OPERADS, OPERADIC CATEGORIES AND THE BLOB COMPLEX

MICHAEL BATANIN & MARTIN MARKL

ABSTRACT. We will show that the Morrison–Walker blob complex appearing in Topological Quantum Field Theory is an operadic bar resolution of a certain operad composed of fields and local relations. As a by-product we develop the theory of unary operadic categories and study some novel and interesting phenomena arising in this context.

CONTENTS

2020 *Mathematics Subject Classification.* 55U15, 18D50, 81T05.
Supported by RVO: 67985840 and Praemium Academiæ.

INTRODUCTION

The blob complex was introduced by S. Morrison and K. Walker in [12]. It associates to an n-dimensional manifold $\mathbb{M}$, equipped with a system of fields $\mathcal{C}$ containing an ideal of local relations $\mathcal{U}$, the *blob complex* $\mathcal{B}_*(\mathbb{M},\mathcal{C})$, which is a chain complex whose salient feature is the isomorphism

$$H_0\big(\mathcal{B}_*(\mathbb{M},\mathcal{C})\big) \cong A(\mathbb{M}) := \mathcal{C}(\mathbb{M})/\mathcal{U}(\mathbb{M}),$$

where $A(\mathbb{M})$ is the skein module associated to $\mathbb{M}$. If the fields come from an n-category $\mathtt{C}$ with strong duality, $A(\mathbb{M})$ is the usual topological quantum field invariant of $\mathbb{M}$ associated to $\mathtt{C}$. The name originated from a *blob*, defined as the standard n-dimensional ball embedded in $\mathbb{M}$.

The initial impulse for the present work was a seminar given by K. Walker at MSRI, Berkeley, in the winter of 2020. The second author noticed a striking similarity between the diagrams drawn by Kevin on the board, and pictures representing elements of free operads over graph-related operadic categories that can be found in [3, Section 5]. This inspired the idea that the blob complex might be the bar resolution of an operad over a suitable operadic category.

That hope indeed turned out to be true; there even exist two related but non-equivalent ways to interpret the blob complex within operad theory. The *first* interpretation produces a complex quasi-isomorphic to the Morrison–Walker blob complex – the bar construction of a certain operad of fields over an operadic category of blob configurations in $\mathbb{M}$. The *second* interpretation identifies the blob complex with Fresse's bar construction of a traditional coloured operad, reminiscent of the little discs operad; the colours are blobs in $\mathbb{M}$ with boundaries decorated by fields. Both approaches thus lead to several versions of 'blob complexes.' Their relations are summarized in Figure 4 at the end of this paper.

Disclaimer. The present work does not bring anything new to the theory of blob complexes per se, neither it adds anything to the explicit calculations given in [12]. The free, acyclic resolution of the skein module in Theorem B on page 41 might however pave way for the study of derived TQFT invariants.

Novelties. The operadic category of blobs is unary, meaning that the cardinalities of all its objects are one. The blob complex thus represents an interesting, highly nontrivial example of an unary operadic category and justifies careful analysis of operads, operadic modules, fibrations and various versions of the bar construction in this context. Several interesting and new phenomena were discovered en route.

Conceptual understanding of the relationship between the decorated version of the unital operadic category of blobs and the un-decorated one in Section 9 inspired the notion of partial discrete operadic fibrations, partial operads and the associated partial Grothendieck construction, given in Section 4. The concrete partial operad that arose in this context is unital in a weak, unexpected sense, formalized in Definition 27 by introducing pseudo-units. Pseudo-unitality is a new, nontrivial concept even in the realm of traditional algebra, as Example 18 shows. We also introduce several versions of the 'standard' unitality condition for operads over operadic categories that are not equipped with the chosen local terminal objects required in [1, Section 1].

While free modules over classical operads have simple structure, cf. e.g. [5, Subsection 2.10.1], this is not true in the world of operads and their modules over general operadic categories, where the structure depends on the shape of the operadic category, as illustrated in Example 64. A structure result can however still be obtained under the condition of rigidity introduced in Definition 62, which has no analog in the standard operad theory. We believe that all the above notions admit generalizations to non-unary operadic categories.

The present paper has two parts. Part 1 develops general theory of unary operadic categories, operads, modules and resolutions, Part 2 is devoted to applications to the blob complex. The main results of the article are Theorem A on page 33, Theorem B on page 41 and Theorem C on page 47. Propositions 41 and 74 have no counterparts in traditional algebra.

Requirements and conventions. We will assume working knowledge of operads; suitable references are the monograph [11] and the overview [10]. Operadic categories and related notions were introduced in [1], but all necessary material from that paper is recalled in Sections 1 and 3. Some preliminary knowledge of [12] may ease reading Part 2.

Categories will be denoted by typewriter letters such as $\mathtt{C}, \mathtt{O}, \mathtt{Q}$, &c, operads and their modules written in script, e.g. $\mathcal{P}$, $\mathcal{S}$, $\mathcal{M}$, &c. From Section 6 on, all algebraic objects will live in the monoidal category R-$\mathtt{Mod}$ of graded modules over a unital commutative associative ring R. Chain complexes will be non-negatively graded, with differentials of degree -1. By a quasi-isomorphism we mean a morphism of chain complexes that induces an isomorphism of homology. Preprints [2, 3] are under permanent revision, so we indicated explicitly which concrete versions we referred to.

Acknowledgment. The second author is indebted to Benoit Fresse for pointing to the results of his impressive book [5] that are relevant to our work.

Part 1. Unary operadic categories

1. Operadic categories and operads

Our immediate aim is to rephrase the definitions of operadic categories and their operads as given in [1, Section 1] to the particular, unary case when the cardinality functor is constant and equals 1. We believe that this would make this article independent of [1].

Unary operadic categories will appear in Definition 3 as categories equipped with fiber functors; Propositions 7 and 9 then describe them also as algebras for a certain monad. Operads over unary operadic categories are introduced in Definition 11. In this section we however, unlike in [1], do not require the existence of chosen local terminal objects in operadic categories, neither we assume units of operads. A refined analysis of these additional structures is given in Section 2.

Lemma 1. *Each family* $\{\mathscr{F}_S : \mathtt{O}/S \to \mathtt{O} \mid S \in \mathtt{O}\}$ *of functors indexed by objects of* $\mathtt{O}$ *canonically induces a family* $\{\mathscr{F}_c : \mathtt{O}/S \to \mathtt{O}/\mathscr{F}_T(c) \mid c : S \to T\}$ *of functors indexed by arrows of* $\mathtt{O}$.

Proof. Assume that $c : S \to T$ is a morphism of $\mathtt{O}$. The functor $\mathscr{F}_c$ acts, by definition, on an object $f : X \to S \in \mathtt{O}/S$ as follows. Embed f in the diagram

$$\begin{array}{ccc} & & X \\ & \overset{h}{\swarrow} & \downarrow f \\ T & \xleftarrow{\;c\;} & S \end{array}$$

in which $h := cf$. Interpreting the arrows h and c as objects of $\mathtt{O}/T$, f appears as a morphism $f : h \to c$ in $\mathtt{O}/T$. We then define

$$\mathscr{F}_c(f) := \mathscr{F}_T(f) : \mathscr{F}_T(h) \longrightarrow \mathscr{F}_T(c) \in \mathtt{O}/\mathscr{F}_T(c).$$

To describe the action of $\mathscr{F}_c$ on a morphism $\Phi : b \to a$ in $\mathtt{O}/S$ given by the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;\phi\;} & X \\ & {\scriptstyle b}\searrow \quad \swarrow{\scriptstyle a} & \\ & S, & \end{array}$$

embed that diagram into

$$\begin{array}{ccc} Y & \xrightarrow{\;\phi\;} & X \\ {\scriptstyle h}\downarrow & \overset{b \quad l}{\searrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle a} \\ T & \xleftarrow[\;c\;]{} & S \end{array} \tag{1}$$

in which $h := bc$ and $l := ca$. Then $\mathscr{F}_c(\Phi) : \mathscr{F}_c(b) \to \mathscr{F}_c(a)$ is given by the commutative diagram

$$\begin{array}{ccc} \mathscr{F}_T(h) & \xrightarrow{\;\mathscr{F}_T(\phi)\;} & \mathscr{F}_T(l) \\ & {\scriptstyle \mathscr{F}_T(b)}\searrow \quad \swarrow{\scriptstyle \mathscr{F}_T(a)} & \\ & \mathscr{F}_T(c). & \end{array} \tag{2}$$

The verification that the above rules define a functor is straightforward. □

Definition 2. A family $\mathscr{F}_S$ of functors as in Lemma 1 is called a family of *fiber functors* if, for each arrow $c : S \to T$ in O, the diagram of functors

$$\begin{array}{ccc} \mathsf{O}/S & \xrightarrow{\mathscr{F}_c} & \mathsf{O}/\mathscr{F}_T(c) \\ & {\scriptstyle \mathscr{F}_S}\searrow \quad \swarrow{\scriptstyle \mathscr{F}_{\mathscr{F}_T(c)}} & \\ & \mathsf{O} & \end{array} \tag{3}$$

commutes.

To expand Definition 2, introduce the following notation and terminology. Given a map $f : X \to S$, we call $\mathscr{F}_S(f)$ the *fiber* of f and denote it simply by $\mathscr{F}(f)$. The fact that $F \in \mathsf{O}$ is the fiber of f will also be expressed by writing $F \triangleright X \xrightarrow{f} S$. For a morphism F in O/S given by the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X'' \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f''} & \\ & S & \end{array}$$

denote by ϕ_S the induced map $\mathscr{F}_S(F) : \mathscr{F}(f') \to \mathscr{F}(f'')$ between fibers. Then the commutativity of (3) on objects is expressed by the equality of the fibers in the diagram

$$\begin{array}{ccccc} F & \triangleright & F' & \xrightarrow{\phi_S} & F'' \\ \| & & \triangledown & & \triangledown \\ F & \triangleright & X' & \xrightarrow{\phi} & X'' \\ & & & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f''} & \\ & & & S. & \end{array} \tag{4}$$

In words, the fiber of a map equals the fiber of the induced map between its fibers.

To expand the commutativity of (3) on morphisms, notice that in the above notation, diagram (2) reads

$$\begin{array}{ccc} \mathscr{F}(h) & \xrightarrow{\phi_T} & \mathscr{F}(l) \\ & {\scriptstyle b_T}\searrow \quad \swarrow{\scriptstyle a_T} & \\ & \mathscr{F}(c). & \end{array}$$

In the situation of (1), diagram (3) requires that

$$(\phi_T)_{\mathscr{F}(c)} = \phi_S.$$

Definition 3. A strict *unary (nonunital) operadic category* is a category O equipped with a family of fiber functors as per Definition 2. A strict *operadic functor* $\Phi : \mathsf{O}' \to \mathsf{O}''$ between strict unary operadic categories is a functor that commutes with the associated fiber functors.

Since all operadic categories and operadic functors in this article will be *strict*, we will for brevity omit this adjective. If not indicated otherwise, by an operadic category we will always mean unary and nonunital one.

Example 4. Denote by $\mathscr{D}(\mathtt{A})$ the coproduct

$$\mathscr{D}(\mathtt{A}) := \coprod_{c\in A} \mathtt{A}/c$$

of the slice categories over the objects of a small category $\mathtt{A}$. It is simple to verify that $\mathscr{D}(\mathtt{A})$ is an unary operadic category, with the fiber functor that assigns to each morphism $\varphi : f' \to f''$ in $\mathscr{D}(\mathtt{A})$ given by the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X'' \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f''} & \\ & c & \end{array}$$

the object $\varphi \in \mathtt{A}/X'' \subset \mathscr{D}(\mathtt{A})$.

It follows from the following more general fact that the rule $A \mapsto \mathscr{D}(\mathtt{A})$ defines a nonunital monad in the category of small categories.

Lemma 5. *Let $M : \mathtt{C} \to \mathtt{C}$ be an endofunctor on a category $\mathtt{C}$ and $d : M \rightsquigarrow \mathbb{1}_{\mathtt{C}}$ a natural transformation of M to the identity functor on $\mathtt{C}$. Then the pair (M,μ) with $\mu := Md$ is a nonunital monad over $\mathtt{C}$, and $d_A : MA \to A$ is an M-algebra for each $A \in \mathtt{C}$.*

Proof. The diagram of natural transformations

(5)
$$\begin{array}{ccc} M^2 & \overset{Md}{\rightsquigarrow} & M \\ {\scriptstyle dM}\wr\downarrow & & \wr\downarrow{\scriptstyle d} \\ M & \overset{d}{\rightsquigarrow} & \mathbb{1}_{\mathtt{C}} \end{array}$$

commutes by the naturality of d. Applying M on that diagram and substituting μ for Md gives the commutative diagram

$$\begin{array}{ccc} M^3 & \overset{M\mu}{\rightsquigarrow} & M^2 \\ {\scriptstyle M\mu}\wr\downarrow & & \wr\downarrow{\scriptstyle \mu} \\ M^2 & \overset{\mu}{\rightsquigarrow} & M \end{array}$$

meaning that M is a nonunital monad. The second part of the lemma is clear. □

Example 6. Let $\mathtt{C}$ be the category $\mathtt{Cat}$ of small categories. The assumptions of Lemma 5 are fulfilled for $M := \mathscr{D}$ and d the transformation given by assembling the domain functors

$$\{\mathrm{dom} : \mathscr{D}(\mathtt{O}) \to \mathtt{O} \mid \mathtt{O} \in \mathtt{Cat}\}.$$

To verify the commutativity of (5) for that choice, consider a commutative diagram in $\mathtt{O}$

$$\begin{array}{ccc} & & X \\ & \overset{h}{\swarrow} & \\ T & & \Big\downarrow f \\ & \underset{c}{\nwarrow} & \\ & & S \end{array} \tag{6}$$

representing an object $F : h \to c \in \mathscr{D}^2(\mathtt{O})$. Then

$$\mathscr{D}d(F) = f \ \text{ and } \ d\mathscr{D}(F) = h,$$

so $d\mathscr{D}d(F) = dd\mathscr{D}(F) = X$. The above formulas show that $Md \neq dM$ in general.

Proposition 7. *Unary (non-unital) operadic categories with small sets of objects are algebras for the non-unital monad $\mathscr{D}$ in* $\mathtt{Cat}$.

Proof. A family $\left\{\mathscr{F}_S : \mathtt{O}/S \to \mathtt{O} \mid S \in \mathtt{O}\right\}$ of functors is the same as a single functor $\mathscr{F} : \mathscr{D}(\mathtt{O}) \to \mathtt{O}$. It is easy to see that $\mathscr{F}$ is a $\mathscr{D}$-algebra if and only if $\{\mathscr{F}_S\}_{S\in\mathtt{O}}$ are the fiber functors in Definition 2. □

Definition 8. The *tautological* unary operadic category generated by a small category $\mathtt{A}$ is the category $\mathscr{T}(\mathtt{A}) := \mathtt{A} \sqcup \mathscr{D}(\mathtt{A})$ with the fiber functors given by the composite

$$\mathscr{D}(\mathscr{T}(\mathtt{A})) = \mathscr{D}(\mathtt{A} \sqcup \mathscr{D}(\mathtt{A})) \cong \mathscr{D}(\mathtt{A}) \sqcup \mathscr{D}^2(\mathtt{A}) \xrightarrow{\mathbb{1}\sqcup\kappa} \mathscr{D}(\mathtt{A}) \hookrightarrow \mathtt{A} \sqcup \mathscr{D}(\mathtt{A}) = \mathscr{T}(\mathtt{A}).$$

Explicitly, the fiber of a morphism $g : S \to T$ in $\mathtt{A} \subset \mathscr{T}(\mathtt{A})$ is g again, but interpreted as an object of $\mathtt{A}/T \subset \mathscr{D}(\mathtt{A}) \subset \mathscr{T}(\mathtt{A})$, while the fiber of a morphism $F : h \to c$ of $\mathscr{D}(\mathtt{A}) \subset \mathscr{T}(\mathtt{A})$ as in (6) is $f \in \mathtt{A}/S \subset \mathscr{D}(\mathtt{A}) \subset \mathscr{T}(\mathtt{A})$.

The operadic category $\mathscr{T}(\mathtt{A})$ is an operadic subcategory of $\mathscr{D}(\mathtt{A}_\odot)$, where $\mathtt{A}_\odot$ is the result of formally adjoining a terminal object $\odot$ to $\mathtt{A}$, which means adding to the morphisms of $\mathtt{A}$ the unit endomorphism of $\odot$ and one new morphism $X \xrightarrow{!} \odot$ for any object $X \in \mathtt{A}$. The inclusion

$$\mathscr{T}(\mathtt{A}) = \mathtt{A} \sqcup \mathscr{D}(\mathtt{A}) \hookrightarrow \mathscr{D}(\mathtt{A}_\odot) \tag{7}$$

sends an object $X \in \mathtt{A}$ to $X \xrightarrow{!} \odot \in \mathscr{D}(\mathtt{A}_\odot)$ and objects $X \to Y$ of $\mathscr{D}(\mathtt{A})$ into the corresponding objects of $\mathscr{D}(\mathtt{A}_\odot)$. The image of (7) covers all objects of $\mathscr{D}(\mathtt{A}_\odot)$ except $\odot \xrightarrow{!} \odot$. Inclusion (7) will be used in Section 9 in our description of the operadic categories of blobs.

Proposition 9. *The assignment $\mathtt{A} \mapsto \mathscr{T}(\mathtt{A})$ gives rise to a (unital) monad in* $\mathtt{Cat}$ *whose algebras are (non-unital) operadic categories with small sets of objects.*

Proof. Direct verification. □

Comparing Propositions 7 and 9, one may wonder why two very different monads have the same algebras. The explanation lies in the unitality of $\mathscr{T}$ versus the non-unitality of $\mathscr{D}$. As a simple example of this phenomenon, consider the nonunital monad $\mathring{\mathsf{T}}$ in $\mathtt{Set}$ that sends a set X to the coproduct $\coprod_{n\geq 2} X^{\times n}$, and the unital monad T given by $\mathsf{T}X := \coprod_{n\geq 1} X^{\times n}$. Both $\mathring{\mathsf{T}}$ and T have the same algebras, namely associative (non-unital) monoids. Ideologically, T is obtained from $\mathring{\mathsf{T}}$ by freely adjoining the monadic unit. The relation between $\mathscr{T}(\mathtt{A})$ and $\mathscr{D}(\mathtt{A})$ is of the same nature. Michael, do the notion augmented monads exist in the literature?

In the following statement, $\Box : \mathtt{OpCat} \to \mathtt{Cat}$ is the obvious forgetful functor from the category of unary (non-unital) operadic categories with small sets of objects to the category of small categories.

Proposition 10. *For an arbitrary small category* $\mathtt{A}$ *and an unary (non-unital) operadic category* $\mathtt{O}$*, one has a natural isomorphism of functor sets*

$$\mathtt{OpCat}(\mathscr{T}(\mathtt{A}),\mathtt{O}) \cong \mathtt{Cat}(\mathtt{A},\Box\mathtt{O}).$$

In other words, $\mathscr{T}(\mathtt{A})$ *is the* free unary non-unital operadic category *generated by a small category* $\mathtt{A}$.

Proof. Direct verification, cf. [4, Theorem 2.2]. □

We are going to introduce (non-unital) operads over (non-unital) unary operadic categories. Our definition is the non-unital, unary version of [1, Definition 1.11]. Unital operadic categories and unital operads will be the subject of Section 2.

Definition 11. Let $\mathtt{V} = (\mathtt{V},\otimes,\mathbf{1})$ be a monoidal, not necessarily symmetric, category and $\mathtt{O}$ an unary operadic category. A (non-unital) *operad* for $\mathtt{O}$ in $\mathtt{V}$, or simply an $\mathtt{O}$-*operad*, is a collection $\mathcal{P} = \{\mathcal{P}(A)\}_A$ of objects of $\mathtt{V}$ indexed by objects of $\mathtt{O}$ together with structure morphisms

$$\gamma_h : \mathcal{P}(F) \otimes \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$$

given for any arrow $h : A \to B$ in $\mathtt{O}$ with fiber F. Moreover, the *associativity* requires that, for any pair of composable arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathtt{O}$, the diagram

$$\begin{array}{ccc} \mathcal{P}(F)\otimes\mathcal{P}(Y)\otimes\mathcal{P}(C) & \xrightarrow{\mathbb{1}\otimes\gamma_g} & \mathcal{P}(F)\otimes\mathcal{P}(B) \\ \Big\downarrow{\scriptstyle \gamma_{f_C}\otimes\mathbb{1}} & & \Big\downarrow{\scriptstyle \gamma_f} \\ \mathcal{P}(X)\otimes\mathcal{P}(C) & \xrightarrow{\gamma_{gf}} & \mathcal{P}(A) \end{array} \tag{8}$$

commutes. The meaning of the symbols in that diagram is explained by the following instance of (4):

(9)
$$
\begin{array}{ccc}
F \triangleright X & \xrightarrow{f_C} & Y \\
\| \quad \triangledown & & \triangledown \\
F \triangleright A & \xrightarrow{f} & B \\
& {\scriptstyle gf}\searrow \quad \swarrow{\scriptstyle g} & \\
& C. &
\end{array}
$$

We will see later in this paper that operads over unary operadic categories share many features with associative algebras, but also ones that do not have analogs in classical algebra. We believe that they in their own rite might present an interesting theme of research.

Definition 12. Let $\Phi : \mathtt{O}' \to \mathtt{O}''$ be an operadic functor and $\mathcal{P}$ an $\mathtt{O}''$-operad. The *restriction* of $\mathcal{P}$ along Φ is the $\mathtt{O}'$-operad $\Phi^*(\mathcal{P})$ with components $\Phi^*(\mathcal{P})(s) := \mathcal{P}\big(\Phi(s)\big)$, $s \in \mathtt{O}''$.

Example 13. Let $\mathtt{A}$ be a small category and $\mathscr{D}(\mathtt{A})$ the associated operadic category of Example 4. A $\mathscr{D}(\mathtt{A})$-operad in $\mathtt{V}$ is the same as a collection $\mathcal{A} = \{\mathcal{A}(f)\}_f$ of objects of $\mathtt{V}$ indexed by morphisms of $\mathtt{A}$ with a 'multiplication' $\mathcal{A}(f) \otimes \mathcal{A}(g) \to \mathcal{A}(gf)$ for any pair of composable arrows $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathtt{A}$. The multiplication is required to be associative in the obvious sense. Intuitively, $\mathcal{A}$ is a non-unital $\mathrm{Mor}(\mathtt{A})$-graded associative algebra in $\mathtt{V}$.

Example 14. An operad over the tautological operadic category $\mathscr{T}(\mathtt{A})$ in Definition 8 is the same as a pair $(\mathcal{A}, \mathcal{M})$ of a $\mathscr{D}(\mathtt{A})$-operad $\mathcal{A}$ as in Example 13 and a collection $\mathcal{M} = \big\{\mathcal{M}(A)\big\}_{A \in \mathtt{A}}$ of objects of $\mathtt{V}$ indexed by objects of $\mathtt{A}$, equipped with the 'actions' $\mathcal{A}(f) \otimes \mathcal{M}(B) \to \mathcal{M}(A)$ given for any morphism $A \xrightarrow{f} B$ in $\mathtt{A}$. It is moreover required that the diagram

$$
\begin{array}{ccc}
\mathcal{A}(f) \otimes \mathcal{A}(g) \otimes \mathcal{M}(C) & \longrightarrow & \mathcal{A}(f) \otimes \mathcal{M}(B) \\
\downarrow & & \downarrow \\
\mathcal{A}(gf) \otimes \mathcal{M}(C) & \longrightarrow & \mathcal{M}(A)
\end{array}
$$

commutes for any pair $A \xrightarrow{f} B \xrightarrow{g} C$ of composable morphisms in $\mathtt{A}$. Intuitively, $\mathcal{M}$ is a left $\mathrm{Ob}(\mathtt{A})$-graded $\mathcal{A}$-module.

2. The shades of unitality

In this section we discuss sundry versions of unitality for operadic categories and their operads, with or without the presence of chosen local terminal objects. This refinement of definitions given in [1] is required in Part 2 by the applications to the blob complex.

Definition 15. Suppose that the set $\pi_0(\mathtt{O})$ of path-connected components of a unary operadic category $\mathtt{O}$ is small and a family

$$\{U_c \in \mathtt{O} \mid c \in \pi_0(\mathtt{O})\} \tag{10}$$

of *chosen local terminal objects* is specified, with U_c belonging to the connected component c.

We say that $\mathtt{O}$ is *left unital* if the fiber functor $\mathscr{F}_S : \mathtt{O}/S \to \mathtt{O}$ sends the identity automorphism $\mathbb{1}: S \to S$ to one of the chosen local terminal objects of $\mathtt{O}$, for each $S \in \mathtt{O}$. The category $\mathtt{O}$ is *right unital* if $\mathscr{F}_{U_c} : \mathtt{O}/U_c \to \mathtt{O}$ is the domain functor for each $c \in \pi_0(\mathtt{O})$. Finally, $\mathtt{O}$ is *unital* if it is both left and right unital. An operadic functor between left and/or right unital operadic categories is assumed to preserve the chosen local terminal objects.

Unital operadic categories in the sense of the above definition are precisely the unary versions of operadic categories introduced in [1, Section 1].

Exercise 16. The operadic category $\mathscr{D}(\mathtt{A})$ is unital, with the set of the chosen local terminal objects $\{c \xrightarrow{\mathbb{1}} c \in \mathscr{D}(\mathtt{A}) \mid c \in \mathtt{A}\}$. The tautological operadic category $\mathscr{T}(\mathtt{A})$ is left unital if and only if each connected component of $\mathtt{A}$ has a terminal object. It is however never right unital.

To see why, assume that $U \in \mathtt{A}$ is a local terminal object of $\mathtt{A}$. By definition, the fiber of a morphism $a \to U$ is $a \to U$ again, but now being an object $\mathscr{D}(\mathtt{A}) \subset \mathscr{T}(\mathtt{A})$. Thus the fiber of $a \to U$ is not a, so $\mathscr{F}_U : \mathscr{T}(\mathtt{A})/U \to \mathscr{T}(\mathtt{A})$ is not the domain functor as required by the right unitality. The inclusion $\mathscr{D}(\mathtt{A}) \subset \mathscr{T}(\mathtt{A})$ is an example of a unital operadic subcategory of a non-unital one.

Example 17. Each unital associative monoid A determines a unital unary operadic category $\mathtt{O}_A$ as follows. Its objects are the elements of A, and morphisms are pairs $(x,a): xa \to a$, for $a, x \in A$. The composition of the chain

$$yxa \xrightarrow{(y,xa)} xa \xrightarrow{(x,a)} a$$

is the morphism $(yx,a) : yxa \to a$. The identity automorphisms are the pairs $(e,a) : a \to a$, where e is the unit of A. The fiber of $(x,a): xa \to x$ is x and the only chosen terminal object is e. This example and also Examples 18 and 19 below are put in more general context in Exercise 36.

Example 18. Suppose that the associative monoid A in Example 17 possesses, instead of a two-sided unit e, a family $\{e_b \mid b \in A\}$ such that each e_b is a right unit for A, i.e.

$$z\, e_t = z \ \text{ for each } \ z, t \in A, \tag{11a}$$

and a form of the left unitality requiring that

$$e_t\, t = t \ \text{ and } \ e_{tb}\, t = t \ \text{ for each } \ t, b \in A, \tag{11b}$$

is fulfilled. Notice that the first equation of (11b) with $b := e_b$ implies the second one, but for the reasons explained in the next paragraph we keep both of them. We will call the structures $\big(A, \{e_t\}_{t\in A}\big)$ as above *pseudo-unital monoids.*

Let $\mathtt{O}_A$ be the modification of the category constructed in Example 17 with the unit automorphisms defined by $(e_a, a) : a \to a$, for $a \in A$. The first equation in (11b) guarantees that (e_a, a) is indeed an automorphism of a, the second one that $\{(e_t, t)\}_{t\in A}$ are the left units for the composition in $\mathtt{O}_A$, and (11a) that they are the right units. We leave as an exercise to prove that

– each e_t is a global terminal object of $\mathtt{O}_A$, and

– with an arbitrary e_t as the chosen terminal object, $\mathtt{O}_A$ is a right unital operadic category.

Moreover,

$$\mathtt{O}_A \text{ is unital} \iff \mathtt{O}_A \text{ is left unital} \iff A \text{ is unital.}$$

Example 19. The set $A := \{u, v\}$ with a binary operation given by

$$uu := u,\ vv := v,\ uv := u \ \text{ and } \ vu := u,$$

and the 'pseudo-units' $e_u := u$ and $e_v := v$, is a pseudo-unital monoid in the sense of Example 18. The associated category $\mathtt{O}_A$ consists of two isomorphic objects u and v, related by the isomorphisms $(u, v) : u \to v$ and $(v, u) : v \to u$.

More generally, an arbitrary set X with a multiplication $X \times X \to X$ given by the projection to the first factor and pseudo-units $e_t := t$ for $t \in X$ is a pseudo-unital monoid. The associated operadic category $\mathtt{O}_X$ is the chaotic groupoid generated by X.

Remark 20. Let us try to 'categorify' pseudo-unital monoids of Example 18 by assembling the pseudo-units $\{e_b \mid b \in A\}$ to a single map $\eta : A \to A$ with $\eta(b) := e_b$, $b \in A$. The right unitality (11a) is expressed by the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\ \mu\ } & A \\ {\scriptstyle \mathbb{1}\times\eta}\uparrow & & \| \\ A \times A & \xrightarrow{\ \pi_1\ } & A \end{array}$$

in which μ is the multiplication in A and π_1 the projection to the second factor. Similarly, (11b) can be expressed via the diagrams

$$\begin{array}{ccc} A \times A & \xrightarrow{\ \eta\times\mathbb{1}\ } & A \times A \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \mu} \\ A & = & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} A \times A \times A & \xrightarrow{\ \mathbb{1}\times\tau\ } & A \times A \times A & \xrightarrow{\ \mu\times\mathbb{1}\ } & A \times A \\ {\scriptstyle \Delta\times\mathbb{1}}\uparrow & & & & \downarrow{\scriptstyle \eta\times\mathbb{1}} \\ A \times A & \xrightarrow{\ \pi_1\ } & A & \xleftarrow{\ \mu\ } & A \times A \end{array}$$

where Δ is the diagonal $a \mapsto a \times a$, $a \in A$. Since the diagrams above involve the projection and diagonal, pseudo-unitality admits a categorification only inside a cartesian monoidal category.

Below we present three versions of the unitality for $\mathtt{O}$-operads. The unitality in the sense of the first one is precisely the unary version of the standard definition [1, Definition 1.11].

Definition 21. Assume that the unary operadic category $\mathtt{O}$ is unital in the sense of Definition 15, with the set (10) of chosen local terminal objects $\{U_c \mid c \in \pi_0(\mathtt{O})\}$. Let $\mathcal{P}$ be an $\mathtt{O}$-operad in $\mathtt{V}$ equipped with a family of morphisms

$$\{\eta_c : \mathbf{1} \to \mathcal{P}(U_c) \mid c \in \pi_0(\mathtt{O})\}. \tag{12}$$

We say that $\mathcal{P}$ is *left unital* if, for any $T \in \mathtt{O}$, the diagram

$$\begin{array}{ccc} \mathcal{P}(U_c) \otimes \mathcal{P}(T) & \xrightarrow{\gamma_{\mathbb{1}}} & \mathcal{P}(T) \\ {\scriptstyle \eta_c \otimes \mathbb{1}} \uparrow & & \| \\ \mathbf{1} \otimes \mathcal{P}(T) & \xrightarrow{\cong} & \mathcal{P}(T) \end{array} \tag{13}$$

in which U_c is the fiber of the identity automorphism $\mathbb{1} : T \to T$, commutes.

The operad $\mathcal{P}$ is *right unital* if, for any $F \in \mathtt{O}$ and the unique morphism $! : F \to U_c$ to some local chosen terminal object U_c, the diagram

$$\begin{array}{ccc} \mathcal{P}(F) \otimes \mathcal{P}(U_c) & \xrightarrow{\gamma_!} & \mathcal{P}(F) \\ {\scriptstyle \mathbb{1} \otimes \eta_c} \uparrow & & \| \\ \mathcal{P}(F) \otimes \mathbf{1} & \xrightarrow{\cong} & \mathcal{P}(F) \end{array}$$

commutes. Finally, $\mathcal{P}$ is *unital* if it is both left and right unital.

If the background monoidal category is the cartesian category $\mathtt{Set}$ of sets, the family (12) is determined by a choice of *units* $e_c := \eta_c(\star) \in \mathcal{P}(U_c)$, $c \in \pi_0(\mathtt{O})$, where $\star$ is the unique element of the monoidal unit $\{\star\}$ of $\mathtt{Set}$.

Example 22. Let $\mathcal{A}$ be the operad over the operadic category $\mathscr{D}(\mathtt{A})$ described in Example 13. Exercise 16 tells us that $\mathscr{D}(\mathtt{A})$ is a unital operadic category with the chosen local terminal objects $\{\mathbb{1}_c : c \to c \in \mathscr{D}(\mathtt{A}) \mid c \in \mathtt{A}\}$. Assume the existence of a family $\{\eta_c : \mathbf{1} \to \mathcal{A}(\mathbb{1}_c)\}_{c \in \mathtt{A}}$ of morphisms in $\mathtt{V}$ indexed by objects of $\mathtt{A}$. The left (resp. right) unitality of $\mathcal{A}$ is then expressed by the left (resp. right) diagram below:

$$\begin{array}{ccc} \mathcal{A}(\mathbb{1}_c) \otimes \mathcal{A}(f) & \longrightarrow & \mathcal{A}(f) \\ {\scriptstyle \eta_c \otimes \mathbb{1}} \uparrow & & \| \\ \mathbf{1} \otimes \mathcal{A}(f) & \xrightarrow{\cong} & \mathcal{A}(f) \end{array} \qquad \begin{array}{ccc} \mathcal{A}(f) \otimes \mathcal{A}(\mathbb{1}_B) & \longrightarrow & \mathcal{A}(f) \\ {\scriptstyle \mathbb{1} \otimes \eta_B} \uparrow & & \| \\ \mathcal{A}(f) \otimes \mathbf{1} & \xrightarrow{\cong} & \mathcal{A}(f) \end{array}$$

which are required to commute for any morphism $f : c \to B$ of $\mathtt{A}$.

The next version of unitality makes sense also if the base operadic category $\mathtt{O}$ is not unital, i.e. when the chosen local terminal objects as in Definition 3 are not available.

Definition 23. Let $\mathcal{P}$ be an $\mathtt{O}$-operad equipped with a family of morphisms

$$\big\{\eta_T : \mathbf{1} \to \mathcal{P}(U_T) \mid T \in \mathtt{O},\ U_T \text{ is the fiber of } \mathbb{1} : T \to T\big\}. \tag{14}$$

We say that $\mathcal{P}$ is *left unital* if, for any $T \in \mathtt{O}$, the diagram

$$\begin{array}{ccc} \mathcal{P}(U_T) \otimes \mathcal{P}(T) & \xrightarrow{\gamma_{\mathbf{1}}} & \mathcal{P}(T) \\ {\scriptstyle \eta_T \otimes \mathbb{1}} \uparrow & & \| \\ \mathbf{1} \otimes \mathcal{P}(T) & \xrightarrow{\cong} & \mathcal{P}(T) \end{array} \tag{15}$$

commutes.

For an arbitrary morphism $f : T \to S$ of $\mathtt{O}$, the axioms of unary operadic categories provide the diagram

$$\begin{array}{ccc} F \triangleright F & \xrightarrow{f_T} & U_T \\ \| \quad \triangledown & & \triangledown \\ F \triangleright S & \xrightarrow{f} & T \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \mathbb{1}} & \\ & T & \end{array} \tag{16}$$

with f_T the induced map between fibers. The *right unitality* requires that the induced diagram

$$\begin{array}{ccc} \mathcal{P}(F) \otimes \mathcal{P}(U_T) & \xrightarrow{\gamma_{f_T}} & \mathcal{P}(F) \\ {\scriptstyle \mathbb{1} \otimes \eta_T} \uparrow & & \| \\ \mathcal{P}(F) \otimes \mathbf{1} & \xrightarrow{\cong} & \mathcal{P}(F) \end{array}$$

commutes for an arbitrary morphism $T \xrightarrow{f} S$ of $\mathtt{O}$. The operad $\mathcal{P}$ is *unital* if it is both left and right unital.

Example 24. Consider the algebra $(\mathcal{A}, \mathcal{M})$ over the (non-unital) tautological operadic category $\mathscr{T}(\mathtt{A})$ described in Example 14. The left unitality of $(\mathcal{A}, \mathcal{M})$ requires that the $\mathscr{D}(\mathtt{A})$-operad $\mathcal{A}$ is left unital, cf. Example 22, and the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbb{1}_c) \otimes \mathcal{M}(c) & \longrightarrow & \mathcal{M}(c) \\ {\scriptstyle \eta_c \otimes \mathbb{1}} \uparrow & & \| \\ \mathbf{1} \otimes \mathcal{M}(c) & \xrightarrow{\cong} & \mathcal{M}(c) \end{array}$$

for all $c \in \mathtt{A}$. The the right unitality of $(\mathcal{A}, \mathcal{M})$ is just the right unitality of $\mathcal{A}$. Notice that when $\mathtt{V}$ is the cartesian category of sets and $\mathcal{A}$ the terminal $\mathscr{D}(\mathtt{A})$-operad, i.e. when if $\mathcal{A}(f)$ is the one-point set for each morphism f of $\mathtt{A}$, unital $\mathscr{T}(\mathtt{A})$-operads are the same as presheaves over $\mathtt{A}$.

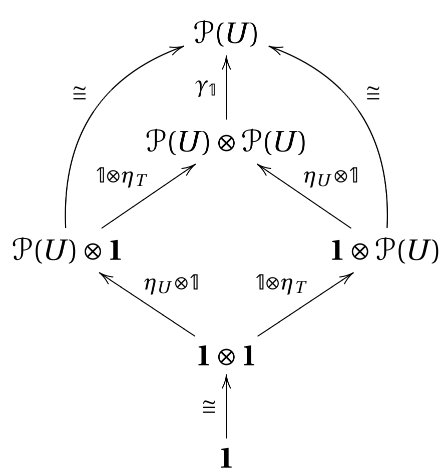
Proposition 25. *Suppose that the operadic category* $\mathtt{O}$ *is unital. Then the left (resp. right) unitality in the sense of Definition 21 implies the left (resp. right) unitality in the sense of Definition 23. In the opposite direction, the (two-sided) unitality of Definition 23 implies the (two-sided) unitality of Definition 21.*

Remark 26. Notice that we do not claim that the left (resp. right) unitality of Definition 23 alone implies the left (resp. right) unitality of Definition 21; the second part of the proposition is true only for the (two-sided) unitality. This shall be compared with the obvious fact that, while an associative algebra admits at most one two-sided unit, it might have several left or right units. An immediate implication of the proposition is that, for operads over unital unary operadic categories, the two definitions provide equivalent notions of (two-sided) unitality.

Proof of Proposition 25. Let $\{U_c \mid c \in \pi_0(\mathtt{O})\}$ be the set (10) of chosen local terminal objects of $\mathtt{O}$. Since the fiber of each identity automorphism in a unital unary operadic category is a chosen local terminal object, each U_T in Definition 23 equals U_c for some $c \in \pi_0(\mathtt{O})$ uniquely determined by T. Therefore the family (12) determines a family (14) which clearly fulfills the left (resp. right) unitality if the family (12) does. Thus the left (resp. right) unitality of Definition 21 implies the left (resp. right) unitality of Definition 23.

Suppose now that $\mathcal{P}$ is unital in the sense of Definition 23 and prove that the map η_T in the family (14) depends only on U_T, not on a concrete T. By this we mean that if T and S are objects of $\mathtt{O}$ such that the identity automorphisms $T \to T$ and $S \to S$ have the same fiber, say U, then $\eta_T = \eta_S$. It clearly suffices to verify this property for $S = U$. To show that $\eta_T = \eta_U$, consider the diagram

(17)



associated to the diagram

$$\begin{array}{ccc} U \triangleright U & \xrightarrow{\mathbb{1}} & U \\ \| \quad \triangledown & & \triangledown \\ U \triangleright T & \xrightarrow{\mathbb{1}} & T \\ & {\scriptstyle \mathbb{1}} \searrow \quad \swarrow {\scriptstyle \mathbb{1}} & \\ & T. & \end{array}$$

The commutativity of (17) follows from the left unitality of η_U and the right unitality of η_T. It implies that the composite of the leftmost arrows equals the composite of the rightmost arrows, i.e. $\eta_U = \eta_T$.

The family (14) thus determines a family (12) by $\eta_c := \eta_{U_c}$ that satisfies the left and right unitality of Definition 21. For instance, (13) is fulfilled with η_T in place of η_c by (15), but $\eta_T = \eta_c$ as proven above. The right unitality is discussed similarly. □

The following definition extends the concept of pseudo-unitality of associative monoids introduced in Example 18 to operads. The background monoidal category will be crucially the cartesian category of sets, cf. Remark 20.

Definition 27. Let $\mathcal{S}$ be an $\mathtt{O}$-operad in $\mathtt{Set}$ equipped with a family of elements

$$\big\{e_t \in \mathcal{S}(U_T) \mid T \in \mathtt{O},\ t \in \mathcal{S}(T),\ U_T \text{ is the fiber of } \mathbb{1} : T \to T\big\}. \tag{18}$$

Then $\mathcal{S}$ is *left pseudo-unital* if, for any $T \in \mathtt{O}$ and $t \in \mathcal{S}(T)$, $\gamma_{\mathbb{1}}(e_t, t) = t$ and

$$\gamma_{\mathbb{1}_C}(e_{\gamma_\xi(\rho,c)}, \rho) = \rho$$

for arbitrary diagram

$$\begin{array}{ccc} U_T \triangleright R & \xrightarrow{\mathbb{1}_C} & R \\ \| \quad \triangledown & & \triangledown \\ U_T \triangleright T & \xrightarrow{\mathbb{1}} & T \\ & {\scriptstyle \xi} \searrow \quad \swarrow {\scriptstyle \xi} & \\ & C & \end{array} \tag{19}$$

of morphism in $\mathtt{O}$ and elements $\rho \in \mathcal{S}(R)$, $c \in \mathcal{S}(C)$. The operad $\mathcal{S}$ is *right pseudo-unital* if, in the situation of diagram (16),

$$\gamma_{f_T}(\varphi, e_t) = \varphi$$

for any $t \in \mathcal{S}(U_T)$ and $\varphi \in \mathcal{S}(F)$. Finally, $\mathcal{S}$ is *pseudo-unital* if it is both left and right pseudo-unital. In this case we call the elements of the collection (18) the *pseudo-units* of $\mathcal{S}$.

Proposition 28. *Assume that $\mathcal{S}$ is an $\mathtt{O}$-operad in the category of sets, left (resp. right) unital in the sense of Definition 23. Then it is left (resp. right) pseudo-unital.*

Proof. The monoidal unit of the category $\mathtt{Set}$ is the one-point set $\{\star\}$. The family (14) determines a family as in (18) by $e_t := \eta_T(\star)$. Notice that this e_t depends only on U_T, not on a concrete $t \in \mathcal{S}(U_T)$. It is simple to verify that if $\mathcal{S}$ is left (resp. right) unital in the sense of Definition 23, then $\{e_t\}$ are left (resp. right) pseudo-units of $\mathcal{S}$. □

Example 29. Let $\mathtt{O}$ be an unital unary operadic category. The operad $\emptyset$ with $\emptyset(T)$ the empty set for each $T \in \mathtt{O}$, is a pseudo-unital operad, which is however not unital. This shows that pseudo-unitality is less demanding than unitality even when $\mathtt{O}$ is unital. Below is a less trivial example.

Example 30. Let $\bigodot$ be the terminal unary unital operadic category, i.e. the category with one object $\odot$ and one morphism $\mathbb{1} : \odot \to \odot$ with fiber $\odot$, which is simultaneously the unique chosen terminal object. Nonunital $\bigodot$-operads in $\mathtt{Set}$ are non-unital monoids, i.e. sets with one binary associative operation. Unital $\bigodot$-operads are unital monoids, and pseudo-unital $\bigodot$-operads are pseudo-unital monoids introduced in Example 18.

3. Discrete operadic fibrations

Discrete operadic fibrations appearing as operadic Grothendieck constructions [1, Section 2] are strong, useful tools for constructing new operadic categories from old ones, as several examples given [2, Section 4] convincingly show. In the first part of this section we recall unital versions of the relevant definitions given in [1], the second part is devoted to the non-unital case required in Part 2 by our applications to the blob complex.

Definition 31. A operadic functor $p : \mathtt{Q} \to \mathtt{O}$ between unary unital operadic categories is a *discrete operadic fibration* if

(i) p induces an epimorphism $\pi_0(\mathtt{Q}) \twoheadrightarrow \pi_0(\mathtt{O})$ of the sets of connected components, and
(ii) for any morphism $f : T \to S$ in $\mathtt{O}$ with fiber F and any two objects $\epsilon, s \in \mathtt{Q}$ such that $p(s) = S$ and $p(\epsilon) = F$, there exists a unique morphism $\sigma : t \to s$ in $\mathtt{Q}$ with fiber ϵ such that $p(\sigma) = f$. Schematically,

$$
\begin{array}{cccccc}
\mathtt{Q}: & \underline{\epsilon} & \triangleright & \underline{t} & \overset{\sigma}{\underset{!}{\dashrightarrow}} & \underline{s} \\
{\scriptstyle p}\downarrow & \wr & & \wr & & \wr \\
\mathtt{O}: & F & \triangleright & T & \xrightarrow{\;f\;} & S.
\end{array}
\tag{20}
$$

The unary version of the operadic *Grothendieck construction* [1, page 1647] associates to a unital $\mathtt{Set}$-valued operad $\mathcal{S}$ over a unary unital operadic category $\mathtt{O}$, cf. Definition 21, a unary unital operadic category $\int_{\mathtt{O}} \mathcal{S}$ together with an operadic functor $p : \int_{\mathtt{O}} \mathcal{S} \to \mathtt{O}$ as follows. Objects of $\int_{\mathtt{O}} \mathcal{S}$ are elements $t \in \mathcal{S}(T)$ for some $T \in \mathtt{O}$. Given $s \in \mathcal{S}(S)$ and $t \in \mathcal{S}(T)$, a morphism $\sigma : s \to t$ in $\int_{\mathtt{O}} \mathcal{S}$ is a pair (ϵ, f) consisting of a morphism $f : S \to T$ in $\mathtt{O}$ and an element $\epsilon \in \mathcal{S}(F)$, where F is the fiber of f, such that $\gamma_f(\epsilon, t) = s$. The fiber of a morphism $\sigma : s \to t$ of this form is $\epsilon \in \mathcal{S}(F)$. The unit automorphism $\mathbb{1}_t : t \to t$ of $t \in \mathcal{S}(T)$ in $\int_{\mathtt{O}} \mathcal{S}$ is the pair $(e_c, \mathbb{1}_T)$, where $e_c := \eta_c(\star) \in \mathcal{S}(U_c)$, U_c is the fiber of the identity automorphism $T \to T$ and $\star$ is the only element of the monoidal unit $\{\star\}$ of the category $\mathtt{Set}$. The chosen local terminal objects of $\int_{\mathtt{O}} \mathcal{S}$ are $\{e_c \in \mathcal{S}(U_c) \mid c \in \pi_0(\mathtt{O})\}$.

The categorical composition in $\int_{\mathtt{O}} \mathcal{S}$ is given as follows. Assume that $a \in \mathcal{S}(A)$, $b \in \mathcal{S}(B)$ and $c \in \mathcal{S}(C)$ are objects of $\int_{\mathtt{O}} \mathcal{S}$, and $\phi : a \to b$, resp. $\psi : b \to c$ their morphisms given by pairs (ω, f), resp. (y, g), where $f : A \to B$, resp. $g : B \to C$ are morphisms of $\mathtt{O}$ with the fibers F resp. Y, and $\omega \in \mathcal{S}(F)$ resp. $y \in \mathcal{S}(Y)$ are such that

$$a = \gamma_f(\omega, b) \ \text{ and } \ b = \gamma_g(y, c).$$

The composite $\psi\phi$ in $\int_{\mathtt{O}} \mathcal{S}$ is defined to be the pair (x, gf), where $x := \gamma_{f_C}(\omega, y)$ and f_C is as in diagram (9).

It turns out that the functor $p : \int_{\mathtt{O}} \mathcal{S} \to \mathtt{O}$ that sends $t \in \mathcal{S}(T)$ to $T \in \mathtt{O}$ is a discrete operadic fibration. The correspondence $\mathcal{S} \mapsto \int_{\mathtt{O}} \mathcal{S}$ is one-to one, as claims the following a unary version of [1, Proposition 2.5].

Proposition 32. *The Grothendieck construction provides an equivalence between the category of unital* $\mathtt{O}$*-operads in the monoidal category of sets, and the category of discrete operadic fibrations of unital unary operadic categories over* $\mathtt{O}$*.*

Given a discrete operadic fibration $p : \mathtt{Q} \to \mathtt{O}$, the corresponding $\mathtt{Set}$-operad $\mathcal{S}$ has the components

$$\mathcal{S}(T) := \{t \in \mathtt{Q} \mid p(t) = T\},\ T \in \mathtt{O}. \tag{21}$$

Any discrete operadic fibration induces and isomorphism $\pi_0(\mathtt{Q}) \xrightarrow{\cong} \pi_0(\mathtt{O})$ by [1, Lemma 2.2]. For each chosen local terminal $U_c \in \mathtt{O}$ therefore exists precisely one chosen local terminal $u_c \in \mathtt{Q}$ with $p(u_c) = U_c$. The units of $\mathcal{S}$ are then defined as $e_c := u_c \in \mathcal{S}(U_c)$, $c \in \pi_0(\mathtt{O})$.

Let us proceed to the non-unital situation. The modification of discrete operadic fibrations is straightforward:

Definition 33. An operadic functor $p : \mathtt{Q} \to \mathtt{O}$ between unary, not necessary unital, operadic categories is a *discrete operadic fibration* if it has the lifting property in item (ii) of Definition 31.

The *non-unital* version of the Grothendieck construction has as its input a pseudo-unital $\mathtt{Set}$-valued operad $\mathcal{S}$ as in Definition 27. The objects of the resulting non-unital operadic category $\int_{\mathtt{O}} \mathcal{S}$ are the same as in the unital case, and also the morphisms and their compositions are defined as before. The unit automorphism $\mathbb{1}_t : t \to t$ of $t \in \mathcal{S}(T)$ in $\int_{\mathtt{O}} \mathcal{S}$ is however now the pair $(e_t, \mathbb{1}_T)$, where $e_t \in \mathcal{S}(U_T)$ is as in (18).

Proposition 34. *The above version of the Grothendieck construction is an equivalence between the category of pseudo-unital* $\mathtt{O}$*-operads in* $\mathtt{Set}$ *and the category of discrete operadic fibrations over an unary operadic category* $\mathtt{O}$.

Proof. Given a discrete operadic fibration $p : \mathtt{Q} \to \mathtt{O}$, the corresponding $\mathtt{Set}$-operad $\mathcal{S}$ has the components as in (21). The pseudo-unit $e_t \in \mathcal{S}(U_T)$ associated to $t \in \mathcal{S}(T)$ is, by definition, the fiber of the identity automorphism $t \to t$ in $\mathtt{Q}$. To verify that this recipe is the inverse of the Grothendieck construction is straightforward. □

Example 35. Let $\mathtt{O}$ be an unital operadic category. The Grothendieck construction $\int_{\mathtt{O}} \mathbf{1}_{\mathtt{O}}$ of the terminal unital $\mathtt{O}$-operad in $\mathtt{Set}$ is isomorphic to $\mathtt{O}$. The Grothendieck construction $\int \emptyset$ of the 'empty' pseudo-unital operad from Example 29 gives the discrete operadic fibration $\mathtt{\emptyset} \to \mathtt{O}$ of non-unital operadic categories, where $\mathtt{\emptyset}$ is the trivial operadic category (no objects).

Exercise 36. Let $\odot$ be the terminal unital operadic category in Example 30. Verify that the operadic category $\mathtt{O}_A$ discussed in Example 17 resp. 18 equals the Grothendieck constructions $\int_{\odot} A$, with A interpreted as an unital, resp. pseudo-unital $\odot$-operad, cf. Example 30. Then show that there are one-to-one correspondences between

– unital associative monoids,

– unital $\odot$-operads, and

– discrete operadic fibrations of unital unary operadic categories over $\odot$.

Likewise, there are one-to-one correspondences between

– pseudo-unital associative monoids,

– pseudo-unital $\odot$-operads, and

– discrete operadic fibrations of operadic categories over $\odot$.

In particular, the chaotic groupoid generated by X is the Grothendieck construction $\int_{\odot} X$ of the pseudo-unital monoid X discussed in Example 19, with the fibers given by the domain functor.

4. Partial operads, partial fibrations

The Grothendieck construction used in (43) of Part 2 to decorate blobs by fields on their boundaries uses a pseudo-unital operad $\mathcal{S}$ whose structure operations are only partially defined. This requires further generalization of the material of Section 3. Namely, we formulate a 'partial' version of Proposition 34 tailored for the context of Proposition 79 in Part 2.

Definition 37. A partial $\mathtt{O}$-operad is a collection of sets $\mathcal{S} = \{\mathcal{S}(A)\}_{A \in \mathtt{O}}$ with structure operations

$$\gamma_h : \mathcal{D}(h) \longrightarrow \mathcal{S}(A),\ h : A \to B \text{ a morphism of } \mathtt{O} \text{ with fiber } F,$$

defined on a subset $\mathcal{D}(h) \subset \mathcal{S}(F) \times \mathcal{S}(B)$. The domains $\{\mathcal{D}(h)\}_h$ are such that, for each diagram as in (9),

$$\big(\mathcal{S}(F) \times \gamma_g(\mathcal{D}(g))\big) \cap \mathcal{D}(f) = \big(\gamma_{f_C}(\mathcal{D}(f_C)) \times \mathcal{S}(C)\big) \cap \mathcal{D}(gf) \tag{22}$$

and $\gamma_f(\mathbb{1} \times \gamma_g) = \gamma_{gf}(\gamma_{f_C} \times \mathbb{1})$ on the set in (22).

Equation (22) means that the composites $\gamma_f(\mathbb{1} \times \gamma_g)$ and $\gamma_{gf}(\gamma_{f_C} \times \mathbb{1})$ are defined on the same subset of $\mathcal{S}(F) \times \mathcal{S}(Y) \times \mathcal{S}(C)$.

Definition 38. Let $\mathcal{S}$ be a partial $\mathtt{O}$-operad as in Definition 37 equipped with a family of elements

$$\big\{e_t \in \mathcal{S}(U_T) \mid T \in \mathtt{O},\ t \in \mathcal{S}(T),\ U_T \text{ is the fiber of } \mathbb{1} : T \to T\big\}. \tag{23}$$

We say that $\mathcal{S}$ is *left pseudo-unital* if, for any $T \in \mathtt{O}$ and $t \in \mathcal{S}(T)$, $\gamma_{\mathbb{1}}(e_t, t)$ is defined and equals t. We moreover require that, for any diagram as in (19) and elements $\rho \in \mathcal{S}(R)$, $c \in \mathcal{S}(C)$ for which $\gamma_\xi(\rho, c)$ is defined, $\gamma_{\mathbb{1}_C}(e_{\gamma_\xi(\rho,c)}, \rho)$ is defined and equals ρ.

We say that $\mathcal{S}$ is *right pseudo-unital* if, in the situation of diagram (16), $\gamma_{f_T}(\varphi, e_t)$ is defined for any $t \in \mathcal{S}(U_T)$ and $\varphi \in \mathcal{S}(F)$, and equals φ. Finally, $\mathcal{S}$ is *pseudo-unital* if it is both left and right pseudo-unital.

Example 39. Partial pseudo-unital operads over the terminal unital operadic category $\odot$ are partial pseudo-unital monoids. We define them as partial associative monoids A equipped with a family $\{e_b \in A \mid b \in A\}$ such that the product $z\,e_t$ is defined for each $z, t \in A$ and equals z and, if tb is defined, then $e_{tb}\,t$ is defined and equals t, for each $t, b \in A$. Such partial pseudo-unital monoids are, of course, partial versions of pseudo-unital monoids introduced in Example 18.

The Grothendieck construction recalled in Section 3 works even when $\mathcal{S}$ is only a partial unital $\mathtt{Set}$-valued operad. The objects of the modified category $\int_{\mathtt{O}} \mathcal{S}$ are elements $t \in \mathcal{S}(T)$, $T \in \mathtt{O}$, as before, but the pair (ϵ, f) with $f : S \to T$, $\epsilon \in \mathcal{S}(F)$ and F the fiber of f, is a morphism $s \to t$ in $\int_{\mathtt{O}} \mathcal{S}$ only if $\gamma_f(\epsilon, t)$ is defined (and equals s).

Let us verify that (22) guarantees that the categorical composition is defined for all pairs of morphisms of $\int_{\mathtt{O}} \mathcal{S}$ whose targets and domains match as usual. Assume that $\phi : a \to b$ and $\psi : b \to c$ are as in the paragraph on page 17, Section 3, where the composition in $\int_{\mathtt{O}} \mathcal{S}$ is described. Since $\gamma_g(y, c)$ is defined and equals b, and $\gamma_f(\omega, b)$ is also defined, the composite $\gamma_f(\omega, \gamma_g(y, c))$ is defined and, thus, $\gamma_{gf}(\gamma_{f_C}(\omega, y), c)$ is defined by (22). In particular, $\gamma_{f_C}(\omega, y)$ must be defined, and we define the composite $\psi\phi$ to be the pair (x, gf), where $x := \gamma_{f_C}(\omega, y)$.

The unit automorphism $\mathbb{1}_t : t \to t$ of $t \in \int_{\mathtt{O}} \mathcal{S}$ is the pair (e_t, t), where e_t is as in (23); notice that $\gamma_{\mathbb{1}}(e_t, t)$ is defined. The projection $\pi : \int_{\mathtt{O}} \mathcal{S} \to \mathtt{O}$ of unary operadic categories sends the object $t \in \mathcal{S}(T)$ of $\int_{\mathtt{O}} \mathcal{S}$ to $T \in \mathtt{O}$. Let us formulate a 'partial' version of Definition 33.

Definition 40. A *partial discrete operadic fibration* is an operadic functor $p:\mathtt{Q}\to\mathtt{O}$ between unary operadic categories, together with a choice of subsets

$$\mathcal{L}(f)\subset p^{-1}(F)\times p^{-1}(S),\ \ f:T\to S \text{ is a morphism in } \mathtt{O} \text{ with fiber } F. \tag{24}$$

The sets $\{\mathcal{L}(f)\}_f$ are such that

(i) for any $(\varepsilon,s)\in\mathcal{L}(f)$ there exists a unique lift σ as in (20),
(ii) for any morphism $\sigma:t\to s$ in $\mathtt{Q}$ with fiber ε, one has $(\varepsilon,s)\in\mathcal{L}\big(p(\sigma)\big)$, and
(iii) for any $T\in\mathtt{O}$ and $t\in p^{-1}(T)$, one has $(u_t,t)\in\mathcal{L}(\mathbb{1}_T)$, where u_t is the fiber of the identity automorphism $\mathbb{1}_t:t\to t$.

Denote the lift σ of $(\varepsilon,s)\in\mathcal{L}(f)$ in item (i) above by $\ell(f,\varepsilon,s)$. Consider the diagram (9) in $\mathtt{O}$ and elements $y\in p^{-1}(Y)$, $r\in p^{-1}(C)$ and $\varepsilon\in p^{-1}(F)$. We require that

$$(y,c)\in\mathcal{L}(g)\ \&\ \big(\varepsilon,\ell(g,y,c)\big)\in\mathcal{L}(f)\iff(\varepsilon,y)\in\mathcal{L}(f_C)\ \&\ \big(\ell(f_C,\varepsilon,y),c\big)\in\mathcal{L}(gf). \tag{25}$$

Equivalence (25) expresses that the lift of the composite gf exists if and only if there exist composable lifts of f and g. We leave the proof of the following 'partial' version of Proposition 34 to the reader.

Proposition 41. *The 'partial' Grothendieck construction is an equivalence between the category of partial pseudo-unital* $\mathtt{O}$*-operads in* $\mathtt{Set}$ *and the category of partial discrete operadic fibrations over a unary operadic category* $\mathtt{O}$.

5. Modules

The inputs of the classical bar resolution [7, Section X.2] are an associative algebra Λ and its (left) Λ-module C. In Section 7 we generalize the input data to an operad $\mathcal{P}$ and its suitably defined $\mathcal{P}$-module $\mathcal{M}$. Operadic modules are the content of the present section; its floor plan is similar to that of Section 1.

While $\mathcal{P}$ is, as before, defined over an operadic category $\mathtt{O}$, $\mathcal{P}$-modules live over a categorical 'module' $\mathtt{M}$ over $\mathtt{O}$. This feature has no analog in the classical algebra. The word 'module' in the rest of this paper might thus mean either a categorical module over an operadic category, or a module over an operad. We believe that the concrete meaning will always be clear from the context. Michael, do you have any comments on the terminology?

Definition 42. Let $\mathtt{C}$ be a category. A (left) *module* $\mathtt{L}$ *over* $\mathtt{C}$, or simply a *left* $\mathtt{C}$*-module*, consists of 'objects' $L\in\mathtt{L}$ and, for each such L and each object S of $\mathtt{C}$, a (possibly empty) set of 'arrows' $\mathtt{L}(L,S)$. These data are equipped with 'actions'

$$\mathtt{L}(L,S)\times\mathtt{C}(S,T)\ni(\alpha,g)\longmapsto g\alpha\in\mathtt{L}(L,T),\ L\in\mathtt{L},\ S\in\mathtt{C},$$

which are associative, i.e. $(fg)\alpha = f(g\alpha)$ for α and g as above and $f \in \mathtt{C}(T,R)$, and unital, meaning that $\mathbb{1}_S\alpha = \alpha$ for each $\alpha \in \mathtt{L}(L,S)$ and the identity automorphism $\mathbb{1}_S \in \mathtt{C}(S,S)$.

Right $\mathtt{C}$-modules as well as $\mathtt{C}$-bimodules can be defined analogously, but we will not need them here.

Remark 43. The rule $(\alpha, g) \mapsto g\alpha$ does not look as a left action, one would expect $(\alpha, g) \mapsto \alpha g$ instead. This unpleasing feature is due to the bad but favored convention of writing 'α followed by g' as $g\alpha$.

Example 44. Given a category $\mathtt{C}$ and a set S, one has the chaotic $\mathtt{C}$-module $\mathtt{Chaos}(S,\mathtt{C})$ with exactly one arrow $L \to T$ for every $L \in S$ and $T \in \mathtt{C}$. A concrete example will be given in Section 9.

Example 45. Each category is a left module over itself. If both $\mathtt{C}$ and $\mathtt{L}$ have just one object, the resulting structure is the standard left module over an associative unital algebra. If $\mathtt{L}$ is a $\mathtt{C}$-module and $c \in \mathtt{C}$, then there exits the left 'overmodule' $\mathtt{L}/c$ over $\mathtt{C}/c$ whose objects are arrows $\alpha : L \to c$ in $\mathtt{L}$. Arrows from α to $g : T \to c \in \mathtt{C}/c$ are diagrams

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & T \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle g} & \\ & c & \end{array}$$

in which $\varphi \in \mathtt{L}(L,T)$ is such that $\alpha = g\varphi$.

Definition 46. We denote by $\mathtt{MOD}$ the category whose objects are pairs $(\mathtt{C},\mathtt{L})$ consisting of a category $\mathtt{C}$ and its left module $\mathtt{L}$. Morphisms from $(\mathtt{C}',\mathtt{L}')$ to $(\mathtt{C}'',\mathtt{L}'')$ are pairs (Φ,Ψ) consisting of a functor $\Phi : \mathtt{C}' \to \mathtt{C}''$ and of a rule that assigns to each object L' of $\mathtt{L}'$ an object $\Psi(L')$ of $\mathtt{L}''$, and to each arrow $\alpha : L' \to S'$ of $\mathtt{L}'$ an arrow $\alpha_* : \Psi(L') \to \Phi(S')$ of $\mathtt{L}''$ such that the diagram

$$\begin{array}{ccc} & & \Phi(S') \\ & \nearrow^{\alpha_*} & \Big\downarrow {\scriptstyle f_*} \\ \Psi(L') & & \\ & \searrow_{(f\alpha)_*} & \Phi(T') \end{array}$$

commutes for an arbitrary morphism $f : S' \to T'$ of the category $\mathtt{C}'$.

In the following analog of Lemma 1, $\mathtt{M}$ is a left module over an unary operadic category $\mathtt{O}$ and, for each $S \in \mathtt{O}$, $\mathtt{M}/S$ denotes the left $\mathtt{O}/S$-module introduced in Example 45.

Lemma 47. *Each family* $\big\{(\mathscr{F}_S,\mathscr{G}_S) : (\mathtt{O}/S,\mathtt{M}/S) \to (\mathtt{O},\mathtt{M}) \mid S \in \mathtt{O}\big\}$ *of morphisms in* $\mathtt{MOD}$ *indexed by objects of* $\mathtt{O}$ *induces a family*

$$\big\{(\mathscr{F}_c,\mathscr{G}_c) : (\mathtt{O}/S,\mathtt{M}/S) \to (\mathtt{O}/\mathscr{F}_T(c),\mathtt{M}/\mathscr{F}_T(c)) \mid c : S \to T\big\} \tag{26}$$

of morphisms in $\mathtt{MOD}$ *indexed by arrows of* $\mathtt{O}$*, with* $\mathscr{F}_c$ *as in Lemma 45.*

Proof. Analogous to the proof of Lemma 1. □

Definition 48. Let $\mathtt{O}$ be an operadic category with the associated family $\mathscr{F}_S$ of fiber functors, and $\mathtt{M}$ a left $\mathtt{O}$-module. We call a family $(\mathscr{F}_S, \mathscr{G}_S)$ in Lemma 47 a family of *fiber morphisms* if the module part of the extension (26) is such that

$$\begin{array}{ccc} \mathtt{M}/S & \xrightarrow{\mathscr{G}_c} & \mathtt{M}/\mathscr{F}_T(c) \\ & {\scriptstyle \mathscr{G}_S}\searrow \quad \swarrow{\scriptstyle \mathscr{G}_{\mathscr{F}_T(c)}} & \\ & \mathtt{M} & \end{array}$$

commutes for any $c : S \to T$.

We use similar notation and terminology as for operadic categories. That is, given an arrow $\alpha : M \to S$ in $\mathtt{M}$, we call $\mathscr{G}_S(\alpha)$ the *fiber* of α and denote it simply by $\mathscr{G}(\alpha)$. The fact that $G = \mathscr{G}(\alpha)$ will be abbreviated by $G \triangleright M \stackrel{\alpha}{\to} S$. For a diagram

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & X \\ & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array} \tag{27}$$

where $\alpha : L \to X$ is an arrow in $\mathtt{M}$, $g : X \to S$ is a morphism of $\mathtt{O}$ and $\beta = g\alpha$, we denote by α_S the induced arrow $\mathscr{G}(\beta) \to \mathscr{F}(g)$ between the fibers. The module analog of diagram (4) associated to (27) reads

$$\begin{array}{ccccc} G & \triangleright & H & \xrightarrow{\alpha_S} & F \\ \shortparallel & & \triangledown & & \triangledown \\ G & \triangleright & L & \xrightarrow{\alpha} & X \\ & & & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle g} & \\ & & & S. & \end{array} \tag{28}$$

Definition 49. An *operadic module* over an operadic category $\mathtt{O}$ is a (left) $\mathtt{O}$-module $\mathtt{M}$ equipped with a family of fiber morphisms as per Definition 48. A *morphism* from an operadic module $\mathtt{M}'$ over $\mathtt{O}'$ to a module $\mathtt{M}''$ over $\mathtt{O}''$ is a morphism $(\Phi, \Psi) : (\mathtt{O}', \mathtt{M}') \to (\mathtt{O}'', \mathtt{M}'')$ in $\mathtt{MOD}$ that commutes with the associated fiber morphisms.

Example 50. Let $\mathtt{A}$ be a small category and $\mathtt{B}$ a left $\mathtt{A}$-module. Since each overmodule in the coproduct $\mathscr{D}_{\mathtt{A}}(\mathtt{B}) := \coprod_{c \in \mathtt{A}} \mathtt{B}/c$ over objects of $\mathtt{A}$ is a module over the category $\mathscr{D}(\mathtt{A})$, cf. Example 45, $\mathscr{D}_{\mathtt{A}}(\mathtt{B})$ is a left $\mathscr{D}(\mathtt{A})$-module. With the fiber functor assigning to an arrow $\psi : \alpha \to f$ in $\mathscr{D}_{\mathtt{A}}(\mathtt{B})$ of the form

$$\begin{array}{ccc} L & \xrightarrow{\psi} & X \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle f} & \\ & c & \end{array}$$

the object $\psi : L \to X$ of $\mathscr{D}_{\mathtt{A}}(\mathtt{B})$, the module $\mathscr{D}_{\mathtt{A}}(\mathtt{B})$ becomes a left operadic module over the operadic category $\mathscr{D}(\mathtt{A})$. It is easy to verify that also $\mathscr{T}_{\mathtt{A}}(\mathtt{B}) := \mathtt{B} \sqcup \mathscr{D}_{\mathtt{A}}(\mathtt{B})$ is a left operadic module over the tautological operadic category $\mathscr{T}(\mathtt{A})$ of Definition 8. We call $\mathscr{T}_{\mathtt{A}}(\mathtt{B})$ the *tautological* $\mathscr{T}(\mathtt{A})$-*module generated by* $\mathtt{B}$.

Let us formulate a monadic description of left modules over operadic categories, analogous to Propositions 7 and 9. Since it is a straightforward generalization of the material in Section 1, we will be telegraphic.

Denote by $\mathtt{Mod}$ the subcategory of $\mathtt{MOD}$ consisting of pairs $(\mathtt{A},\mathtt{B})$ with $\mathtt{A}$ a small category. One has the functor $\mathscr{D}_{\mathtt{Mod}} : \mathtt{Mod} \to \mathtt{Mod}$ given by $\mathscr{D}_{\mathtt{Mod}}(\mathtt{A},\mathtt{B}) := \big(\mathscr{D}(\mathtt{A}), \mathscr{D}_{\mathtt{A}}(\mathtt{B})\big)$ which turns out to be a nonunital monad. Likewise, the functor $\mathscr{T}_{\mathtt{Mod}} : \mathtt{Mod} \to \mathtt{Mod}$ given by $\mathscr{T}_{\mathtt{Mod}}(\mathtt{A},\mathtt{B}) := \big(\mathscr{T}(\mathtt{A}), \mathscr{T}_{\mathtt{A}}(\mathtt{B})\big)$ is a (unital) monad.

Proposition 51. *Algebras for the nonunital monad* $\mathscr{D}_{\mathtt{Mod}}$*, resp. for the unital monad* $\mathscr{T}_{\mathtt{Mod}}$ *are pairs* $(\mathtt{O},\mathtt{M})$*, where* $\mathtt{O}$ *is a non-unital unary operadic category with a small set of objects, and* $\mathtt{M}$ *its left operadic module.*

Definition 52. Let $\mathtt{M}$ be a left operadic module over an unary operadic category $\mathtt{O}$, and $\mathcal{P}$ an $\mathtt{O}$-operad in $\mathtt{V}$. A (right) $\mathcal{P}$*-module* in $\mathtt{V}$ is a collection $\mathcal{M} = \{\mathcal{M}(M)\}_M$ of objects of $\mathtt{V}$ indexed by objects of $\mathtt{M}$ along with the 'actions'

$$\nu = \nu_\alpha : \mathcal{M}(G) \otimes \mathcal{P}(L) \to \mathcal{M}(X)$$

given for any arrow $\alpha : L \to S$ in $\mathtt{M}$ with fiber $G := \mathscr{G}(\alpha)$. Moreover, for any arrow $L \xrightarrow{\alpha} B$ in $\mathtt{M}$ and a morphisms $B \xrightarrow{g} C$ in $\mathtt{O}$, the diagram

$$\begin{array}{ccc}
\mathcal{M}(G) \otimes \mathcal{P}(F) \otimes \mathcal{P}(C) & \xrightarrow{\mathbb{1} \otimes \gamma_g} & \mathcal{M}(G) \otimes \mathcal{P}(B) \\
{\scriptstyle \nu_{\alpha_C} \otimes \mathbb{1}} \downarrow & & \downarrow {\scriptstyle \nu_\alpha} \\
\mathcal{M}(X) \otimes \mathcal{P}(C) & \xrightarrow{\nu_{g\alpha}} & \mathcal{M}(L)
\end{array}$$

is required to commute. The symbols in that diagram are explained by the following instance of (28):

$$\begin{array}{ccc}
G \triangleright X & \xrightarrow{\alpha_C} & F \\
\shortparallel \quad \triangledown & & \triangledown \\
G \triangleright L & \xrightarrow{\alpha} & B \\
& {\scriptstyle g\alpha} \searrow \quad \swarrow {\scriptstyle g} & \\
& C. &
\end{array}$$

Definition 53. Let $(\Phi,\Psi) : (\mathtt{O}',\mathtt{M}') \to (\mathtt{O}'',\mathtt{M}'')$ be a morphism of left operadic modules, $\mathcal{P}$ an $\mathtt{O}''$-operad, $\Phi^*(\mathcal{P})$ the restriction of $\mathcal{P}$ along Φ as in Definition 12, and $\mathcal{M}$ a $\mathcal{P}$-module. The *restriction* $\Psi^*(\mathcal{M})$ of $\mathcal{M}$ along (Φ,Ψ) is the $\Phi^*(\mathcal{P})$-module with the components $\Psi^*(\mathcal{M})(m) := \mathcal{M}\big(\Psi(m)\big)$, for $m \in \mathtt{M}''$.

Example 54. Any operadic category $\mathtt{O}$ is a left operadic module over itself. Having this in mind, each $\mathtt{O}$-operad is a module over itself.

Definition 55. Suppose that $\mathcal{P}$ is left unital in the sense of Definition 23 and $\mathcal{M}$ a $\mathcal{P}$-module. An arbitrary arrow $\alpha : M \to S$ of $\mathtt{M}$ induces the diagram

$$
\begin{array}{ccc}
G \triangleright G & \xrightarrow{\alpha_T} & U_T \\
\parallel \quad \triangledown & & \triangledown \\
G \triangleright M & \xrightarrow{\alpha} & T \\
& {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \mathbb{1}} & \\
& T &
\end{array}
$$

with α_T the induced map between the fibers. The $\mathcal{P}$-module $\mathcal{M}$ is *unital* if the diagram

$$
\begin{array}{ccc}
\mathcal{M}(G) \otimes \mathcal{P}(U_T) & \xrightarrow{\nu_{\alpha_T}} & \mathcal{M}(G) \\
{\scriptstyle \mathbb{1}\otimes\eta_T} \uparrow & & \parallel \\
\mathcal{M}(G) \otimes \mathbf{1} & \xrightarrow{\cong} & \mathcal{M}(G)
\end{array}
$$

in which η_T is as in (14), commutes for an arbitrary arrow $M \stackrel{\alpha}{\to} T$ in (16).

Example 56. A unital operad $\mathcal{P}$ is a unital module over itself, cf. Example 54.

Definition 57. Let $(\mathcal{M}',\nu')$ and $(\mathcal{M}'',\nu'')$ be left operadic $\mathcal{P}$-modules. A *morphism* $\Omega : \mathcal{M}' \to \mathcal{M}''$ is a family $\Omega = \big\{\Omega_M : \mathcal{M}'(M) \to \mathcal{M}''(M)\big\}$ of morphisms in $\mathtt{V}$ indexed by objects of $\mathtt{M}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{M}'(G) \otimes \mathcal{P}(L) & \xrightarrow{\nu'_\alpha} & \mathcal{M}'(X) \\
{\scriptstyle \Omega_G\otimes\mathbb{1}} \downarrow & & \downarrow {\scriptstyle \Omega_X} \\
\mathcal{M}''(G) \otimes \mathcal{P}(L) & \xrightarrow{\nu''_\alpha} & \mathcal{M}''(X)
\end{array}
$$

commutes for each $\alpha : X \to L$ with fiber G. We denote by $\mathtt{Mod}_{\mathtt{M}}(\mathcal{P})$ the corresponding category.

6. Free modules

In this section we study the structure of free operadic modules. The main result, Proposition 63, requires a certain rigidity property that has no analog in the classical algebra. The base monoidal category will be from this point on the category R-Mod of (graded) vector spaces over a commutative unital ring R though any closed monoidal category would do as well.

To warm up, we recall the following simple classical facts. Let E be a vector space and Λ a non-unital associative algebra. Then

$$\mathring{\mathbb{F}}(E) := E \oplus (\Lambda \otimes E)$$

with the left Λ-action given by $\lambda(e \oplus a \otimes f) := 0 \oplus (\lambda \otimes e + \lambda a \otimes f)$, for $\lambda, a \in \Lambda$ and $e, f \in E$, is the free left Λ-module generated by E.

Assume now that Λ possesses a two-sided unit $1 \in \Lambda$ and restrict to the subcategory of left Λ-modules on which 1 acts as the identity endomorphism. The free Λ-module in this category is obtained by identifying, in $\mathring{\mathbb{F}}(E)$, $e \oplus 0$ with $0 \oplus 1 \otimes e$ for each $e \in E$, explicitly

$$\mathbb{F}(E) := \frac{\mathring{\mathbb{F}}(E)}{\big(e \oplus 0 = 0 \oplus (1 \otimes e)\big)} \cong \Lambda \otimes E \tag{29}$$

with the left Λ-action on the right hand side given by $\lambda(a \otimes e) := \lambda a \otimes e$.

Remark 58. Let us act on both sides of the equality $e \oplus 0 = 0 \oplus (1 \otimes e)$ in the denominator of (29) by some $\lambda \in \Lambda$. By the definition of the left Λ-action, we get the equality $0 \oplus (\lambda \otimes e) = 0 \oplus (\lambda \cdot 1 \otimes e)$, which implies the relation

$$\lambda \cdot 1 \otimes e \sim \lambda \otimes e \ \text{ for each } \lambda \in \Lambda,\ e \in E \tag{30}$$

that is $\lambda \cdot 1 \sim \lambda$ for each $\lambda \in \Lambda$. The assumption that 1 is also a right, not only the left unit of Λ guarantees that the 'unexpected' relation in (30) is satisfied automatically.

Free modules in the operadic context have a similarly simple structure only when the following unary version of the weak blow-up axiom [2, Section 2], abbreviated WBU, is fulfilled.

Weak blow-up (category version). For each morphism $f' : X' \to S$ in O with fiber F', and another morphism $\phi : F' \to F''$, the left diagram in (WBU) below can be *uniquely* completed to the diagram in the right hand side so that ϕ will became the map between the fibers induced by φ:

$$\begin{array}{ccc} F' & \xrightarrow{\ \phi\ } & F'' \\ \triangledown & & \\ X' & & \\ & \searrow^{f'} & \\ & & S \end{array} \qquad\qquad \begin{array}{ccc} F' & \xrightarrow{\ \phi\ } & F'' \\ \triangledown & & \triangledown \\ X' & \xrightarrow{\ \varphi\ } & X''. \\ {\scriptstyle f'}\searrow & & \swarrow{\scriptstyle f''} \\ & S & \end{array} \tag{WBU}$$

Weak blow-up (module version). A straightforward modification of the operadic blow-up with ϕ, φ and f' arrows of M and f'' a morphism in O.

Exercise 59. The operadic categories $\mathscr{D}(\mathtt{A})$ and $\mathscr{T}(\mathtt{A})$, as well as the left modules $\mathscr{D}_{\mathtt{M}}(\mathtt{A})$ and $\mathscr{T}_{\mathtt{M}}(\mathtt{A})$, satisfy $\mathtt{WBU}$.

Let $\mathcal{E} = \{\mathcal{E}(M)\}_M$ be a collection of graded vector spaces indexed by objects of $\mathtt{M}$. For a given object $M \in \mathtt{M}$, put

$$\mathring{\mathbb{F}}(\mathcal{E})(M) := \mathcal{E}(M) \oplus \bigoplus\nolimits_\alpha \big(\mathcal{P}(T) \otimes_\alpha \mathcal{E}(G)\big), \tag{31}$$

where α runs over arrows $M \xrightarrow{\alpha} T$ in $\mathtt{M}$, G is the fiber of α, and $\mathcal{P}(T) \otimes_\alpha \mathcal{E}(G) := \mathcal{P}(T) \otimes \mathcal{E}(G)$, the subscript α of the tensor product symbol indicating the summand corresponding to this concrete α.

For $\alpha : M \to T$ with the fiber G, the action $\nu_\alpha : \mathcal{P}(T) \otimes \mathring{\mathbb{F}}(\mathcal{E})(G) \to \mathring{\mathbb{F}}(\mathcal{E})(M)$ is described as follows. If $e \in \mathcal{E}(G) \subset \mathring{\mathbb{F}}(\mathcal{E})(G)$, the action is 'tautological,' i.e.

$$\nu_\alpha(e,t) := t \otimes e \in \mathcal{P}(T) \otimes_\alpha \mathcal{E}(G) \in \mathring{\mathbb{F}}(\mathcal{E})(M).$$

Let $s \otimes h \in \mathcal{P}(S) \otimes_\beta \mathcal{E}(H) \in \mathring{\mathbb{F}}(\mathcal{E})(G)$, where $\beta : G \to S$ has the fiber H, and α be as before. In this situation we have the diagram

$$\begin{array}{ccc} S & \xleftarrow{\ \beta\ } & G \triangleleft H \\ \triangledown & & \triangledown \\ X & \xleftarrow{\ \omega\ } & M \triangleleft H \\ & \searrow^{g} & \downarrow{\scriptstyle \alpha} \\ & & T \end{array}$$

constructed from the initial data $\alpha : M \to T$ and $\beta : G \to S$ invoking the $\mathtt{WBU}$. We define the action of $t \in \mathcal{P}(T)$ by

$$\nu_\alpha(s \otimes h, t) := \gamma_g(s,t) \otimes h \in \mathcal{P}(X) \otimes_\omega \mathcal{E}(H) \in \mathring{\mathbb{F}}(\mathcal{E})(M).$$

Proposition 60. *The above structure makes $\mathring{\mathbb{F}}(\mathcal{E})$ an operadic $\mathcal{P}$-module. It is the free left operadic module generated by $\mathcal{E}$.*

Proof. The first part of the proposition is a simple exercise. Let us attend to the freeness. Denote by $\mathtt{Coll}_{\mathtt{M}}$ the category of collections $\mathcal{E} = \{\mathcal{E}(M)\}_M$ of vector spaces indexed by objects of $\mathtt{M}$ and their component-wise morphisms, and recall from Definition 57 the category $\mathtt{Mod}_{\mathtt{M}}(\mathcal{P})$ of left operadic $\mathcal{P}$-modules. There is an obvious forgetful functor $\square : \mathtt{Mod}_{\mathtt{M}}(\mathcal{P}) \to \mathtt{Coll}_{\mathtt{M}}$. The freeness of $\mathring{\mathbb{F}}(\mathcal{E})$ means that, for each $\mathcal{E} \in \mathtt{Coll}_{\mathtt{M}}$, $\mathcal{M} \in \mathtt{Mod}_{\mathtt{M}}(\mathcal{P})$ and a morphism $\omega : \mathcal{E} \to \square\mathcal{M}$ in $\mathtt{Coll}_{\mathtt{M}}$, there exists a unique morphism $\Omega : \mathring{\mathbb{F}}(\mathcal{E}) \to \mathcal{M}$ in $\mathtt{Mod}_{\mathtt{M}}(\mathcal{P})$ such that the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \omega\ } & \square\mathcal{M} \\ {\scriptstyle \iota}\downarrow & \nearrow_{\square\Omega} & \\ \square\mathring{\mathbb{F}}(\mathcal{E}) & & \end{array}$$

in which $\iota : \mathcal{E} \to \Box\mathring{\mathbb{F}}(\mathcal{E})$ is the obvious inclusion, commutes in $\mathtt{Coll}_{\mathtt{M}}$. We prove this claim by giving an explicit formula for Ω. Namely, for

$$e \oplus (t \otimes g) \in \mathcal{E}(M) \oplus (\mathcal{P}(T) \otimes_\alpha \mathcal{E}(G)) \subset \mathring{\mathbb{F}}(\mathcal{E})(M),\ M \in \mathtt{M},$$

we put

$$\Omega_M(e \oplus (t \otimes g)) := \omega_M(e) + (-1)^{|g||t|} \cdot \nu_\alpha(\omega_G(g), t) \in \mathcal{M}(M).$$

It is not difficult to verify that the above formula defines the required morphism in $\mathtt{Mod}_{\mathtt{M}}(\mathcal{P})$, and that such a morphism is unique. □

Let us discuss the unital version of the above constructions, assuming that that the operad $\mathcal{P}$ is left unital in the sense of Definition 23 . In the situation captured by diagram (16) denote $1_T := \eta_T(1) \in \mathcal{P}(U_T)$. For each $\alpha : M \to T$ in (16) we identify, in (31), $e \oplus 0 \in \mathcal{E}(M) \subset \mathring{\mathbb{F}}(\mathcal{E})(\Upsilon)$ with

$$0 \oplus (1_T \otimes_{\alpha_T} e) \in \big(\mathcal{P}(U_T) \otimes_{\alpha_T} \mathcal{E}(M)\big) \subset \mathring{\mathbb{F}}(\mathcal{E})(M).$$

Finally, we denote by $\mathbb{F}(\mathcal{E})$ the quotient of the free nonunital $\mathcal{P}$-module $\mathring{\mathbb{F}}(\mathcal{E})$ by the relation generated by the above identifications.

Proposition 61. *The $\mathcal{P}$-module $\mathbb{F}(\mathcal{E})$ is the free unital operadic $\mathcal{P}$-module generated by $\mathcal{E}$.*

Proof. Follows from the freeness of $\mathring{\mathbb{F}}(\mathcal{E})$ established in Proposition 60 combined with the definition of unitality. □

A structure result for free unital $\mathcal{P}$-modules similar to the isomorphism in (29) holds only at objects of $\mathtt{M}$ that are rigid in the sense of

Definition 62. An object M of an operadic $\mathtt{O}$-module $\mathtt{M}$ is *rigid* if there is precisely one object $\odot$ of $\mathtt{O}$ and one arrow $M \to \odot$ with fiber M.

Proposition 63. *For a rigid object M of a left operadic $\mathtt{O}$-module $\mathtt{M}$, one has the isomorphism*

$$\mathbb{F}(\mathcal{E})(M) \cong \bigoplus_\alpha \big(\mathcal{P}(T) \otimes_\alpha \mathcal{E}(G)\big), \tag{32}$$

where the direct sum runs over all arrows $\alpha : M \to T$ of $\mathtt{M}$ and where G is the fiber of α.

Proof. The rigidity of M guarantees that each $e \oplus 0 \in \mathcal{E}(M)$ is identified in $\mathbb{F}(\mathcal{E})(M)$ with precisely one element of the sum in the right hand side of (32). □

Example 64. Let $\mathtt{O}$ be the terminal unary operadic category with one object $\odot$ and $\mathtt{M}$ the left $\mathtt{O}$-module with one object $\star$ and one arrow $\star \to \odot$. A left unital $\mathtt{O}$-operad is a classical left unital associative algebra Λ, and a unital $\mathcal{P}$-module is the classical right Λ-module on which the left unit $1 \in \Lambda$ acts trivially. The object $\star \in \mathtt{M}$ is rigid and (32) recovers the isomorphism (29).

Let $\mathtt{O}$ be as before, and let $\mathtt{M}$ have one object $\star$ but *no* arrow. A unital $\mathcal{P}$-module is just a vector space. The free unital $\mathcal{P}$-module generated by $\mathcal{E}$ is thus $\mathcal{E}$ again, while the right hand side of (32) is trivial. This shows that (32) need not to hold without the rigidity assumption.

7. The bar resolution

Our aim will be to introduce an operadic analog of the following classical construction. Let Λ be a (graded) unital associative algebra and C a left Λ-module. An (un-normalized) bar resolution of C, cf. [7, Section X.2], is an augmented chain complex $\beta_*(\Lambda, C) \stackrel{\epsilon}{\to} C$ of the form

$$\cdots \xrightarrow{\partial_{n+2}} \beta_{n+1}(\Lambda, C) \xrightarrow{\partial_{n+1}} \beta_n(\Lambda, C) \xrightarrow{\partial_n} \beta_{n-1}(\Lambda, C) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \beta_0(\Lambda, C) \xrightarrow{\epsilon} C,$$

where $\beta_n(\Lambda, C) := \Lambda \otimes \Lambda^{\otimes n} \otimes C$ and the differential $\partial_n : \beta_n(\Lambda, C) \to \beta_{n-1}(\Lambda, C)$ is defined as the sum $\partial_n := \sum_0^n (-1)^i d_i$ with

$$d_i(\lambda_0 \otimes \cdots \otimes \lambda_n \otimes c) := \lambda_0 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \lambda_n \otimes c$$

if $0 \leq i \leq n-1$, while

$$d_n(\lambda_0 \otimes \cdots \otimes \lambda_n \otimes c) := \lambda_0 \otimes \cdots \otimes \lambda_n c,$$

for $\lambda_0, \ldots, \lambda_n \in \Lambda,\ c \in C$. The augmentation $\beta_0(\Lambda, C) \xrightarrow{\epsilon} C$ is defined using the left action of Λ on C by $\epsilon(\lambda_0 \otimes c) := \lambda_0 c$. The following theorem is classical, cf. again [7, Section X.2].

Theorem 65. *The augmented chain complex $\beta_*(\Lambda, C) \xrightarrow{\epsilon} C$ is an acyclic resolution of C via free left Λ-modules.*

Let $1 \in \Lambda$ be the unit of Λ. For each $n \geq 1$ define linear operators $s_j : \beta_n(\Lambda, C) \to \beta_{n+1}(\Lambda, C)$, $0 \leq j \leq n$, by

$$s_j(\lambda_0 \otimes \cdots \otimes \lambda_n \otimes c) := \lambda_0 \otimes \cdots \otimes \lambda_i \otimes 1 \otimes \lambda_{i+1} \otimes \cdots \lambda_n \otimes c$$

for $0 \leq j \leq n$, and

$$s_n(\lambda_0 \otimes \cdots \otimes \lambda_n \otimes c) := \lambda_0 \otimes \cdots \otimes \lambda_n \otimes 1 \otimes c.$$

The following statement is also classical.

Proposition 66. *The family $\beta_\bullet(\Lambda, C) = \{\beta_n(\Lambda, C)\}_{n \geq 0}$ of vector spaces with the operators d_i and s_j defined above is a simplicial vector space. The bar resolution $\beta_\bullet(\Lambda, C)$ is its associated chain complex.*

The operadic analog of $\beta_*(\Lambda, C)$ will possess the similar properties. The input data will be a left module $\mathtt{M}$ over an operadic category $\mathtt{O}$, an $\mathtt{O}$-operad $\mathcal{P}$ and a $\mathcal{P}$-module $\mathcal{M}$. The basic building blocks will be the diagrams $\mathfrak{T}_M$ of the form

$$\begin{array}{cccccccccccc} \mathfrak{T}_M : & T_0 & \xleftarrow{f_1} & T_1 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_{n-1}} & T_{n-1} & \xleftarrow{f_n} & T_n & \xleftarrow{\alpha} & M \\ & & & \triangle & & \cdots & & \triangle & & \triangle & & \triangle \\ & & & F_1 & & \cdots & & F_{n-1} & & F_n & & N \end{array} \tag{33}$$

where $T_0,\dots,T_n$ are objects of $\mathtt{O}$, $f_1,\dots,f_n$ are morphisms of $\mathtt{O}$, and $M \xrightarrow{\alpha} T_n$ is an arrow of $\mathtt{M}$. Further, $F_1,\dots,F_n$ are the fibers of $f_1,\dots,f_n$, respectively, and N is the fiber of α. We will call $(T_0,F_1,\dots,F_n,N)$ the *fiber sequence* of $\mathfrak{T}_M$. For $n \geq 0$ denote

$$\beta_n(\mathcal{P},\mathcal{M})(M) := \bigoplus \mathcal{P}(T_0)\otimes\mathcal{P}(F_1)\otimes\cdots\otimes\mathcal{P}(F_n)\otimes\mathcal{M}(N) \tag{34}$$

with the direct sum running over all towers $\mathscr{T}_M$ in (33). We assemble the above vector spaces into an augmented chain complex

$$\cdots \xrightarrow{\partial_{n+1}} \beta_n(\mathcal{P},\mathcal{M})(M) \xrightarrow{\partial_n} \beta_{n-1}(\mathcal{P},\mathcal{M})(M) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \beta_0(\mathcal{P},\mathcal{M})(M) \xrightarrow{\epsilon} \mathcal{M}(M).$$

Its nth differential ∂_n is the sum $\sum_0^n(-1)^i d_i$, with d_i acting on an element

$$t_0\otimes p_1\otimes\cdots\otimes p_i\otimes p_{i+1}\otimes\cdots\otimes p_n\otimes n \tag{35}$$

of $\mathcal{P}(T_0)\otimes\mathcal{P}(F_1)\otimes\cdots\otimes\mathcal{P}(F_i)\otimes\mathcal{P}(F_{i+1})\otimes\cdots\otimes\mathcal{P}(F_n)\otimes\mathcal{M}(N)$ as follows. If $1 \leq i \leq n-1$, replace first the piece

$$\begin{array}{ccccc} T_{i-1} & \xleftarrow{f_i} & T_i & \xleftarrow{f_{i+1}} & T_{i+1} \\ & & \vartriangle & & \vartriangle \\ & & F_i & & F_{i+1} \end{array}$$

in the tower (33) by

$$\begin{array}{ccc} T_{i-1} & \xleftarrow{f_i f_{i+1}} & T_{i+1} \\ & & \vartriangle \\ & & F', \end{array}$$

where F' is the fiber of $f_i f_{i+1}$. This situation gives rise to the following instance of (4):

$$\begin{array}{ccccc} F_{i+1} & \triangleright\, F' & & \xrightarrow{(f_{i+1})_{T_{i-1}}} & F_i \\ \| & \triangledown & & & \triangledown \\ F_{i+1} & \triangleright\, T_{i+1} & & \xrightarrow{f_{i+1}} & T_i \\ & & \searrow^{f_{i+1}f_i} & & \swarrow_{f_i} \\ & & & T_{i-1}. & \end{array} \tag{36}$$

The operation d_i now replaces $p_i\otimes p_{i+1}$ in (35) by $\gamma(p_{i+1},p_i)\in\mathcal{P}(F')$, where γ is the operadic composition induced by the subdiagram $F_{i+1}\triangleright F' \to F_i$ of (36). To define d_0, we cut the left end

$$\begin{array}{ccccccc} T_0 & \xleftarrow{f_1} & T_1 & \xleftarrow{f_2} & T_2 & \xleftarrow{f_3} & \cdots \\ & & \vartriangle & & \vartriangle & & \\ & & F_1 & & F_2 & & \end{array}$$

of the tower (33) to

$$\begin{array}{ccccc} T_1 & \xleftarrow{f_2} & T_2 & \xleftarrow{f_3} & \cdots \\ & & \vartriangle & & \\ & & F_2 & & \end{array}$$

and replace $t_0 \otimes p_1$ in (35) by $\gamma_{f_1}(p_1, t_0) \in \mathcal{P}(T_1)$. To define d_n, we replace

$$\begin{array}{ccccc} T_{n-1} & \xleftarrow{f_n} & T_n & \xleftarrow{\alpha} & M \\ & & \vartriangle & & \vartriangle \\ & & F_n & & N \end{array}$$

in (33), by

$$\begin{array}{ccc} T_{n-1} & \xleftarrow{f_n\alpha} & M \\ & & \vartriangle \\ & & N' \end{array}$$

which gives rise to

(37)
$$\begin{array}{ccccc} N \rhd N' & & \xrightarrow{\alpha_{T_{n-1}}} & & F_n \\ \parallel \quad \triangledown & & & & \triangledown \\ N \rhd M & & \xrightarrow{\alpha} & & T_n \\ & {\scriptstyle f_n\alpha}\searrow & & \swarrow{\scriptstyle f_n} & \\ & & T_{n-1}. & & \end{array}$$

We finally replace $p_n \otimes m$ in (35) by the composite $\nu(n, p_n) \in \mathcal{M}(N')$ associated to the subdiagram $N \rhd N' \to F_n$ of (37).

It remains to attend to $\epsilon : \beta_0(\mathcal{P},\mathcal{M})(M) \to \mathcal{M}(M)$. By definition, $\beta_0(\mathcal{P},\mathcal{M})(M)$ is the direct sum $\bigoplus \mathcal{P}(T_0) \otimes \mathcal{M}(N)$ over diagrams $N \rhd M \xrightarrow{\alpha} T_0$. For $t_0 \otimes n \in \mathcal{P}(T_0) \otimes \mathcal{M}(N)$ we define $\epsilon(t_0 \otimes n)$ to be $\nu_\alpha(n, t_0) \in \mathcal{M}(M)$.

Proposition 67. *One has* $\partial_n \partial_{n+1} = 0$ *for all* $n \geq 1$*, and also* $\epsilon\partial_1 = 0$.

Proof. An exercise on the axioms of operads and their modules. □

Definition 68. The augmented chain complex $\epsilon : \beta_*(\mathcal{P},\mathcal{M})(M) \to \mathcal{M}(M)$ is the (un-normalized) *bar resolution* of the $\mathcal{P}$-module $\mathcal{M}$ at the object M of $\mathtt{M}$.

Suppose that $\mathcal{P}$ is unital as per Definition 23 and $\mathcal{M}$ is a unital $\mathcal{P}$-module. For $M \in \mathtt{M}$ define linear maps

$$s_j : \beta_n(\mathcal{P},\mathcal{M})(M) \to \beta_{n+1}(\mathcal{P},\mathcal{M})(M),\ 0 \leq j \leq n, \tag{38}$$

as follows. Consider an element $u \in \beta_n(\mathcal{P},\mathcal{M})(M)$ in (35) associated to the tower $\mathfrak{T}_M$ in (33). Then modify $\mathfrak{T}_M$ by inserting the identity automorphism of T_j to it, which results in the tower

$$\begin{array}{ccccccccccccccccc} T_0 & \xleftarrow{f_1} & T_1 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_j} & T_j & \xleftarrow{\mathbb{1}} & T_j & \longleftarrow & \cdots & \xleftarrow{f_{n-1}} & T_{n-1} & \xleftarrow{f_n} & T_n & \xleftarrow{\alpha} & M \\ & & \vartriangle & & \cdots & & \vartriangle & & \vartriangle & & \cdots & & \vartriangle & & \vartriangle & & \vartriangle \\ & & F_1 & & \cdots & & F_j & & U_{T_j} & & \cdots & & F_{n-1} & & F_n & & N \end{array}$$

with the fiber sequence $(T_0, F_1, \dots, F_j, U_{T_j}, F_{j+1}, \dots, F_n, N)$. Then $s_j(u) \in \beta_{n+1}(\mathcal{P}, \mathcal{M})$ is defined as

$$s_j(u) := t_0 \otimes p_1 \otimes \cdots \otimes p_j \otimes 1_j \otimes p_{j+1} \otimes \cdots \otimes p_n \otimes n$$

where $1_j := \eta_{U_{T_j}}(1) \in \mathcal{P}(U_{T_j})$ is given by the unitality of the operad $\mathcal{P}$, cf. (14). We formulate the following analog of Proposition 66.

Proposition 69. *If $\mathcal{P}$ is unital, the family $\beta_\bullet(\mathcal{P}, \mathcal{M})(M) = \{\beta_n(\mathcal{P}, \mathcal{M})(M)\}_{n \geq 0}$ of vector spaces with the operators d_i and s_j defined above is a simplicial vector space for each $M \in \mathtt{M}$. The piece $\beta_*(\mathcal{P}, \mathcal{M})(M)$ of the bar resolution is its associated chain complex.*

Proof. Direct verification. □

Definition 70. For $M \in \mathcal{M}$ denote by $B_*(\mathcal{P}, \mathcal{M})(M)$ the normalization of the simplicial abelian group $\beta_\bullet(\mathcal{P}, \mathcal{M})(M)$, i.e. the quotient of $\beta_*(\mathcal{P}, \mathcal{M})(M)$ by the images of the degeneracy operators (38). The augmented chain complex $\epsilon : B_*(\mathcal{P}, \mathcal{M})(M) \to \mathcal{M}(M)$ is the *normalized bar resolution* of the $\mathcal{P}$-module $\mathcal{M}$ at the object M of $\mathtt{M}$.

In the rest of this section we assume that the operadic category $\mathtt{O}$ and the left operadic $\mathtt{O}$-module $\mathtt{M}$ satisfies the weak blow-up axiom recalled in Section 5.

Proposition 71. *The direct sums $\mathcal{M} \oplus \beta_0(\mathcal{P}, \mathcal{M})$ and $\beta_n(\mathcal{P}, \mathcal{M}) \oplus \beta_{n+1}(\mathcal{P}, \mathcal{M})$, $n \geq 0$, are free left $\mathcal{P}$-modules.*

Proof. It follows directly from the definition of $\beta_0(\mathcal{P}, \mathcal{M})$ and (31) that $\mathcal{M} \oplus \beta_0(\mathcal{P}, \mathcal{M}) \cong \mathbb{F}(\Box\mathcal{M})$. It thus remains to prove that

$$\beta_n(\mathcal{P}, \mathcal{M})(N) \oplus \beta_{n+1}(\mathcal{P}, \mathcal{M})(N) \cong \mathbb{F}(\Box\beta_n(\mathcal{P}, \mathcal{M}))(N),$$

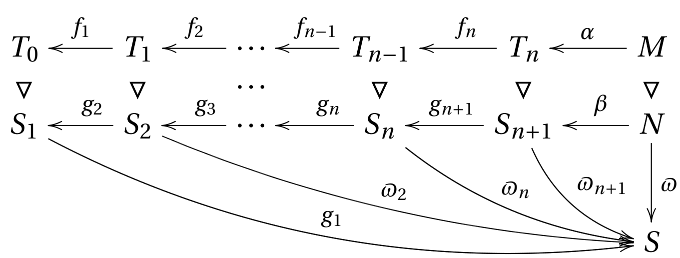
for each $n \geq 0$ and $N \in \mathtt{M}$. Invoking (31) again, we conclude that it suffices to show that

$$\beta_{n+1}(\mathcal{P}, \mathcal{M})(N) \cong \bigoplus_{\varpi} \big(\mathcal{P}(S) \otimes_\varpi \beta_n(\mathcal{P}, \mathcal{M})(M)\big), \tag{39}$$

where the direct sum is taken over all arrows $\varpi : N \to S$, and where M is the fiber of ϖ. Consider the component

$$\mathcal{P}(T_0) \otimes \mathcal{P}(F_1) \otimes \cdots \otimes \mathcal{P}(F_n) \otimes \mathcal{M}(N)$$

in the direct sum (34) defining $\beta_n(\mathcal{P}, \mathcal{M})(M)$ associated to the tower $\mathfrak{T}_M$ in (33). Using $\mathtt{WBU}$ $(n+1)$-times, we embed $\mathfrak{T}_M$ as the tower of morphisms between the fibers into the diagram



which contains the tower

$$\varpi(\mathfrak{T}_M):\ S \xleftarrow{g_1} S_1 \xleftarrow{g_2} S_2 \xleftarrow{g_3} \cdots \xleftarrow{g_n} S_n \xleftarrow{g_{n+1}} S_{n+1} \xleftarrow{\beta} N$$

with the fiber sequence $(S, T_0, F_1, \ldots, F_n, N)$. We may thus interpret the summand

$$\mathcal{P}(S) \otimes \mathcal{P}(T_0) \otimes \mathcal{P}(F_1) \otimes \cdots \otimes \mathcal{P}(F_n) \otimes \mathcal{M}(N) \subset \mathcal{P}(S) \otimes_{\varpi} \beta_n(\mathcal{P}, \mathcal{M})(M)$$

in the right hand side of (39) as belonging to the component of $\beta_{n+1}(\mathcal{P}, \mathcal{M})(N)$ associated to the tower $\varpi(\mathfrak{T}_M)$. It is easy to show that the map

$$\bigoplus_{\varpi} \big(\mathcal{P}(S) \otimes_{\varpi} \beta_n(\mathcal{P}, \mathcal{M})(M)\big) \longrightarrow \beta_{n+1}(\mathcal{P}, \mathcal{M})(N)$$

thus defined is an isomorphism. □

Let us move to the issue of acyclicity of the bar resolution. For an object $\Upsilon \in \mathtt{M}$ denote by $\Upsilon/\mathtt{O}$ the category whose objects are arrows $\alpha : \Upsilon \to X$, $X \in \mathtt{O}$, and morphisms $\alpha' \to \alpha''$ are commutative diagrams

$$\begin{array}{ccc} & \Upsilon & \\ {\scriptstyle\alpha'}\swarrow & & \searrow{\scriptstyle\alpha''} \\ X' & \longrightarrow & X'' \end{array}$$

where the horizontal arrow is a morphism of $\mathtt{O}$. It will turn out that the bar resolution is acyclic at objects Υ of $\mathtt{M}$ with the property that

(P1) the category $\Upsilon/\mathtt{O}$ has a global terminal object $\Upsilon \xrightarrow{!} \odot$ such that

(P2) the fiber of $\Upsilon \xrightarrow{!} \odot$ is Υ and the fiber functor $\mathscr{F} : \mathtt{O}/\odot \to \mathtt{O}$ is the domain functor.

The terminality in (P1) means that each arrow $\alpha : \Upsilon \to X$ in $\mathtt{M}$ is uniquely left divisible:

$$\Upsilon \xrightarrow{\ \alpha\ } X \overset{\exists\,!}{\dashrightarrow} \odot\ , \qquad \Upsilon \xrightarrow{\ !\ } \odot .$$

By (P2), the fiber of the unique arrow $X \xrightarrow{!} \odot$ is X.

Assume that $\mathcal{P}$ is left unital the sense of Definition 23 and $\mathcal{M}$ a unital $\mathcal{P}$-module as in Definition 55. Since the fiber of the identity $\mathbb{1} : \odot \to \odot$ is $\odot$, by the left unitality of $\mathcal{P}$ there exists a morphism $\eta_\odot : \mathbf{1} \to \mathcal{P}(\odot)$ for which the diagram

$$\begin{array}{ccc} \mathcal{P}(\odot) \otimes \mathcal{P}(X) & \xrightarrow{\ \gamma_1\ } & \mathcal{P}(X) \\ {\scriptstyle \eta_\odot \otimes \mathbb{1}}\big\uparrow & & \big\| \\ \mathbf{1} \otimes \mathcal{P}(X) & \xrightarrow{\ \cong\ } & \mathcal{P}(X) \end{array}$$

commutes. Likewise, the unitality of $\mathcal{M}$ implies the commutativity of

$$\begin{array}{ccc} \mathcal{M}(\Upsilon) \otimes \mathcal{P}(\odot) & \xrightarrow{\ \nu_!\ } & \mathcal{M}(\Upsilon) \\ \big\uparrow{\scriptstyle \mathbb{1} \otimes \eta_\odot} & & \big\| \\ \mathcal{M}(\Upsilon) \otimes R & \xrightarrow{\ \cong\ } & \mathcal{M}(\Upsilon). \end{array}$$

We are finally able to formulate the operadic version of Theorem 65. Recall that we assume that the operadic category $\mathtt{O}$ and the left operadic $\mathtt{O}$-module $\mathtt{M}$ satisfy the weak blow-up axiom, $\mathcal{P}$ is a unital $\mathtt{O}$-operad in R-$\mathtt{Mod}$ and $\mathcal{M}$ a unital $\mathcal{P}$-module.

Theorem A. *The augmented chain complex $\beta_*(\mathcal{P},\mathcal{M})(\Upsilon) \xrightarrow{\epsilon} \mathcal{M}(\Upsilon)$ is acyclic whenever $\Upsilon \in \mathtt{M}$ fulfills properties (P1)–(P2) above. If Υ is rigid in the sense of Definition 62, then $\beta_n(\mathcal{P},\mathcal{M})(\Upsilon)$ is a piece of a free unital $\mathcal{P}$-module for each $n \geq 0$. An obvious similar statement holds also for the normalized bar construction $B_*(\mathcal{P},\mathcal{M})$.*

Proof. It follows from the definition of $\beta_0(\mathcal{P},\mathcal{M})$ and (32) that $\beta_0(\mathcal{P},\mathcal{M})(\Upsilon) \cong \mathbb{F}(\Box\mathcal{M})(\Upsilon)$. We need to prove that

$$\beta_{n+1}(\mathcal{P},\mathcal{M})(\Upsilon) \cong \mathbb{F}(\Box\beta_n(\mathcal{P},\mathcal{M}))(\Upsilon),$$

for each $n \geq 0$ which, by (32), amounts to proving that

$$\beta_{n+1}(\mathcal{P},\mathcal{M})(\Upsilon) \cong \bigoplus_{\varpi} \big(\mathcal{P}(S) \otimes_{\varpi} \beta_n(\mathcal{P},\mathcal{M})(M)\big), \tag{40}$$

where the direct sum is taken over all $\varpi : \Upsilon \to S$, and where M is the fiber of ϖ. But this isomorphism was established in the proof of Proposition 71, cf. (39) with $N = \Upsilon$.

To prove the acyclicity of the augmented complex $\beta_*(\mathcal{P},\mathcal{M})(\Upsilon) \xrightarrow{\epsilon} \mathcal{M}(\Upsilon)$, we construct the contracting homotopies $h : \mathcal{M}(\Upsilon) \to \beta_0(\mathcal{P},\mathcal{M})(\Upsilon)$ and $h_n : \beta_n(\mathcal{P},\mathcal{M})(\Upsilon) \to \beta_{n+1}(\mathcal{P},\mathcal{M})(\Upsilon)$, $n \geq 0$, as follows. For $u \in \mathcal{M}(\Upsilon)$ we put

$$h(u) := 1_{\odot} \otimes u \in \mathcal{P}(\odot) \otimes_! \mathcal{M}(\Upsilon) \in \beta_0(\mathcal{P},\mathcal{M})(\Upsilon).$$

To construct h_n for $n \geq 0$, we consider a tower $\mathfrak{T}_\Upsilon$ as in (33) with $M = \Upsilon$ and a related element $u \in \beta_n(\mathcal{P},\mathcal{M})(\Upsilon)$ in (35). Since $\Upsilon \in \pi_0(\odot)$, all T_i's and T_0 in particular belong to $\pi_0(\odot)$, so there exists a unique $! : T_0 \to \odot$, hence $\mathfrak{T}_\Upsilon$ can be uniquely extended to

$$\begin{array}{cccccccccccccc} h(\mathfrak{T}_\Upsilon): & \odot & \xleftarrow{!} & T_0 & \xleftarrow{f_1} & T_1 & \xleftarrow{f_2} & \cdots & \xleftarrow{f_{n-1}} & T_{n-1} & \xleftarrow{f_n} & T_n & \xleftarrow{\alpha} & M \\ & & & \triangle & & \triangle & & \cdots & & \triangle & & \triangle & & \triangle \\ & & & T_0 & & F_1 & & \cdots & & F_{n-1} & & F_n & & N \end{array}$$

with the associated fiber sequence $(\odot, T_0, F_1, \dots, F_n, N)$. We finally define $h_n(u)$ to be $1_\odot \otimes u$ in the component of $\beta_{n+1}(\mathcal{P},\mathcal{M})(\Upsilon)$ corresponding to the tower $h(\mathfrak{T}_\Upsilon)$. The desired property of the contracting homotopies for $h, h_0, h_1, \dots$ constructed above is easy to verify. □

We note that the right unitality of $\mathcal{P}$ only is sufficient for the acyclicity of $\beta_*(\mathcal{P},\mathcal{M})(\Upsilon) \xrightarrow{\epsilon} \mathcal{M}(\Upsilon)$. Theorem A has the following obvious

Corollary 72. *For each Υ fulfilling (P1)–(P2) above, one has*

$$H_0(\beta_*(\mathcal{P},\mathcal{M}))(\Upsilon) = \frac{\mathcal{M}(\Upsilon)}{\mathrm{Span}\{\nu_\alpha(n,x)\}},$$

where $\mathrm{Span}\{\nu_\alpha(n,x)\} \subset \mathcal{M}(\Upsilon)$ *is the subspace spanned by all* $\nu_\alpha(n,x)$*'s with* $\alpha:\Upsilon \to X$*,* $n \in \mathcal{M}(N)$ *and* $x \in \mathcal{P}(X)$*, where* N *is the fiber of* α*. The higher homology of* $\beta_*(\mathcal{P},\mathcal{M})(\Upsilon)$ *is trivial.*

Remark 73. If $\mathtt{O}$ is the terminal operadic category with one object $\odot$, and the left $\mathtt{O}$-module $\mathtt{M}$ has of one object $\star$ and one arrow $\star \to \odot$ as in Example 64, then the above operadic constructions reduce to the classical machinery à la MacLane [7] recalled at the beginning of this section.

The following kind of functoriality has no analog in classical homological algebra.

Proposition 74. *Let* $(\Phi,\Psi):(\mathtt{O}',\mathtt{M}') \to (\mathtt{O}'',\mathtt{M}'')$ *be a morphism of operadic left modules, let* $\mathcal{P}$ *be an* $\mathtt{O}''$*-operad,* $\Phi^*(\mathcal{P})$ *its restriction along* Φ*,* $\mathcal{M}$ *a* $\mathcal{P}$*-module and* $\Psi^*(\mathcal{M})$ *its restriction along* (Φ,Ψ)*. Then, for each* $M \in \mathtt{M}'$*, there exists a natural chain map*

$$\beta_*(\Phi,\Psi)(M) : \beta_*\big(\Phi^*(\mathcal{P}),\Psi^*(\mathcal{M})\big)(M) \to \beta_*(\mathcal{P},\mathcal{M})\big(\Psi(M)\big)$$

such that the diagram

$$\begin{array}{ccc} \beta_*\big(\Phi^*(\mathcal{P}),\Psi^*(\mathcal{M})\big)(M) & \longrightarrow & \Psi^*(\mathcal{M})(M) \\ {\scriptstyle \beta_*(\Phi,\Psi)(M)}\big\downarrow & & \big\| \\ \beta_*(\mathcal{P},\mathcal{M})\big(\Psi(M)\big) & \longrightarrow & \mathcal{M}\big(\Psi(M)\big), \end{array}$$

in which the horizontal arrows are the augmentations, commutes.

Proof. The component

$$\Phi^*(\mathcal{P})(T_0) \otimes \Phi^*(\mathcal{P})(F_1) \otimes \cdots \otimes \Phi^*(\mathcal{P})(F_n) \otimes \Psi^*(\mathcal{M})(N)$$

of $\beta_*\big(\Phi^*(\mathcal{P}),\Psi^*(\mathcal{M})\big)(M)$ corresponding to the tower in (33) is mapped to the component

$$\mathcal{P}\big(\Phi(T_0)\big) \otimes \mathcal{P}\big(\Phi(F_1)\big) \otimes \cdots \otimes \mathcal{P}\big(\Phi(F_n)\big) \otimes \mathcal{M}\big(\Psi(N)\big)$$

of $\beta_*(\mathcal{P},\mathcal{M})\big(\Psi(M)\big)$ corresponding to the tower

$$\begin{array}{ccccccccccc} \Phi(T_0) & \xleftarrow{\Phi(f_1)} & \Phi(T_1) & \xleftarrow{\Phi(f_2)} & \cdots & \xleftarrow{\Phi(f_{n-1})} & \Phi(T_{n-1}) & \xleftarrow{\Phi(f_n)} & \Phi(T_n) & \xleftarrow{\Psi(\alpha)} & \Psi(M)\,. \\ & & \vartriangle & & \cdots & & \vartriangle & & \vartriangle & & \vartriangle \\ & & \Phi(F_1) & & \cdots & & \Phi(F_{n-1}) & & \Phi(F_n) & & \Psi(N) \end{array}$$

It is simple to check that the morphism thus constructed has the desired properties. □

Part 2. Fields and blobs

8. Basic notions

We will deal with manifolds, their boundaries, embeddings, &c. The precise meanings of these nouns will depend on the setup in which we choose to work – manifolds could be topological, smooth, piecewise linear, with some additional structures, &c. Since all constructions below are of combinatorial and/or algebraic nature, we allow ourselves to be relaxed about the nomenclature; compare the intuitive approach in the related sections of [13]. We reserve d for a non-negative integer.

Definition 75. A *system of fields* is a rule that to each manifold X of dimension $\leq d+1$ assigns a set $\mathcal{C}(X)$. This assignment should satisfy properties listed e.g. in [12, Section 2]. We will in particular need the following:

(i) For each codimension-zero submanifold Z of ∂X one has a functorial restriction

$$\mathcal{C}(X) \ni c \mapsto c|_Z \in \mathcal{C}(Z).$$

(ii) Let $X' \sqcup_Z X''$ be a manifold obtained by glueing manifolds X' and X'' along a common piece Z of their boundaries, and $c' \in \mathcal{C}(X')$, resp. $c'' \in \mathcal{C}(X'')$ be fields whose restrictions to Z agree. Then c' and c'' can be glued into a field $c' \sqcup_Z c'' \in \mathcal{C}(X' \sqcup_Z X'')$ which restricts to the original fields on X' resp. X''.

We will denote by $\mathcal{C}(X;c)$ the subset of $\mathcal{C}(X)$ consisting of fields that restrict to $c \in \mathcal{C}(Z)$. We will always be in the situation when Z in (ii) is closed, so we will not need 'glueing with corners' described in [12, Section 2]. We allow Z to be empty, in which case we denote the result of the glueing by $c' \sqcup c'' \in \mathcal{C}(X' \sqcup X'')$. Standard examples of fields are $\mathcal{C}(X)$ the set of maps from X to some fixed space B, or $\mathcal{C}(X)$ the set of equivalence classes of G-bundles with connection over X.

Definition 76. Let X be a $(d+1)$-dimensional, not necessarily connected, manifold, with the (possibly empty) boundary ∂X and the interior $\mathring{X}$. A *blob* in X is the image D of the standard closed $(d+1)$-dimensional ball $\mathbb{D}^{d+1} \subset \mathbb{R}^{\times(d+1)}$ embedded in X in so that either

(i) $D \subset \mathring{X}$, and $X \setminus \mathring{D}$ is a $(d+1)$-dimensional manifold with the boundary $\partial X \cup \partial D$, or
(ii) D is one of the connected components of X.

A *configuration of blobs* in X is a nonempty unordered finite set $\mathscr{D} = \{D_1, \dots, D_r\}$ of pairwise disjoint blobs in X.

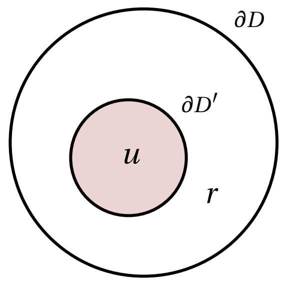
We will sometimes use $\mathscr{D}$ also to denote the union $\bigcup \mathscr{D} := \bigcup_{i=1}^r D_i$ if the meaning is clear from the context. It is a manifold with the boundary $\partial\mathscr{D} := \bigcup_{i=1}^r \partial D_i$ and the interior $\mathring{\mathscr{D}} := \bigcup_{i=1}^r \mathring{D}_i$, and $X \setminus \mathring{\mathscr{D}}$ is a $(d+1)$-dimensional manifold with the boundary $\partial X \cup \partial\mathscr{D}$. If some of the blobs in $\mathscr{D}$ happen to be some of the components of X, then the corresponding components of $X \setminus \mathring{\mathscr{D}}$ are d-dimensional embedded spheres, interpreted as degenerate $(d+1)$-dimensional manifolds with empty interiors.

In the rest of this paper we assume that the space of fields on codimension zero submanifolds of X is *enriched* in R-`Mod`. This is the case e.g. when the fields come from a $(d+1)$-category whose spaces of top-dimensional cells are linearly enriched, cf. [12, Subsection 2.2].

Definition 77. A *local relation* is a collection of subspaces $\mathcal{U} = \{\mathcal{U}(D; c) \subset \mathcal{C}(D; c)\}$ specified for any blob $D \subset X$ and any field $c \in \mathcal{C}(\partial D)$, which is an ideal in the following sense.

Suppose we are given blobs D' and D such that $\mathring{D} \supset D'$, and a field $c \sqcup c' \in \mathcal{C}(\partial D \sqcup \partial D')$. Then for any local relation $u \in \mathcal{U}(D'; c')$ and any field $r \in \mathcal{C}(D \setminus \mathring{D}'; c \sqcup c')$, the glueing $u \sqcup_{\partial D'} r$ is a local relation in $\mathcal{U}(D; c)$.

It is a simplified version of [12, Definition 2.3.1], sufficient for our purposes. The configuration of balls and fields in Definition 77 is depicted in the following schematic picture:



Definition 78. For local relations as in Definition 77, denote by $\mathcal{U}(X)$ the subspace of $\mathcal{C}(X)$ spanned by fields of the form $u \sqcup_{\partial D} r$, where $D \subset X$ is a blob, $r \in \mathcal{C}(X \setminus \mathring{D}; c)$ a field and $u \in \mathcal{U}(D; c)$ a local relation. A TQFT invariant of X called the *skein module* associated to a system of fields $\mathcal{C}$ and local relations $\mathcal{U}$ is the quotient

$$A(X) := \mathcal{C}(X)/\mathcal{U}(X).$$

If X has a non-empty boundary ∂X and $b \in \mathcal{C}(\partial X)$, one has the obvious restricted version

$$A(X; b) := \mathcal{C}(X; b)/\mathcal{U}(X; b).$$

9. Blobs via unary operadic categories

We are going to introduce various operadic categories and operadic modules together with the related operads and their modules, arising from the blobs and fields in Section 8. Our aim is to describe the associated bar resolutions, cf. the second half of Section 7. In Section 10 we show that they are quasi-isomorphic to the original blob complex in [13]. The notation is summarized at the end of this section.

Let us fix a connected $(d+1)$-dimensional, not necessarily closed, non-empty manifold $\mathbb{M}$. If its boundary $\partial\mathbb{M}$ is non-empty, some constructions below will depend on a fixed field b on $\partial\mathbb{M}$. We will however often omit such a boundary condition from the notation.

We denote by `blob` the category opposite to the category of configurations of blobs in $\mathbb{M}$ and their 'well-behaved' inclusions. More precisely, objects of `blob` are configurations $\mathscr{D}$ of blobs in $\mathbb{M}$ as in Definition 76, and a unique map $\mathscr{D}' \to \mathscr{D}''$ exists if and only if $\mathscr{D}''$ is a blob configuration in the union of blobs in $\mathscr{D}'$, cf. Definition 76 again. In what follows, by an inclusion of blob configurations we will *always mean* a well-behaved inclusion in this sense.

Let us turn our attention to the operadic category $\mathscr{D}(\mathtt{blob})$ associated to the small category $\mathtt{blob}$ via the recipe of Example 4. Its objects are inclusions $i' : \mathscr{D}' \hookleftarrow \mathscr{D}$ of blob configurations in $\mathbb{M}$. Notice that there is a one-to-one correspondence between these inclusions, i.e. objects of $\mathscr{D}(\mathtt{blob})$, and *blob complements*, which we define as closed submanifolds of $\mathbb{M}$ of the form $\bigcup \mathscr{D}' \setminus \bigcup \mathring{\mathscr{D}}$ with $\mathscr{D}$ a blob configuration in $\mathscr{D}'$. Morphisms $i' \to i''$ of $\mathscr{D}(\mathtt{blob})$ are diagrams

(41)
$$\begin{array}{ccc} \mathscr{D}' & \hookleftarrow & \mathscr{D}'' \\ {\scriptstyle i'}\nwarrow & & \nearrow{\scriptstyle i''} \\ & \mathscr{D} & \end{array}$$

of inclusions of blob configurations. The fiber of the above morphism is the inclusion $\mathscr{D}' \hookleftarrow \mathscr{D}''$ interpreted as an object of $\mathtt{blob}/\mathscr{D}'' \subset \mathscr{D}(\mathtt{blob})$.

Let $\overline{\mathtt{blob}}$ be the category $\mathtt{blob}$ with a terminal object $\varnothing$ formally added. In this particular case, $\varnothing$ can be viewed as the empty configuration of blobs, whence the notation. Inclusion (7) describes the tautological operadic category $\mathtt{Blob} := \mathscr{T}(\mathtt{blob})$ as the subcategory of $\overline{\mathtt{Blob}} := \mathscr{D}(\overline{\mathtt{blob}})$ whose objects are inclusions $i' : \mathscr{D}' \hookleftarrow \mathscr{D}$, where $\mathscr{D}$ is allowed to be *empty*. If this is so, we identify i' with $\mathscr{D}' \in \mathtt{blob}$. Morphisms of $\mathtt{Blob}$ then arise as diagrams in (41) with possibly empty $\mathscr{D}$.

Every system of fields $\mathcal{C}$ in Definition 75 leads to the decorated version $\mathtt{blob}(\mathcal{C})$ of the operadic category $\mathtt{blob}$. Its objects are pairs $(\mathscr{D}; c)$ consisting of blob configurations $\mathscr{D}$ in $\mathbb{M}$ and of a field $c \in \mathcal{C}(\partial \mathscr{D})$. Morphisms $(\mathscr{D}'; c') \to (\mathscr{D}''; c'')$ are inclusions $\mathscr{D}' \hookleftarrow \mathscr{D}''$ of blob configurations subject to the condition:

if the blob configurations $\mathscr{D}'$ and $\mathscr{D}''$ share a common blob D, then $c'|_{\partial D} = c''|_{\partial D}$.

We will tacitly assume that all inclusions of decorated blobs satisfy the above condition. Denoting by $\overline{\mathtt{blob}}(\mathcal{C})$ the category $\mathtt{blob}(\mathcal{C})$ extended by the empty blob, the tautological operadic category $\mathtt{Blob}(\mathcal{C}) := \mathscr{T}(\mathtt{blob}(\mathcal{C}))$ becomes the full subcategory of $\overline{\mathtt{Blob}}(\mathcal{C}) := \mathscr{D}(\overline{\mathtt{blob}}(\mathcal{C}))$ whose objects are 'extended' morphisms $i' : (\mathscr{D}'; c') \hookleftarrow (\mathscr{D}; c)$ in $\mathtt{blob}(\mathcal{C})$ with $\mathscr{D}$ allowed to be empty, and morphisms the diagrams

(42)
$$\begin{array}{ccc} (\mathscr{D}'; c') & \hookleftarrow & (\mathscr{D}''; c'') \\ {\scriptstyle i'}\nwarrow & & \nearrow{\scriptstyle i''} \\ & (\mathscr{D}; c) & \end{array}$$

with $\mathscr{D}$ allowed to be empty.

It turns out that $\mathtt{Blob}(\mathcal{C})$ is the partial operadic Grothendieck construction, in the sense of Section 4, over its un-decorated version. Explicitly

$$\mathtt{Blob}(\mathcal{C}) \cong \int_{\mathtt{Blob}} \mathcal{S}, \tag{43}$$

where $\mathcal{S}$ is the following partial pseudo-unital $\mathtt{Blob}$-operad in $\mathtt{Set}$. The component of $\mathcal{S}$ corresponding to $\mathscr{D}' \hookrightarrow \mathscr{D} \in \mathtt{Blob}$ is the set $\mathcal{C}(\partial\mathscr{D}' \cup \partial\mathscr{D})$. We denote this component by $\mathcal{S}\binom{\mathscr{D}'}{\mathscr{D}}$. The partial composition

$$\gamma : \mathcal{S}\binom{\mathscr{D}'}{\mathscr{D}''} \times \mathcal{S}\binom{\mathscr{D}''}{\mathscr{D}} \longrightarrow \mathcal{S}\binom{\mathscr{D}'}{\mathscr{D}} \tag{44}$$

associated to the morphism in (42) is defined for the pairs (x, y) of fields

$$x \in \mathcal{C}(\partial\mathscr{D}' \cup \partial\mathscr{D}'') = \mathcal{S}\binom{\mathscr{D}'}{\mathscr{D}''} \ \text{ and } \ y \in \mathcal{C}(\partial\mathscr{D}'' \cup \partial\mathscr{D}) = \mathcal{S}\binom{\mathscr{D}''}{\mathscr{D}}$$

such that

$$x|_{\partial\mathscr{D}''} = y|_{\partial\mathscr{D}''} \tag{45}$$

in which case

$$\gamma(x, y) := x|_{\partial\mathscr{D}'} \cup y|_{\partial\mathscr{D}} \in \mathcal{C}(\partial\mathscr{D}' \cup \partial\mathscr{D}) = \mathcal{S}\binom{\mathscr{D}'}{\mathscr{D}}.$$

Let $T \in \mathtt{Blob}$ be the inclusion $\mathscr{D}' \hookrightarrow \mathscr{D}$. The fiber U_T of the identity $T \to T$ is the inclusion $\mathscr{D}' \hookrightarrow \mathscr{D}'$. By definition,

$$\mathcal{S}(T) = \mathcal{C}(\partial\mathscr{D}' \cup \partial\mathscr{D}) \ \text{ and } \ \mathcal{S}(U_T) = \mathcal{C}(\partial\mathscr{D}').$$

The pseudo-unit e_t in (23) associated to a field $t \in \mathcal{S}(T)$ is the restriction $e_t := t|_{\partial\mathscr{D}'} \in \mathcal{S}(U_T)$.

Proposition 79. *The isomorphism (43) holds for the partial pseudo-unital operad $\mathcal{S}$ defined above. The natural projection* $\mathtt{Blob}(\mathcal{C}) \to \mathtt{Blob}$ *that forgets the decorating fields is thus a partial discrete operadic Grothendieck fibration.*

The proposition is easy to check. The subspace $\mathcal{L}(f)$ in (24) associated to the partial discrete operadic Grothendieck fibration $\mathtt{Blob}(\mathcal{C}) \to \mathtt{Blob}$ equals

$$\mathcal{L}(f) = \left\{(\varepsilon, s) \in \mathcal{C}(\partial\mathscr{D}' \cup \partial\mathscr{D}'') \times \mathcal{C}(\partial\mathscr{D} \cup \partial\mathscr{D}'') \;\middle|\; \varepsilon|_{\partial\mathscr{D}''} = s|_{\partial\mathscr{D}''}\right\}$$

when f is the morphism (42).

We will also need modules arising from blobs and fields. Let us denote by $\mathtt{m}$ the left $\mathtt{blob}$-module with one object $\mathbb{M}$ and the unique arrow $\mathbb{M} \to \mathscr{D}$ for each configuration $\mathscr{D}$ of blobs in $\mathbb{M}$. In the terminology of Example 44, $\mathtt{m}$ is the chaotic module $\mathtt{Chaos}(\{\mathbb{M}\}, \mathtt{blob})$. Likewise, let $\overline{\mathtt{m}} := \mathtt{Chaos}(\{\mathbb{M}\}, \overline{\mathtt{blob}})$ be the left $\overline{\mathtt{blob}}$-module with one object $\mathbb{M}$ and one arrow $\mathbb{M} \to \mathscr{D}$ for each configuration $\mathscr{D}$ of blobs, plus one arrow $\mathbb{M} \to \varnothing$.

Referring to Example 50, we introduce the tautological operadic $\mathtt{Blob}$-module $\mathtt{M} := \mathscr{T}_{\mathtt{blob}}(\mathtt{m})$. It is, by the obvious analog of the inclusion (7), the operadic submodule of the $\overline{\mathtt{Blob}}$-module $\overline{\mathtt{M}} := \mathscr{D}_{\overline{\mathtt{blob}}}(\overline{\mathtt{m}})$. Objects of $\mathtt{M}$ appear in $\overline{\mathtt{M}}$ as inclusions $\mathbb{M} \hookleftarrow \mathscr{D}$ of blob configurations, where $\mathscr{D}$ might be empty, and the diagram

$$\begin{array}{ccc} \mathbb{M} & \hookleftarrow & \mathscr{D}' \\ & {\scriptstyle i}\nwarrow \quad \nearrow{\scriptstyle i'} & \\ & \mathscr{D} & \end{array}$$

in $\overline{\mathtt{M}}$ represents an arrow from $\mathbb{M} \hookleftarrow \mathscr{D} \in \mathtt{M}$ to $\mathscr{D}' \hookleftarrow \mathscr{D} \in \mathtt{Blob}$ with fiber $\mathbb{M} \hookleftarrow \mathscr{D}' \in \mathtt{M}$. In the above diagram, $\mathscr{D}$ is again allowed to be empty.

Every system of fields in Definition 75 gives rise to the decorated versions of the above modules. Namely, we have the left $\mathtt{blob}(\mathcal{C})$-module $\mathtt{m}(\mathcal{C})$ with single object the pair $(\mathbb{M}; b)$, where $b \in \mathcal{C}(\partial\mathbb{M})$ is the fixed boundary condition. By definition, $\mathtt{m}(\mathcal{C})$ has one arrow $(\mathbb{M}; b) \to (\mathscr{D}; c)$ for each configuration $(\mathscr{D}; c)$ of decorated blobs such that $\mathscr{D} \subset \mathring{\mathbb{M}}$. If $\mathscr{D}$ consists of a single blob D and $\mathbb{M} = D$, then the arrow $(\mathbb{M}; b) \to (\mathscr{D}; c)$ exists only if and only if $b = c$. Thus $\mathtt{m}(\mathcal{C})$ is a chaotic module unless $\mathbb{M}$ is a ball.

Let $\overline{\mathtt{m}}(\mathcal{C})$ be the left $\overline{\mathtt{blob}}(\mathcal{C})$-module obtained from $\mathtt{m}(\mathcal{C})$ by adding one arrow $(\mathtt{M}; b) \to \varnothing$ for each $(\mathtt{M}; b) \in \mathtt{m}(\mathcal{C})$. The $\mathtt{Blob}(\mathcal{C})$-module $\mathtt{M}(\mathcal{C}) := \mathscr{T}_{\mathtt{Blob}(\mathcal{C})}(\mathtt{m}(\mathcal{C}))$ is then a natural submodule of $\overline{\mathtt{M}}(\mathcal{C}) := \mathscr{D}_{\overline{\mathtt{blob}}(\mathcal{C})}(\overline{\mathtt{m}}(\mathcal{C}))$ whose objects are inclusions $(\mathbb{M}; b) \hookleftarrow (\mathscr{D}; c)$ where $\mathscr{D}$ is allowed to be empty.

Lemma 80. *All objects of $\overline{\mathtt{M}}(\mathcal{C})$ that have the form $(\mathbb{M}; b) \to \varnothing$ are rigid in the sense of Definition 62 and satisfy (P1)–(P2) on page 32. On the contrary, none of the objects of $\mathtt{M}(\mathcal{C})$ is rigid and none of them satisfies (P1)–(P2).*

Notice that $(\mathbb{M}; b) \to \varnothing$ is the image of $(\mathbb{M}; b)$ under the natural inclusion $\mathtt{M}(\mathcal{C}) \hookrightarrow \overline{\mathtt{M}}(\mathcal{C})$. Thus $(\mathbb{M}; b)$ becomes rigid and satisfying (P1)–(P2) when considered as an object of $\overline{\mathtt{M}}(\mathcal{C})$. Therefore $\overline{\mathtt{M}}(\mathcal{C})$ is a kind of completion of $\mathtt{M}(\mathcal{C})$, whence the notation.

Proof of Lemma 80. The object $(\mathbb{M}; b) \to \varnothing$ of $\overline{\mathtt{M}}(\mathcal{C})$ has the desired properties for $\odot := \varnothing \to \varnothing$. The second part of the lemma is easy to check. □

Assume, as in the previous section, that the fields on codimension-zero submanifolds of $\mathbb{M}$ are linearly enriched. We are going to define a $\overline{\mathtt{Blob}}(\mathcal{C})$-operad $\overline{\mathcal{F}}$ with values in R-$\mathtt{Mod}$ as follows. If $\mathscr{D}$ and $\mathscr{D}'$ are nonempty blob configurations, we put

$$\overline{\mathcal{F}}\begin{pmatrix} (\mathscr{D}'; c') \\ \hookuparrow \\ (\mathscr{D}; c) \end{pmatrix} := \big\{ f \in \mathcal{C}(\mathscr{D}' \setminus \mathring{\mathscr{D}}) \ \big| \ f|_{\partial\mathscr{D}} = c,\ f|_{\partial\mathscr{D}'} = c' \big\}.$$

The definition is competed by setting

$$\overline{\mathcal{F}}\begin{pmatrix}(\mathscr{D};c)\\ \uparrow\\ \emptyset\end{pmatrix} := \mathcal{U}(\mathscr{D};c),$$

where $\mathcal{U}(\mathscr{D};c) \subset \mathcal{C}(\mathscr{D};c)$ is the subspace of local relations, cf. Definition 77, that restrict to c at $\partial\mathscr{D}$. Finally,

$$\overline{\mathcal{F}}\begin{pmatrix}\emptyset\\ \uparrow\\ \emptyset\end{pmatrix} := R, \ \text{ the ground ring.}$$

Thus $\overline{\mathcal{F}}$ is composed of fields that extend the given ones on the boundary. The structure operation

$$\overline{\mathcal{F}}\begin{pmatrix}(\mathscr{D}';c')\\ \uparrow\\ (\mathscr{D}'';c'')\end{pmatrix} \otimes \overline{\mathcal{F}}\begin{pmatrix}(\mathscr{D}'';c'')\\ \uparrow\\ (\mathscr{D};c)\end{pmatrix} \longrightarrow \overline{\mathcal{F}}\begin{pmatrix}(\mathscr{D}';c')\\ \uparrow\\ (\mathscr{D};c)\end{pmatrix}$$

associated to the morphism in (42) is given by the glueing of fields along $\partial\mathscr{D}''$. The operad $\overline{\mathcal{F}}$ is unital, with the units

$$c \in \overline{\mathcal{F}}\begin{pmatrix}(\mathscr{D};c)\\ \uparrow\\ (\mathscr{D};c)\end{pmatrix} \ \text{ if } \mathscr{D} \neq \emptyset, \text{ and } 1 \in \overline{\mathcal{F}}\begin{pmatrix}\emptyset\\ \uparrow\\ \emptyset\end{pmatrix}.$$

We will also use the $\mathtt{Blob}(\mathcal{C})$-operad $\mathcal{F}$ defined as the restriction of the $\overline{\mathtt{Blob}}(\mathcal{C})$-operad $\overline{\mathcal{F}}$ along the inclusion $\mathtt{Blob}(\mathcal{C}) \hookrightarrow \overline{\mathtt{Blob}}(\mathcal{C})$. We define an $\overline{\mathcal{F}}$-module $\overline{\mathcal{M}}$ by

$$\overline{\mathcal{M}}\begin{pmatrix}(\mathbb{M};b)\\ \uparrow\\ (\mathscr{D};c)\end{pmatrix} := \big\{ f \in \mathcal{C}(\mathbb{M} \setminus \mathring{\mathscr{D}}) \bigm| f|_{\partial\mathscr{D}} = c,\ f|_{\partial\mathbb{M}} = b \big\}$$

if $\mathscr{D} \neq \emptyset$, and

$$\overline{\mathcal{M}}\begin{pmatrix}(\mathbb{M};b)\\ \uparrow\\ \emptyset\end{pmatrix} := \mathcal{C}(\mathbb{M};b).$$

Denote finally by $\mathcal{M}$ the $\mathcal{F}$-module which is the restriction, cf. Definition 53, of the $\overline{\mathcal{F}}$-module $\overline{\mathcal{M}}$ along the pair $(\iota,\jmath) : (\mathtt{Blob}(\mathcal{C}), \mathtt{M}(\mathcal{C})) \to (\overline{\mathtt{Blob}}(\mathcal{C}), \overline{\mathtt{M}}(\mathcal{C}))$ of the natural inclusions.

Referring to Definition 68, we will consider for a fixed field b on the boundary of $\mathbb{M}$ two augmented complexes, namely

$$\beta_*(\overline{\mathcal{F}},\overline{\mathcal{M}})\big((\mathbb{M};b)\to\emptyset\big) \longrightarrow \mathcal{C}(\mathbb{M};b) \ \text{ and } \ \beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b) \longrightarrow \mathcal{C}(\mathbb{M};b).$$

By the functoriality of Proposition 74, the pair $(\iota,\jmath)$ induces the commutative diagram

$$\begin{array}{ccc}
\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b) & \longrightarrow & \mathcal{C}(\mathbb{M};b)\\
\big\downarrow & & \big\|\\
\beta_*(\overline{\mathcal{F}},\overline{\mathcal{M}})\big((\mathbb{M};b)\to\emptyset\big) & \longrightarrow & \mathcal{C}(\mathbb{M};b)
\end{array}$$

of augmented complexes. The next theorem follows from Theorem A, Lemma 80 and an easy computation.

Theorem B. *The augmented complex $\beta_*(\overline{\mathcal{F}},\overline{\mathcal{M}})\big((\mathbb{M};b)\to\varnothing\big)\to\mathcal{C}(\mathbb{M};b)$ is a component of an acyclic resolution of $\overline{\mathcal{M}}$ via unital free $\overline{\mathcal{F}}$-modules. In particular*

$$H_0\big(\beta_*(\overline{\mathcal{F}},\overline{\mathcal{M}})\big((\mathbb{M};b)\to\varnothing\big)\big)\cong\mathcal{C}(\mathbb{M};b).$$

The complex $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)\to\mathcal{C}(\mathbb{M};b)$ resolves the skein module in Definition 78, namely

$$H_{-1}\big(\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)\to\mathcal{C}(\mathbb{M};b)\big)\cong A(\mathbb{M};b).$$

An obvious similar statement holds also for the normalized bar construction.

The subscript -1 of H refers to the homology at $\mathcal{C}(\mathbb{M};b)$. In the next section we prove that the complex $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)\to\mathcal{C}(\mathbb{M};b)$ is quasi-isomorphic, but not isomorphic(!), to the blob complex introduced in [12].

Having in mind the comparison with other 'blob complexes' in the next section, we describe the complex $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ more explicitly. The corresponding towers in (33) are in this particular situation the same as the towers of admissible inclusions

$$\mathfrak{T}_{\mathbb{M}}:\ (\mathscr{D}^0;c^0)\overset{\iota^0}{\hookrightarrow}(\mathscr{D}^1;c^1)\overset{\iota^1}{\hookrightarrow}(\mathscr{D}^2;c^2)\overset{\iota^2}{\hookrightarrow}\cdots\overset{\iota^{n-2}}{\hookrightarrow}(\mathscr{D}^{n-1};c^{n-1})\overset{\iota^{n-1}}{\hookrightarrow}(\mathscr{D}^n;c^n)\overset{\iota}{\hookrightarrow}(\mathbb{M};b) \tag{46}$$

of decorated blob configurations.

For $0\le k\le n-1$ and $D\in\mathscr{D}^{k+1}$ denote by $\mathscr{D}^k_D$ the sub-configuration of $\mathscr{D}^k_D$ of blobs which are subsets of D, i.e.

$$\mathscr{D}^k_D:=\big\{D'\in\mathscr{D}^k\mid D'\subset D\big\}$$

with the order induced from $\mathscr{D}^k$. Denote also by $(\mathscr{D}^k_D,c^k_D)$ the configuration $\mathscr{D}^k_D$ with the decoration on the boundary inherited from $\mathscr{D}^k$. The product in the right hand side of (34) then equals

$$\bigotimes_{D\in\mathscr{D}^0}\mathcal{U}(D;c_D)\otimes\bigotimes_{k=0}^{n-1}\left\{\bigotimes_{\substack{D\in\mathscr{D}^{k+1}\\ \mathscr{D}^k_D\neq\emptyset}}\mathcal{C}\big(D\setminus\mathring{\mathscr{D}}^k_D;c_D\sqcup c^k_D\big)\otimes\bigotimes_{\substack{D\in\mathscr{D}^{k+1}\\ \mathscr{D}^k_D=\emptyset}}\mathcal{U}(D;c_D)\right\}\otimes\mathcal{C}(\mathbb{M}\setminus\mathring{\mathscr{D}}^n;b\sqcup c^n). \tag{47}$$

Since all inclusions in (46) are admissible, if $\mathscr{D}^k_D=(D)$, then $c_D=c^k_D$ and thus in (47)

$$\mathcal{C}\big(D\setminus\mathring{\mathscr{D}}^k_D;c_D\sqcup c^k_D\big)=\mathrm{Span}(c).$$

An analogous formula for the piece $B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ of the normalized bar construction in Definition 70 can be obtained by restricting in (47) the first tensor product in the curly braces to $D\in\mathscr{D}^{k+1}$ such that $\mathscr{D}^k_D\neq(D)$.

Tower (46) determines the planar rooted tree $\mathrm{T}_{\mathbb{M}}$ with $n+2$ levels. Its root is at level zero, level ℓ has one vertex for each blob $D\in\mathscr{D}^{n-\ell+1}$, $1\le\ell\le n+1$. There is one oriented edge $D'\to D$ for each pair (D',D) with $D\in\mathscr{D}^{k+1}$ and $D'\in\mathscr{D}^k_D$. Since blob configurations are linearly ordered sets by definition, the set of input edges of each vertex is linearly ordered too, so $\mathrm{T}_{\mathbb{M}}$ is planar, cf. [2, Example 4.10]. We invite the reader to draw a picture.

The product (47) is thus the space of all vertex-decorations of $\mathrm{T}_{\mathbb{M}}$ such that the root is decorated by the fields on $\mathbb{M}$, the twigs (= the vertices with no input edges) by the local relations, and the remaining vertices by the fields on the blob complements, all subject to the matching of the fields on the boundaries. This description will play an important rôle in the next section.

Notation recall

Operadic categories:

$\mathtt{blob}$ . . . the category opposite to the category of blob configurations and their inclusions

$\overline{\mathtt{blob}}$ . . . the category $\mathtt{blob}$ extended by the empty configuration

$\mathtt{Blob}$ . . . the tautological operadic category $\mathscr{T}(\mathtt{blob})$ associated to $\mathtt{blob}$

$\overline{\mathtt{Blob}}$ . . . the operadic category $\mathscr{D}(\overline{\mathtt{blob}})$

$\mathtt{blob}(\mathcal{C})$, $\overline{\mathtt{blob}}(\mathcal{C})$, $\mathtt{Blob}(\mathcal{C})$, $\overline{\mathtt{Blob}}(\mathcal{C})$ . . . the $\mathcal{C}$-decorated versions of the above categories

Operadic modules:

$\mathtt{m}$ . . . the chaotic left $\mathtt{blob}$-module $\mathtt{Chaos}(\{\mathbb{M}\},\mathtt{blob})$ with one object $\mathbb{M}$

$\overline{\mathtt{m}}$ . . . the chaotic left $\overline{\mathtt{blob}}$-module $\mathtt{Chaos}(\{\mathbb{M}\},\overline{\mathtt{blob}})$ with one object $\mathbb{M}$

$\mathtt{M}$ . . . the tautological $\mathtt{Blob}$-module $\mathscr{T}_{\mathtt{blob}}(\mathtt{m})$ associated to $\mathtt{m}$

$\overline{\mathtt{M}}$ . . . the $\overline{\mathtt{Blob}}$-module $\mathscr{D}_{\overline{\mathtt{blob}}}(\overline{\mathtt{m}})$

$\mathtt{m}(\mathcal{C})$, $\overline{\mathtt{m}}(\mathcal{C})$, $\mathtt{M}(\mathcal{C})$, $\overline{\mathtt{M}}(\mathcal{C})$ . . . the $\mathcal{C}$-decorated versions of the above modules

Operads and their modules:

$\overline{\mathcal{F}}$ . . . the $\overline{\mathtt{Blob}}(\mathcal{C})$-operad of fields

$\mathcal{F}$ . . . the restriction of $\overline{\mathcal{F}}$ to $\mathtt{Blob}(\mathcal{C})$

$\overline{\mathcal{M}}$ . . . the $\overline{\mathcal{F}}$-module of fields

$\mathcal{M}$ . . . the $\mathcal{F}$-module defined as the restriction of $\overline{\mathcal{M}}$ to $\mathtt{M}(\mathcal{C})$

10. Blobs via colored operads, and comparison theorems

The assumptions about the base manifold, blobs, fields, local relations &c., are the same as in Section 9. We start by showing that these data determine a traditional R-linear unital colored operad $\mathcal{F}_c$, cf. [8, Section 2] for the definition of colored operads. Operad modules were introduced in [9, Definition 1.3], cf. also more recent [5, Subsections 2.1.5–6].

Colored operads require colors. In our case, colors will be pairs (D,c) consisting of a blob D in $\mathbb{M}$ with a field $c \in \mathcal{C}(\partial D)$ on its boundary. Suppose that $\mathscr{D} = \{D_1,\dots,D_r\}$ is a configuration of blobs in D as in Definition 76, and $r \geq 2$. Then

$$\mathcal{F}_c\binom{(D,c)}{(D_1,c_1)\cdots(D_r,c_r)} := \mathcal{C}\big(D \setminus \bigcup_{i=1}^{r} \mathring{D}_i; c \sqcup c_1 \sqcup \cdots \sqcup c_r\big), \tag{48}$$

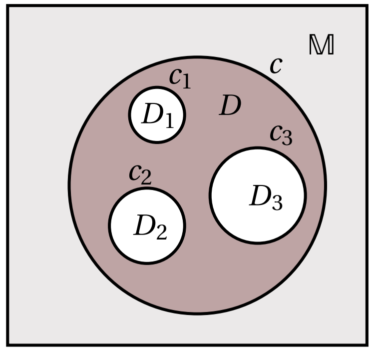


FIGURE 1. A piece of the colored operad $\mathcal{F}_c$ – a schematic picture of a 'punctured blob.'

the span of fields in $\mathcal{C}(D \setminus \mathring{\mathscr{D}})$ that restrict to the field $c \sqcup c_1 \sqcup \cdots \sqcup c_r$ on $\partial D \cup \partial \mathscr{D}$, cf. Figure 1. To define the component

$$\mathcal{F}_c\binom{(D,c)}{(D',c')}, \tag{49}$$

we distinguish two cases. If $D' \subset \mathring{D}$, then (49) equals $\mathcal{C}(D \setminus \mathring{D}'; c \sqcup c')$ as expected. If $D = D'$, we moreover require that $c = c'$, and then

$$\mathcal{F}_c\binom{(D,c)}{(D,c)} := \mathrm{Span}(\{c\}),$$

the R-linear span of the one-point set $\{c\}$. If $r = 0$, we define

$$\mathcal{F}_c\binom{(D,c)}{\emptyset} := \mathcal{U}(D;c),$$

the space of local relations. They thus appear as operations with no input and one output, that is, the 'constants.'

Warning. The symbol '$\emptyset$' in the above display is not an input color, but indicates that the set of inputs is empty.

Proposition 81. *The structure $\mathcal{F}_c$ defined above is a unital colored R-linear operad.*

Proof. Let $x \in \mathcal{F}_c\binom{(D;c)}{(D_1;c_1)\cdots(D_r;c_r)}$ and $x_i \in \mathcal{F}_c\binom{(D_i;c_i)}{(D^i_1;c^i_1)\cdots(D^i_{k_i};c^i_{k_i})}$, $1 \le i \le r$. Then the operadic composite

$$x(x_1,\dots,x_n) \in \mathcal{F}_c\binom{(D,c)}{(D^1_1;c^1_1)\cdots(D^1_{k_1};c^1_{k_1})\cdots(D^r_1;c^r_1)\cdots(D^r_{k_r};c^r_{k_r})} \tag{50}$$

is the field obtained by glueing the fields $x, x_1,\dots,x_r$ along the boundaries of the balls $D_1,\dots,D_r$. The color-matching guarantees that this glueing is possible. The image of the glueing

$$\mathcal{F}_c\binom{(D;c)}{(D_1;c_1)\cdots(D_r;c_r)} \otimes \mathcal{F}_c\binom{(D_1,c_1)}{\emptyset} \otimes \cdots \otimes \mathcal{F}_c\binom{(D_r,c_r)}{\emptyset} \longrightarrow \mathcal{F}_c\binom{(D,c)}{\emptyset}$$

consists of local relations as expected due to the ideal property of Definition 77. The operad axioms are immediately clear, including the unit property of $c \in \mathcal{F}_c\binom{(D,c)}{(D,c)}$. □

Fields on the base manifold $\mathbb{M}$ that restrict to a given $b \in \mathcal{C}(\partial\mathbb{M})$ form a *left $\mathcal{F}_c$-module* $\mathcal{M}_c$ with the components

$$\mathcal{M}_c\big((D_1,c_1)\cdots(D_r,c_r)\big) := \mathcal{C}\big(\mathbb{M}\setminus\bigcup_{i=1}^{r}\mathring{D}_i; c_1\sqcup\cdots\sqcup c_r\sqcup b\big),$$

with the left $\mathcal{F}_c$-action assigning to each $m\in\mathcal{M}_c\big((D_1,c_1)\cdots(D_r,c_r)\big)$ and to x_i's as in the proof of Proposition 81 the element

$$m(x_1,\ldots,x_n)\in\mathcal{M}_c\big((D^1_1;c^1_1)\cdots(D^1_{k_1};c^1_{k_1})\cdots(D^r_1;c^r_1)\cdots(D^r_{k_r};c^r_{k_r})\big),$$

given by the glueing of fields as before. In the rest of this section, by 'colors' we mean the colors used in the definition of the operad $\mathcal{F}_c$ and its module $\mathcal{M}_c$.

The operad $\mathcal{F}_c$ and its module $\mathcal{M}_c$ can be used to write formula (47) for $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ and for its normalized modification $B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ in a nice compact form. Given $D\in\mathscr{D}^{k+1}$, denote

$$(\mathscr{D}^k_D,c^k_D)=\big((D^1_D,c^1_D),\ldots,(D^{i_D}_D,c^{i_D}_D)\big)\ \text{ and }\ \mathscr{D}^n=\big((D^1_n,c^1_n),\ldots,(D^{i_n}_n,c^{i_n}_n)\big).$$

The right hand side of (47) then becomes

$$\bigotimes_{D\in\mathscr{D}^0}\mathcal{F}_c\binom{(D;c_D)}{\varnothing}\otimes\bigotimes_{k=0}^{n+1}\bigotimes_{D\in\mathscr{D}^{k+1}}\mathcal{F}_c\binom{(D;c_D)}{(D^1_D,c^1_D),\ldots,(D^{i_D}_D,c^{i_D}_D)}\otimes\mathcal{M}_c\big((D^1_n,c^1_n),\ldots,(D^{i_n}_n,c^{i_n}_n)\big). \tag{51}$$

Notice that the two tensor products in the curly brackets of (47) have been absorbed by one tensor product, thanks to the convenient definition of the operad $\mathcal{F}_c$. The normalized variant is obtained by assuming that in the 'big' tensor product either $i_D\geq 2$, or $i_D=1$ but $D^1_D\neq D$.

As in the paragraph following formula (47) we interpret the product (51) as the space of all vertex-decorations of a planar rooted tree with $n+2$ levels and edges colored by fields, such that the root is decorated by $\mathcal{M}_c$ and the other vertices with $\mathcal{F}_c$ in such a way that the output and the inputs of the decorations match the colors of the adjacent edges.

Comparing the above with the material in [5, Subsection 4.3.2] we identify (51) with the constant part of the colored version of Fresse's *simplicial bar construction* $C_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)$, resp. with its normalization $N_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)$. We therefore have

Proposition 82. *There are natural isomorphisms of chain complexes*

$$\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)\cong C_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)(\varnothing)\ \textit{ and }\ B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)\cong N_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)(\varnothing) \tag{52}$$

compatible with the augmentations.

Proposition 82 provides a bridge between blob complexes viewed from the perspective of unary operadic categories and blob complexes based on colored operads. Fact 4.1.7 of [5] applied to $P = \mathcal{F}_c$ and $R = \mathcal{M}_c$ may suggest that the complexes $C_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)$ and $N_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{M}_c)$ are acyclic in positive dimensions. This is however not true, because the crucial assumption of connectivity required by Fact 4.1.7 is violated in our situation.

B. Fresse introduced in [5, Section 4] the *differential bar construction* $B_*(L,P,R)$ of an augmented P-operad with coefficients in a right P-module L and a left P-module R. We will use the obvious colored version of his construction with $P = \mathcal{F}_c$, $L = \mathcal{M}_c$ and $R = \mathcal{F}_c$. Let $\mathcal{I}_c$ be the colored operad whose only nontrivial component is

$$\mathcal{I}_c\binom{(D,c)}{(D;c)} := \mathrm{Span}(\{c\}),$$

the R-linear span of the field $\{c\}$. The operad of fields $\mathcal{F}_c$ is augmented by the obvious morphism $\varepsilon : \mathcal{F}_c \to \mathcal{I}_c$ of colored operads. We denote by $\widehat{\mathcal{F}}_c := \mathrm{Ker}(\varepsilon)$ the augmentation ideal.

Let $\overline{B}(\mathcal{F}_c)$ be the cofree conilpotent cooperad generated by the component-wise suspension of the colored collection $\widehat{\mathcal{F}}_c$. Mimicking Fresse's definition we consider

$$B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c) := \mathcal{M}_c \circ \overline{B}(\mathcal{F}_c) \circ \mathcal{F}_c, \tag{53}$$

where $\circ$ is the straightforward colored version of the composition product [5, §1.3.5]. The iterated product (53) bears the differential given by the operad structure of $\mathcal{F}_c$ and the right $\mathcal{F}_c$-action on $\mathcal{M}_c$. The differential bar construction $B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)$ is thus a colored collection with components

$$B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)\big((D_1,c_1)\cdots(D_r,c_r)\big).$$

We will be particularly interested in the component with $r = 0$ (i.e. 'no inputs'), which we denote by $B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$.

The elements of $B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$ can be visualized as finite linear combinations of forests growing from $\mathcal{M}_c$, whose trees have forks (= vertices) decorated by the fields in $\widehat{\mathcal{F}}_c$, branches (= edges) colored by blobs with fields on the boundaries, and twigs (= leaves) by the fields in the local relations, cf. Figure 2. We must however be careful, since the fields (= decorations of the vertices) are assigned degree +1, cf. (53), so we are in fact dealing with the *equivalence classes* of forests in Figure 2 with vertices linearly ordered compatibly with the partial order given by the distance from the soil (= root). We identify a forest F' with the forest $\epsilon \cdot F''$, where $\epsilon \in \{+1,-1\}$ is the signum of the permutation that brings the order of vertices of F' to the order of vertices of F''. The differential contracts the edges, one at a time, and decorates the new vertex thus created by the glued field. Notice that this description is practically identical with the definition of the blob complex in [12].

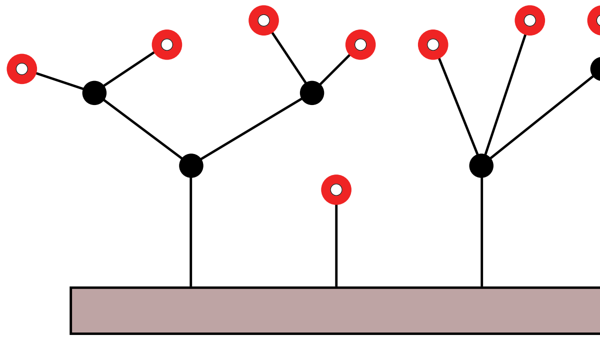


FIGURE 2. Viewing elements of $B_*(\mathcal{M}_c, \mathcal{F}_c, \mathcal{I}_c)(\emptyset)$ as forests. Internal vertices are decorated by fields on punctured blobs, leaves by generators of local relations, the soil by a field on $\mathbb{M}$.

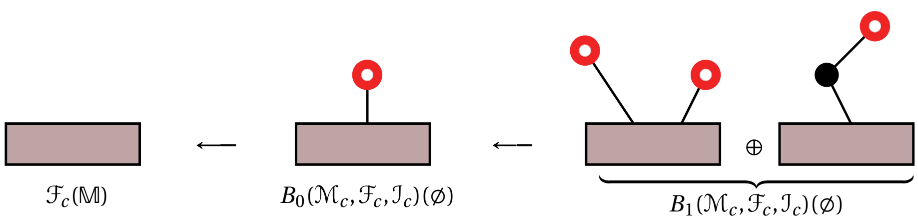


FIGURE 3. The initial part of the augmented bar construction.

In the 'forest representation' of $B_*(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset)$, the homological degree is the number of vertices. In particular,

$$B_0(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset) \cong \bigoplus_{(D;c)} \mathcal{M}_c\big((D;c)\big) \otimes \mathcal{F}_c \binom{(D;c)}{\emptyset} = \bigoplus_{(D;c)} \mathcal{C}(\mathbb{M} \setminus \mathring{D}; c \sqcup b) \otimes \mathcal{U}(D;c)$$

with the augmentation $\epsilon : B_0(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset) \to \mathcal{C}(\mathbb{M}; b)$ given by gluing the fields in $\mathcal{C}(\mathbb{M} \setminus \mathring{D}; c \sqcup b)$ with the fields from $\mathcal{U}(D, c)$.

Example 83. Figure 3 symbolizes the types of terms in the initial part

$$\mathcal{C}(\mathbb{M}; b) \xleftarrow{\epsilon} B_0(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset) \xleftarrow{\partial} B_1(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset) \xleftarrow{\partial} \cdots$$

of the augmented bar construction. This should be compared to the explicit description of the initial terms of the blob complex given on pages 1500–1502 of [12].

We finally arrive at

Proposition 84. *The blob complex* $\mathcal{B}(\mathbb{M}, \mathcal{C})$ *of* [12, Section 3] *is isomorphic to the piece*

$$\mathcal{C}(\mathbb{M}; b) \xleftarrow{\epsilon} B_*(\mathcal{M}_c, \mathcal{F}_c, \mathcal{F}_c)(\emptyset) \tag{54}$$

of the augmented differential bar construction.

Proof. Comparing the respective definitions, we easily construct the required isomorphism

$$\begin{array}{ccccccccc}
\mathcal{C}(\mathbb{M};b) & \xleftarrow{\epsilon} & B_0(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset) & \xleftarrow{\partial} & B_1(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset) & \xleftarrow{\partial} & B_2(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset) & \xleftarrow{\partial} & \cdots \\
\| & & \updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong & & \\
\mathcal{B}_0(\mathbb{M},\mathcal{C}) & \xleftarrow{\partial} & \mathcal{B}_1(\mathbb{M},\mathcal{C}) & \xleftarrow{\partial} & \mathcal{B}_2(\mathbb{M},\mathcal{C}) & \xleftarrow{\partial} & \mathcal{B}_3(\mathbb{M},\mathcal{C}) & \xleftarrow{\partial} & \cdots
\end{array}$$

of chain complexes. Notice the degree shift. □

Recall that Fresse introduced, for a 'traditional' operad P, a right P-module L and a left module R, the *levelization morphism*

$$\phi_*(L,P,R): B_*(L,P,R) \to N_*(L,P,R)$$

from the differential bar construction to the normalization of the simplicial bar construction. While the elements of $B_*(L,P,R)$ are represented by decorated trees, the elements of $N_*(L,P,R)$ are decorated trees equipped with levels. The chain map $\phi_*(L,P,R)$ sends a given decorated tree to the sum, with appropriate signs, of all decorated trees with levels whose underlying non-leveled decorated tree equals the given one. Fresse then proved in [5, Theorem 4.1.8] that $\phi_*(L,P,R)$ is a quasi-isomorphism. Although he assumed simple connectivity, his theorem holds without this assumption, which expresses the folklore fact that the space of levels of a given tree is a contractible groupoid, cf. also [6]. Michael, do you want to say anything more specific about that citation?Combining this with isomorphisms (52) and (54) results in the following comparison between the original blob complex $\mathcal{B}_*(\mathbb{M},\mathcal{C})$ defined in [13, Section 3] and the normalized bar resolution $B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$, cf. Definition 70.

Theorem C. *The levelization morphism of* [5, Theorem 4.1.8] *induces a quasi-isomorphism*

$$\begin{array}{ccc}
\mathcal{B}_{*+1}(\mathbb{M},\mathcal{C}) & \longrightarrow & \mathcal{C}(\mathbb{M};b) \\
\ell_*\downarrow\sim & & \| \\
B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b) & \longrightarrow & \mathcal{C}(\mathbb{M};b)
\end{array}$$

of augmented chain complexes.

The colored operad $\mathcal{F}_c$ is a right module over itself, so one may also consider $B_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{F}_c)$ instead of $B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)$. Since the pieces of $\mathcal{F}_c$ possess also the output color, the components of $B_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{F}_c)$ are

$$B_*(\mathcal{F}_c,\mathcal{F}_c,\mathcal{F}_c)\binom{(D;c)}{(D_1;c_1)\cdots(D_r;c_r)}. \tag{55}$$

Although the connectivity assumption of [5, Lemma 4.1.3] is not fulfilled, a simple explicit contracting homotopy which was in fact constructed in the proof of [13, Proposition 3.2.1] shows

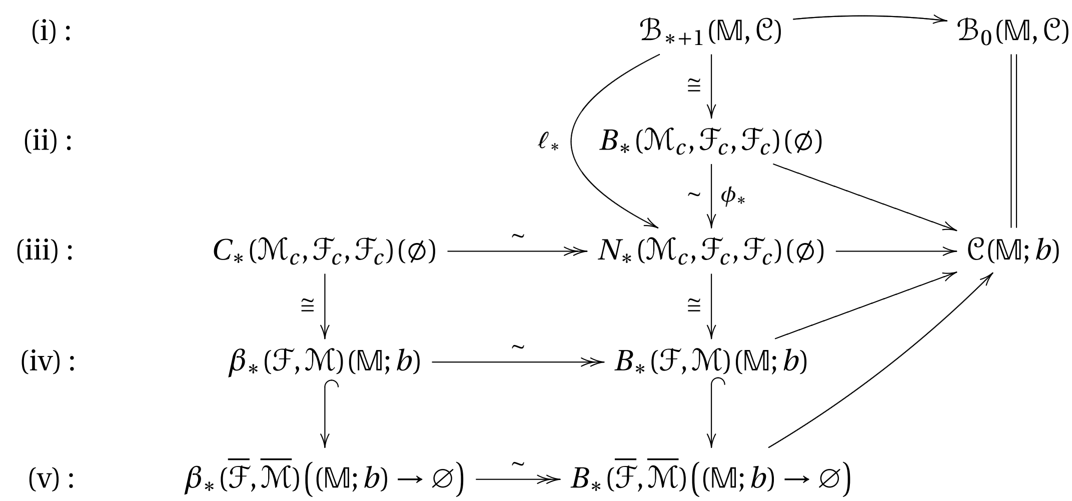


FIGURE 4. Sundry chain complexes and their maps.

that (55) is acyclic in positive dimensions for each choice of colors. If the base manifold $\mathbb{M}$ is isomorphic to a ball D, we easily verify that

$$B(\mathcal{M}_c,\mathcal{F}_c,\mathcal{I}_c) \cong \bigoplus_{c\in\mathcal{C}(\partial D)} B(\mathcal{F}_c,\mathcal{F}_c,\mathcal{I}_c)\begin{pmatrix}(\mathbb{M};b)\\ \emptyset\end{pmatrix}.$$

The acyclicity of (55) combined with Proposition 84 gives

Corollary 85 (Corollary 3.2.2 of [12])**.** *If $\mathbb{M}$ is isomorphic to a $(d+1)$-dimensional ball, then the chain complex $\mathcal{B}(\mathbb{M},\mathcal{C})$ is contractible.*

SUMMARY

Chain complexes featured in Part 2 together with the connecting maps are summarized in Figure 4. In that figure:

– $\mathcal{B}_{*+1}(\mathbb{M},\mathcal{C})$ in row (i) is the original blob complex of [13],
– $B_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$ in row (ii) is the piece of Fresse's bar construction,
– $C_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$ in row (iii) is the piece of Fresse's simplicial bar construction,
– $N_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$ in row (iii) is the normalization of $C_*(\mathcal{M}_c,\mathcal{F}_c,\mathcal{F}_c)(\emptyset)$,
– $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ in row (iv) is the piece of the un-normalized bar resolution in Definition 68,
– $B_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$ in row (iv) is the normalization of $\beta_*(\mathcal{F},\mathcal{M})(\mathbb{M};b)$, and
– the items in row (v) are as in row (iv) but this time applied on $\overline{\mathcal{F}}$ and $\overline{\mathcal{M}}$.

The vertical map ϕ_* is Fresse's levelization morphism, ℓ_* is the map in Theorem C. The remaining maps are either natural isomorphisms, or augmentations, or inclusions, or projections. The vertical isomorphism between row (i) and (ii) comes from Proposition 84, the two vertical isomorphism between rows (iii) and (iv) are that of Proposition 82.

REFERENCES

[1] M.A. Batanin and M. Markl. Operadic categories and duoidal Deligne's conjecture. *Adv. Math.*, 285:1630–1687, 2015.
[2] M.A. Batanin and M. Markl. Operadic categories as a natural environment for Koszul duality. Preprint `arXiv:1812.02935`, version 4, July 2022.
[3] M.A. Batanin and M. Markl. Koszul duality for operadic categories. Preprint `arXiv:arXiv:2105.05198`, version 2, July 2022.
[4] S. Eilenberg and J.C. Moore. Adjoint functors and triples. *Illinois J. Math.*, 9(3):381–398, 1965.
[5] B. Fresse. Koszul duality of operads and homology of partition posets. *Contemp. Math.*, 346: 115–215, 2004.
[6] G. Heuts, I. Moerdijk. Partition complexes and trees. Preprint `arXiv:2112.08043`, December 2021.
[7] S. MacLane. *Homology.* Springer Verlag, 1963
[8] M. Markl. Homotopy algebras are homotopy algebras. *Forum Matematicum* 16(1): 129–160, 2004.
[9] M. Markl. Models for operads. *Communications in Algebra*, 24(4):1471–1500, 1996.
[10] M. Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008.
[11] M. Markl, S. Shnider, and J.D. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
[12] S. Morrison and K. Walker. Blob homology. *Geometry & Topology*, 16:1481–1607, 2012.
[13] K. Walker. *TQFTs.* Early incomplete draft, version 1h, May 2006. Available from the author's web homepage.

THE CZECH ACADEMY OF SCIENCES, INSTITUTE OF MATHEMATICS, ŽITNÁ 25, 115 67 PRAGUE 1, THE CZECH REPUBLIC