Distributive laws between the three graces

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Definition (John Beck). Assume $T_1 = (T_1, \mu_1, \eta_1)$ and $T_2 = (T_2, \mu_2, \eta_2)$ are triples on a category $\mathtt{C}$. A distributive law guarantees that for every T_2-algebra A in $\mathtt{C}$, the object $T_2(A) \in \mathtt{C}$ is a T_1-algebra in a very specific way. More precisely, a distributive law is a natural transformation

$$\lambda : T_1T_2 \to T_2T_1,$$

such that, for every T_2-algebra $A = (A, \alpha\colon T_2(A) \to A)$, the object $T_2(A) \in \mathtt{C}$ is a T_1-algebra with structure morphism

$$T_1T_2A \xrightarrow{\lambda} T_2T_1A \xrightarrow{T_2\alpha} T_2A.$$

This imposes certain conditions on λ whose explicit form can be found in the literature. The endofunctor $T = T_2T_1$ is then again a triple, with structure transformations

$$\mu = T_2\mu_1 \circ \mu_2T_1^2 \circ T_2\lambda T_1, \qquad \eta = \eta_1 \circ \eta_2 \circ T_1.$$

The equality

$$T(X) = T_2\big(T_1(X)\big), \qquad X \in \mathtt{C},$$

may be interpreted as saying that the free T-algebra on X is (as an object of $\mathtt{C}$) naturally isomorphic to the free T_2-algebra generated by the free T_1-algebra on X.

'Definition.' A distributive law is given by a rewriting rule

$$\mathcal{D} : \bullet(\circ) \rightsquigarrow \circ(\bullet)$$

specifying how to rewrite iterated products involving $\bullet$ and $\circ$ to expressions where all $\bullet$'s go first ($\bullet$ is 'heavier' so it goes down). The rewriting rule determines a distributive law if it is coherent in that it does not generate unexpected relations. We say that the rewriting rule (and the induced distributive law) is

(i) operadic if it does not involve repetition of variables (it is *injective* in the terminology of universal algebra).
(ii) homogeneous if all terms have the same bidegree in $(\bullet, \circ)$, and
(iii) quadratic if all terms contain precisely two operations.

Notice that (ii)+(iii) means that all terms are of bidegree $(1, 1)$.

Example. An *inhomogeneous* quadratic operadic distributive law is e.g. the one which describes associative algebras as algebras with two operations, a commutative nonassociative multiplication $- \cdot -$ and a Lie bracket $[-,-]$, with the 'Leibniz rule'

$$[a \cdot b, c] = a \cdot [b,c] + [a,b] \cdot c \tag{1}$$

plus

$$[b,[a,c]] = (a \cdot b) \cdot c - a \cdot (b \cdot c)$$

whose left hand side has bidegree $(2,0)$ while the right hand side bidegree $(0,2)$. Notice that the green term is the deviation from the associativity of $- \cdot -$.

Example. Poisson algebras have a Lie bracket $[-,-]$ and a commutative associative multiplication $- \cdot -$ tied by the rewrite rule (1) of the form

$$\mathcal{D} : \mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie).$$

Coherence is manifested by the isomorphism

$$\mathbf{Pois}(X) \cong \mathbf{Com}(\mathbf{Lie}(X)).$$

This and all the remaining rewrite rules (distributive laws) will be homogeneous, quadratic and operadic.

Example. Non-symmetric Poisson algebras – two associative noncommutative multiplications $\bullet$ and $\circ$ tied by the distributive law:

$$(x \circ y) \bullet z \ = \ x \circ (y \bullet z), \ x \bullet (y \circ z) \ = \ (x \bullet y) \circ z.$$

or, denoting $\langle a, b \rangle := a \bullet b$, $ab := a \circ b$,

$$\langle x, yz \rangle = \langle x, y \rangle z, \ \langle xy, z \rangle = x \langle y, z \rangle.$$

It is a perfectly self-dual rewrite rule $\mathcal{A}ss(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{A}ss)$.

Example of the coherence check:

The compatibility of the distributive law for Poisson algebras with the anticommutativity of $[-,-]$:

$$[ab, cd] = -[cd, ab]. \tag{2}$$

Expanding the left-hand side using the distributive law twice gives:

$$\begin{aligned} [ab, cd] &= a[b, cd] + [a, cd]b \\ &= ac[b, d] + a[b, c]d + c[a, d]b + [a, c]db. \end{aligned}$$

Expanding the right-hand side using the distributive law twice gives:

$$\begin{aligned} -[cd, ab] &= -c[d, ab] - [c, ab]d \\ &= -ca[d, b] - c[d, a]b - a[c, b]d - [c, a]bd. \end{aligned}$$

Applying the anti-symmetry of $[-,-]$ leads to:

$$-[cd, ab] = ca[b, d] + c[a, d]b + a[b, c]d + [a, c]bd.$$

Rearranging the terms, we get the tautological equality

$$\begin{gathered} ac[b, d] + c[a, d]b + a[b, c]d + [a, c]db \\ \| \\ ca[b, d] + c[a, d]b + a[b, c]d + [a, c]bd \end{gathered}$$

showing that (2) is compatible with the distributive law.

Notice that in order to compare the first and the last terms, we need to know that $ac = ca$ and that $bd = db$, i.e. that the $\cdot$ -product is commutative, cf. the non-example of non-commutative Poisson algebras!

The compatibility with the associativity and commutativity of the $\cdot$ -product, and the Jacobi identity can be verified similarly.

Why it is reasonable to restrict to *operadic homogeneous quadratic* (abbrev. OHQ) distributive laws? Some 20 years ago in Chapel Hill, while visiting Jim, I proved:

Theorem. For an OHQ distributive law (rewrite rule) it is enough to check the coherence for products of 4 elements only. Therefore we face only a finite, though possibly very big, number of equations.

Moreover, one has the following superimportant:

Theorem. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are quadratic Koszul operads and

$$\mathcal{D}\colon \mathcal{P}_1(\mathcal{P}_2) \rightsquigarrow \mathcal{P}_2(\mathcal{P}_1)$$

a OHQ, then the combination $\mathcal{P}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ via $\mathcal{D}$ is Koszul, too.

Corollary. Poisson, non-commutative Poisson (properly defined), non-symmetric Poisson, Gerstenhaber, &c. are Koszul.

Theorem. To each OHQ

$$\mathcal{D}\colon \mathcal{P}_1(\mathcal{P}_2) \rightsquigarrow \mathcal{P}_2(\mathcal{P}_1)$$

one has the canonical dual OHQ distributive law

$$\mathcal{D}^!\colon \mathcal{P}_2^!(\mathcal{P}_1^!) \rightsquigarrow \mathcal{P}_1^!(\mathcal{P}_2^!),$$

such that the resulting combined operad $\mathcal{Q}$ is the Koszul dual of the operad $\mathcal{P}$ which is the combination of $\mathcal{P}_1$ and $\mathcal{P}_2$ using $\mathcal{D}$.

Thanks to this theorem, in order to describe *all* OHQ distributive laws between the Three graces, it is enough to investigate the following cases:

- distributive laws $\mathcal{A}ss(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{A}ss)$,
- distributive laws $\mathcal{A}ss(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{A}ss)$,
- distributive laws $\mathcal{A}ss(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{A}ss)$,
- distributive laws $\mathcal{C}om(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{C}om)$,
- distributive laws $\mathcal{C}om(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{C}om)$, and
- distributive laws $\mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie)$.

The remaining cases, i.e.

- distributive laws $\mathcal{L}ie(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{L}ie)$,
- distributive laws $\mathcal{C}om(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{C}om)$, and
- distributive laws $\mathcal{L}ie(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{L}ie)$

follow by taking the Koszul dual of the appropriate cases of the first list. The colors indicate the correspondence, the black ones are self-dual.

Distributive laws $\mathcal{A}ss(\mathcal{A}ss) \rightsquigarrow \mathcal{A}ss(\mathcal{A}ss)$.

Analyzing the coherence leads to quadratic equations in 24 variables. Solving them requires at some point a 960×960 -matrix. Murray's computer found the following:

Theorem. The only distributive laws between two associative multiplications are given by

$$
\begin{array}{lrclrcl}
(1) & (x \circ y) \bullet z & = & 0, & x \bullet (y \circ z) & = & 0 \\
(2) & (x \circ y) \bullet z & = & 0, & x \bullet (y \circ z) & = & (x \bullet y) \circ z \\
(3) & (x \circ y) \bullet z & = & x \circ (y \bullet z), & x \bullet (y \circ z) & = & 0 \\
(4) & (x \circ y) \bullet z & = & x \circ (y \bullet z), & x \bullet (y \circ z) & = & (x \bullet y) \circ z \\
(5) & (x \circ y) \bullet z & = & 0, & x \bullet (y \circ z) & = & y \circ (x \bullet z) \\
(6) & (x \circ y) \bullet z & = & (x \bullet z) \circ y, & x \bullet (y \circ z) & = & 0 \\
(7) & (x \circ y) \bullet z & = & (x \bullet z) \circ y, & x \bullet (y \circ z) & = & y \circ (x \bullet z).
\end{array}
$$

Laws of the same color are isomorphic modulo the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action given by $\bullet \to \bullet^{\mathrm{op}}$ resp. $\circ \to \circ^{\mathrm{op}}$.

The black law is the trivial one, the blue laws the truncated ones, and the red laws are the laws for non-symmetric Poisson algebras.

So, up to isomorphism, there are only *three* distributive laws between two associative multiplications.

Corollary. Up to isomorphism, there is only *one* distributive law between two associative multiplications which lives in the category `Sets` of sets, namely the one for non-symmetric Poisson algebras.

What we know about the remaining cases

(1) distributive laws $\mathcal{A}ss(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{A}ss)$,
(2) distributive laws $\mathcal{A}ss(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{A}ss)$,
(3) distributive laws $\mathcal{C}om(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{C}om)$,
(4) distributive laws $\mathcal{C}om(\mathcal{L}ie) \rightsquigarrow \mathcal{L}ie(\mathcal{C}om)$, and
(5) distributive laws $\mathcal{L}ie(\mathcal{C}om) \rightsquigarrow \mathcal{C}om(\mathcal{L}ie)$?

(1), (2) – only the trivial distributive law exists.
(3), (4) – ?
(5) – only Poisson and the trivial one exists, cf. [Bremner-Dotsenko].

Outside the Three Graces anything may happen.

Example. The formulas

$$\begin{aligned}(x\circ y)\bullet z &= 0\\ x\bullet(y\circ z) &= -\gamma\,(x\bullet y)\circ z + \sqrt{\gamma^2+\gamma}\,(x\bullet z)\circ y\\ &\quad + (\gamma+1)\,y\circ(x\bullet z) - \sqrt{\gamma^2+\gamma}\,z\circ(x\bullet y)\end{aligned}$$

represent an one-parametric family of distributive laws between an associative bilinear multiplication $\bullet$ and a free (= satisfying no axioms) bilinear operation $\circ$.

For instance, taking $\gamma = -\frac{1}{2}$ we get

$$(x\circ y)\bullet z = 0$$

$$x\bullet(y\circ z) = \frac{1}{2}(x\bullet y)\circ z + \sqrt{-\frac{1}{2}}(x\bullet z)\circ y + \frac{1}{2}y\circ(x\bullet z) - \sqrt{-\frac{1}{2}}z\circ(x\bullet y).$$

We see a distributive law whose existence depends on the existence of the square root of $-\frac{1}{2}$ in the ground field!

0.1. **Example.** Let us consider the rewriting law

$$(4a)\qquad (x\circ y)\bullet z = 0$$

$$(4b)\qquad x\bullet(y\circ z) = -\gamma\,(x\bullet y)\circ z + \sqrt{\gamma^2+\gamma}\,(x\bullet z)\circ y + (\gamma+1)\,y\circ(x\bullet z) - \sqrt{\gamma^2+\gamma}\,z\circ(x\bullet y)$$

between an associative bilinear multiplication $\bullet$ and a free (= satisfying no axioms) bilinear operation $\circ$. To verify that it defines a distributive law, we need to consider the following three equalities that follow from the associativity of $\bullet$:

$$(5a)\qquad ((u\circ v)\bullet a)\bullet b = (u\circ v)\bullet(a\bullet b),$$

$$(5b)\qquad (u\bullet(v\circ a))\bullet b = u\bullet((v\circ a)\bullet b), \quad \text{and}$$

$$(5c)\qquad (u\bullet v)\bullet(a\circ b) = u\bullet(v\bullet(a\circ b)),$$

and apply the rewrite rule to both sides of them. In all cases we must obtain equalities again. Let us illustrate this on (5c). Expanding its left hand side gives

$$\begin{aligned}
(u\bullet v)\bullet(a\circ b) &= -\,\gamma\,\big((u\bullet v)\bullet a\big)\circ b + \sqrt{\gamma^2+\gamma}\,\big((u\bullet v)\bullet b\big)\circ a\\
&\quad + (\gamma+1)\,a\circ\big((u\bullet v)\bullet b\big) - \sqrt{\gamma^2+\gamma}\,b\circ\big((u\bullet v)\bullet a\big)\\
&= -\,\gamma\,(u\bullet v\bullet a)\circ b + \sqrt{\gamma^2+\gamma}\,\boxed{(u\bullet v\bullet b)\circ a}\\
&\quad + (\gamma+1)\,a\circ(u\bullet v\bullet b) - \sqrt{\gamma^2+\gamma}\,b\circ(u\bullet v\bullet a),
\end{aligned}$$

while its right hand side leads to

$$\begin{aligned}
u\bullet\big(v\bullet(a\circ b)\big) &= -\,\gamma\,u\bullet\big((v\bullet a)\circ b\big) + \sqrt{\gamma^2+\gamma}\,u\bullet\big((v\bullet b)\circ a\big)\\
&\quad + (\gamma+1)\,u\bullet\big(a\circ(v\bullet b)\big) - \sqrt{\gamma^2+\gamma}\,u\bullet\big(b\circ(v\bullet a)\big)\\
&= \gamma^2\big(u\bullet(v\bullet a)\big)\circ b - \gamma\sqrt{\gamma^2+\gamma}\,(u\bullet b)\circ(v\bullet a)\\
&\quad - \gamma\,(\gamma+1)\,(v\bullet a)\circ(u\bullet b) + \gamma\sqrt{\gamma^2+\gamma}\,b\circ\big(u\bullet(v\bullet a)\big)\\
&\quad - \gamma\sqrt{\gamma^2+\gamma}\,\big(u\bullet(v\bullet b)\big)\circ a + (\gamma^2+\gamma)\,(u\bullet a)\circ(v\bullet b)\\
&\quad + (\gamma+1)\sqrt{\gamma^2+\gamma}\,(v\bullet b)\circ(u\bullet a) - (\gamma^2+\gamma)\,a\circ\big(u\bullet(v\bullet b)\big)\\
&\quad - \gamma\,(\gamma+1)\,(u\bullet a)\circ(v\bullet b) + (\gamma+1)\sqrt{\gamma^2+\gamma}\,\big(u\bullet(v\bullet b)\big)\circ a\\
&\quad + (\gamma+1)^2 a\circ\big(u\bullet(v\bullet b)\big) - (\gamma+1)\sqrt{\gamma^2+\gamma}\,(v\bullet b)\circ(u\bullet a)\\
&\quad + \gamma\sqrt{\gamma^2+\gamma}\,(u\bullet b)\circ(v\bullet a) - (\gamma^2+\gamma)\,\big(u\bullet(v\bullet a)\big)\circ b\\
&\quad - (\gamma+1)\sqrt{\gamma^2+\gamma}\,b\circ\big(u\bullet(v\bullet a)\big) + (\gamma^2+\gamma)\,(v\bullet a)\circ(u\bullet b)\\
&= \gamma^2(u\bullet v\bullet a)\circ b - \gamma\sqrt{\gamma^2+\gamma}\,(u\bullet b)\circ(v\bullet a)\\
&\quad - (\gamma^2+\gamma)\,(v\bullet a)\circ(u\bullet b) + \gamma\sqrt{\gamma^2+\gamma}\,b\circ(u\bullet v\bullet a)\\
&\quad - \gamma\sqrt{\gamma^2+\gamma}\,\boxed{(u\bullet v\bullet b)\circ a} + (\gamma^2+\gamma)\,(u\bullet a)\circ(v\bullet b)\\
&\quad + (\gamma+1)\sqrt{\gamma^2+\gamma}\,(v\bullet b)\circ(u\bullet a) - (\gamma^2+\gamma)\,a\circ(u\bullet v\bullet b)\\
&\quad - (\gamma^2+\gamma)\,(u\bullet a)\circ(v\bullet b) + (\gamma+1)\sqrt{\gamma^2+\gamma}\,\boxed{(u\bullet v\bullet b)\circ a}
\end{aligned}$$

$$+ (\gamma+1)^2 a\circ(u\bullet v\bullet b) - (\gamma+1)\sqrt{\gamma^2+\gamma}\,(v\bullet b)\circ(u\bullet a)$$
$$+ \gamma\sqrt{\gamma^2+\gamma}\,(u\bullet b)\circ(v\bullet a) - (\gamma^2+\gamma)\,(u\bullet v\bullet a)\circ b$$
$$- (\gamma+1)\sqrt{\gamma^2+\gamma}\,b\circ(u\bullet v\bullet a) + (\gamma^2+\gamma)\,(v\bullet a)\circ(u\bullet b).$$

What we obtained is indeed an equality. For instance, $\sqrt{\gamma^2+\gamma}$ appears as the coeffocient at the term $(u\bullet v\bullet b)\circ a$ (the boxed term) in the expansion of $(u\bullet v)\bullet(a\circ b)$, while in the expansion of $u\bullet\big(v\bullet(a\circ b)\big)$ we see this term twice, once with coefficient $-\gamma\sqrt{\gamma^2+\gamma}$, once with coefficient $(\gamma+1)\sqrt{\gamma^2+\gamma}$. Since

$$\sqrt{\gamma^2+\gamma} = -\gamma\sqrt{\gamma^2+\gamma} + (\gamma+1)\sqrt{\gamma^2+\gamma},$$

these terms cancel. We leave a similar (and in fact, easier) analysis of (4a) and (4b) to the reader. The last property to be verified is that the resuts of succesive applications of the rewriting rule to $(u\circ v)\bullet(a\circ b)$ rule does not depend on the order of applications.

Applying (4a) with $x = u$, $y = v$ and $z = a\circ b$ gives $(u\circ v)\bullet(a\circ b) = 0$ immediately. Rule (4b) with $x = u\circ v$, $y = a$ and $z = b$ leads to

$$(u\circ v)\bullet(a\circ b) = -\gamma\big((u\circ v)\bullet y\big)\circ z + \sqrt{\gamma^2+\gamma}\big((u\circ v)\bullet z\big)\circ y$$
$$+ (\gamma+1)\,y\circ\big((u\circ v)\bullet z\big) - \sqrt{\gamma^2+\gamma}\,z\circ\big((u\circ v)\bullet y\big).$$

Its right hand side equals zero by (4a), as required.