ -structures over a field of characteristic zero, for any quadratic Koszul operad  $\mathcal{P}$ .

### 1. INTRODUCTION

An  $A_\infty$ -algebra is a homotopical generalization of a differential graded associative algebra [21]. It is a chain complex  $(A, d)$  equipped with a binary operation  $\mu_2$  for which the associative law only holds up to a specified chain homotopy  $\mu_3$ . This homotopy is taken to be part of the structure; it too must satisfy a law, but only up to another specified homotopy  $\mu_4$ , which satisfies yet another law and so forth. See [16, Sec. 2] for the precise definition and terminology. This seemingly complicated generalization is in fact quite natural, and it endows  $A_\infty$ -algebras with many desirable homological properties.

For example, given a dg associative algebra  $A$  and a chain homotopy equivalent complex  $A'$ , there is in general no dg associative algebra structure on  $A'$  such that the given chain homotopy equivalence becomes a morphism of dg associative algebras. On the other hand, for  $A_\infty$ -algebras one has:

**Homotopy Transfer Theorem** ([14, Theorem 10.3.1]). *Let the chain complex  $(A', d')$  be a homotopy retract of  $(A, d)$ , i.e. there exists a diagram*

$$(1) \quad h \circlearrowleft (A, d) \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{g_1} \end{array} (A', d'), \quad g_1 f_1 - \text{id}_A = dh + hd$$

*in which  $f_1$  and  $g_1$  are chain maps, with  $g_1$  inducing an isomorphism on homology, and  $h$  is a chain homotopy between  $g_1 f_1$  and the identity endomorphism of  $A$ .<sup>1</sup> Then any  $A_\infty$ -algebra structure on  $(A, d)$  can be transferred to an  $A_\infty$ -algebra structure on  $(A', d')$  such that  $g_1$  extends to a weak  $A_\infty$ -morphism.*

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Received by the editors June 17, 2020, and, in revised form, February 17, 2021, May 11, 2021, and June 10, 2021.

2020 *Mathematics Subject Classification.* Primary 13D99, 55S20.

*Key words and phrases.*  $A_\infty$ -algebra, transfer, quasi-isomorphism, weak equivalence.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. The first author was also supported by grant GA ĆR 18-07776S, Praemium Academiæ and RVO: 67985840. The second author was also supported by a grant from the Simons Foundation/SFARI (585631,CR).

<sup>1</sup>In other words,  $g_1$  is a left homotopy inverse of  $f_1$ .

The first author proved in [16] a much stronger result, providing simple explicit formulas not only for the transferred  $A_\infty$ -structure and an extension of  $g_1$ , but also for extensions of  $f_1$  and  $h$ . Furthermore,  $g_1$  is not required in [16] to induce an isomorphism on homology. The extensions of both  $f_1$  and  $g_1$  will play a crucial role in this work.

The transfers of  $A_\infty$ -structures over a chain map admitting a left homotopy inverse, as given by the formulas presented in [10, 16, 17] and recalled in Sec. 3, have found applications in many contexts. For example, they have been used in geometry [1, 4–6, 19], homological algebra [2, 3] and mathematical physics [11, 20]. It is therefore natural to ask which  $A_\infty$ -structures are equivalent to a given transferred one. This was the question posed to the first author by Dennis Sullivan during his visit to the Simons Center in June 2019. The aim of this note is to give an answer for the case when the chain map  $f_1$  in (1) over which the transfer is performed is a *chain homotopy equivalence* as in (7). That is, in addition to the hypothesis that  $g_1 f_1$  is chain homotopic to the identity  $\text{id}_A$ , we also assume that  $f_1 g_1$  is chain homotopic to the identity  $\text{id}_{A'}$ . If the ground ring is a field, then this is the same as being a quasi-isomorphism.<sup>2</sup>

**Conventions.** All algebraic objects in Sec. 2–5 are defined over a fixed commutative unital ring  $R$ , except Sec. 4.3 where  $R$  is a field. In Sec. 6, we restrict to the case  $R = \mathbb{k}$ , where  $\mathbb{k}$  is a field of characteristic zero. All graded objects are  $\mathbb{Z}$ -graded and unbounded; we use homological conventions for all dg objects. Given graded  $R$ -modules  $V$  and  $W$ , we denote by  $\text{Hom}_R(V, W)$  the graded  $R$ -module  $\text{Hom}_R(V, W)_n := \prod_{k \in \mathbb{Z}} \text{Hom}_{R\text{Mod}}(V_k, W_{k+n})$ , where  $\text{Hom}_{R\text{Mod}}(-, -)$  denotes the internal hom in the category of  $R$ -modules. We denote by  $\mathbf{s}V$  and  $\mathbf{s}^{-1}V$  the suspension and desuspension, respectively, of the graded module  $V$ . Concretely,  $(\mathbf{s}V)_n := V_{n-1}$  and  $(\mathbf{s}^{-1}V)_n := V_{n+1}$ .

Conventions and notations for  $A_\infty$ -algebras and their weak and strict morphisms are taken from [16, Sec. 2]. In Sec. 6, which is separate from the rest of the paper, we assume some familiarity with Koszul operads and homotopy operadic algebras as in [14, Ch. 10].

## 2. SUMMARY OF RESULTS

Suppose that  $(A, d, \boldsymbol{\mu}) = (A, d, \mu_2, \mu_3, \dots)$  is an  $A_\infty$ -algebra,  $(A', d')$  a chain complex, and  $f_1 : (A, d) \rightarrow (A', d')$  a chain map which is a chain homotopy equivalence. Then it is well known (see Sec. 3) that there exists a transferred  $A_\infty$ -structure

$$(A', d', \boldsymbol{\nu}) = (A', d', \nu_2, \nu_3, \dots)$$

on  $(A', d')$  and a lift of  $f_1$  to a weak  $A_\infty$ -morphism  $\mathbf{f} = (f_1, f_2, f_3, \dots) : (A, d, \boldsymbol{\mu}) \rightarrow (A', d', \boldsymbol{\nu})$ . Now suppose that

$$(A', d', \boldsymbol{\mu}') = (A', d', \mu'_2, \mu'_3, \dots)$$

is another  $A_\infty$ -structure on  $(A', d')$ . For the purposes of exposition, let us begin with an approximation to Sullivan’s question.

**Question 1.** In the situation above, is the  $A_\infty$ -structure  $\boldsymbol{\mu}' = \{\mu'_2, \mu'_3, \dots\}$  on the complex  $(A', d')$  equivalent to a transferred structure?

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<sup>2</sup>That is a chain map inducing a homology isomorphism.

We need to specify what precisely “is equivalent” and the adjective “transferred” mean in the above sentence. Let us start with the former one; transferred structures will be treated in the next section. First, by “is equivalent”, we could mean that there exists a weak  $A_\infty$ -morphism

$$(2) \quad \phi = (\phi_1, \phi_2, \phi_3, \dots): (A', d', \nu) \rightarrow (A', d', \mu')$$

such that one of the following cases is satisfied:

Case	Relationship between $(A', d', \nu)$ and $(A', d', \mu')$	Criterion for chain map $\phi_1: (A', d') \rightarrow (A', d')$	Criteria for higher maps $\phi_{k \geq 2}$
<i>Strict Cases</i>			
1	equality	$\phi_1 = \text{id}_{A'}$	$\phi_k = 0 \ \forall k \geq 2$
2	strictly isomorphic	$\phi_1$ is an automorphism	$\phi_k = 0 \ \forall k \geq 2$
<i>Weak Cases</i>			
3	isotopic	$\phi_1 = \text{id}_{A'}$	none
4	weakly isomorphic	$\phi_1$ is an automorphism	none

Other relationships are possible. Recall that a weak  $A_\infty$ -morphism such as  $\phi$  in (2) is an  $A_\infty$ -**quasi-isomorphism**, or  $A_\infty$ -**quism** for short, if  $\phi_1$  is a quasi-isomorphism of chain complexes. We say  $(A', d', \nu)$  and  $(A', d', \mu')$  are **weakly equivalent** or, depending on the context, that they have the same **homotopy type** if they are connected by a zigzag of  $A_\infty$ -quisms. Then “is” in Question 1 could also mean:

<i>Homotopical Cases</i>		
Case	Relationship between $(A', d', \nu)$ and $(A', d', \mu')$	Criteria for weak morphisms between $(A', d', \nu)$ and $(A', d', \mu')$
5a	$A_\infty$ -quasi-isomorphic	$\exists A_\infty$ -quism $\phi: (A', d', \nu) \rightarrow (A', d', \mu')$
5b		$\exists A_\infty$ -quism $\psi: (A', d', \mu') \rightarrow (A', d', \nu)$
6	weakly equivalent	$\exists A_\infty$ -quisms $(A', d', \nu) \leftarrow \bullet \rightarrow (A', d', \mu')$

**2.1. Main results.** There are seven variations of Question 1 to consider. Cases 1 and 2 involve comparing the isomorphism class of  $(A', d', \mu')$  to that of  $(A', d', \nu)$  in the category of  $A_\infty$ -algebras and strict morphisms. As we show in Sec. 4.1, these turn out to be the only cases in which explicit formulas for the transferred structure actually matter.

Cases 3 and 4 involve comparing isomorphism classes in the category of  $A_\infty$ -algebras and weak morphisms, while the remaining three concern isomorphism classes in the corresponding “homotopy category”. The characterization via isotopy, Case 3, is perhaps the most interesting. In Thm. 2, we exhibit a precise relationship between the isotopy class of a transfer and the homotopy type of  $(A, d, \mu)$ . In particular, any  $A_\infty$ -structure which is a target of an  $A_\infty$ -quism is isotopic to a transferred one (Cor. 3). Our proofs of these results are based on an obstruction theory for  $A_\infty$ -morphisms, which we develop in Sec. 5.

When  $R$  is a field, we prove in Sec. 4.3 that the three homotopical cases, 5a, 5b, and 6, are all equivalent to the existence of an  $A_\infty$ -quism between the original  $A_\infty$ -structure  $(A, d, \mu)$  and  $(A', d', \mu')$ .

Furthermore, as we show in Cor. 8, our results for Case 3 also provide a positive answer to Sullivan’s original question, which we can now state precisely:

**Question 2.** Can one formulate, in terms of the initial data  $\mu$  and  $f_1$  as above, the necessary and sufficient conditions for the  $A_\infty$ -structure  $\mu'$  to be isotopic to a structure transferred over the chain homotopy equivalence  $f_1$ ?

Finally, in Sec. 6, we generalize Thm. 2 to the transfer of  $\mathcal{P}_\infty$ -structures over a field  $\mathbb{k}$  with  $\text{char } \mathbb{k} = 0$ , for any quadratic Koszul operad  $\mathcal{P}$ .

### 3. A REMINDER ON TRANSFERS

We recall some basic features of transferred  $A_\infty$ -structures. The initial data are an  $A_\infty$ -algebra  $(A, d, \mu) = (A, d, \mu_2, \mu_3, \dots)$ , a chain complex  $(A', d')$ , and a chain map  $f_1 : (A, d) \rightarrow (A', d')$ . A **transfer** of  $(A, d, \mu)$  **over a chain map**  $f_1$  is an  $A_\infty$ -structure  $(A', d', \nu) = (A', d', \nu_2, \nu_3, \dots)$  on  $A'$  and an extension

$$(3) \quad \mathbf{f} = (f_1, f_2, f_3, \dots) : (A, d, \mu_2, \mu_3, \dots) \longrightarrow (A', d', \nu_2, \nu_3, \dots)$$

of the chain map  $f_1$  into a weak  $A_\infty$ -morphism.

There are two standard situations in which such transfers are known to exist: the “homology setup” and the “homotopy setup”. In the former scenario, the transferred structure  $\nu$  and the extension  $\mathbf{f}$  are built inductively via homological obstruction theory, so that the end result is non-canonical. A prototype of transfer theorems of this kind was established by T. Kadeišvili in his seminal paper [12]. A very general formulation [18, Theorem 2] together with a historical account can be found in the recent paper of D. Petersen. In that work,  $f_1$  is assumed to induce a quasi-isomorphism of certain hom complexes.

**3.1. The homotopy setup.** This is the formalism which we will use in the present work. It was thoroughly developed in [16], with special cases and partial results appearing earlier in the work of M. Kontsevich and Y. Soibelman [10], and S. Merkulov [17]. In this approach, which is valid over an arbitrary commutative ring, a transfer exists provided that we have a left homotopy inverse  $g_1$  to  $f_1$ , and a chain homotopy  $h : g_1 f_1 \simeq \text{id}_A$ , as in Eq. 1.

The homotopy setup, in fact, yields explicit formulas for the transfer and much more. Fix a left homotopy inverse  $g_1$  of  $f_1$ , and a chain homotopy  $h$ , as above. Then the formulas in [16] produce an *explicit*  $A_\infty$ -structure  $(A', d', \nu) = (A', d', \nu_2, \nu_3, \dots)$  on  $A'$ , an *explicit* extension  $\mathbf{f} : (A, d, \mu) \rightarrow (A', d', \nu)$  of the chain map  $f_1$ , as well as an *explicit* extension

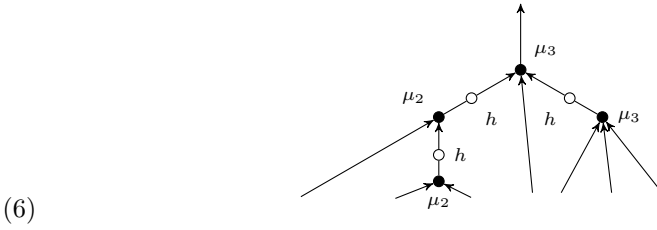
$$(4) \quad \mathbf{g} = (g_1, g_2, g_3, \dots) : (A', d', \nu) \longrightarrow (A, d, \mu)$$

of the chain map  $g_1$ , and an *explicit* extension  $\mathbf{h} = (h, h_2, h_3, \dots)$  of the homotopy  $h$ . The extension  $\mathbf{g}$  plays a crucial role in Sec. 4, but  $\mathbf{h}$  will not be needed.

We recall the formulas for the transferred structure  $(A', d', \nu)$ . According to the Ansatz [16, Eq. 1], the structure operations  $\nu_n$  are of the form

$$(5) \quad \nu_n := f_1 \circ \mathbf{p}_n \circ g_1^{\otimes n}, \quad n \geq 2,$$

where the **p-kernels** [16, Section 4]  $\mathbf{p}_n : A^{\otimes n} \rightarrow A$  are defined as follows. Let  $\mathbf{P}_n$  denote the set of planar rooted trees whose vertices all have at least two incoming edges, with internal edges decorated by the symbol  $\circlearrowleft$ , and which have  $n$  leaves. Elements of  $\mathbf{P}_n$  encode maps and their compositions. For example, the tree



is an element of  $\mathcal{P}_7$ . We assign to every tree  $T \in \mathcal{P}_n$  a map  $F_T: A^{\otimes n} \rightarrow A$  such that each  $\circ$  corresponds to the homotopy  $h: A \rightarrow A$ , and each vertex with  $k$  incoming edges corresponds to the map  $\mu_k: A^{\otimes k} \rightarrow A$ . For example, the tree  $T$  in (6) is assigned to the degree 5 map  $F_T = \mu_3(h \circ \mu_2(\text{id}_A \otimes h \circ \mu_2) \otimes \text{id}_A \otimes h \circ \mu_3): A^{\otimes 7} \rightarrow A$ . The  $\mathfrak{p}$ -kernels in (5) are then given by

$$\mathfrak{p}_n := \sum_{T \in \mathcal{P}_n} (-1)^{\vartheta(T)} \cdot F_T, \quad n \geq 2,$$

where the sign  $(-1)^{\vartheta(T)}$  depends<sup>3</sup> on the number of subtrees in  $T$  of a certain type. Notice that, while  $\nu_2 = f_1 \circ \mu_2 \circ (g_1 \otimes g_1)$ , the higher arity transfer operations  $\nu_{n \geq 3}$  depend on the homotopy  $h$ , as well as  $f_1$  and  $g_1$ .

In the rest of the paper,  $f_1$  will always be a chain homotopy equivalence. We will call the explicit transfer given by (5) the **transfer over a chain homotopy equivalence**  $f_1$ , in contrast to a less specific transfer over a chain map  $f_1$  defined at the beginning of this section.

#### 4. CLASSIFYING TRANSFERRED STRUCTURES

In this section, we present the main results previously summarized in Sec. 2.1. In 4.1–4.3, we address the strict isomorphism, weak isomorphism, and homotopical variations of Question 1, giving us seven cases in total to consider. In Sec. 4.3, we also address Question 2, the precise version of D. Sullivan’s original query.

The starting point for all results in this section is an  $A_\infty$ -algebra  $(A, d, \mu)$ , a *chain homotopy equivalence*  $f_1: (A, d) \rightarrow (A', d')$ , and an  $A_\infty$ -algebra  $(A', d', \mu')$ . The goal is to compare the latter  $A_\infty$ -algebra to a transfer  $(A', d', \nu)$  of the former over  $f_1$  via the “homotopy setup” from Sec. 3.1.

**4.1. Strict isomorphism: Cases 1 and 2.** These are the only variations of Question 1 in which explicit formulas for the transfer matter. We simply check whether the operations  $\mu'_n$  of  $(A', d', \mu')$  are either: (1) equal to the operations  $\nu_n$  defined via Eq. (5) or (2) equal to a twist of these operations by the automorphism  $\phi_1: (A', d') \xrightarrow{\cong} (A', d')$ .

*Remark 1.* Characterizing transfers via strict isomorphism leads to an interesting side question which is also related to Thm. 2. Suppose that we are given a weak  $A_\infty$ -morphism  $\mathbf{F}: (A, d, \mu) \rightarrow (A', d', \mu')$  which extends a *quasi-isomorphism* of chain complexes  $f_1$ . That is,  $(A', d', \mu')$  is a transfer of  $(A, d, \mu)$  over a chain map  $f_1$ . Can one enhance  $f_1$  into a homotopy data such that  $(A', d', \mu')$  equals, or is strictly isomorphic to, the transfer (5) of  $(A, d, \mu)$  over a chain homotopy equivalence  $f_1$ ?

The answer is no in general, as the following example shows. Let  $(A, d, \mu)$  be the free associative  $R$ -algebra  $R\langle x \rangle$  generated by an element  $x$  of degree 0. Interpret

<sup>3</sup>We will not need the precise definition of  $(-1)^{\vartheta(T)}$ ; see [16, Prop. 6] for details.

$R\langle x \rangle$  as an  $A_\infty$ -algebra with the trivial differential and all structure operations except  $\mu_2$  trivial. Let  $(A', d', \boldsymbol{\mu}')$  be the free associative  $R$ -algebra  $R\langle x, u, \bar{u} \rangle$  generated by  $x$  of degree 0,  $u$  of degree 2 and  $\bar{u}$  of degree 1, with the differential given by  $d'x = d'\bar{u} := 0$ , and  $d'u := \bar{u}$ . Finally, let  $\mathbf{F}: (R\langle x \rangle, d = 0) \rightarrow (R\langle x, \bar{u}, u \rangle, d')$  be the dg algebra morphism  $\mathbf{F}(x) := x$ , viewed as a strict  $A_\infty$ -morphism  $\mathbf{F} = (f_1, 0, 0, \dots)$  with  $f_1 := \mathbf{F}$ .

Consider the possible left homotopy inverses  $g_1$  of  $f_1$ . Since the differential of  $R\langle x \rangle$  is trivial,  $g_1$  must be a strict inverse, and we easily see that the only possibility is that  $g_1(x^k) := x^k$  for  $k \geq 0$ , while  $g_1$  is trivial on the remaining elements of  $R\langle x, \bar{u}, u \rangle$ . Moreover, the homotopy  $h$  witnessing  $g_1 \circ f_1 \simeq \text{id}$  must be zero for degree reasons. Hence, the formulas (5) for the transferred  $A_\infty$ -structure give us

$$\nu_2(a, b) := \begin{cases} \mu'_2(a, b) & \text{if } a = x^k, b = x^l \text{ for some } k, l \geq 0, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

while  $\nu_n := 0$  for  $n \geq 3$ . It is easy to check that this transferred structure is neither equal to nor strictly isomorphic to  $R\langle x, \bar{u}, u \rangle$ .

**4.2. Weak isomorphism: Cases 3 and 4.** This is the most interesting variation of Question 1. The main technical tool used here is the obstruction theory developed in the next section (Sec. 5). We start with Case 3, which concerns the isotopy class of a transferred structure  $(A', d', \boldsymbol{\nu})$ .

**Theorem 2.** *Given a chain homotopy equivalence  $f_1 : (A, d) \rightarrow (A', d')$ , an  $A_\infty$ -structure  $(A', d', \boldsymbol{\mu}')$  on  $(A', d')$  is a transfer over the chain map  $f_1$  if and only if it is isotopic to the transfer over the chain homotopy equivalence  $f_1$ .*

*Proof.* Assume that  $(A', d')$  is a transfer over the chain map  $f_1$ , i.e.  $f_1$  extends to a weak  $A_\infty$ -morphism  $\mathbf{F} = (f_1, F_2, F_3, \dots) : (A, d, \boldsymbol{\mu}) \rightarrow (A', d', \boldsymbol{\mu}')$  and promote the chain homotopy equivalence  $f_1$  to the data

$$(7) \quad h \circlearrowleft (A, d) \xrightleftharpoons[g_1]{f_1} (A', d') \circlearrowright l \quad g_1 f_1 - \text{id}_A = dh + hd, \quad f_1 g_1 - \text{id}_{A'} = d'l + ld'.$$

Let  $(A', d', \boldsymbol{\nu})$  be the structure transferred over  $f_1$  using  $f_1, g_1$  and  $h$ . Then  $f_1$  can be extended to a weak  $A_\infty$ -morphism  $\mathbf{f} : (A, d, \boldsymbol{\mu}) \rightarrow (A', d', \boldsymbol{\nu})$  as in (3) and  $g_1$  can also be extended to  $\mathbf{g} : (A', d', \boldsymbol{\nu}) \rightarrow (A, d, \boldsymbol{\mu})$  as in (4). The linear term  $(\mathbf{F} \circ \mathbf{g})_1$  of the composition  $\mathbf{F} \circ \mathbf{g} : (A', d', \boldsymbol{\nu}) \rightarrow (A, d, \boldsymbol{\mu})$  equals  $f_1 \circ g_1$ , which is homotopic to the identity  $\text{id}_{A'}$  via the homotopy  $l$  in (7). It then follows from Prop. 14 in Sec. 5 that there exists a weak  $A_\infty$ -morphism of the form

$$\boldsymbol{\phi} := (\text{id}_{A'}, \phi_2, \phi_3, \dots) : (A', d', \boldsymbol{\nu}) \rightarrow (A', d', \boldsymbol{\mu}').$$

Hence,  $\boldsymbol{\phi}$  is our desired isotopy.

The opposite implication is simple. If  $\boldsymbol{\phi} : (A', d', \boldsymbol{\nu}) \rightarrow (A', d', \boldsymbol{\mu}')$  is an isotopy, then  $\mathbf{F} := \boldsymbol{\phi} \circ \mathbf{f}$  is a weak  $A_\infty$ -morphism extending  $f_1$ . □

**Corollary 3.** *The isotopy type of the transfer over a given chain homotopy equivalence  $f_1$  does not depend on the choices of  $g_1$  and  $h$  in (1). If  $R$  is a field, then any  $A_\infty$ -structure which is a target of an  $A_\infty$ -quism enhancing  $f_1$  is isotopic to a structure transferred over a chain homotopy equivalence enhancing  $f_1$ .*

In light of Thm. 2, we invite the reader to verify that the two  $A_\infty$ -structures on the complex  $R\langle x, \bar{u}, u \rangle$  considered in Remark 1 are indeed isotopic. On the other

hand, if  $f_1: 0 \rightarrow (A', d')$  is the trivial chain map, then the class of transfers over  $f_1$  consists of all  $A_\infty$ -structures on  $(A', d')$ . Hence, the class of transfers over a chain map will not equal the isotopy class of a given transfer, in general.

The analogous result for Case 5, which involves weak isomorphism classes, is:

**Theorem 4.** *Given a chain homotopy equivalence  $f_1: (A, d) \rightarrow (A', d')$ , an  $A_\infty$ -structure  $(A', d', \mu')$  is a transfer of  $(A, d, \mu)$  over the chain map  $\phi_1 \circ f_1$  for some automorphism  $\phi_1: (A', d') \xrightarrow{\cong} (A', d')$  if and only if it is weakly isomorphic to the transfer of  $(A, d, \mu)$  over the chain homotopy equivalence  $f_1$ .*

The proof is a simple modification of the one given for Thm. 2, so we omit it.

**4.3. The homotopical Cases 5 and 6.** To give sensible answers for these cases, we assume that  $R$  is a field, so that  $f_1$  is a chain homotopy equivalence if and only if it is a quasi-isomorphism of complexes. It then turns out that Cases (5a), (5b), and (6) are equivalent. As before, let  $(A', d', \nu)$  denote a transfer of  $(A, d, \mu)$  over the quasi-isomorphism of complexes  $f_1$ , and let  $(A', d', \mu')$  be an arbitrary  $A_\infty$ -structure on  $(A', d')$ . Below, the relation “ $\simeq$ ” denotes weak equivalence of  $A_\infty$ -algebras.

**Proposition 5.** *The following six conditions are equivalent:*

- (i)  $\exists A_\infty$ -quism  $(A', d', \nu) \rightarrow (A', d', \mu')$ ,
- (ii)  $\exists A_\infty$ -quism  $(A', d', \mu') \rightarrow (A', d', \nu)$ ,
- (iii)  $(A', d', \nu) \simeq (A', d', \mu')$ ,
- (iv)  $\exists A_\infty$ -quism  $(A', d', \mu') \rightarrow (A, d, \mu)$ ,
- (v)  $\exists A_\infty$ -quism  $(A, d, \mu) \rightarrow (A', d', \mu')$ ,
- (vi)  $(A, d, \mu) \simeq (A', d', \mu')$ .

We need Lemma 6, which is [14, Theorem 10.4.4] with  $\mathcal{P}$  the operad for associative algebras; it also follows from an abstract homotopy theoretic argument [9, Sec. 3.7].

**Lemma 6.** *Let  $(B, d, \omega)$  and  $(B', d', \omega')$  be  $A_\infty$ -algebras. Then there exists an  $A_\infty$ -quism  $\alpha: (B, d, \omega) \rightarrow (B', d', \omega')$  if and only if there exists an  $A_\infty$ -quism  $\beta: (B', d', \omega') \rightarrow (B, d, \omega)$  in the opposite direction.*

*Proof of Prop. 5.* Since  $f_1$  is a quasi-isomorphism of chain complexes by assumption,  $(A, d, \mu)$  is weakly equivalent to its transfer  $(A', d', \nu)$ . Weak equivalence is an equivalence relation, therefore (iii) is equivalent to (vi). On the other hand, by definition, a weak equivalence is a zigzag of  $A_\infty$ -quisms, and each arrow of this zigzag can be inverted by Lemma 6. This makes the remaining equivalences clear.  $\square$

**4.4. Necessary and sufficient conditions for isotopy.** We now address Question 2, the precise version of D. Sullivan’s original query. It is motivated by an observation concerning the characterization of weak isomorphism classes, based on a conjecture communicated to the authors by Sullivan. (Recall that isotopy is a special case of a weak isomorphism.) A form of this conjecture is proven as Thm. 7. Cor. 8 is then our answer to Question 2 given in the language of obstruction theory.

Let  $\mathcal{A}_\infty(A, d)$  denote the set of weak isomorphism classes of  $A_\infty$ -structures on a given chain complex  $(A, d)$ . Following [16, Sec. 6], since  $f_1$  is assumed to be a chain homotopy equivalence, choosing homotopy data as in (7) induces maps of sets

$$(8) \quad \text{Tr}_{f,g,h}: \mathcal{A}_\infty(A, d) \rightarrow \mathcal{A}_\infty(A', d'), \quad \text{Tr}_{g,f,l}: \mathcal{A}_\infty(A', d') \rightarrow \mathcal{A}_\infty(A, d).$$



Proposition 10 in [16] implies that the functions  $\text{Tr}_{f,g,h}$  and  $\text{Tr}_{g,f,l}$  are mutually inverse bijections. In Thm. 7, if  $(A, d)$  is a subcomplex of  $(A', d')$ , then an **extension** of an  $A_\infty$ -structure  $\boldsymbol{\mu} = (\mu_2, \mu_3, \dots)$  on  $(A, d)$  to  $(A', d')$  is an  $A_\infty$ -structure  $\boldsymbol{\nu} = (\nu_2, \nu_3, \dots)$  on  $(A', d')$  such that the restriction  $\nu_n|_{A^{\otimes n}}$  equals  $\mu_n$  for  $n \geq 2$ . Note that we formulate the second statement of the theorem in terms of weak isomorphism classes since we do not know whether bijections analogous to those in (8) exist for isotopy classes.

**Theorem 7.** *Suppose  $R$  is a field, and that  $(A, d)$  is a subcomplex of  $(A', d')$  such that the inclusion  $\iota : (A, d) \hookrightarrow (A', d')$  is a quasi-isomorphism. Then*

- (1) *The isotopy class of a transfer of an  $A_\infty$ -structure  $(A, d, \boldsymbol{\mu})$  over the chain map  $\iota$  contains an extension of the family  $\boldsymbol{\mu} = (\mu_2, \mu_3, \dots)$  to  $A'$ .*
- (2) *Moreover, the  $A_\infty$ -structure  $(A, d, \boldsymbol{\mu})$  is characterized, up to weak isomorphism, by the weak isomorphism class of its extension.*

*Proof.* Since we are working over a field, we may promote the initial setup into the data in (7), with  $f_1 := \iota$ ,  $g_1$  a strict left inverse  $\pi$  of  $\iota$ ,  $h := 0$  and  $l$  an arbitrary chain homotopy between  $\iota\pi$  and  $\text{id}_{A'}$ . Formulas (5) then clearly determine the pieces  $\nu_n$ ,  $n \geq 2$ , of the transferred structure as the extensions  $\nu_n := \iota \circ \mu_n \circ \pi^{\otimes n}$  of  $\mu_n$ . Part (1) then follows from Thm. 2.

If the same weak isomorphism class of  $A_\infty$ -structures on  $(A', d')$  contains extensions of two  $A_\infty$ -structures on  $(A, d)$ , then these structures must be weakly isomorphic since the maps (8) of weak isomorphism classes are bijections. This proves part (2) of the theorem.  $\square$

Returning to the general situation over an arbitrary commutative ring, Thm. 2 combined with Cor. 11 provides a characterization of transfers up to isotopy.

**Corollary 8.** *The obstruction to exhibit an isotopy between  $(A', d', \boldsymbol{\mu}')$  and the transfer of  $(A, d, \boldsymbol{\mu})$  over a chain homotopy equivalence  $f_1 : (A, d) \rightarrow (A', d')$  is an infinite sequence of homology classes determined by  $\boldsymbol{\mu}$ ,  $f_1$ , and  $\boldsymbol{\mu}'$ :*

$$\left\{ [\kappa_n] \in H_{n-2}(\text{Hom}_R(A^{\otimes n}, A')) \mid n \geq 2 \right\},$$

where the differential on the complex  $\text{Hom}_R(A^{\otimes n}, A')$  is the canonical one induced by  $d$  and  $d'$ .

### 5. OBSTRUCTION THEORY FOR $A_\infty$ -MORPHISMS

We develop in this section the tools needed to prove Thm. 2, Thm. 4, and Cor. 8. We begin by recalling some basic facts concerning dg coalgebras and  $A_\infty$ -algebras, following [13, Sec. 2] and [14, Sec. 1.26].

**5.1. Coalgebras and the bar construction.** Let  $V$  be a graded  $R$ -module. We denote by  $(\bar{T}^c(V), \bar{\Delta})$  the reduced cofree conilpotent coassociative coalgebra generated by  $V$ . Recall that this is the graded coalgebra with underlying  $R$ -module  $\bigoplus_{n \geq 1} V^{\otimes n}$  equipped with the comultiplication  $\bar{\Delta}(v_1 \otimes v_2 \otimes \dots \otimes v_n) := \sum_{i=1}^{n-1} (v_1 \otimes v_2 \otimes \dots \otimes v_i) \otimes (v_{i+1} \otimes \dots \otimes v_n)$ . We denote by  $\bar{\Delta}_{(n)} : \bar{T}^c(V) \rightarrow \bar{T}^c(V)^{\otimes n+1}$  the  $n$ th reduced diagonal: the  $R$ -linear map defined recursively as  $\bar{\Delta}_{(0)} := \text{id}$ ,  $\bar{\Delta}_{(1)} := \bar{\Delta}$ , and  $\bar{\Delta}_{(n)} := (\bar{\Delta} \otimes \text{id}^{\otimes (n-1)}) \circ \bar{\Delta}_{(n-1)}$  for  $n > 1$ . By construction, for  $k < n$  we have

$$(9) \quad v_1 \otimes v_2 \otimes \dots \otimes v_k \in \ker \bar{\Delta}_{(n-1)} \quad \forall v_1, \dots, v_k \in V.$$



Given a linear map  $F: \bar{T}^c(V) \rightarrow \bar{T}^c(W)$  and integers  $m, n \geq 1$ , we denote by  $F_n^m: V^{\otimes n} \rightarrow W^{\otimes m}$  the restriction  $F|_{V^{\otimes n}}$  composed with the projection  $\bar{T}^c(W) \rightarrow W^{\otimes m}$ . In addition, linear maps corresponding to elements of the graded  $R$ -module

$$(10) \quad \text{Hom}_R(\bar{T}^c(V), W) \cong \prod_{n \geq 1} \text{Hom}_R(V^{\otimes n}, W)$$

will be denoted as  $F^1 = (F_1^1, F_2^1, \dots)$ . Recall that there is a one-to-one correspondence [13, Sec. 2.1] between degree  $-1$  linear maps  $D^1 \in \text{Hom}_R(\bar{T}^c(V), V)$  and degree  $-1$  coderivations  $D: \bar{T}^c(V) \rightarrow \bar{T}^c(V)$  given explicitly by

$$(11) \quad D_n^m := \sum_{\substack{i+j=m-1 \\ i, j \geq 0}} \text{id}^{\otimes i} \otimes D_{n-m+1}^1 \otimes \text{id}^{\otimes j}$$

for each  $n \geq 1$ . Note that  $D_n^m = 0$  if  $m > n$ . A **codifferential** on  $\bar{T}^c(V)$  is a degree  $-1$  coderivation  $D$  as above satisfying  $D \circ D = 0$ , or equivalently, for all  $n \geq 1$ :

$$(12) \quad \sum_{k=1}^n D_k^1 \circ D_n^k = 0.$$

Analogously, there is a one-to-one correspondence [13, Sec. 2.2] between degree 0 linear maps  $F^1 \in \text{Hom}_R(\bar{T}^c(V), V')$  and coalgebra morphisms  $F: \bar{T}^c(V) \rightarrow \bar{T}^c(V')$ , given explicitly by the formulas

$$(13) \quad F_n^m := \sum_{i_1+i_2+\dots+i_m=n} F_{i_1}^1 \otimes F_{i_2}^1 \otimes \dots \otimes F_{i_m}^1,$$

for each  $n \geq 1$ . In particular,  $F_n^m = 0$  if  $m > n$ . If  $D$  and  $D'$  are codifferentials on  $\bar{T}^c(V)$  and  $\bar{T}^c(V')$ , respectively, then a coalgebra morphism  $F: \bar{T}^c(V) \rightarrow \bar{T}^c(V')$  satisfies  $D' \circ F = F \circ D$  if and only if for all  $n \geq 1$ :

$$(14) \quad \sum_{k=1}^n D'_k{}^1 \circ F_n^k = \sum_{k=1}^n F_k^1 \circ D_n^k.$$

In this case,  $F: (\bar{T}^c(V), D) \rightarrow (\bar{T}^c(V'), D')$  is a **morphism** of dg-coalgebras.

5.1.1. *The bar construction.* Lastly, we recall the functorial assignment of an  $A_\infty$ -algebra  $(A, d, \mu)$  to the coalgebra  $C(A) := \bar{T}^c(\mathfrak{s}A)$  equipped with the codifferential  $\delta: C(A) \rightarrow C(A)$  defined as  $\delta_1^1 := \mathfrak{s} \circ d \circ \mathfrak{s}^{-1}$ , and  $\delta_n^1 := \mathfrak{s} \circ \mu_n \circ (\mathfrak{s}^{-1})^{\otimes n}$ , for  $n \geq 2$ . The assignment is fully faithful: there is a one-to-one correspondence [13, Sec. 2.3] between weak  $A_\infty$ -morphisms  $\mathbf{f}: (A, d, \mu) \rightarrow (A', d', \mu')$  and dg coalgebra morphisms  $F: (C(A), \delta) \rightarrow (C(A'), \delta')$  given by the formulas  $F_n^1 := \mathfrak{s} \circ f_n \circ (\mathfrak{s}^{-1})^{\otimes n}$ , for all  $n \geq 1$ . In what follows,  $C^m(A)$  and  $C^{\leq n}(A)$  denote the graded  $R$ -modules  $(\mathfrak{s}A)^{\otimes n}$  and  $\bigoplus_{k \geq 1}^n (\mathfrak{s}A)^{\otimes k}$ , respectively.

5.2. **Operations on the Hom complex.** Let  $(A, d, \mu)$  and  $(A', d', \mu')$  be  $A_\infty$ -algebras; let  $(C(A), \delta)$  and  $(C(A'), \delta')$  denote their corresponding dg coalgebras. Consider the graded  $R$ -module  $\mathcal{H} := \text{Hom}_R(C(A), \mathfrak{s}A')$ , as defined in (10), equipped with the differential

$$\partial F^1 := \delta_1^1 \circ F^1 - (-1)^m F^1 \circ \delta,$$

where  $F^1: C(A) \rightarrow \mathfrak{s}A'$  is a degree  $m$   $R$ -linear map. Observe that  $(\mathcal{H}, \partial)$  admits a descending filtration of dg submodules  $\mathcal{H} = \mathcal{F}_1\mathcal{H} \supseteq \mathcal{F}_2\mathcal{H} \supseteq \dots$

$$\mathcal{F}_r\mathcal{H} := \{F^1 \in \text{Hom}_R(C(A), \mathfrak{s}A') \mid F^1|_{C^{\leq r-1}(A)} = 0\}.$$

Via the isomorphisms

$$\mathcal{F}_{r-1}\mathcal{H}/\mathcal{F}_r\mathcal{H} \cong \text{Hom}_R(C^{r-1}(A), \mathfrak{s}A') \text{ and } \mathcal{H}/\mathcal{F}_r\mathcal{H} \cong \text{Hom}_R(C^{\leq r-1}(A), \mathfrak{s}A'),$$

it is easy to see that  $(\mathcal{H}, \partial)$  is **complete** with respect to the topology induced by above filtration, i.e.  $\mathcal{H} \cong \varprojlim_r \mathcal{H}/\mathcal{F}_r\mathcal{H}$ .

5.2.1. *A codifferential on  $\bar{T}^c(\mathcal{H})$ .* Given elements  $F_{(1)}^1, F_{(2)}^1, \dots, F_{(n)}^1 \in \mathcal{H}$ , let  $F_{(1)}^1 \otimes F_{(2)}^1 \otimes \dots \otimes F_{(n)}^1 \in \mathcal{H}^{\otimes n}$  denote the usual corresponding tensor.<sup>4</sup> In particular, we denote by  $F^1 \otimes^n$   $n$ -fold tensor product of  $F^1 \in \mathcal{H}$ .

The next result concerns the properties of the linear maps  $\mathcal{Q}_n^1: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$  defined as  $\mathcal{Q}_1^1(F) := \partial F$ , and for  $n \geq 2$

$$(15) \quad \mathcal{Q}_n^1(F_{(1)}^1 \otimes F_{(2)}^1 \otimes \dots \otimes F_{(n)}^1) := \delta'^1_n \circ (F_{(1)}^1 \otimes F_{(2)}^1 \otimes \dots \otimes F_{(n)}^1) \circ \bar{\Delta}_{(n-1)}.$$

Note that (9) implies that the maps  $\mathcal{Q}_{n \geq 2}^1$  are compatible with the filtration on  $\mathcal{H}$ , i.e.

$$(16) \quad \mathcal{Q}_n^1(\mathcal{F}_{j_1}\mathcal{H}, \mathcal{F}_{j_2}\mathcal{H}, \dots, \mathcal{F}_{j_n}\mathcal{H}) \subseteq \mathcal{F}_{j_1+j_2+\dots+j_n}\mathcal{H}.$$

A variation of Lemma 9 was given in [7, Sec. 4] and [7, Sec. 7.2] for the case when  $R$  is a field.

**Lemma 9.**

- (1) *The linear maps  $\{\mathcal{Q}_n^1\}_{n \geq 1}$  induce, via the formulas (11), a degree  $-1$  codifferential  $\mathcal{Q}$  on the coalgebra  $\bar{T}^c(\mathcal{H})$ .*
- (2) *Given a degree 0 element  $F^1 \in \mathcal{H}$ , the assignment*

$$(17) \quad F^1 \mapsto \mathcal{R}(F) := \sum_{n=1}^{\infty} \mathcal{Q}_n^1(F^1 \otimes^n) \in \mathcal{H}_{-1}$$

*induces a well-defined set-theoretic function  $\mathcal{R}: \mathcal{H}_0 \rightarrow \mathcal{H}_{-1}$ . Moreover,  $\mathcal{R}(F) = 0$  if and only if  $F^1$  corresponds, via the formulas (13), to a dg coalgebra morphism  $F: (C(A), \delta) \rightarrow (C(A'), \delta')$*

- (3) *For all  $F^1 \in \mathcal{H}_0$ , the following identity holds:*

$$(18) \quad \mathcal{Q}_1^1 \mathcal{R}(F) + \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} \mathcal{Q}_n^1(F^1 \otimes^{(n-1)-k} \otimes \mathcal{R}(F) \otimes F^1 \otimes^k) = 0.$$

*Proof.* (1) Note that  $\mathcal{Q}_1^1 \circ \mathcal{Q}_1^1 = 0$ , since  $\mathcal{Q}_1^1 = \partial$  is a differential. Let  $n > 1$ . Since  $\delta'$  is a codifferential on  $C(A')$ , we have  $\sum_{k=1}^n \delta'^1_k \circ \delta'^1_n = 0$ , and the coLeibniz rule implies that  $\bar{\Delta}_{(n-1)} \circ \delta' = \sum_{i=1}^n (\text{id}^{i-1} \otimes \delta' \otimes \text{id}^{n-i}) \circ \bar{\Delta}_{(n-1)}$ . A direct computation using these equalities, along with Eq. (11), gives  $\sum_{k=1}^n \mathcal{Q}_k^1 \circ \mathcal{Q}_k^1 = 0$ .

(2) Since  $\mathcal{H} = \mathcal{F}_1\mathcal{H}$ , Eq. (16) implies that  $\mathcal{Q}_n^1(F^1 \otimes^n) \in \mathcal{F}_n\mathcal{H}_{-1}$  for all  $n \geq 1$ . Hence, the infinite summation in the definition of  $\mathcal{R}(F)$  converges, since  $\mathcal{H}$  is complete, and so  $\mathcal{R}: \mathcal{H}_0 \rightarrow \mathcal{H}_{-1}$  is a well-defined function. From combining Eq. (13) and Eq. (15) along with the identity  $F_n^m = (F^1)^{\otimes m} \circ \bar{\Delta}_{(m-1)}|_{C^n(A)}$ , it follows that

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<sup>4</sup>The notation  $F_{(i)}^1$  should not be confused with  $F_i^1$ , i.e. the  $i$ th component of an element  $F^1 = (F_1^1, F_2^1, F_3^1, \dots) \in \mathcal{H}$ .

$F^1$  is in the zero locus of  $\mathcal{R}$  if and only if the corresponding coalgebra map  $F$  satisfies Eq. (14).

(3) Let  $F^1 \in \mathcal{H}_0$ . The left-hand side of Eq. (18) is a sum of terms of the form  $s_{m,\ell} := \sum_{k=0}^{m-1} F^1 \otimes^{(m-1)-k} \otimes \mathcal{Q}_\ell^1(F^1 \otimes^\ell) \otimes F^1 \otimes^k$  for  $m, \ell \geq 1$ . From Eq. (11), we deduce that  $s_{m,\ell} = \mathcal{Q}_m^1 \circ \mathcal{Q}_{m+\ell-1}^m(F^1 \otimes^{m+\ell-1})$ . The desired equality (18) will then follow from Eq. (12), or equivalently, the fact that  $\mathcal{Q} \circ \mathcal{Q} = 0$ .  $\square$

Prop. 10 shows that  $(\bar{T}^c(\mathcal{H}), \mathcal{Q})$  encodes the obstruction theory for dg coalgebra morphisms between  $(C(A), \delta)$  and  $(C(A'), \delta')$ .

**Proposition 10.** *Let  $m > 1$  and suppose  $\{F_1^1, \dots, F_{m-1}^1\}$  is a collection of degree 0 linear maps  $F_k^1 : C^k(A) \rightarrow \mathfrak{s}A'$  such that the corresponding coalgebra morphism  $F : C(A) \rightarrow C(A')$  satisfies*

$$(\delta' \circ F - F \circ \delta)|_{C^{\leq m-1}(A)} = 0.$$

Then the linear map  $c_m(F) : C^m(A) \rightarrow \mathfrak{s}A'$  defined as

$$(19) \quad c_m(F) := \sum_{k=2}^m \delta_k^{\prime 1} \circ F_m^k - \sum_{k=1}^{m-1} F_k^1 \circ \delta_m^k$$

is a degree  $-1$  cycle in the quotient

$$(\text{Hom}_R(C^m(A), \mathfrak{s}A'), \bar{\partial}) \cong (\mathcal{F}_m \mathcal{H}, \partial) / (\mathcal{F}_{m+1} \mathcal{H}, \partial).$$

Moreover, there exists a linear map  $\tilde{F}_m^1 : C^m(A) \rightarrow \mathfrak{s}A'$  such that the coalgebra morphism  $\tilde{F} : C(A) \rightarrow C(A')$  corresponding to the collection  $\{F_1^1, \dots, F_{m-1}^1, \tilde{F}_m^1\}$  satisfies

$$(\delta' \circ \tilde{F} - \tilde{F} \circ \delta)|_{C^{\leq m}(A)} = 0$$

if and only if  $c_m(F) = -\bar{\partial} \tilde{F}_m^1$ .

*Proof.* The definition of the differential  $\partial = \mathcal{Q}_1^1$  on  $\mathcal{H}$  implies that we may write the induced differential on the quotient as  $\bar{\partial} \tilde{F}_m^1 = (\mathcal{Q}_1^1 \tilde{F}_m^1)|_{C^m(A)} = \delta_1^{\prime 1} \circ \tilde{F}_m^1 - \tilde{F}_m^1 \circ \delta_m^m$ , for any degree 0 map  $\tilde{F}_m^1 \in \text{Hom}_R(C^m(A), \mathfrak{s}A')$ . Hence, the second statement of the proposition follows directly from Eq. (14). It then remains to show that  $\bar{\partial} c_m(F) = 0$ , or, equivalently, that  $\mathcal{Q}_1^1 c_m(F) \in \mathcal{F}_{m+1} \mathcal{H}$ .

Let  $F^1 = (F_1^1, F_2^1, \dots, F_{m-1}^1, 0, 0 \dots) \in \mathcal{H}_0$ . For  $\ell \geq 1$ , let  $\mathcal{R}(F)_\ell := \mathcal{R}(F)|_{C^\ell(A)}$  denote the restriction of the map (17) to the submodule  $C^\ell(A)$ . Then  $\mathcal{R}(F)_\ell = \sum_{k=1}^\ell (\delta_k^{\prime 1} \circ F_\ell^k - F_k^1 \circ \delta_\ell^k)$ . Note that  $F_k^1$  makes no contribution to  $\mathcal{R}_\ell(F^1)$  if  $k > \ell$ . Hence, the hypothesis for the collection  $\{F_1^1, \dots, F_{m-1}^1\}$  implies that  $\mathcal{R}(F)_{\ell \leq m-1} = 0$ , and so we have  $\mathcal{R}(F) \in \mathcal{F}_m \mathcal{H}$ . Since the linear maps  $\{\mathcal{Q}_n^1\}$  are compatible with the filtration on  $\mathcal{H}$ , it follows from Eq. (18) that  $\mathcal{Q}_1^1 \mathcal{R}(F) \in \mathcal{F}_{m+1} \mathcal{H}$ . On the other hand,  $\mathcal{R}(F)_m = c_m(F) \in \mathcal{F}_m \mathcal{H}$ , since the  $k$ th components of  $F^1$  vanish for  $k \geq m$ . Therefore,  $\mathcal{R}(F) - c_m(F) \in \mathcal{F}_{m+1} \mathcal{H}$ , and so we conclude that  $\mathcal{Q}_1^1 c_m(F) \in \mathcal{F}_{m+1} \mathcal{H}$ .  $\square$

Using, for each  $n \geq 2$ , the  $R$ -module isomorphisms  $\text{Hom}_R(C^n(A), \mathfrak{s}A')_{-1} \cong \text{Hom}_R(A^{\otimes n}, A')_{n-2}$ , we obtain as a corollary the obstruction theory for weak  $A_\infty$ -morphisms.

**Corollary 11.** *The obstruction to lifting a chain map  $f_1: (A, d) \rightarrow (A', d')$  to a weak  $A_\infty$ -morphism  $\mathbf{f} = (f_1, f_2, f_3, \dots): (A, d, \boldsymbol{\mu}) \rightarrow (A', d', \boldsymbol{\mu}')$  is an infinite sequence of homology classes*

$$\{[\kappa_n] \in H_{n-2}(\text{Hom}_R(A^{\otimes n}, A')) \mid n \geq 2\},$$

where the differential on the hom complex  $\text{Hom}_R(A^{\otimes n}, A')$  is the canonical one induced by  $d$  and  $d'$ .

**5.3. Algebraic models for the interval.** We will need a notion of homotopy between morphisms of dg coalgebras.

**Definition 12.** A unital dg associative  $R$ -algebra  $(\mathfrak{J}, d, \cdot)$  is a **model for the interval** if there exist unital dg algebra morphisms  $\varepsilon_0, \varepsilon_1: \mathfrak{J} \rightarrow R$ , and  $j: R \rightarrow \mathfrak{J}$  such that

- (1) The composition  $R \xrightarrow{j} \mathfrak{J} \xrightarrow{(\varepsilon_0, \varepsilon_1)} R \times R$  is the diagonal.
- (2) As chain maps, the morphisms  $R \xrightarrow{j} \mathfrak{J} \xrightarrow{\varepsilon_i} R$  are deformation retractions for  $i = 0, 1$ .
- (3) Given two maps between chain complexes  $f, g: (V, d_V) \rightarrow (W, d_W)$ , and a chain homotopy between them, there exists a corresponding chain map  $\tilde{h}: V \rightarrow W \otimes_R \mathfrak{J}$  such that  $(\text{id} \otimes \varepsilon_0) \circ \tilde{h} = f$  and  $(\text{id} \otimes \varepsilon_1) \circ \tilde{h} = g$ .

We recall two examples. The first is the normalized cochain algebra  $N(I) := (N(\Delta^1), d_N, \cup)$  on the 1-simplex with coefficients in  $R$ . As a graded  $R$ -module,  $N(I)_{-1} := R\varphi_I$ , and  $N(I)_0 := R\varphi_0 \oplus R\varphi_1$ . The differential is  $d_N\varphi_0 := \varphi_I$ , and  $d_N\varphi_1 := -\varphi_I$ , and  $\cup$  denotes the usual cup product. Lemma 13 is well known; the proof follows from a straightforward verification, so we omit it.

**Lemma 13.** *The dg  $R$  algebra  $N(I)$  is an algebraic model for the interval over  $R$  when equipped with the morphisms  $j: R \rightarrow N(I)$ , and  $\varepsilon_0, \varepsilon_1: N(I) \rightarrow R$  defined as:  $j(1_R) := \varphi_0 + \varphi_1$ ,  $\varepsilon_0(\varphi_0) := \varepsilon_1(\varphi_1) := 1_R$ , and  $\varepsilon_0(\varphi_1) := \varepsilon_1(\varphi_0) := 0$ .*

The second example is a graded commutative model for the case when  $R = \mathbb{k}$  is a field of characteristic zero. This will be used in Sec. 6. We denote by  $\Omega(I) := (\mathbb{k}[t, dt], d_{\text{dR}}, \wedge)$  the polynomial de Rham algebra on the 1-simplex. As a graded vector space,  $\Omega(I)_{-1} := \mathbb{k}[t]dt$ , and  $\Omega(I)_0 = \mathbb{k}[t]$ . The differential is

$$d_{\text{dR}}(f(t) + g(t)dt) = \frac{df}{dt}dt,$$

and  $\wedge$  is the usual wedge product. The obvious analog of Lemma 13 holds for  $\Omega(I)$ , in which  $\varepsilon_0, \varepsilon_1: \Omega(I) \rightarrow \mathbb{k}$  are the evaluation maps at  $t = 0$ , and  $t = 1$ , respectively.

**5.3.1. Tensoring  $A_\infty$ -algebras with dg algebras.** Recall that if  $(A, d, \boldsymbol{\mu})$  is an  $A_\infty$ -algebra, and  $(B, d_B, \cdot)$  is a dg associative algebra, then the tensor product  $(A \otimes_R B, d_\otimes, \boldsymbol{\mu}_\otimes)$  is an  $A_\infty$ -algebra with  $d_\otimes := d \otimes \text{id} + \text{id} \otimes d_B$ , and

$$\mu_{\otimes k}(x_1 \otimes b_1, x_2 \otimes b_2, \dots, x_k \otimes b_k) := (-1)^\varepsilon \mu_k(x_1, \dots, x_k) \otimes b_1 \cdot b_2 \cdots b_k$$

for  $k \geq 2$ , where  $(-1)^\varepsilon$  is the usual Koszul sign. Note that this construction is functorial: if  $\phi: B \rightarrow B'$  is a morphism of dg algebras then  $\text{id}_A \otimes \phi$  is a strict morphism of  $A_\infty$ -algebras.

**5.4. Lifting chain maps to weak  $A_\infty$ -morphisms.** A special case of the proposition below, valid when  $R$  is a field of characteristic zero, was given in [15, Prop. 35] using different methods. In what follows,  $\theta = (\theta_1, \theta_2, \dots): (A, d, \mu) \rightarrow (A', d', \mu')$  is a given weak  $A_\infty$ -morphism, and  $\Theta: (C(A), \delta) \rightarrow (C(A'), \delta')$  denotes the corresponding morphism of dg coalgebras.

**Proposition 14.** *Suppose  $\psi: (A, d) \rightarrow (A', d')$  is a chain map that is chain homotopic to  $\theta_1: (A, d) \rightarrow (A', d')$ . Then there exists a weak  $A_\infty$ -morphism  $\psi = (\psi_1, \psi_2, \dots): (A, d, \mu) \rightarrow (A', d', \mu')$  such that  $\psi_1 = \psi$ .*

*Proof.* Let  $(\mathcal{J}, d_{\mathcal{J}}, \cdot)$  be an algebraic model of the interval with  $\varepsilon_{(0)}, \varepsilon_{(1)}: \mathcal{J} \rightarrow R, j: R \rightarrow \mathcal{J}$  as described in Def. 12. Let  $J: (C(A'), \delta') \rightarrow (C(A' \otimes \mathcal{J}), \delta'_{\otimes})$  and  $E_{(i)}: (C(A' \otimes \mathcal{J}), \delta'_{\otimes}) \rightarrow (C(A'), \delta')$  for  $i = 0, 1$  denote the dg coalgebra morphisms corresponding to the strict  $A_\infty$ -morphisms  $\text{id}_{A'} \otimes j$ , and  $\text{id}_{A'} \otimes \varepsilon_{(i)}$  respectively.

To prove the proposition, we will use the obstruction theory developed in Prop. 10 to inductively construct a dg coalgebra morphism  $H: (C(A), \delta) \rightarrow (C(A' \otimes \mathcal{J}), \delta'_{\otimes})$  such that the linear component  $\Psi_1^1$  of the composition  $\Psi := E_{(1)} \circ H: (C(A), \delta) \rightarrow (C(A'), \delta')$  satisfies  $\Psi_1^1 = \mathbf{s} \circ \psi \circ \mathbf{s}^{-1}$ .

For the base case, let  $h: A \rightarrow \mathbf{s}^{-1}A'$  be a chain homotopy satisfying  $\psi - \theta_1 = d'h + hd$ . Let  $\tilde{h}: (A, d) \rightarrow (A' \otimes \mathcal{J}, d'_{\otimes})$  be the corresponding chain map as in Def. 12, and denote by  $H: C(A) \rightarrow C(A' \otimes \mathcal{J})$  the coalgebra morphism associated to the linear map  $H_1^1 := \mathbf{s} \circ \tilde{h} \circ \mathbf{s}^{-1}$ . Then by construction

$$(\delta'_{\otimes} \circ H - H \circ \delta)|_{C^1(A)} = 0, \quad (E_{(1)} \circ H)_1^1 = \mathbf{s} \circ \psi \circ \mathbf{s}^{-1}, \quad (E_{(0)} \circ H)_1^1 = \Theta_1^1.$$

Now the inductive step. Let  $m \geq 2$ . Suppose  $H: C(A) \rightarrow C(A' \otimes \mathcal{J})$  is a coalgebra morphism such that

$$(20) \quad \begin{aligned} (\delta'_{\otimes} \circ H - H \circ \delta)|_{C^{\leq m-1}(A)} &= 0, & (E_{(1)} \circ H)_1^1 &= \mathbf{s} \circ \psi \circ \mathbf{s}^{-1}, \\ (E_{(0)} \circ H)_k^1 &= \Theta_k^1 & \text{for } k &= 1, \dots, m-1. \end{aligned}$$

Consider the cycle  $c_m(H) \in (\text{Hom}_R(C^m(A), \mathbf{s}(A \otimes \mathcal{J})), \bar{\partial})$  as defined in Eq. (19). We will show that it is a boundary. Composition with  $E_{(0)1}^1$  and  $J_1^1$  gives chain maps

$$\begin{aligned} E_{(0)*}^1: \text{Hom}_R(C^m(A), \mathbf{s}(A' \otimes \mathcal{J})) &\rightarrow \text{Hom}_R(C^m(A), \mathbf{s}A'), \\ J_*^1: \text{Hom}_R(C^m(A), \mathbf{s}A') &\rightarrow \text{Hom}_R(C^m(A), \mathbf{s}(A' \otimes \mathcal{J})), \end{aligned}$$

respectively. Since  $E_{(0)}$  corresponds to a strict  $A_\infty$ -morphism,  $E_{(0)k}^1 = 0$  for  $k \geq 2$ . Therefore, it follows from the induction hypothesis (20) and the definition of  $c_m(H)$  that  $E_{(0)*}^1(c_m(H)) = c_m(\Theta)$ . Since  $\Theta$  is a dg coalgebra morphism, the cycle  $c_m(\Theta)$  is a boundary. In particular,  $c_m(\Theta) = -\bar{\partial}\Theta_m^1$ . Since  $\mathcal{J}$  is a model for the interval, and tensor product preserves chain homotopy equivalence, there exists a chain homotopy  $\lambda: \mathbf{s}(A' \otimes \mathcal{J}) \rightarrow (A' \otimes \mathcal{J})$  such that the chain maps  $J_1^1$  and  $E_{(0)1}^1$  satisfy  $J_1^1 \circ E_{(0)1}^1 - \text{id}_{\mathbf{s}(A' \otimes \mathcal{J})} = \delta'_{\otimes 1}^1 \circ \lambda + \lambda \circ \delta'_{\otimes 1}^1$ , in addition to  $E_{(0)1}^1 \circ J_1^1 = \text{id}_{\mathbf{s}A'}$ . Moreover,  $\lambda$  induces a chain homotopy equivalence  $\text{Hom}_R(C^m(A), \mathbf{s}(A' \otimes \mathcal{J})) \simeq \text{Hom}_R(C^m(A), \mathbf{s}A')$ . Indeed,  $E_{(0)*}^1 \circ J_* = \text{id}_{\text{Hom}}$  and

$$J_*^1 \circ E_{(0)*}^1 - \text{id}_{\text{Hom}} = \bar{\partial} \circ \lambda_* + \lambda_* \circ \bar{\partial}.$$

The above equation implies that  $-\bar{\partial}J_*^1(\Theta_m^1) - c_m(H) = \bar{\partial}K_m^1$ , where  $K_m^1 := \lambda \circ c_m(H)$ . By construction,  $\bar{\partial}K_m^1$  is a cycle in the dg submodule  $(\ker E_{(0)*}^1, \bar{\partial})$ . Since

$E_{(0)*}^1$  is a deformation retraction,  $\ker E_{(0)*}^1$  is acyclic, and so there exists  $\tilde{K}_m^1 \in \ker E_{(0)*}^1$  such that  $\bar{\partial}\tilde{K}_m^1 = \bar{\partial}K_m^1$ .

Finally, let  $\tilde{H}_m^1 := J_1^1 \circ \Theta_m^1 + \tilde{K}_m^1$ , and denote by  $\tilde{H}: C(A) \rightarrow C(A' \otimes \mathfrak{J})$  the coalgebra morphism corresponding to the collection of linear maps  $\{H_1^1, \dots, H_{m-1}^1, \tilde{H}_m^1\}$ . Then, by construction,  $(E_{(1)} \circ \tilde{H})_1^1 = (E_{(1)} \circ H)_1^1 = \mathfrak{s} \circ \psi \circ \mathfrak{s}^{-1}$ . Furthermore,  $(E_{(0)} \circ \tilde{H})_k^1 = \Theta_k^1$  for  $k = 1, \dots, m$ , and  $c_m(H) = -\bar{\partial}\tilde{H}_m^1$ . By Prop. 10, the latter equation implies that  $(\delta'_\otimes \circ \tilde{H} - \tilde{H} \circ \delta)|_{C^{\leq m}(A)} = 0$ . This completes the induction step, and hence the proof.  $\square$

6. CLASSIFYING TRANSFERS OF  $\mathcal{P}_\infty$ -ALGEBRAS FOR A QUADRATIC KOSZUL OPERAD  $\mathcal{P}$

We describe how to generalize Thm. 2, the classification of transfers up to isotopy, to  $\mathcal{P}_\infty$ -algebras, where  $\mathcal{P}$  is a symmetric operad in graded vector spaces over a field  $\mathbb{k}$  with  $\text{char } \mathbb{k} = 0$ . Furthermore, we assume  $\mathcal{P}$  is a quadratic Koszul operad [14, Sec. 7.2.3]. Examples of such  $\mathcal{P}_\infty$ -algebras include  $L_\infty$ -algebras and  $C_\infty$ -algebras (i.e., homotopy Lie and homotopy commutative algebras, respectively).

Let  $(A, d, \mu_{\mathcal{P}})$  be a  $\mathcal{P}_\infty$ -algebra and  $f_1: (A, d) \rightarrow (A', d')$  a quasi-isomorphism of chain complexes. Let  $h, g_1$ , and  $l$  be homotopy data as in (7). A version of the transfer theorem via the “homotopy setup” from Sec. 3.1 exists in this context provided that  $g_1 f_1 = \text{id}_A$ , and that  $h$  satisfies the *side conditions*:  $h^2 = 0, f_1 h = 0, h g_1 = 0$  [8, Theorem 5]. Assuming that this is the case, we obtain formulas for a transferred structure  $(A', d', \nu_{\mathcal{P}})$ , and weak  $\mathcal{P}_\infty$ -morphisms

$$\mathbf{f}: (A, d, \mu_{\mathcal{P}}) \rightleftarrows (A', d', \nu_{\mathcal{P}}): \mathbf{g},$$

as in (3) and (4). In particular,  $\mathbf{f}$  and  $\mathbf{g}$  are lifts of  $f_1$  and  $g_1$ , respectively, and  $f_1 \circ g_1 \simeq \text{id}_A$ . We now suppose that  $(A', d', \mu'_{\mathcal{P}})$  is another  $\mathcal{P}_\infty$ -algebra on  $(A', d')$ . Our goal is to determine, as in Thm. 2, whether or not it is isotopic to the transferred structure  $(A', d', \nu_{\mathcal{P}})$ .

To address this, we proceed exactly as in the proof of Thm. 2. All that we need is a suitable version of the lifting result from Prop. 14, and the associated obstruction theory behind it, which we now provide. First, in analogy with Sec. 5.1.1, weak  $\mathcal{P}_\infty$ -morphisms  $(A, d, \mu_{\mathcal{P}}) \rightarrow (A', d', \mu'_{\mathcal{P}})$  are equivalent to dg  $\mathcal{P}^i$ -coalgebra morphisms  $(\mathcal{C}(A), \delta) \rightarrow (\mathcal{C}(A'), \delta')$  [14, Sec. 10.2.2]. Here  $\mathcal{C}(V)$  denotes the “cofree”  $\mathcal{P}^i$ -coalgebra generated by the graded vector space  $\mathfrak{s}V$ , where  $\mathcal{P}^i$  is the Koszul dual cooperad of  $\mathcal{P}$  [14, Sec. 10.1.8]. Conveniently, an exact replica of Prop. 10 for  $\mathcal{P}_\infty$ -algebras is given in [22, Thm. A.1]. Next, we recall [22, Sec. 3.1] that the tensor product of any  $\mathcal{P}_\infty$ -algebra with a dg commutative algebra, is also a  $\mathcal{P}_\infty$ -algebra (cf. Sec. 5.3.1). In particular, if  $\Omega(I)$  is the commutative model of the interval from Sec. 5.3, then  $(A' \otimes \Omega(I), d'_\otimes, \mu'_{\mathcal{P}_\otimes})$  is a  $\mathcal{P}_\infty$ -algebra in the obvious way.

Finally, we observe that by setting the dg algebra  $\mathfrak{J}$  to  $\Omega(I)$  in the proof of Prop. 14, we obtain a proof of the analogous statement for lifting chain maps to  $\mathcal{P}_\infty$ -morphisms. All of the required pieces are now in place to extend the proof of Thm. 2 to the  $\mathcal{P}_\infty$ -case:

**Theorem 15.** *Given a surjective chain homotopy equivalence  $f_1: (A, d) \rightarrow (A', d')$  with homotopy data satisfying the aforementioned “side conditions,” there exists a weak  $\mathcal{P}_\infty$ -morphism  $\mathbf{F}: (A, d, \mu_{\mathcal{P}}) \rightarrow (A', d', \mu'_{\mathcal{P}})$  extending  $f_1$  if and only if the*

$\mathcal{P}_\infty$ -algebra  $(A', d', \mu'_{\mathcal{P}})$  is isotopic to a transfer of  $(A, d, \mu_{\mathcal{P}})$  over the chain homotopy equivalence  $f_1$ .

## ACKNOWLEDGMENTS

We express our gratitude to Jim Stasheff, Dennis Sullivan and the referee for useful suggestions and comments that led to substantial improvement of our paper.

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