

## ASSOCIAHEDRA, CELLULAR $W$ -CONSTRUCTION AND PRODUCTS OF $A_\infty$ -ALGEBRAS

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ABSTRACT. The aim of this paper is to construct a functorial tensor product of  $A_\infty$ -algebras or, equivalently, an explicit diagonal for the operad of cellular chains, over the integers, of the Stasheff associahedron. These constructions in fact already appeared (Saneblidze and Umble, 2000 and 2002); we will try to give a more conceptual presentation. We also prove that there does not exist a coassociative diagonal.

### 1. INTRODUCTION

In this paper we study tensor products of  $A_\infty$ -algebras. More precisely, given two  $A_\infty$ -algebras  $A = (V, \partial^V, \mu_2^V, \mu_3^V, \dots)$  and  $B = (W, \partial^W, \mu_2^W, \mu_3^W, \dots)$ , we will be looking for a functorial definition of an  $A_\infty$ -structure  $A \odot B$  that would extend the standard (non-associative) dg-algebra structure on the tensor product  $A \otimes B$ . This means that the  $A_\infty$ -algebra  $A \odot B$  will be of the form  $(V \otimes W, \partial, \mu_2, \mu_3, \dots)$ , where  $\partial$  is the usual differential on the tensor product,

$$(1.1) \quad \partial(v \otimes w) := \partial^V(v) \otimes w + (-1)^{\deg v} v \otimes \partial^W(w),$$

and the bilinear product  $\mu_2$  is given by another standard formula

$$(1.2) \quad \mu_2(v' \otimes w', v'' \otimes w'') := (-1)^{\deg v'' \deg w'} \mu^V(v', v'') \otimes \mu^W(w', w''),$$

where  $v, v', v'' \in V$  and  $w, w', w'' \in W$ .

A “coordinate-free” formulation of the problem is the following. Let  $\underline{\mathcal{A}}$  be the non- $\Sigma$  operad describing  $A_\infty$ -algebras (see [8, page 45]), that is, the minimal model of the non- $\Sigma$  operad  $\underline{\mathcal{A}}_{ss}$  for associative algebras. The above product is equivalent to a morphism of dg-operads (a *diagonal*)  $\Delta : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$  such that  $\Delta$  induces the usual diagonal  $\Delta_{\underline{\mathcal{A}}_{ss}}$  on the non- $\Sigma$  operad  $\underline{\mathcal{A}}_{ss} = H_*(\underline{\mathcal{A}})$ .

The existence of such a diagonal is not surprising and follows from properties of minimal models for operads; see [8, Proposition 3.136]. On the other hand, there is no way to control the coassociativity of diagonals constructed using this general argument, and we will see below, in Theorem 6.1, that there, surprisingly enough, *does not exist* a coassociative diagonal.

For practical purposes, such as applications in open string theory [2], one needs a tensor product (and therefore also a diagonal) given by an explicit formula. Such an explicit diagonal was constructed by Umble and Saneblidze in [9]. Our work

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Received by the editors January 5, 2004.

2000 *Mathematics Subject Classification*. Primary 18D50, 55U99.

The first author was supported by the grant GA ČR 201/02/1390.

The second author was supported by the Israel Academy of Sciences.

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was in fact motivated by our unsuccessful attempts to understand their paper. We will denote this diagonal by  $\Delta^{\text{su}}$  and call it the *SU-diagonal*. In this article we recall the definition of this diagonal and give a conceptual explanation of why it is well defined. The operad  $\mathcal{A}$  can be identified with the operad of the cellular chain complexes of the non- $\Sigma$  operad of associahedra,  $\mathcal{A} \cong \mathcal{C}_*(\mathcal{K})$  (see [8, page 45]). Therefore, the required diagonal is given by a family of chain maps

$$\Delta_{K_n} : \mathcal{C}_*(K_n) \rightarrow \mathcal{C}_*(K_n) \otimes \mathcal{C}_*(K_n), \quad n \geq 1,$$

commuting with the induced operad structures and such that  $H_*(\Delta_{K_n}) = \Delta_{\mathcal{A}ss}$ .

The cells of the associahedra are not conducive to the definition of a diagonal. There is, however, a cubical decomposition of the associahedra provided by the *W-construction* of Boardman and Vogt [1], which is a homotopically equivalent non- $\Sigma$  operad  $\mathcal{W} = \{W_n\}_{n \geq 1}$ , for which there is a canonical diagonal

$$\Delta_{W_n} : \mathcal{C}_*(W_n) \rightarrow \mathcal{C}_*(W_n) \otimes \mathcal{C}_*(W_n), \quad n \geq 1,$$

induced by the cubical structure (see (5.1)). A suitable diagonal on the associahedra can be then obtained by transferring  $\Delta_{W_n}$  from  $\mathcal{W}$  to  $\mathcal{K}$ . More precisely, let

$$(1.3) \quad \mathcal{C}_*(W_n) \xrightarrow{p_n} \mathcal{C}_*(K_n) \xrightarrow{q_n} \mathcal{C}_*(W_n), \quad n \geq 1,$$

be arbitrary operadic maps such that  $H_*(p_n)$  and  $H_*(q_n)$  are identity endomorphisms of  $\mathcal{A}ss(n)$ , via the canonical identifications

$$H_*(\mathcal{C}_*(W_n)) \cong \mathcal{A}ss(n) \cong H_*(\mathcal{C}_*(K_n)), \quad n \geq 1.$$

Then the formula

$$(1.4) \quad \Delta_{K_n} := (p_n \otimes p_n) \circ \Delta_{W_n} \circ q_n$$

clearly defines a diagonal. In fact, it can be proved that the operadic maps  $p = \{p_n\}_{n \geq 1}$  and  $q = \{q_n\}_{n \geq 1}$  with the above properties are homotopy inverses, but we will not need this statement.

Our task is to define the maps in (1.3). While there is an obvious and simple definition of  $q_n$ , finding a suitable formula for  $p_n$  is much less obvious. We give an explicit and very natural definition inspired by a formula in [9].

We will see that the operad of cellular chains  $\mathcal{C}_*(\mathcal{W})$  can be described in terms of metric trees. Similar *cellular W-constructions* on a given dg-operad were considered by Kontsevich and Soibelman in [4]. In this terminology, the chain maps  $p$  and  $q$  are explicit homotopy equivalences, defined over the integers, between the chain *W-construction* on the operad  $\mathcal{A}ss$  and the minimal model  $\mathcal{A}$  of  $\mathcal{A}ss$ , which give rise to explicit equivalences of the categories of algebras over these dg-operads.

*Remark.* In a more recent follow-up [10] of [9], Sanedidze and Umble first constructed a diagonal on the permutahedron. The SU-diagonal  $\Delta^{\text{su}}$  was then obtained as the transfer of this permutahedral diagonal to the associahedron, via Tonk’s projection [12].

## 2. CATEGORIAL PROPERTIES OF DIAGONALS AND TENSOR PRODUCTS

Recall [5] that there are two notions of morphisms of  $A_\infty$ -algebras. A *strict morphism* of  $A_\infty$ -algebras  $(X, \partial, \mu_2, \mu_3, \dots)$  and  $(Y, \partial, \nu_2, \nu_3, \dots)$  is a linear map  $f : X \rightarrow Y$  that commutes with all structure operations. A weaker notion is that of a strongly homotopy (sh) morphism, given by a sequence of maps  $f_n : X^{\otimes n} \rightarrow Y$ ,  $n \geq 1$ , satisfying a rather complicated set of axioms (see, for example, [5, 7]). Such

a map is invertible if and only if  $f_1 : X \rightarrow Y$  is an isomorphism. The category of  $A_\infty$ -algebras and their strict morphisms will be denoted by  $\mathbf{strA}_\infty$ ; the category of  $A_\infty$ -algebras and their sh morphisms will be denoted by  $\mathbf{shA}_\infty$ .

As proved in [8, Proposition 3.136], any two diagonals  $\Delta', \Delta'' : \underline{A} \rightarrow \underline{A} \otimes \underline{A}$  are homotopic as maps of operads. Let  $\odot'$  (resp.  $\odot''$ ) denote the tensor product induced by  $\odot'$  (resp.  $\odot''$ ). Although  $A \odot' B$  and  $A \odot'' B$  are, in general, not strictly isomorphic, the homotopy between  $\Delta'$  and  $\Delta''$  can be shown to induce a strongly homotopy isomorphism between  $A \odot' B$  and  $A \odot'' B$ . Therefore we obtain the following statement of uniqueness:

**Proposition 2.1.** *For any two  $A_\infty$ -algebras  $A, B$ , the  $A_\infty$ -algebras  $A \odot' B$  and  $A \odot'' B$  are isomorphic in  $\mathbf{shA}_\infty$ .*

In Theorem 6.1 we prove that there are no coassociative diagonals. This means that in general

$$A \odot (B \odot C) \not\cong (A \odot B) \odot C$$

in the ‘strict’ category  $\mathbf{strA}_\infty$ . On the other hand, as argued in [8, Proposition 3.136], each diagonal  $\Delta$  is *homotopy associative* in the sense that the maps  $(\Delta \otimes 1)\Delta$  and  $(1 \otimes \Delta)\Delta$  are homotopic maps of operads, from which we infer:

**Proposition 2.2.** *For any three  $A_\infty$ -algebras  $A, B$  and  $C$ ,*

$$A \odot (B \odot C) \cong (A \odot B) \odot C$$

*in the ‘weak’ category  $\mathbf{shA}_\infty$ .*

By the same argument, one can also prove

**Proposition 2.3.** *For any two  $A_\infty$ -algebras  $A$  and  $B$ ,*

$$A \odot B \cong B \odot A$$

*in  $\mathbf{shA}_\infty$ .*

This naturally raises the question of whether  $\mathbf{shA}_\infty$  with a product  $\odot$  based on an appropriate diagonal is a (possibly symmetric) monoidal category. Formulating this question precisely requires one more step.

While it is clear that  $\odot$  is a functor  $\mathbf{strA}_\infty \times \mathbf{strA}_\infty \rightarrow \mathbf{strA}_\infty$ , to make it a functor  $\mathbf{shA}_\infty \times \mathbf{shA}_\infty \rightarrow \mathbf{shA}_\infty$ , one should define a ‘product’  $f \odot g : A' \odot A'' \rightarrow B' \odot B''$  for given sh morphisms  $f : A' \rightarrow A''$  and  $g : B' \rightarrow B''$ . One should then consider a functorial ‘associator’  $\Phi_{A,B,C} : A \odot (B \odot C) \rightarrow (A \odot B) \odot C$  and a ‘symmetry’  $\sigma_{A,B} : A \odot B \rightarrow B \odot A$ .

The above objects exist by general nonsense, but it is not clear whether they fulfill the axioms of a (symmetric) monoidal category (the pentagon and the hexagons), although it is quite possible that these axioms are satisfied for certain special choices of the data above. On a more abstract level, the ‘full’ functorial monoidal product  $A, B \mapsto A \odot B, f, g \mapsto f \odot g$  in  $\mathbf{shA}_\infty$  requires us to construct a ‘diagonal’ in the minimal model of the two-colored operad  $\underline{Ass}_{\bullet, \rightarrow, \bullet}$  describing homomorphisms of associative algebras satisfying additional properties that do not follow from general nonsense.

3. CALCULUS OF ORIENTED CELL COMPLEXES OF  $K_n$  AND  $W_n$

All operads  $\mathcal{P}$  considered in this paper are such that  $\mathcal{P}(0)$  is trivial and that  $\mathcal{P}(1)$  is isomorphic to the ground field. The category of operads with this property is equivalent to the category of *pseudo-operads*  $\mathcal{P}$  such that  $\mathcal{P}(0) = \mathcal{P}(1) = 0$ , where the equivalence is given by forgetting the  $n = 1$  piece. Roughly speaking, this means that we may ignore operadic units; see [6, Observation 1.2] for details. *Therefore, for the rest of this paper, an operad means a pseudo-operad with  $\mathcal{P}(0) = \mathcal{P}(1) = 0$ .*

First, we establish some notation. Let  $\underline{K} = \{K_n\}_{n \geq 2}$  be the non- $\Sigma$  operad of associahedra. The topological cell complex  $K_n$  can be realized as a convex polytope in  $\mathbf{R}^{n-2}$ , with  $k$ -cells labeled by planar rooted trees with  $n$  leaves and  $n - k - 2$  internal edges, or equivalently by  $(n - k - 2)$ -fold bracketings of  $n$  elements; see [8, II.1.6]. For example, 0-cells correspond to binary trees with  $n$  leaves, or equivalently, full bracketings of  $n$  elements. All of our constructions will be expressed in terms of rooted planar trees although there is clearly an underlying geometric meaning based on the polytope realization of  $K_n$ .

Boardman and Vogt have defined in [1] a cubical subdivision of the cells of  $K_n$ , for  $n \geq 2$ , giving rise to a cubical cell complex known as the  $W$ -construction,  $W_n$ . See Figure 6 of [8, Section II.2.8] for  $W_4$  represented as a cubical subdivision of  $K_4$ . The cells of  $W_n$  are in one-to-one correspondence with “metric  $n$ -trees,” that is, planar rooted trees with  $n$  leaves and with internal edges labeled either “metric” or “non-metric.” Metric  $n$ -trees with  $k$  metric edges label the topological  $k$ -cells of  $W_n$ . A cubical cell is called *an interior cell* if the labeling tree has only metric edges. In the geometric realization the interior cells are in the interior of the convex polytope.

In order to define the boundary operators on the cell complexes  $\mathcal{C}_*(\underline{K}) := \{\mathcal{C}_*(K_n)\}_{n \geq 2}$  and  $\mathcal{C}_*(\underline{W}) := \{\mathcal{C}_*(W_n)\}_{n \geq 2}$  (non- $\Sigma$  operads in the category of chain complexes), we have to introduce an orientation on the cells. Let  $T$  be a planar rooted tree with internal edges labeled  $e_1, \dots, e_m$ . Two orderings  $e_{i_1}, \dots, e_{i_m}$  and  $e_{j_1}, \dots, e_{j_m}$  will be called *equivalent* if they are related by an even permutation. The equivalence class corresponding to an ordering  $e_{i_1}, \dots, e_{i_m}$  will be called an orientation and denoted  $e_{i_1} \wedge \dots \wedge e_{i_m}$ .

**Definition 3.1.** An oriented  $k$ -cell in  $K_n$  is a pair  $(T, \omega)$ , where  $T$  is a planar rooted tree with  $n$  leaves and  $n - k - 2$  internal edges and  $\omega$  is an orientation. Let  $\mathcal{C}_k(K_n)$  be the vector space spanned by the oriented  $k$ -cells in  $K_n$  modulo the relation  $(T, \omega) = -(T, \omega')$ , where  $\omega$  and  $\omega'$  are the two distinct orientations.

An oriented metric  $k$ -cell in  $W_n$  is a pair  $(T, \omega)$ , where  $T$  is a metric tree with  $n$  leaves and  $k$  metric edges and  $\omega$  is an orientation of the metric edges. Let  $\mathcal{C}_k(W_n)$  be the vector space spanned by the oriented  $k$ -cells in  $W_n$  modulo the relation  $(T, \omega) = -(T, \omega')$ , where  $\omega$  and  $\omega'$  are the two distinct orientations.

The operad composition law

$$\circ_i : \mathcal{C}_k(K_r) \otimes \mathcal{C}_l(K_s) \longrightarrow \mathcal{C}_{k+l}(K_{r+s-1})$$

is defined on basis elements by

$$(3.1) \quad (T, \omega) \circ_i (T', \omega') := (-1)^{rl+i(s+1)}(T \circ_i T', \omega \wedge \omega' \wedge e),$$

where  $\circ_i$  is defined on planar rooted trees in the standard way, grafting the second tree onto the  $i$ -th leaf of the first, and  $\omega \wedge \omega' \wedge e$  is the concatenation of the

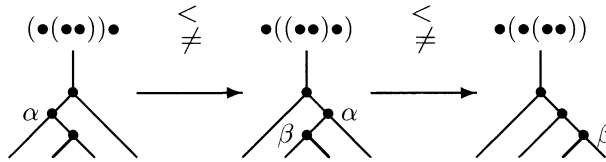


FIGURE 1. The partial order on the set of binary trees. The left-hand arrow moves the vertex  $\alpha$ ; the right-hand arrow moves the vertex  $\beta$ . Therefore the left-hand tree is strictly less than the middle tree, which is strictly less than the right-hand tree.

two orientations, with  $e$  labeling the new edge created by grafting. The operad composition law

$$\circ_i : \mathcal{C}_k(W_r) \otimes \mathcal{C}_l(W_s) \longrightarrow \mathcal{C}_{k+l}(W_{r+s-1})$$

is defined on basis elements by

$$(3.2) \quad (T, \omega) \circ_i (T', \omega') := (T \circ_i T', \omega \wedge \omega').$$

A heuristic explanation of why signs in display (3.2) are unnecessary is that the orientation of the cells of  $\underline{\mathcal{W}}$  defined in terms of metric edges is geometric in the sense that the number of metric edges is the same as the dimension of the cell. In the case of  $\mathcal{C}_*(\underline{\mathcal{W}})$  the new edge created by grafting is non-metric and does not appear in the ordering of metric vertices.

The boundary operator on  $\mathcal{C}_*(K_n)$  is defined by

$$(3.3) \quad \partial_K(T, e_1 \wedge \cdots \wedge e_m) := \sum_{\{T' \mid T'/e' = T\}} (T', e' \wedge e_1 \wedge \cdots \wedge e_m),$$

where the sum is over all trees  $T'$  with an edge  $e'$  such that  $T'$  collapses to  $T$  when  $e'$  is collapsed. The condition  $\partial_K^2 = 0$  follows immediately from the identities

$$(T'', e'' \wedge e' \wedge e_1 \wedge \cdots \wedge e_m) = -(T'', e' \wedge e'' \wedge e_1 \wedge \cdots \wedge e_m).$$

Next we define the boundary operator on the complex  $\mathcal{C}_*(W_n)$ . Let  $T$  be a metric tree and  $e_1 \wedge \cdots \wedge e_k$  an orientation:

$$(3.4) \quad \partial_W(T, e_1 \wedge \cdots \wedge e_k) := \sum_{1 \leq i \leq k} (-1)^i [(T/e_i, e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k) - (T_i, e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_k)],$$

where  $T_i$  is the same (unlabeled) tree as  $T$  but with the metric edge  $e_i$  changed to a non-metric edge. As above, the condition  $\partial_W^2 = 0$  follows from the relations for the orientation elements.

In the rest of this section we introduce ‘standard orientations’ for top-dimensional cells of  $W_n$  and 0-dimensional cells of  $K_n$ . The associator gives a partial order relation on rooted planar binary trees that moves a vertex to the right and changes the outgoing edge from a right-leaning position to a left-leaning position, as shown in Figure 1. The *standard orientation*  $\omega_{\bar{b}(n)}$  of the maximal fully metric binary tree  $\bar{b}(n)$  (all internal edges are left-leaning) is given by enumerating the internal edges in sequence, starting with  $e_1$  (the edge adjacent to the root) and continuing in sequence with  $e_2, \dots, e_{n-2}$  moving away from the root (see Figure 2).

The standard orientation  $\omega_T$  of a non-maximal fully metric binary tree  $T$  is determined by a sequence of sign changes and relabelings along a path from  $\bar{b}(n)$

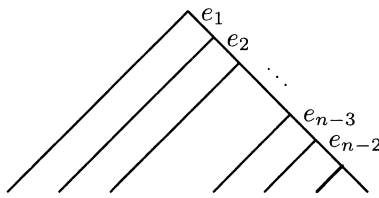


FIGURE 2. The maximal binary tree  $\bar{b}(n)$ .

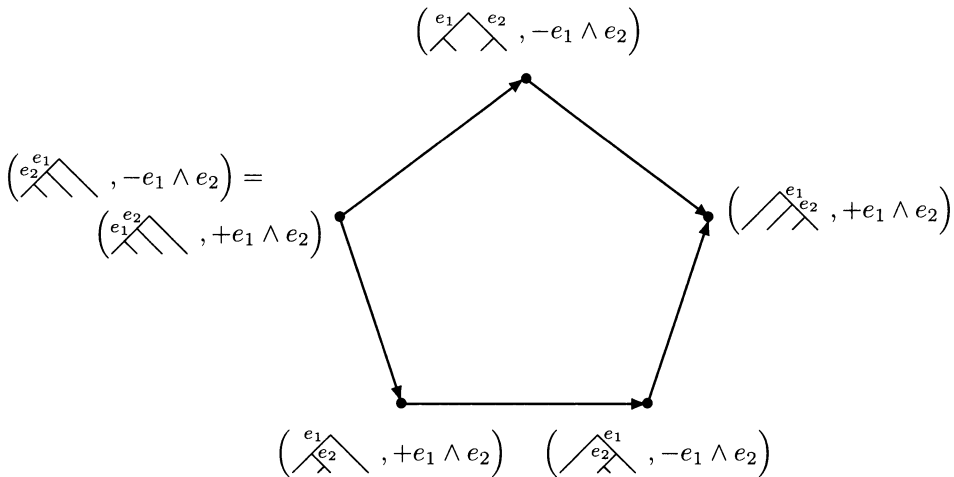


FIGURE 3. The rule for defining the standard orientation of fully metric trees is illustrated for the pentagon; this verifies that the definition is independent of the path.

to  $T$  in the associahedron. Figure 3 displays the standard orientations of binary trees with four leaves. To check that this rule gives an unambiguous definition of the orientation, it is sufficient (thanks to Mac Lane’s Coherence Theorem) to verify that the definition is independent of the path in the pentagon (expressing coherence of the associator) and in the square (expressing naturality). The verification for the pentagon is given in Figure 3. The verification for the square is straightforward and follows from functoriality. Therefore each fully metric binary  $n$ -tree  $T$  together with its standard orientation  $\omega_T$  determines an element  $(T, \omega_T) \in \mathcal{C}_{n-2}(W_n)$ .

We also define the *standard orientation*  $\xi_T$  of a binary  $n$ -tree  $T$  representing a 0-cell of  $\mathcal{C}_0(K_n)$  inductively as follows. The single binary 2-tree representing a 0-cell of  $\mathcal{C}_0(K_2)$  has no internal edges, and its canonical orientation is given by assigning the sign  $+1$  to this tree. The canonical orientation of any binary tree is then determined by the formula

$$(S, \xi_S) \circ (T, \xi_T) = (S \circ_i T, \xi_{S \circ_i T}),$$

having checked that there is no ambiguity. This can be done exactly as in the previous paragraph for  $\omega_T$ . For example, we immediately get the following standard orientations:

$$\left( \begin{array}{c} a \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} , -a \right) \text{ and } \left( \begin{array}{c} e \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} , e \right).$$

As an exercise we recommend that the reader verify that the standard orientation  $\xi_{\bar{b}(n)}$  of the maximal binary tree in Figure 3, this time considered as a 0 cell of  $K_n$ ,  $n > 2$ , is

$$(\bar{b}(n), \xi_{\bar{b}(n)}) := (-1)^{(n-2)(n-3)/2} \cdot (\bar{b}(n), e_1 \wedge \cdots \wedge e_{n-2}) \in \mathcal{C}_0(K_n)$$

and that the standard orientation of the minimal binary  $n$ -tree  $\underline{b}(n)$  with interior edges (all right-leaning) enumerated in sequence moving away from the root is given by

$$(\underline{b}(n), \xi_{\underline{b}(n)}) := (-1)^n \cdot (\underline{b}(n), e_1 \wedge \cdots \wedge e_{n-2}) \in \mathcal{C}_0(K_n).$$

#### 4. THE CHAIN MAPS $p$ AND $q$

The goal of this section is to construct maps  $p : \mathcal{C}_*(\mathcal{K}) \rightarrow \mathcal{C}_*(\mathcal{W})$  (Definition 4.3) and  $q : \mathcal{C}_*(\mathcal{W}) \rightarrow \mathcal{C}_*(\mathcal{K})$  (Definition 4.1) with the properties discussed in Section 1. The proofs that these maps are indeed chain maps (Proposition 4.6 and Proposition 4.2) are postponed until Section 7.

As an operad in the category of vector spaces,  $\mathcal{C}_*(\mathcal{K})$  is a free operad generated by the collection with arity  $n$  component, a one-dimensional subspace concentrated in degree  $n - 2$  spanned by the  $n$ -leaf corolla, and  $\mathcal{C}_*(\mathcal{W})$  is a free operad generated by the collection with arity  $n$  component, the vector space with basis the set of purely metric planar rooted trees with  $n$  leaves. Since an operadic map of a free operad is determined by its value on generators, the operadic chain map  $\mathcal{C}_*(\mathcal{K}) \xrightarrow{q} \mathcal{C}_*(\mathcal{W})$  is determined by its value on corollae, and the operadic chain map  $\mathcal{C}_*(\mathcal{W}) \xrightarrow{p} \mathcal{C}_*(\mathcal{K})$  is determined by its value on purely metric trees.

Let  $c(n)$  be the  $n$ -leaf corolla; since there are no internal edges, we denote the orientation by the symbol 1, and adopt the convention that

$$1 \wedge e_1 \wedge \cdots \wedge e_k := e_1 \wedge \cdots \wedge e_k.$$

**Definition 4.1.** Let  $mBin(n)$  be the set of  $n - 2$  cells of  $W_n$  corresponding to the fully metric planar rooted binary trees with  $n$  leaves and standard orientation. Then  $q(c(n), 1)$  is defined as a sum over  $mBin(n)$ :

$$(4.1) \quad q(c(n), 1) := \sum_{T \in mBin(n)} (T, \omega_T).$$

The operadic extension of  $q$  to the free operad  $\mathcal{C}(\mathcal{K})$ , also denoted by  $q$ , defines a morphism of operads in the category of graded vector spaces.

**Proposition 4.2.** *The morphism  $q$  defined in Definition 4.1 commutes with boundary operators, i.e.,*

$$(4.2) \quad q(\partial_K(c(n), 1)) = \partial_W q(c(n), 1),$$

*and is therefore a morphism of operads in the category of chain complexes.*

The proof of Proposition 4.2 is postponed until Section 7. The operad chain map  $p : \mathcal{C}_*(\mathcal{W}) \rightarrow \mathcal{C}_*(\mathcal{K})$  is determined by its value on fully metric trees. Before stating the precise definition, we give a conceptual description. As a topological cell complex, the associahedron can be realized as a convex polytope  $K_n \subset \mathbf{R}^{n-2}$ .

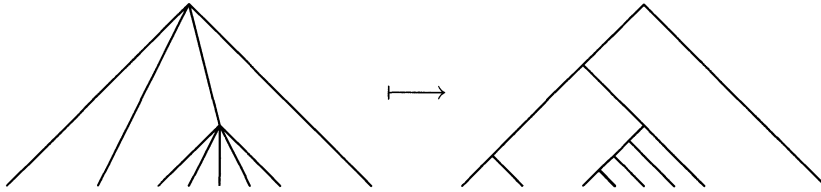


FIGURE 4. An example of the filling-in procedure that passes from a fully metric tree  $T$  to the binary tree  $T_{min}$ .

The cubical cell complex  $W_n$  is a decomposition of the associahedral  $k$ -cells into  $k$ -cubes. The interior  $k$ -cell of  $W_n$  labeled by a purely metric tree  $T$  with  $k$  edges is transverse to the  $(n - 2 - k)$ -cell of  $K_n$  labeled by the same tree. Let  $T_{min}$  be the binary tree labeling the minimal vertex of this transverse cell in  $K_n$ .

The image  $p(T)$  is defined as the sum with appropriate signs of all the  $k$ -cells in  $\mathcal{C}_k(K_n)$  all of whose vertices are labeled by binary trees less than or equal to  $T_{min}$  relative to the partial order on binary trees.

The tree  $T_{min}$  is created by “filling-in” the non-binary vertices of  $T$ . A vertex in  $T$  with  $r$  input edges,  $r > 2$ , is replaced in  $T_{min}$  by the minimal binary tree with  $r$  leaves; this introduces  $r - 2$  new right-leaning edges. When performed at each non-binary vertex of  $T$ , this procedure adds  $n - 2 - k$  new right-leaning edges. See Figure 4 for an example of this procedure. In exactly the same way, one defines  $T_{max}$  as the binary tree obtained from  $T$  by filling-in the non-binary vertices by left-leaning edges.

Since an associativity move applied to a binary tree replaces a right-leaning edge with a left-leaning edge (see Figure 1) and the tree labeling the maximal vertex of a  $k$ -cell is the output of at least  $k$  distinct associativity moves corresponding to the  $k$  one-cells of the associahedron that meet at the given vertex, the maximal vertex of a  $k$ -cell in  $K_n$  corresponds to a binary tree  $S$  containing at least  $k$  left-leaning edges.

If  $T$  is an interior  $k$ -cell in  $W_n$ , then  $T_{min}$  can have at most  $k$  left-leaning edges, since the new edges in  $T_{min}$  are all right-leaning. Since the number of left-leaning edges in a binary tree is a non-decreasing function relative to the partial order, we put  $p(T) = 0$  whenever  $T_{min}$  has less than  $k$  left-leaning edges (there are no  $k$ -cells less than  $T_{min}$ ).

Given  $(T, e_1 \wedge \cdots \wedge e_k) \in \mathcal{C}_k(W_n)$  such that  $T_{min}$  has  $k$  left-leaning edges, each edge  $e_i$  corresponds to an edge in  $T_{min}$  which we also denote  $e_i$ . Choose any labeling  $f_1 \wedge \cdots \wedge f_{n-k-2}$  of the new edges, and let  $\xi_{T_{min}}$  be the standard orientation of  $T_{min}$  considered as the label for a 0-cell of  $K_n$ . Then

$$\xi_{T_{min}} = \eta \cdot e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge f_{n-k-2}, \quad \eta \in \{-1, +1\};$$

the leading term of  $p(T, e_1 \wedge \cdots \wedge e_k)$  is

$$(4.3) \quad (T_{min} / \{e_1, \dots, e_k\}, e_1 \wedge \cdots \wedge e_k \lrcorner \xi_{T_{min}}) = \eta \cdot (-1)^{k(k-1)/2} (T_{min} / \{e_1, \dots, e_k\}, f_1 \wedge \cdots \wedge f_{n-k-2}).$$

In the above display,  $\lrcorner$  is the contraction relative to the pairing  $\langle e_i, e_j \rangle := \delta_j^i$ .

Given a binary tree  $S < T_{min}$  with  $k$  left-leaning edges, we now describe a method (analogous to the definition of the standard orientation) for assigning the labels  $e_1, \dots, e_k$  to the left-leaning edges in a unique way. First, we describe a rule which determines the labeling of the left-leaning edges in a tree, given the labeling of the left-leaning edges in an adjacent tree (related by one associativity move). Trees adjacent to a binary tree with labels on left-leaning edges only are related by replacing a configuration of two edges  $\triangleright$  by the configuration  $\triangleleft$  (going from the greater tree to the lesser tree). If both edges are internal, use the same label for the left-leaning edge in both configurations. The ambiguity of the path connecting two trees resolves into a sequence of pentagons and squares, and the validity of the definition is checked by considering these two figures. On the other hand, if the lower edge is a leaf, a left-leaning edge is replaced by a right-leaning edge in the new configuration, and the resulting binary tree has less than  $k$  left-leaning edges. Consequently there are no  $k$ -cells below it in the partial ordering, and the contribution to  $p(T)$  is zero. In Figure 1, for example, the first associativity move preserves the number of left-leaning edges, while the second increases this number by 1.

We can now give the full definition of  $p$ :

**Definition 4.3.** Define a function on oriented fully metric trees  $(T, e_1 \wedge \dots \wedge e_k)$  by

$$(4.4) \quad p(T, e_1 \wedge \dots \wedge e_k) := \sum (S / \{e_1, \dots, e_k\}, e_1 \wedge \dots \wedge e_k \lrcorner \xi),$$

where the sum is over binary trees  $S$  less than or equal to  $T_{min}$  with  $k$  left-leaning edges labeled  $e_1, \dots, e_k$  according to the procedure described above,  $\xi$  is the standard orientation of the binary tree  $S$  and  $\lrcorner$  is the same contraction as in (4.3).

The function  $p$  has a unique extension to a morphism (also denoted by the same symbol)  $p : \mathcal{C}_*(W_*) \rightarrow \mathcal{C}_*(K_*)$  of operads in the category of graded vector spaces.

**Exercise 4.4.** Verify that

$$p(\bar{b}(n), \omega_{\bar{b}(n)}) = (c(n), 1) \text{ and } p(c(n), 1) = (\underline{b}(n), \xi_b).$$

Note that the first equation involves  $(n - 2)$ -cells and the second involves 0-cells. Observe also that, modulo orientations, (4.4) is the sum of all trees  $U$  with  $n - k$  interior edges such that  $U_{max} \leq T_{min}$ .

**Example 4.5.** Let us explicitly describe the map  $p : \mathcal{C}_*(W_n) \rightarrow \mathcal{C}_*(K_n)$  for some small  $n$ . For  $n = 1$  and 2,  $p$  is given by

$$p\left(\mid, 1\right) := \left(\mid, 1\right) \text{ and } p\left(\wedge, 1\right) := \left(\wedge, 1\right).$$

For  $n = 3$ ,

$$p\left(\bigwedge, 1\right) := \left(\overset{a}{\bigwedge}, -a\right), p\left(\bigwedge^e, e\right) := \left(\bigwedge, 1\right) \text{ and } p\left(\overset{e}{\bigwedge}, e\right) := 0,$$

where  $e$  denotes a metric edge of  $W_3$ . Finally, for  $n = 4$ ,

$$\begin{aligned}
 p\left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, 1\right) &:= \left(\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \\ \wedge \end{array}, a \wedge b\right), \\
 p\left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, e\right) &:= \left(\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \\ \wedge \end{array}, a\right) + \left(\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \\ \wedge \end{array}, a\right), \\
 p\left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, e\right) &:= \left(\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \\ \wedge \end{array}, a\right), \\
 p\left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, e\right) &:= \left(\begin{array}{c} \begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array} \\ \wedge \end{array}, -a\right) \text{ and} \\
 p\left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, e \wedge f\right) &:= \left(\begin{array}{c} \diagup \quad \diagdown \\ \wedge \end{array}, 1\right),
 \end{aligned}$$

where  $e$  and  $f$  are metric edges of  $W_4$ . The above equations can be written in a more condensed form as

$$\begin{aligned}
 p(\mid) &= \mid, \quad p(\wedge) = \wedge, \quad p(\wedge) = \wedge, \quad p(\wedge) = \wedge, \quad p(\wedge) = 0, \\
 p(\wedge) &= \wedge, \quad p(\wedge) = \wedge + \wedge, \quad p(\wedge) = \wedge, \quad p(\wedge) = -\wedge \text{ and } p(\wedge) = \wedge,
 \end{aligned}$$

with the convention that binary trees are endowed with their canonical orientations, corollas are oriented with the sign  $+1$  and trees  $T$  with one binary and one ternary vertex are oriented as  $(T, e)$ , where  $e$  denotes the unique interior edge of  $T$ .

Let us close this section with the following proposition whose proof appears in Section 7.

**Proposition 4.6.** *Let  $(T, \omega_T)$  be an oriented fully metric tree; then*

$$(4.5) \quad p(\partial_W(T, \omega_T) = \partial_K p(T, \omega_T).$$

Since  $p$  is a operad morphism, this implies that  $p$  commutes with the differential on  $C_*(W_*)$  and therefore is a morphism of operads in the category of chain complexes.

### 5. THE SANEBLIDZE-UMBLE DIAGONAL

In this section we define the SU-diagonal [9]. Let us start with a definition of the cubical diagonal  $\Delta_W$  adapted from [11, Section 2]:

$$\begin{aligned}
 (5.1) \quad \Delta_W(T, e_1 \wedge \dots \wedge e_k) &:= \sum_{L,R} (-1)^{\rho_{L,R}} (T/e_L, e_1 \wedge \dots \wedge \hat{e}_{i_1} \wedge \dots \wedge \hat{e}_{i_r} \wedge \dots \wedge e_k) \\
 &\quad \otimes (T_R, e_1 \wedge \dots \wedge \hat{e}_{j_1} \wedge \dots \wedge \hat{e}_{j_r} \wedge \dots \wedge e_k),
 \end{aligned}$$

where the summation runs over all disjoint decompositions  $L \sqcup R = \{i_1, \dots, i_r\} \sqcup \{j_1, \dots, j_r\}$  of  $\{1, \dots, k\}$  into ordered subsets,  $T/e_L$  is the tree obtained from  $T$  by contracting edges  $\{e_i; i \in L\}$ ,  $T_R$  is the tree obtained by changing the metric edges  $\{e_j; j \in R\}$  to non-metric ones, and  $\rho_{L,R}$  is the number of couples  $i \in L, j \in R$  such that  $i < j$ . We leave the proof of the following proposition as an exercise.

**Proposition 5.1.** *The diagonal (5.1) is coassociative and commutes with the  $\circ_i$ -operations introduced in (3.2), therefore the  $W$ -construction  $(\underline{W}, \Delta_W)$  is a Hopf non- $\Sigma$  operad.*

The SU-diagonal is then defined by formulas (1.4) as

$$(5.2) \quad \Delta^{\text{su}} := (p \otimes p) \circ \Delta_W \circ q.$$

**Exercise 5.2.** Derive from definition that, in the shorthand of Example 4.5,

$$(5.3) \quad \begin{aligned} \Delta^{\text{su}}(\wedge) &= \wedge \otimes \wedge, \\ \Delta^{\text{su}}(\lrcorner) &= \lrcorner \otimes \lrcorner + \lrcorner \otimes \lrcorner, \text{ and} \\ \Delta^{\text{su}}(\blacktriangleright) &= \blacktriangleright \otimes \blacktriangleright + \blacktriangleright \otimes \blacktriangleright + \blacktriangleright \otimes \blacktriangleright + \blacktriangleright \otimes \blacktriangleright - \blacktriangleright \otimes \blacktriangleright + \blacktriangleright \otimes \blacktriangleright. \end{aligned}$$

Also prove that  $\Delta^{\text{su}}(c(n), 1)$  always contains the terms

$$(\underline{b}(n), \xi_{\underline{b}(n)}) \otimes (c(n), 1) \text{ and } (c(n), 1) \otimes (\bar{b}(n), \xi_{\bar{b}(n)}).$$

Let us analyze formula (5.2) applied to  $(c(n), 1)$ . The map  $q_n$  applied to the oriented corolla  $(c(n), 1) \in \mathcal{C}_{n-2}(K_n)$  is, by definition, the sum of all fully metric binary trees with standard orientations. The diagonal  $\Delta_{W_n}$  acts on such a tree  $(S, \omega_S)$  as follows. Divide interior edges of  $S$  into two disjoint groups,  $\{f_1, \dots, f_s\}$ ,  $\{e_1, \dots, e_t\}$ ,  $t + s = n - 2$ , and let

$$\omega_S = \eta \cdot e_1 \wedge \dots \wedge e_t \wedge f_1 \wedge \dots \wedge f_s,$$

with some  $\eta \in \{-1, 1\}$ , be the standard orientation.

Then  $\Delta_{W_n}(S)$  contains the term  $(S_L, e_1 \wedge \dots \wedge e_t) \otimes (S_R, f_1 \wedge \dots \wedge f_s)$ , where  $S_L = S/\{f_1, \dots, f_s\}$  and  $S_R$  is obtained by replacing edges  $\{e_1, \dots, e_t\}$  of  $S$  by non-metric ones. We must then evaluate

$$(5.4) \quad p_n(S_L, e_1 \wedge \dots \wedge e_t) \otimes p_n(S_R, f_1 \wedge \dots \wedge f_s).$$

One can also describe the pair  $S_L, S_R$  as follows:  $S_R$  is the same binary tree as  $S$ , but with only a subset of the edges retaining the metric label, and  $S_L$  is the fully-metric tree formed from  $S$  by collapsing the same subset of edges.

Let us pause a little and observe that the expression in (5.4) is non-zero only for trees  $S$  of a very special form. Since the value  $p(U, \omega)$  is, for a binary fully metric tree  $U$ , non-zero only when  $U$  is maximal,  $p_n(S_R, f_1 \wedge \dots \wedge f_s)$  is non-trivial only when  $S_R$  is built from maximal binary fully metric trees, using the  $\circ$ -operation  $t$ -times. Similarly, as we saw in Section 4,  $p_n(S_L, e_1 \wedge \dots \wedge e_t)$  is non-zero if and only if  $(S_L)_{\text{min}}$  has exactly  $t$  left-leaning edges.

A moment's reflection convinces us that the above two conditions are satisfied if and only if  $S_R$  is built from  $t + 1$  fully metric maximal binary trees, using  $t$  times  $\circ_i$ -operations with  $i \geq 2$  (that is,  $\circ_1$  is forbidden). Clearly  $p_n(S_R, f_1 \wedge \dots \wedge f_s)$  is then an  $n$ -tree created from  $t + 1$  corollas using  $\circ_i$  with  $i \geq 2$ . Let  $M_n^t$  denote the set of such  $n$ -trees and let  $M_n := M_n^0 \sqcup \dots \sqcup M_n^{n-2}$ . A more formal definition is that  $T \in M_n$  if and only if  $T = c(k) \circ_i S$  for some  $S \in M_l$ , where  $k + l = n - 1$  and  $i \geq 2$ . For example,

$$M_1 = \{|\}, M_2 = \{\lrcorner\}, M_3 = \{\lrcorner, \blacktriangleright\}, M_4 = \{\blacktriangleright, \blacktriangleright, \blacktriangleright, \blacktriangleright, \blacktriangleright\}, \text{ etc.}$$

As an exercise, we recommend that the reader prove the following fact:  $M_n$  is the set of all  $n$  trees  $T$  whose number of interior edges is the same as the number of left-leaning edges of  $T_{\text{min}}$ .

Let us reverse the process and start with an oriented  $n$ -tree  $(T, \xi) \in \mathcal{C}_s(K_n)$  such that  $T \in M_n^t$  and  $\xi = e_1 \wedge \dots \wedge e_t$ . Let  $\tilde{T}$  be the tree obtained from  $T$  by filling all non-binary vertices by left-leaning metric edges. Let us denote these newly created metric edges  $f_1, \dots, f_s$ . Observe that

$$p_n(\tilde{T}_R, f_1 \wedge \dots \wedge f_s) = \epsilon \cdot (T, \xi)$$

for some  $\epsilon \in \{-1, +1\}$ . Define  $\eta_T \in \{-1, +1\}$  by demanding that  $\eta_T \cdot \epsilon \cdot e_1 \wedge \cdots \wedge e_t \wedge f_1 \wedge \cdots \wedge f_s$  be the standard orientation of  $\tilde{T}$ . Indeed, it is not hard to prove that  $\eta_T$  depends only on  $T$  and not on the choices of the labels  $e_1, \dots, e_t, f_1, \dots, f_s$ , as suggested by the notation. For example,  $\eta_T = 1$  for all trees from  $T \in M_n$  with  $n \leq 1$  except  $T = \blacktriangleright = b(4)$  for which  $\eta_T = -1$ . More generally,  $\eta_{\tilde{b}(n)} = (-1)^{(n-2)(n-3)/2}$ .

Finally observe that  $\tilde{T}_L = T$ . Equation (5.2) can then be rewritten as

$$(5.5) \quad \Delta^{\text{su}}(c(n), 1) = \sum_{\substack{T \in M_n^t \\ 0 \leq t \leq n-1}} \eta_T \cdot p_n(T, e_1 \wedge \cdots \wedge e_t) \otimes (T, e_1 \wedge \cdots \wedge e_t).$$

Let us note that the above display contains the symbol  $(T, e_1 \wedge \cdots \wedge e_t)$  twice. The first occurrence of this symbol denotes a cell of  $\mathcal{C}_t(W_n)$ ; the second a cell of  $\mathcal{C}_s(K_n)$ . The sign  $\eta_T$  then accounts for the difference between these two interpretations of the same symbol.

We already observed in Exercise 4.4 that, modulo orientations,  $p_n(T, e_1 \wedge \cdots \wedge e_t)$  in (5.5) is the sum of all  $n$ -trees  $U$  with  $s$  interior edges such that  $U_{\max} \leq T_{\min}$ . This leads to the following formula for the SU-diagonal whose spirit is closer to [9]:

$$(5.6) \quad \Delta^{\text{su}}(c(n), 1) = \sum \vartheta \cdot (U, \omega_U) \otimes (T, \omega_T),$$

where, as usual,  $c(n)$  is the  $n$ -corolla representing the top-dimensional cell of  $K_n$ , the summation is taken over all  $(U, \omega_U), (T, \omega_T)$  with  $U_{\max} \leq T_{\min}$  and  $\dim(S, \omega_S) + \dim(T, \omega_T) = n$ , and  $\vartheta$  is a sign obtained by comparing formulas (5.5) and (5.6).

### 6. NON-EXISTENCE OF A COASSOCIATIVE DIAGONAL

As we already indicated, the SU-diagonal is not coassociative, that is,

$$(\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}} \neq (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}.$$

While

$$(\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}}(\blacktriangleright) = (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}(\blacktriangleright) \text{ and } (\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}}(\blacktriangleleft) = (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}(\blacktriangleleft),$$

the coassociativity already breaks for  $\blacktriangleright$ , explicitly

$$(6.1) \quad (\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}}(\blacktriangleright) - (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}(\blacktriangleright) = \partial(\blacktriangleright \otimes \blacktriangleright \otimes \blacktriangleright).$$

The SU diagonal is also not cocommutative, i.e.,

$$T(\Delta^{\text{su}}) \neq \Delta^{\text{su}},$$

where  $T : \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$  interchanges factors. More explicitly, while  $T(\Delta^{\text{su}})(\blacktriangleright) = \Delta^{\text{su}}(\blacktriangleright)$ ,

$$\Delta^{\text{su}}(\blacktriangleright) - T(\Delta^{\text{su}})(\blacktriangleright) = \partial(\blacktriangleright \otimes \blacktriangleright).$$

In the rest of this section we show that the non-coassociativity of  $\Delta^{\text{su}}$  is not due to bad choices in the definition, but follows from a deeper principle, namely:

**Theorem 6.1.** *The operad  $\underline{\mathcal{A}}$  does not admit a coassociative diagonal. Therefore the operad  $\underline{\mathcal{A}}$  for  $A_\infty$ -algebras is not a Hopf operad in the sense of [3].*

*Proof.* The proof is boring, and the reader is warmly encouraged to skip it. The idea is to try to inductively construct a coassociative diagonal  $\Delta$  and observe that at a certain stage there is a non-trivial coassociativity constraint. Let us begin the construction. For  $\wedge$  we are forced to take

$$\Delta(\wedge) := \wedge \otimes \wedge.$$

The most general form of  $\Delta(\wedge)$  is

$$\Delta(\wedge) = (a\wedge + b\wedge) \otimes \wedge + \wedge \otimes (c\wedge + d\wedge),$$

with some  $a, b, c, d \in \mathbf{k}$ . The compatibility with the differential  $\partial$  of  $\underline{\mathcal{A}}$  means that

$$\partial\Delta(\wedge) = (a\wedge + b\wedge) \otimes (\wedge - \wedge) + (\wedge - \wedge) \otimes (c\wedge + d\wedge)$$

must be the same as

$$\Delta(\partial\wedge) = \Delta(\wedge - \wedge) = \wedge \otimes \wedge - \wedge \otimes \wedge.$$

This is clearly equivalent to

$$a + c = 1, \quad b + d = 1, \quad a = d \text{ and } b = c.$$

It is equally easy to verify that the coassociativity

$$(\Delta \otimes \mathbb{1})\Delta(\wedge) = (\mathbb{1} \otimes \Delta)\Delta(\wedge)$$

is equivalent to

$$a = a^2, \quad b = b^2, \quad c = c^2, \quad d = d^2, \quad ab = 0 \text{ and } cd = 0.$$

We conclude from this that the only two non-trivial coassociative solutions are either  $(a, b, c, d) = (1, 0, 0, 1)$  or  $(a, b, c, d) = (0, 1, 1, 0)$ , that is either

$$(6.2) \quad \Delta(\wedge) = \wedge \otimes \wedge + \wedge \otimes \wedge.$$

or

$$(6.3) \quad \Delta(\wedge) = \wedge \otimes \wedge + \wedge \otimes \wedge.$$

Let us assume solution (6.2) which coincides with the SU-diagonal (compare (5.3)) – solution (6.3) is just the flip  $T(\Delta^{\text{su}}(\wedge))$  and this case can be discussed by flipping all the steps below. We will be looking for  $\Delta$  of the form  $\Delta = \Delta^{\text{su}} + \delta$  with some perturbation  $\delta : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$  satisfying, of course,  $\delta(\wedge) = \delta(\wedge) = 0$ . Since we know that  $\Delta^{\text{su}}$  is a chain map,  $\delta$  must be a chain map as well.

Observe that  $\delta(\wedge)$  depends on 35 parameters. Therefore the coassociativity of  $\Delta$  and the chain condition on  $\delta$  is expressed by a system of linear equations in 35 variables! We claim that this system has no solution. This could be a formidable task, but we will simplify it by making some wise guesses. Let us write

$$(6.4) \quad \delta(\wedge) = A \otimes \wedge + \sum_i J_{(1)}^i \otimes J_{(2)}^i + \wedge \otimes B,$$

where  $A, B \in \underline{\mathcal{A}}_0(4)$  and  $J_{(1)}^i \otimes J_{(2)}^i \in \underline{\mathcal{A}}_1(4) \otimes \underline{\mathcal{A}}_1(4)$ . Let us also denote

$$LHS := [(\delta \otimes \mathbb{1})\delta + (\Delta^{\text{su}} \otimes \mathbb{1})\delta + (\delta \otimes \mathbb{1})\Delta^{\text{su}} + (\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}}](\wedge)$$

and

$$RHS := [(\mathbb{1} \otimes \delta)\delta + (\mathbb{1} \otimes \Delta^{\text{su}})\delta + (\mathbb{1} \otimes \delta)\Delta^{\text{su}} + (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}](\wedge).$$

The coassociativity of  $\Delta$  at  $\blacktriangleright$  of course means that  $LHS = RHS$ . An easy calculation shows that the only term of  $LHS$  of the form  $\blacktriangleright \otimes something$  is

$$\blacktriangleright \otimes (B \otimes B + \blacktriangleright \otimes B + B \otimes \blacktriangleright + \blacktriangleright \otimes \blacktriangleright),$$

while the only term of  $RHS$  of the same form is

$$\blacktriangleright \otimes (B \otimes B + \blacktriangleright \otimes \blacktriangleright).$$

Associativity  $RHS = LHS$  then evidently means

$$\blacktriangleright \otimes B + B \otimes \blacktriangleright = 0,$$

which, since  $char(\mathbf{k}) \neq 2$ , clearly implies  $B = 0$ . Using the same trick we also see that  $A = 0$ , therefore  $\delta(\blacktriangleright)$  must be of the form

$$\delta(\blacktriangleright) = \sum_i J_{(1)}^i \otimes J_{(2)}^i.$$

Since  $\delta$  is a chain map, trivial on  $\blacktriangleleft$  and  $\blacktriangleright$ ,  $\partial\delta(\blacktriangleright) = 0$ , which means that

$$0 = \partial\delta(\blacktriangleright) = \sum_i \partial J_{(1)}^i \otimes J_{(2)}^i - \sum_i J_{(1)}^i \otimes \partial J_{(2)}^i.$$

Looking separately at the components of bidegrees  $(1, 0)$  and  $(0, 1)$ , respectively, and assuming, without loss of generality, that the elements  $J_{(1)}^i$  (resp.  $J_{(2)}^i$ ) are linearly independent, we conclude that  $\partial J_{(1)}^i = \partial J_{(2)}^i = 0$ . Because each cycle in  $\underline{\mathcal{A}}_1(4)$  is a scalar multiple of  $\partial(\blacktriangleright)$ , we see that

$$\delta(\blacktriangleright) = \alpha(\partial(\blacktriangleright) \otimes \partial(\blacktriangleright))$$

for some scalar  $\alpha \in \mathbf{k}$ . Thus we have reduced the 35 parameters in (6.4) to one! Now

$$LHS = \alpha\{\Delta^{su}(\partial(\blacktriangleright)) \otimes \partial(\blacktriangleright) + \partial(\blacktriangleright) \otimes \partial(\blacktriangleright) \otimes \blacktriangleright\} + (\Delta^{su} \otimes \mathbb{1})\Delta^{su}(\blacktriangleright)$$

and

$$RHS = \alpha\{\partial(\blacktriangleright) \otimes \Delta^{su}(\partial(\blacktriangleright)) + \blacktriangleright \otimes \partial(\blacktriangleright) \otimes \partial(\blacktriangleright)\} + (\mathbb{1} \otimes \Delta^{su})\Delta^{su}(\blacktriangleright).$$

The only terms of the  $LHS$  of the form  $\blacktriangleleft \otimes something$  are

$$\alpha\{\blacktriangleleft \otimes \blacktriangleright \otimes \partial(\blacktriangleright) + \blacktriangleleft \otimes \partial(\blacktriangleright) \otimes \blacktriangleright\},$$

while in the  $RHS$ , there is only one term of this form, namely

$$\alpha(\blacktriangleleft \otimes \Delta^{su}(\partial(\blacktriangleright))).$$

The only term of the form  $\blacktriangleleft \otimes \blacktriangleright \otimes something$  in the above two displays is

$$\alpha(\blacktriangleleft \otimes \blacktriangleright \otimes \partial(\blacktriangleright))$$

coming from the first term of the first display. This implies that  $\alpha = 0$ , therefore  $\delta = 0$  and  $\Delta = \Delta^{su}$ . But this is not possible, because the coassociativity of  $\Delta^{su}$  is violated already on  $\blacktriangleright$ , as we saw in (6.1). □

7. REMAINING PROOFS

In this section we prove Propositions 4.2 and 4.6. Let us start with the

*Proof of Proposition 4.2.* By definition of  $\partial_K$ ,

$$\begin{aligned} \partial_K(c(n), 1) &= \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (c(r) \circ_i c(s), e) \\ &= \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{r(s-2)+i(s+1)} (c(r), 1) \circ_i (c(s), 1) \\ &= \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{(r+i)s+i} (c(r), 1) \circ_i (c(s), 1). \end{aligned}$$

The sign comes from formula (3.1) setting  $l = s - 2$ , since  $c(s) \in \mathcal{C}_{s-2}(K_s)$ . Applying  $q$  gives

$$(7.1) \quad q(\partial_K(c(n), 1)) = \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{(r+i)s+i} q(c(r), 1) \circ_i q(c(s), 1).$$

The expression on the right is a sum over all binary rooted planar metric trees with  $n$  leaves and one non-metric edge. On the other hand,

$$\partial_W(q(c(n), 1)) = \sum_{T \in mBin(n)} \partial_W(T, \omega_T).$$

According to (3.4), the terms in  $\partial_W(T, \omega_T)$  are of two types:

- Type A, in which a metric edge has been changed to a non-metric edge and
- Type B, in which a metric edge has been collapsed, creating a fully metric tree which is binary except for one ternary vertex.

In the sum of Type B terms the same cell appears twice with opposite signs, since there are exactly two binary trees which give rise to the same tree with a unique ternary vertex. The terms of Type A, with one non-metric edge, run over the set of all binary rooted planar metric trees with one non-metric edge, which is the same set that appears in the sum on the right of equation (7.1). It only remains to compare the orientations of the corresponding terms on the two sides of (4.2). According to Definition 4.1,

$$(7.2) \quad \begin{aligned} &(-1)^{(r+i)s+i} q(c(r), 1) \circ_i q(c(s), 1) \\ &= (-1)^{(r+i)s+i} (\bar{b}(r) \circ_i \bar{b}(s), e_1 \wedge \cdots \wedge e_{r-2} \wedge f_1 \wedge \cdots \wedge f_{s-2}) + \cdots, \end{aligned}$$

where the term shown explicitly on the right is the leading order term relative to the order relation on binary trees,  $e_1, \dots, e_{r-2}$  label the edges in  $\bar{b}(r)$  and  $f_1, \dots, f_{s-2}$  label the edges in  $\bar{b}(s)$ . Since the definition of the standard orientation on an arbitrary fully metric binary tree involves the same associativities independent of the size of the tree, it is sufficient to compare the orientation of the leading order term in (7.2) with the orientation of the corresponding term in  $\partial_W(q(c(n), 1))$ . If these orientations agree, so will the orientations of all the other terms.

Assume  $i < r$ . Applying  $\partial_W$  to the fully metric binary tree with standard orientation appearing in Figure 5, we get (among others) the term

$$(-1)^{(i+s)} (\bar{b}(r) \circ_i \bar{b}(s), e_1 \wedge \cdots \wedge \hat{e}_i \cdots \wedge e_{n-2})$$

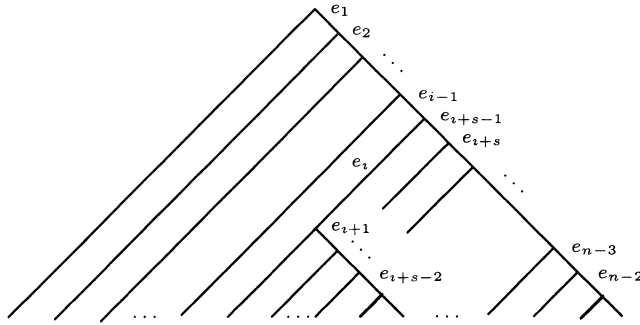


FIGURE 5. The binary tree in the figure is derived from the maximal binary tree  $\bar{b}(n)$  by moving the  $s - 1$  adjacent vertices between edges  $e_i$  and  $e_{i+s-1}$ ; therefore, its standard orientation is  $(-1)^{s-1}e_1 \wedge \cdots \wedge e_{n-2}$ .

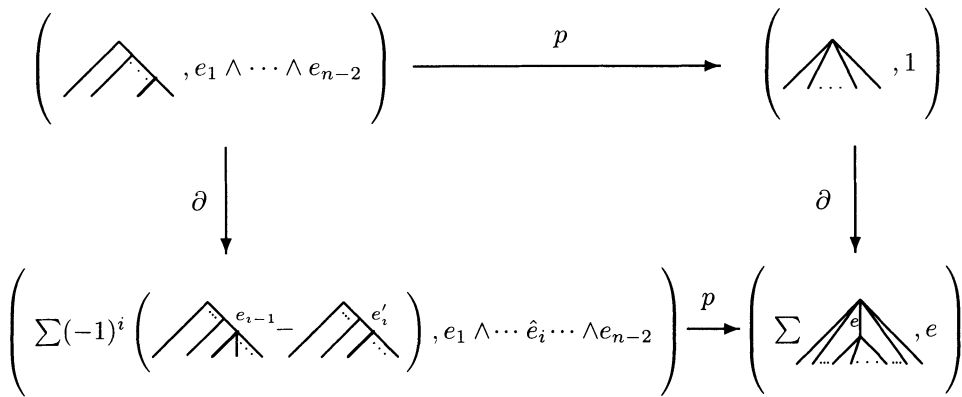


FIGURE 6. The commutative diagram for the chain map  $p$  in degree  $n - 2$  acting on the maximal binary metric tree.

with  $e_i$  changed to a non-metric edge, and the edges labeled  $e_{i+1}, \dots, e_{i+s-2}$  corresponding to the edges in  $\bar{b}(s)$ . Reordering the terms in the orientation element appearing in (7.2) so that  $f_1, \dots, f_{s-2}$  appear in sequence between  $e_{i-1}$  and  $e_i$  introduces a sign factor  $(-1)^{(s-2)(r-i-1)}$ . But

$$(-1)^{(s-2)(r-i-1)}(-1)^{(r+i)s+i} = (-1)^{i+s},$$

since the signs agree. For  $i = r$ , when  $\bar{b}(r) \circ_r \bar{b}(s) = \bar{b}(r + s - 1)$ , the analysis is much easier, and we leave it to the reader.  $\square$

*Proof of Proposition 4.6.* The case  $n = 2$  is trivial. Assuming the conclusion holds for fully metric trees  $(T, \omega_T) \in \mathcal{C}_*(W_m)$  with  $m < n$ , we prove the proposition for  $\mathcal{C}_k(W_n)$ , starting with  $k = n - 2$  and descending. In the case  $\mathcal{C}_{n-2}(W_n)$ , which involves binary fully metric trees, we begin with the maximal binary metric tree. We need to prove the commutativity of the diagram in Figure 6, which follows from the equations in Figure 7 once we check the signs.

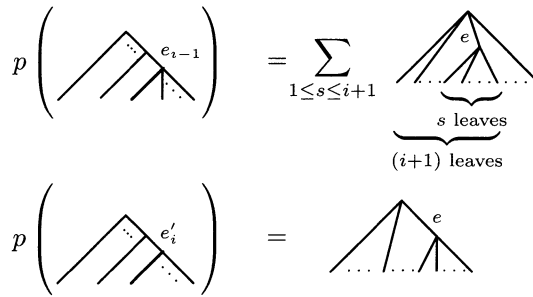


FIGURE 7. Each of the trees labeling the faces of  $K_n$  appears precisely once as a term in  $\partial\bar{b}(n)$ . The orientation elements (not shown in the figure) are  $(-1)^{i-1}e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{n-2}$  in the upper left,  $(-1)^i e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{n-2}$  in the lower left, and  $e$  for both trees on the right.

Let us start with the second equation in Figure 7. The tree in parentheses on the left with orientation element  $(-1)^i e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{n-2}$  corresponding to one of the terms appearing in  $\partial_W(\bar{b}(n), \omega_{\bar{b}(n)})$ , is equal to  $(-1)^i(\bar{b}(i+1), \omega_{\bar{b}(i+1)}) \circ_{i+1}(\bar{b}(n-i), \omega_{\bar{b}(n-i)})$ . Therefore its image under  $p$  is

$$\begin{aligned} &(-1)^i p(\bar{b}(i+1), \omega_{\bar{b}(i+1)}) \circ_{i+1} p(\bar{b}(n-i), \omega_{\bar{b}(n-i)}) \\ &= (-1)^i (c(i+1), 1) \circ_{i+1} (c(n-i), 1) \\ &= (-1)^{i+(i+1)(n-i-2)+(i+1)(n-i+1)} (c(i+1) \circ_{i+1} c(n-i), e) \\ &= (c(i+1) \circ_{i+1} c(n-i), e), \end{aligned}$$

as required. The orientation element for the trees on the right-hand side of the first equation in Figure 7 with  $s$  leaves is

$$(-1)^{i+(n-3)(n-4)/2} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{n-2} \lrcorner (-1)^{s-1} e_1 \wedge \cdots \wedge e_{i-s+1} \wedge e \wedge e_{i-s+2} \wedge \cdots \wedge e_{n-2},$$

which clearly equals  $e$ . Thus  $p$  commutes with  $\partial$  on the maximal binary fully metric tree.

Next we show that  $p$  commutes with  $\partial$  for all binary fully metric trees. To simplify notation, we suppress the orientation element. For a non-maximal fully metric binary tree  $T$ ,  $p(T) = 0$ , because  $T_{min} = T$  has less than  $n - 2$  left-leaning edges. The only fully metric binary trees for which  $p(\partial T) \neq 0$  are trees of the type appearing in Figure 5 with only one right-leaning internal edge.

Let  $T^{i,s}$  be the tree in Figure 5, and let  $T_j^{i,s}$  be the term in  $\partial T^{i,s}$  with edge  $e_j$  non-metric. Then, for  $i \neq j$ ,  $T_j^{i,s}$  is a  $\circ$ -composition of two fully metric binary trees, one of which is not maximal. Since  $p$  is an operad map, the image  $p(T_j^{i,s})$  is also a  $\circ$ -composition, but one of the two components is zero, since  $p(T) = 0$  when  $T$  is fully metric binary but not maximal.

For  $j \neq i, i - 1$  we also have  $p(T^{i,s}/e_j) = 0$ , because the binary tree  $(T^{i,s})_{min}$  has two right-leaning edges. Thus the only terms in  $\partial T^{i,s}$  whose image under  $p$  is not zero are  $T^{i,s}/e_{i-1}$ ,  $T^{i,s}/e_i$ , and  $T_i^{i,s}$ . It follows immediately from the definition

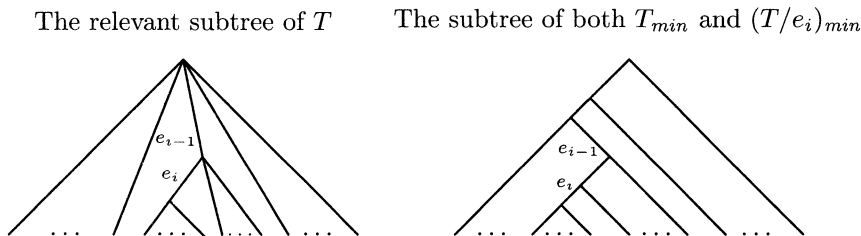


FIGURE 8. The only configuration of edges in  $T$  that is relevant in the calculation of  $p(\partial T)$  may occur at any vertex, not necessarily at the root.

of  $p$  that

$$(7.3) \quad p(T^{i,s}/e_i) = p(T_i^{i,s}) + p(T^{i,s}/e_{i-1}).$$

In fact,  $p(T_i^{i,s})$  is the single term appearing in  $p(T^{i,s}/e_i)$  but not appearing in  $p(T^{i,s}/e_{i-1})$ . Therefore,

$$p(\partial T^{i,s}) = p((-1)^{(i-1)}T^{i,s}/e_{i-1} + (-1)^{i-1}T_i^{i,s} + (-1)^i T^{i,s}/e_i) = 0 = \partial p(T^{i,s}).$$

This completes the proof of (4.5) for  $T \in \mathcal{C}_{n-2}(W_n)$ .

Now, assuming that (4.5) is true for all  $T \in \mathcal{C}_j(W_n)$  with  $k < j \leq n-2$  and all  $T \in \mathcal{C}_*(W_m)$  with  $m < n$ , we prove it for  $T \in \mathcal{C}_k(W_n)$ . Given a fully metric tree  $T$  with  $k$  edges labeled  $e_1, \dots, e_k$ , let  $T_{min}$  be the binary tree obtained by filling in and labeling its  $k$  edges corresponding to the original edges with the same labels. All other edges of  $T_{min}$  are right-leaning. If less than  $k-1$  of the edges  $e_1, \dots, e_k$  in  $T_{min}$  are left-leaning, then  $p(T) = 0 = p(\partial T)$  and therefore,  $\partial p(T) = p(\partial T)$ . First suppose that  $T_{min}$  has  $k-1$  left-leaning edges, and  $e_i$  is right-leaning. Just as for the binary metric trees, the only trees in  $\partial T$  for which the image under  $p$  is non-zero are  $T/e_{i-1}, T/e_i$  and  $T_i$ , where  $e_{i-1}$  and  $e_i$  are adjacent edges in  $T$ . The configuration is illustrated in Figure 8. The right-hand subtree in Figure 8 appears as a subtree in both  $T_{min}$  and  $(T/e_i)_{min}$ , and the right-hand tree in Figure 9 appears as a subtree in  $(T/e_{i-1})_{min}$ . The following equation analogous to (7.3) applies in this case

$$(7.4) \quad p(T/e_i) = p(T_i) + p(T/e_{i-1}),$$

and the remainder of the proof of (4.5) is the same as before.

Next we consider the case when there are exactly  $k$  left-leaning edges in  $T_{min}$ . In general, for any  $k$  cell  $T$  such that  $T_{min}$  has  $k$  left-leaning edges, we can choose a  $k+1$  cell  $T^+$  such that  $\partial T^+$  contains  $T$  as a summand. All other summands of the type  $R_i := T^+/e_i$  for  $i = 1, \dots, k$  have the property that  $(R_i)_{min}$  has  $k-1$  left-leaning edges. The tree  $T^+$  can be defined as follows: Pick any non-binary vertex  $v$  in  $T$  with  $r \geq 3$  incoming edges and replace the corolla with vertex  $v$  by a subtree of type  $c(2) \circ_1 c(r-1)$  with the new edge labeled  $e_{k+1}$ . Then  $T^+/e_{k+1} = T$  and for  $i = 1, \dots, k$ ,  $(T^+/e_i)_{min}$  has  $k-1$  left-leaning edges, since the edge corresponding to  $e_{k+1}$  in  $(T^+/e_i)_{min}$  is right-leaning. Denote the faces  $T_j^+$  of Type A in which a metric edge is changed to a non-metric edge by  $S_j$ ,  $j = 1, \dots, k+1$ . By the operad morphism property and the induction assumption we know that (4.5) holds for each  $S_j$ .  $\square$

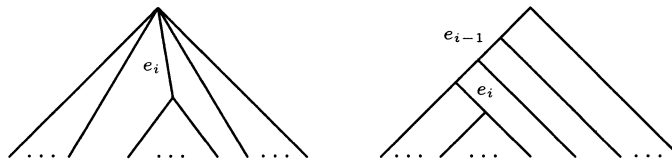


FIGURE 9. Collapsing the edge  $e_{i-1}$  from the left-hand subtree in Figure 8 creates the subtree  $S$  shown here on the left. Filling in to get a binary subtree  $S_{min}$  creates the subtree of  $(T/e_{i-1})_{min}$  shown here on the right, with left-leaning edge  $e_i$  and right-leaning edge  $e_{i-1}$ .

**Lemma 7.1.** *The validity of (4.5) for the faces  $R_i$ ,  $i = 1, \dots, k$ , implies its validity for  $T$ .*

*Proof.* By definition of  $R_i$  and  $S_j$  and the property  $\partial^2 = 0$ ,

$$\begin{aligned} \partial T^+ &= T + \sum R_i + \sum S_j, \\ 0 = \partial \partial T^+ &= \partial T + \sum \partial R_i + \sum \partial S_j. \end{aligned}$$

Therefore,

$$-\partial T = \sum \partial R_i + \sum \partial S_j.$$

Moreover,

$$p(\partial R_i) = \partial p(R_i) \quad \text{and} \quad p(\partial S_j) = \partial p(S_j).$$

Thus

$$\begin{aligned} p(-\partial T) &= \sum p(\partial R_i) + \sum p(\partial S_j) = \sum \partial p(R_i) + \sum \partial p(S_j) \\ &= \partial p(\sum R_i + \sum S_j) = \partial p(\partial T^+ - T) \\ &= \partial p(\partial T^+) - \partial p(T) = \partial \partial(T^+) - \partial p(T) = -\partial p(T). \end{aligned}$$

The summations in the above displays are taken over  $1 \leq i \leq k$  and  $1 \leq j \leq k + 1$ . This completes the proof of Lemma 7.1 and the induction in the proof of (4.5).  $\square$

#### ACKNOWLEDGEMENT

The authors are indebted to the referee for careful reading of the manuscript and for many useful comments and suggestions.

#### REFERENCES

- [1] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer-Verlag, 1973. MR0420609 (54:8623a)
- [2] M.R. Gaberdiel and B. Zwiebach. Tensor constructions of open string theories I: Foundations. *Nucl. Phys. B*, 505(3):569–624, 1997. MR1490781 (99m:81185a)
- [3] E. Getzler and J.D.S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint [hep-th/9403055](https://arxiv.org/abs/hep-th/9403055), March 1994.
- [4] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. Preprint [math.SG/0011041](https://arxiv.org/abs/math.SG/0011041), November 2000.
- [5] M. Markl. A cohomology theory for  $A(m)$ -algebras and applications. *J. Pure Appl. Algebra*, 83:141–175, 1992. MR1191090 (94a:18012)

- [6] M. Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996. MR1380606 (96m:18012)
- [7] M. Markl. Homotopy algebras are homotopy algebras. *Forum Mathematicum*, 16(1):129–160, 2004. MR2034546 (2004k:18010)
- [8] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2002. MR1898414 (2003f:18011)
- [9] S. Saneblidze and R. Umble. A diagonal on the associahedra. Preprint `math.AT/0011065`, November 2000.
- [10] S. Saneblidze and R. Umble. Diagonals on the permutahedra, multiplihedra and associahedra. *Homology, Homotopy and Applications*, 6(1):363–411, 2004. MR2118493 (2005j:55023)
- [11] J.-P. Serre. Homologie singulière des espaces fibreés. *Ann. of Math.*, 54:425–505, 1951. In French. MR0045386 (13:574g)
- [12] A. Tonks. Relating the associahedron and the permutohedron. In J.L. Loday, J.D Stasheff, and A.A. Voronov, editors, *Operads: Proceedings of Renaissance Conferences*, volume 202 of *Contemporary Math.*, pages 33–36. Amer. Math. Soc., 1997. MR1436915 (98c:52015)

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