Distributive Laws, Bialgebras, and Cohomology

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Abstract
The advent of quantum group theory has led to a proliferation of new algebra types, each calling for its own deformation theory and attendant cohomology theory. We here explore two methods of unifying such constructions, using triples and operads.

1 Introduction

There are many ways of describing a class of algebraic structures. The classical approach is to define groups, rings et al. as sets having structure maps satisfying certain axioms. An individual object is given by a presentation in terms of generators and relations. Homological algebra is then a way of gleaning from a particular presentation information about the structure of the given object. However, in this setting there is no natural homology theory attached to each class of algebras. The constructions of homology for groups, rings, associative algebras, commutative algebras, and Lie algebras were each developed separately in an ad hoc manner. As new classes of algebras and bialgebras have arisen in the study of quantum groups, the appropriate homology theories had to be created, but to do so in the classical manner seems to be crude when more unified approaches are available.

Triples and cotriples give a unified way of presenting classes of algebras defined on sets or modules. They lead to a canonical homology theory for each class of algebras, and this may be used to study deformations of algebraic structures. Distributive laws are generalizations of the distributive law of multiplication over addition that we all learned in kindergarten. They were introduced in the context of triples to codify the rules governing abstract algebras having more than one operation. They may be extended to include mixed distributive laws involving triples and cotriples, and this opens the door to the study of the cohomology and deformations of bialgebras, Lie-bialgebras, Poisson Hopf-algebras, etc.

The problem with the categorical approach using triples is that for all their abstract elegance, the complexes used to define homology groups are too unwieldy for computations and do not look like the complexes used to define the corresponding classical cohomology groups. However, they point the way towards an understanding of distributive laws and cohomology in the theory of operads. With a few restrictions, algebras of general type can be presented in this setting, and a natural cohomology theory may be constructed. This has the great advantage of yielding resolutions of classical appearance. These notes are meant as a self-contained introduction to these subjects, with an emphasis on the algebras and bialgebras arising in quantum group theory.

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We start by briefly reviewing the concepts needed for triple cohomology. We then discuss distributive laws between triples and between a triple and a cotriple and show how to compute the cohomology of the resulting algebras and bialgebras using double complexes. In section 6 we give examples, including Poisson algebras and Lie bialgebras, which help to elucidate all our abstract machinery. We then rework everything from the ground up using operads, which yields computationally useful cohomology theories. The interplay between the abstraction of categorical triples and the classicism of operads is the heart of the paper.

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2 Algebras defined by triples

Throughout this paper, $\mathbf{k}$ will denote a commutative ring or field and $\mathcal{M}$ will denote the category of graded modules over $\mathbf{k}$. We will be interested in algebras X over $\mathbf{k}$. The usual approach to abstract algebra is to define a class or category of algebras as modules X equipped with a set of structure maps $\{\alpha_i\}$ each of the form $\alpha_i : \otimes_{\mathbf{k}}^{n_i} X \longrightarrow X$ (we will often drop the $\mathbf{k}$) and satisfying some set of equations. Triples, or "monads", give an alternate way of describing a class of algebras. The basic observation is this: An algebra is always a quotient of a free algebra in a canonical way, so to build a class of algebras one may start with the construction of free algebras and then define general algebras as quotients of frees (with certain restrictions). A triple is just a functor that acts like a free-algebra functor. It turns out that all classical categories of algebras may be defined by triples on the category $\mathcal{M}$. The free algebra construction may then be iterated to yield a canonical resolution to use when defining homology. This not only unifies extant homology theories but builds a homology theory for each new type of algebras.

Recall that a *triple* $\mathbf{T} = (T, \mu, \eta)$ on $\mathcal{M}$ is a functor $T : \mathcal{M} \longrightarrow \mathcal{M}$ equipped with natural transformations $\eta : 1 \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ satisfying $\mu \cdot \mu T = \mu \cdot T\mu$ and $\mu \cdot T\eta = \mu \cdot \eta T = 1$. A $\mathbf{T}$-*algebra* (A, α) is a $\mathbf{k}$-module A equipped with a multiplication map $\alpha : TA \longrightarrow A$ satisfying

$$(1) \qquad \alpha \cdot \mu = \alpha \cdot T\alpha \qquad \alpha \cdot \eta = 1$$

A $\mathbf{T}$-algebra map $(A, \alpha) \longrightarrow (B, \beta)$ is a $\mathbf{k}$-linear map $\psi : A \longrightarrow B$ satisfying $\beta \cdot T\psi = \psi \cdot \alpha$, so a map of algebras is just a linear map that preserves the operations, as one would expect. We will use $\mathbf{T}$-*alg* to denote the category of all $\mathbf{T}$-algebras.

If we want to construct associative algebras, TA is just the tensor algebra generated by the module A. Note that we view TA as just a module – the module underlying the free algebra generated by A. Then $\alpha : TA \longrightarrow A$ defines a product on A, while equations (1) ensure that α is unitary, associative, and defined by its action on $A \otimes A$, so $\mathbf{T}$-algebras are just associative algebras in the normal sense. If we let T be the symmetric algebra functor, we will get commutative algebras, while the free Lie algebra functor yields the category of Lie algebras, etc. Hence any construction carried out in this general setting applies equally well to all these classical special cases, as well as to newly created classes of algebras (see e.g. [1]). We will also need the duals of triples. A *cotriple* $\mathbf{G} = (G, \delta, \varepsilon)$ is an endofunctor G with natural transformations $\delta : G \longrightarrow G^2$ and $\varepsilon : G \longrightarrow 1$ satisfying $G\delta \cdot \delta = \delta G \cdot \delta$ and $G\varepsilon \cdot \delta = \varepsilon G \cdot \delta = 1$.

Suppose we are given a category of algebras $\mathcal{A}$ with a pair of adjoint functors

$F : \mathcal{M} \longrightarrow \mathcal{A}$ and $U : \mathcal{A} \longrightarrow \mathcal{M}$, the free algebra and underlying module respectively. These give rise to a triple $\mathbf{T}$ on $\mathcal{M}$ with $T = UF$, and a cotriple $\mathbf{G}$ on $\mathcal{A}$ with $G = FU$. Notice that the natural transformation $\varepsilon : GA \longrightarrow A$ is the multiplication on A. The question of whether or not $\mathcal{A}$ is equivalent to $\mathbf{T}$-*alg* is the question of "tripleability" [7], with which we will not concern ourselves since all naturally occurring categories of algebras are tripleable.

Triples and cotriples lend themselves to the construction of homology and cohomology groups. The usual definition of the triple cohomology groups $H^n(A, B)$ is as follows: The cotriple $\mathbf{G}$ on $\mathcal{A}$ yields the following cotriple-generated simplicial resolution of an algebra A:

$$A \leftarrow GA \leftleftarrows G^2A \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; G^3A \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; G^4A \ldots \tag{2}$$

where the i-th face $G^{n+1}A \longrightarrow G^nA$ is $G^{n-i}\varepsilon G^i$, for $0 \le i \le n$. In the general theory, one applies a contravariant abelian group-valued functor E to this complex, yielding a complex of abelian groups

$$EA \longrightarrow EGA \rightrightarrows EG^2A \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; EG^3A \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \ldots \tag{3}$$

The homology of the associated cochain complex defines the cotriple cohomology groups (denoted $H^n(A, E)_G$ in [6]).

We are interested in the cohomology of A with coefficients in an A-module B, so the appropriate candidate for E is the functor $\mathrm{Der}(-, B)$ ([6] 1.3,1.4). Thus the cohomology groups $H^n(A, B)$ are defined by the "homogeneous" cochain complex of abelian groups given below:

$$0 \longrightarrow \mathrm{Der}(GA, B) \longrightarrow \mathrm{Der}(G^2A, B) \longrightarrow \mathrm{Der}(G^3A, B) \ldots \tag{4}$$

$$\partial^{n-1} f = \sum_{i=0}^{n} (-1)^i f \cdot G^i \varepsilon G^{n-i}$$

However, since F is the left adjoint of U, $G = FU$ and $T = UF$, we can convert this to a complex of modules using

$$\mathrm{Der}(G^{n+1}A, B) = \mathrm{Der}(FUG^nA, B) \cong \mathrm{Hom}(UG^nA, UB) = \mathrm{Hom}(UT^nA, UB)$$

We will drop the superfluous applications of U, and denote $\mathrm{Hom}(-, -)$ by $(-, -)$. If the switch from Der to Hom seems strange, remember that any linear map lifts to a unique derivation from the free algebra. In fact T acts as a linear map $(T^nA, B) \longrightarrow (T^{n+1}A, TB)$ taking a cochain $f : T^mA \longrightarrow B$ to its unique lifting as a derivation $T^{m+1}A \longrightarrow TB$. This action is extended to α by defining $T\alpha : T^2A \longrightarrow TA$ to be the unique lifting of α as an algebra map. We now find that the complex (4) above is isomorphic to Beck's "non-homogeneous" complex [8]:

$$0 \longrightarrow (A, B) \longrightarrow (TA, B) \longrightarrow (T^2A, B) \longrightarrow \ldots \tag{5}$$

It is easier to give the boundaries for this complex if we first define the following compositions:

2.1 Definition: Let $\alpha : TA \longrightarrow A$ and $\beta : TB \longrightarrow B$ be two $\mathbf{T}$-algebras over a triple $\mathbf{T} = (T, \mu, \eta)$. For $f \in (T^mA, B)$ define $f \circ \alpha$, $f \circ \mu T^k$ $(0 \le k < m)$ and $\beta \circ f$ in $(T^{m+1}A, B)$ by $f \circ \alpha = f \cdot T^m\alpha$, $f \circ \mu T^k = f \cdot T^{m-k-1}\mu T^k$, and $\beta \circ f = \beta \cdot Tf$.

2.2 Definition: If (A, α) and (B, β) are $\mathbf{T}$-algebras, and B is an A-module, the cotriple cohomology groups $H^m(A, B)$ are the homology groups of the cochain complex (5) with the boundaries $\partial^m : (T^m A, B) \longrightarrow (T^{m+1}A, B)$ given by

$$\partial^m f = (-1)^m f \circ \alpha + \sum_{k=1}^{m} (-1)^{m+k} f \circ \mu T^{k-1} - \beta \circ f \tag{6}$$

In particular $\partial^0 f = f \cdot \alpha - \beta \cdot Tf$, so the 0-cocycles are precisely the derivations from A to B, while $H^1(A, A)$ classifies infinitesimal deformations of the $\mathbf{T}$-algebra A.

2.3 Remarks: We have presumed that B is an A-module, and the reader may take this at face value, but the categorical approach is necessary for the generalization to follow. We could ask that B be an abelian group object in the category $\mathbf{T}$-*alg*, but there are none in the classical categories of algebras. To get from $\mathbf{k}$-algebras to A-modules in the usual sense, we replace $\mathcal{A}$ by the slice category $\mathcal{A}/A$ of algebras over A. Then an A-module is an abelian group object $B \longrightarrow A$ in this category [7]. In particular, we have an algebra map $\varphi : A \longrightarrow B$ (the identity of the group structure) splitting $B \longrightarrow A$. Hence $B \cong A \oplus M$, where M is the kernel of $B \longrightarrow A$. Now $\mathrm{Der}(A, B)$ may be thought of as the $\mathbf{k}$-derivations from A to B along φ with image in the A-module M.

This is only the classical piece of a more general construction of cohomology groups which deals with higher order derivations between algebras, [13]. There the group (A, B) is the cofree coalgebra generated by $\mathrm{Hom}_{\mathbf{k}}(A, B)$. The definition of $T : (A, B) \longrightarrow (TA, TB)$ is quite delicate in this setting, but there is a composition of these "enriched" cochains defined as follows: If $f \in (T^m A, B)$ and $g \in (T^n B, C)$, then define $g \circ f$ in $(T^{m+n}A, C)$ by

$$g \circ f = g \cdot T^n f$$

It is quite easy to see that composition satisfies the Leibniz formula $\partial(g \circ f) = \partial g \circ f + (-1)^n g \circ \partial f$ and thus lifts to the level of homology. Hence there is a "composition" (circle product) making $H^*(A, A)$ a graded ring, the "cohomology ring" of the algebra A. There is also an internal grading, since we are dealing with graded modules to begin with, but this plays no role in the triple theoretic cohomology. It will play a substantial role in the operadic theory below.

3 Triples and distributive laws

$$x(y + z) = xy + xz$$

From an abstract point of view, this distributive law says that the operation of multiplication by an element of a ring is a homomorphism of the abelian group underlying the ring. If T_1 is the commutative monoid triple on *Sets* and T_2 is the abelian group triple, then the equation above generates a natural transformation $T_1 T_2 \longrightarrow T_2 T_1$ taking a product of sums to a sum of products. This allows the functor T_2 to act on the category of monoids as we will see below, so one can then think of a ring as a monoid endowed with an abelian group structure, that is, a ring is an abelian group in the category of monoids. Of course one can also think of a ring as a set with a single structure map from the free ring generated by the set, i.e. you can build the ring structure in steps or all at once, and it is the interplay between these approaches that is arbitrated by the distributive law.

There are many other examples of the same sort of relationship between two algebraic structures. For example, in a Poisson algebra the commutative multiplication and Lie bracket are related by the distributive law

$$[x, yz] = [x, y]z + y[x, z]$$

If $\mathbf{T}_1$ is the Lie algebra triple on a category of modules and $\mathbf{T}_2$ is the commutative algebra triple (the symmetric algebra), then the distributive law $T_1T_2 \longrightarrow T_2T_1$ takes a bracket of products to a product of brackets. This allows one to think of a Poisson algebra as a Lie algebra equipped with a commutative multiplication, that is, a commutative ring in the category of Lie algebras. The "distributive laws" of [9] generalize these ideas to algebra types defined by more than one operation.

Given triples $\mathbf{T}_1 = (T_1, \mu_1, \eta_1)$ and $\mathbf{T}_2 = (T_2, \mu_2, \eta_2)$ on $\mathcal{M}$, we want to consider $\mathbf{T}_2$ objects in $\mathbf{T}_1$-*alg* . This only makes sense if we may extend $\mathbf{T}_2$ to act as a triple on the category $\mathbf{T}_1$-*alg*, so given $\alpha : T_1A \longrightarrow A$ in $\mathbf{T}_1$-*alg* we would like T_2A to also be a $\mathbf{T}_1$-algebra, hence we need a map $T_1T_2A \longrightarrow T_2A$. This may be achieved by supposing that there is a natural transformation $\lambda : T_1T_2A \longrightarrow T_2T_1A$ and letting $T_2'(A, \alpha) = (T_2A, T_2\alpha \cdot \lambda)$. This T_2' will be the functor part of the lifting of $\mathbf{T}_2$ to $\mathbf{T}_1$-*alg*, so $(T_2A, T_2\alpha \cdot \lambda)$ must be a $\mathbf{T}_1$-algebra, and equations (1) put two conditions on λ. First we need $T_2\alpha \cdot \lambda \cdot \mu_1T_2 = T_2\alpha \cdot \lambda \cdot T_1T_2\alpha \cdot T_1\lambda$. The latter equals $T_2\alpha \cdot T_2T_1\alpha \cdot \lambda T_1 \cdot T_1\lambda$ (by naturality) $= T_2\alpha \cdot T_2\mu_1 \cdot \lambda T_1 \cdot T_1\lambda$ (α is a $\mathbf{T}_1$-algebra, meaning that $\alpha \cdot \mu_1 = \alpha \cdot T_1\alpha$, and T_2 is natural), so we would like $\lambda \cdot \mu_1T_2 = T_2\mu_1 \cdot \lambda T_1 \cdot T_1\lambda$. Second, we need $T_2'\alpha \cdot \eta_1T_2 = 1$, which can be ensured by assuming $\lambda \cdot \eta_1T_2 = T_2\eta_1$. These conditions on λ make T_2' a functor $\mathbf{T}_1$-*alg*$\longrightarrow$$\mathbf{T}_1$-*alg* by defining $T_2'f = T_2f$ on maps. That this carries algebra maps to algebra maps follows from the naturality of λ.

Now we will get pretty tired of writing expressions like $\lambda T_1 \cdot T_1\lambda$, so we use Λ to denote the natural transformation $\Lambda : T_1^mT_2^n \longrightarrow T_2^nT_1^m$ built from repeated applications of λ. This is well defined, since all ways of moving the T_1s to the back are equal by the naturality of λ. With this notation, the two conditions on λ are

(7) $$\Lambda \cdot \mu_1T_2 = T_2\mu_1 \cdot \Lambda \qquad \Lambda \cdot \eta_1T_2 = T_2\eta_1$$

(T_2A is a $\mathbf{T}_1$-algebra)

To make the functor T_2' into a triple $\mathbf{T}_2' = (T_2', \mu_2', \eta_2')$, we let $\mu_2' = \mu_2$ and $\eta_2' = \eta_2$. However, since these must be $\mathbf{T}_1$-algebra maps, we need two more conditions on λ

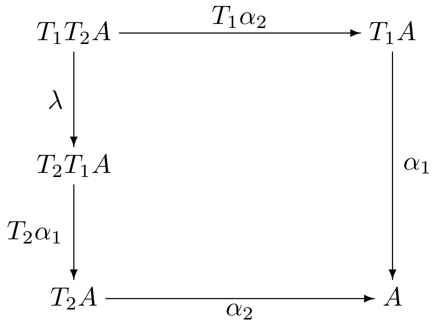
(8) $$\Lambda \cdot T_1\mu_2 = \mu_2T_1 \cdot \Lambda \qquad \Lambda \cdot T_1\eta_2 = \eta_2T_1$$

(μ_2 and η_2 are $\mathbf{T}_1$-algebra maps)

3.1 Definition: A *distributive law* of $\mathbf{T}_1$ over $\mathbf{T}_2$ is a natural transformation $\lambda : T_1T_2 \longrightarrow T_2T_1$ satisfying (7) and (8).

We may now construct the category of $\mathbf{T}_2'$-algebras in $\mathbf{T}_1$-*alg*. This is denoted $\mathbf{T}_2\mathbf{T}_1$-*alg*. Its elements (A, α_1, α_2) are $\mathbf{T}_1$-algebras $\alpha_1 : T_1A \longrightarrow A$ equipped with a second structure map $\alpha_2 : T_2A \longrightarrow A$ making it a $\mathbf{T}_2'$-algebra, so α_2 must satisfy $\alpha_2\mu_2 = \alpha_2 \cdot T_2\alpha_2$ and $\alpha_2 \cdot \eta_2 = 1$. Hence (A, α_2) is just a $\mathbf{T}_2$-algebra that has been lifted to $\mathbf{T}_1$-*alg*. Since α_2 must be a $\mathbf{T}_1$-algebra map from $T_2'(A, \alpha_1)$ to (A, α_1), we also have

(9) $$\alpha_2 \cdot T_2\alpha_1 \cdot \lambda = \alpha_1 \cdot T_1\alpha_2$$



Note that the triple $\mathbf{T}_2$ may be lifted to $\mathbf{T}_1$-*alg only* if such a distributive law exists, [7]. The distributive law λ also allows one to construct the composite triple $\mathbf{T}_{21} = (T_{21}, \mu_{21}, \eta_{21})$ on $\mathcal{M}$ [4]. It is defined by

$$T_{21}A = T_2T_1A \qquad \mu_{21} = \mu_2T_1 \cdot T_2^2\mu_1 \cdot \Lambda \qquad \eta_{21} = \eta_2T_1 \cdot \eta_1$$

That this does define a triple follows from the conditions we have already put on λ. One now may look at the category $\mathbf{T}_{21}$-*alg* of $\mathbf{T}_{21}$-algebras in $\mathcal{M}$. A $\mathbf{T}_{21}$-algebra (A, ξ) has a single structure map $\xi : T_2T_1A \longrightarrow A$, but as we can see from the last diagram, any element of $\mathbf{T}_2\mathbf{T}_1$-*alg* already has such a map. The following theorem is exactly what one would expect — it doesn't matter if you define algebras step by step or all at once.

3.2 Theorem: $\mathbf{T}_{21}$-alg *is isomorphic to* $\mathbf{T}_2\mathbf{T}_1$-alg.

Proof: ([9]) $\mathbf{T}_2\mathbf{T}_1$-*alg* $\longrightarrow$ $\mathbf{T}_{21}$-*alg* is defined by $(A, \alpha_1, \alpha_2) \mapsto (A, \alpha_2 \cdot T_2\alpha_1)$, while its inverse $\mathbf{T}_{21}$-*alg* $\longrightarrow$ $\mathbf{T}_2\mathbf{T}_1$-*alg* is defined by $(A, \xi) \mapsto (A, \xi \cdot \eta_2T_1, \xi \cdot T_2\eta_1)$ □

4 Effect on triple cohomology

The interplay between composite structures has its effect on cohomology and deformations, where one might want to consider the two operations separately. Suppose (A, α_1, α_2) is a $\mathbf{T}_2\mathbf{T}_1$-algebra, and (B, β_1, β_2) is an A-module (or an abelian-group object in the category $\mathbf{T}_2\mathbf{T}_1$-*alg* – see Remarks 2.3); both $\mathbf{T}_1$ and $\mathbf{T}_2'$ yield non-homogeneous complexes

$$0 \longrightarrow (A, B) \longrightarrow (T_1A, B) \longrightarrow (T_1^2A, B) \longrightarrow \dots$$
$$0 \longrightarrow (A, B) \longrightarrow (T_2A, B) \longrightarrow (T_2^2A, B) \longrightarrow \dots$$

Furthermore, each T_2^nA is a $\mathbf{T}_1$-algebra, and so there is a non-homogeneous complex

$$0 \longrightarrow (T_2^nA, B) \longrightarrow (T_1T_2^nA, B) \longrightarrow (T_1^2T_2^nA, B) \longrightarrow \dots$$

This looks like the beginning of a double complex (below): However, T_1A is not, in general, a $\mathbf{T}_2'$-algebra (the free monoid generated by a ring is not naturally a group), so the vertical arrows in the complex below are not really the usual boundaries for a non-homogeneous complex.

$$
(10)\qquad
\begin{array}{ccccccccc}
\uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \\
(T_2^2 A, B) & \xrightarrow{\partial_1} & (T_1 T_2^2 A, B) & \xrightarrow{\partial_1} & (T_1^2 T_2^2 A, B) & \xrightarrow{\partial_1} & T_1^3 T_2^2 A, B) & \xrightarrow{\partial_1} & \\
\uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \\
(T_2 A, B) & \xrightarrow{\partial_1} & (T_1 T_2 A, B) & \xrightarrow{\partial_1} & (T_1^2 T_2 A, B) & \xrightarrow{\partial_1} & T_1^3 T_2 A, B) & \xrightarrow{\partial_1} & \\
\uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \\
(A, B) & \xrightarrow[\partial_1]{} & (T_1 A, B) & \xrightarrow[\partial_1]{} & (T_1^2 A, B) & \xrightarrow[\partial_1]{} & T_1^3 A, B) & \xrightarrow[\partial_1]{} &
\end{array}
$$

Once again it is much easier to give the boundaries for this complex if we first define a circle product involving the cochains, μ_2, α_2, and β_2.

4.1 Definition: Let $f \in (T_1^j T_2^i A, B)$. Then define

$$
\begin{array}{rcl}
f \circ \alpha_2 & = & f \cdot T_1^j \cdot T_2^{i-1} \alpha_2 \\
\beta_2 \circ f & = & \beta_2 \cdot T_2 f \cdot \Lambda^{ij} \\
f \circ \mu_2 & = & f \cdot T_1^j T_2^{i-1} \mu_2
\end{array}
$$

As before, T takes a cochain to its unique lifting as a derivation. We have denoted by Λ^{ij} the variant of Λ which keeps the first i instances of T_2 fixed and shifts j instances of T_1 to the front of the remaining T_2 :

$$\Lambda^{ij} : T_1^j T_2^{i+1} \longrightarrow T_2 T_1^j T_2^i$$

The vertical complexes now have the same boundary formula as the usual non-homogeneous complex, though they have been twisted, a fact that is hidden in the definition of the circle product. One may easily check that these are indeed cochain complexes, the key being that this new circle product is associative (as are all the circle products discussed in this paper). It is then a formality to check that this circle product satisfies the Leibniz formula. Note that there is a circle product on the entire double complex, which (of course) also satisfies the Leibniz formula.

4.2 Definition: Let $f \in (T_1^j T_2^i A, B)$ and $g \in (T_1^n T_2^m B, C)$. Then define

$$g \circ f = (-1)^{ij} g \cdot T_1^n T_2^m f \cdot \Lambda^{ij} \in (T_1^{n+j} T_2^{m+i} A, C)$$

The homology of the total complex of this double complex above defines the cohomology of the $\mathbf{T}_2\mathbf{T}_1$-algebra A with coefficients in B. By Beck's theorem above, we could also consider A and B as $\mathbf{T}_{21}$-algebras and look at the usual triple cohomology using the cotriple $\mathbf{T}_{21}$. The chain complex defining these cohomology groups looks like

$$(11)\qquad 0 \longrightarrow (A, B) \longrightarrow (T_{21} A, B) \longrightarrow (T_{21}^2 A, B) \longrightarrow (T_{21}^3 A, B) \ldots$$

This looks a far cry from the double complex (10), but the main result of [9] is that (11) is homotopy equivalent to the diagonal complex of (10), i.e.

$$0 \longrightarrow (A, B) \longrightarrow (T_2T_1A, B) \longrightarrow (T_2^2T_1^2A, B) \longrightarrow (T_2^3T_1^3A, B)\dots$$

This combined with the Eilenberg-Zilber theorem shows that the cohomology of these three complexes are the same. Note that all of these chain complexes are those associated to obvious simplicial complexes, this being necessary for the application of the Eilenberg-Zilber theorem [18].

5 Bialgebras using triples

So far we have looked at objects with *algebraic* structures, i.e. their operations are maps of the form $\otimes^n A \longrightarrow A$. Bialgebras also have a *coalgebraic* operation — a map $A \longrightarrow A \otimes A$ — so we must deal with these types of structures before considering bialgebras. In general an abstract coalgebra has operations of the form $C \longrightarrow \otimes^n C$. Coalgebras are defined by cotriples on $\mathcal{M}$ just as algebras are defined by triples.

For example, let $\mathcal{C}$ denote the category of coassociative coalgebras in $\mathcal{M}$. There is a pair of adjoint functors $K : \mathcal{M} \longrightarrow \mathcal{C}$, $U : \mathcal{C} \longrightarrow \mathcal{M}$, where U is the obvious forgetful functor, and K is the cofree-coalgebra functor [14]. The composite $S = UK$ induces a cotriple $\mathbf{S} = (S, \delta, \varepsilon)$ on $\mathcal{M}$, and the category $\mathcal{C}$ is equivalent to the category of $\mathbf{S}$-coalgebras. An $\mathbf{S}$-coalgebra (C, c) is a module C equipped with a map $c : C \longrightarrow SC$ satisfying the usual identities: $\varepsilon \cdot c = 1$ and $\delta \cdot c = Sc \cdot c$. These ensure that an $\mathbf{S}$-coalgebra is determined by a coassociative "comultiplication" or "diagonal" $C \longrightarrow C \otimes C$. Of course, using the cocommutative variant of $\mathbf{S}$ gives cocommutative coalgebras, using the Lie version of $\mathbf{S}$ gives Lie coalgebras, etc. Note that these are all subtriples of the triple which defines non-associative non-cocommutative coalgebras, just as free algebras of special types are generally quotients of the free non-associative non-commutative algebra.

The cohomology of coalgebras is defined in a manner dual to that used for algebras [32]. The simplicial cocomplex used is generated by repeated applications of S, that is if $b : B \longrightarrow SB$ and $c : C \longrightarrow SC$ are coalgebras the groups $H^n(B, C)$ are defined by a complex $(B, S^*C) = \mathrm{Hom}_k(B, S^*C)$ (S^* denotes the iterated functor S) whose boundary maps depend on the comultiplications on B and C. As in the algebraic case, the cochains form abelian groups. This is achieved by taking B to be a C-comodule, i.e. an abelian cogroup in the category $C/\mathcal{C}$. Such an object has a map $B \longrightarrow C$ and the function $S : (B, C) \longrightarrow (SB, SC)$ takes a linear map to its unique lifting as a coderivation along that map, though $Sc : SC \longrightarrow S^2C$ is the unique lifting of c to a map of coalgebras. The group $H^0(B, C)$ is the group of coderivations from B to C, while $H^1(C, C)$ classifies infinitesimal deformations of the $\mathbf{S}$-coalgebra C (see [12]). Once again there is a circle product defined on the level of enriched cochains. It satisfies the usual Leibniz formula and is defined by

$$g \circ f = S^m g \cdot f$$

where $f \in (B, S^mC)$ and $g \in (C, S^nE)$. The boundaries for the coalgebra cochain complex above are then given by

$$d^n f = c \circ f + \sum_{i=1}^{n} (-1)^i \delta S^{i-1} \circ f + (-1)^{n+1} f \circ b$$

Now suppose that we are also given a triple $\mathbf{T} = (T, \mu, \eta)$ on $\mathcal{M}$, hence the category $\mathbf{T}$-*alg* of $\mathbf{T}$-algebras in $\mathcal{M}$. For example, T could be the tensor algebra triple or the Lie algebra triple, yielding the category of associative or Lie algebras respectively. To ensure harmony between $\mathbf{S}$ and $\mathbf{T}$, that is, to make sure that the triple $\mathbf{T}$ lifts to $\mathbf{S}$-*coalg* and $\mathbf{S}$ lifts to $\mathbf{T}$-*alg*, we insist that there be a "mixed" distributive law between them [33].

5.1 Definition: If $\mathbf{T}$ is a triple and $\mathbf{S}$ is a cotriple, a *mixed distributive law* $\lambda : TS \longrightarrow ST$ *between* $\mathbf{T}$ and $\mathbf{S}$ is a natural transformation satisfying the following four conditions:

$$\text{(i)} \quad \Lambda \cdot \mu S = S\mu \cdot \Lambda \qquad \text{(ii)} \quad \delta T \cdot \Lambda = \Lambda \cdot T\delta$$

$$\text{(iii)} \quad \lambda \cdot \eta S = S\eta \qquad \text{(iv)} \quad \varepsilon T \cdot \lambda = T\varepsilon$$

We have again used Λ to denote the variant of λ which commutes appropriate compositions of T and S. These equations have the following consequences: Given a $\mathbf{T}$-algebra A, equations *(i)* and *(iii)* ensure that SA is again a $\mathbf{T}$-algebra, while *(ii)* and *(iv)* ensure that δ and ε are $\mathbf{T}$-algebra maps respectively. Dually, given an $\mathbf{S}$-coalgebra C, equations *(ii)* and *(iv)* ensure that TC is again an $\mathbf{S}$-coalgebra, while *(i)* and *(iii)* ensure that μ and η are $\mathbf{S}$-coalgebra maps.

Given $\mathbf{T}$ and $\mathbf{S}$ and the distributive law λ as above, a $\mathbf{TS}$-bialgebra (B, β, b) is a module B equipped with two structure maps $\beta : TB \longrightarrow B$ and $b : B \longrightarrow SB$, making it a $\mathbf{T}$-algebra and an $\mathbf{S}$-coalgebra. Further, the structure maps β and b are required to be compatible in the sense that they must satisfy $S\beta \cdot \lambda \cdot Tb = b \cdot \beta$, that is, the following diagram must commute:

$$\begin{array}{ccc} TSB & \xleftarrow{\;Tb\;} & TB \\ \downarrow{\scriptstyle \lambda} & & \\ STB & & \downarrow{\scriptstyle \beta} \\ \downarrow{\scriptstyle S\beta} & & \\ SB & \xleftarrow[\;b\;]{} & B \end{array} \tag{12}$$

$\mathbf{TS}$-bialgebras may be thought of as $\mathbf{S}$-coalgebras in $\mathbf{T}$-*alg*, or as $\mathbf{T}$-algebras in $\mathbf{S}$-*coalg*. A map between two $\mathbf{TS}$-bialgebras is just a linear map that is both a $\mathbf{T}$-algebra map and an $\mathbf{S}$-coalgebra map. The resulting category is denoted $\mathbf{TS}$-*bialg* or $\mathbf{ST}$-*bialg*. Note that $S\beta \cdot \lambda : TSB \longrightarrow SB$ makes SB a $\mathbf{T}$-algebra, and $\lambda \cdot Tb : TB \longrightarrow STB$ makes TB an $\mathbf{S}$-coalgebra. Then (12) says β is a coalgebra map and b is an algebra map. The reader should compare this with (9) above.

If (A, α, a) is also a bialgebra, the bialgebra cohomology groups of A with coefficients in B are defined via a double complex (T^*A, S^*B) (below) whose boundaries depend on the structure maps of A and B, as well as λ, [33]. The cochains in this case are biderivations between the two bialgebras, while the boundaries of the double complex are just the usual boundaries for algebra and coalgebra cohomology if we put the $\mathbf{S}$-coalgebra structure on T^mA and the $\mathbf{T}$-algebra structure on S^nB as above. To define the cohomology groups for a particular category of bialgebras, one need only define the cotriple $\mathbf{S}$, the triple $\mathbf{T}$, and the distributive law λ.

$$
\begin{array}{ccccccccc}
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
(A, S^2B) & \xrightarrow{\partial} & (TA, S^2B) & \xrightarrow{\partial} & (T^2A, S^2B) & \xrightarrow{\partial} & (T^3A, S^2B) & \xrightarrow{\partial} & \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
(A, SB) & \xrightarrow{\partial} & (TA, SB) & \xrightarrow{\partial} & (T^2A, SB) & \xrightarrow{\partial} & (T^3A, SB) & \xrightarrow{\partial} & \\
\uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \\
(A, B) & \xrightarrow[\partial]{} & (TA, B) & \xrightarrow[\partial]{} & (T^2A, B) & \xrightarrow[\partial]{} & (T^3A, B) & \xrightarrow[\partial]{} &
\end{array}
$$

As in the algebraic and coalgebraic situations, there is a circle product on the double complex and it carries over to the total complex. Let $C^k(A, B)$ and $\mathbf{d}$ respectively denote the k-cochains and boundary of the total complex. We have from [12]

$$
\begin{gathered}
C^k(A, B) = \oplus_{m+n=k}(T^mA, S^nB) \\
\mathbf{d}^{m,n} = (\partial + (-1)^{n+1}d) : (T^mA, S^nB) \longrightarrow (T^{m+1}A, S^nB) \oplus (T^mA, S^{n+1}B)
\end{gathered}
$$

There is again a circle product on the total complex defined by

$$
\begin{gathered}
\circ : (T^mB, S^nC) \otimes (T^iA, S^jB) \longrightarrow (T^{i+m}A, S^{j+n}C) \\
g \circ f = (-1)^{in} S^j g \cdot \Lambda \cdot T^m f
\end{gathered}
$$

This allows us to write the equations for a **TS** bialgebra (B, β, b) as follows:

$$
\beta \circ \mu = \beta \circ \beta \qquad \delta \circ b = b \circ b \qquad \beta \circ b = b \circ \beta
$$

For the reader's convenience we note the following useful equations, which follow by naturality and the equations defining a bialgebra. Here $f \in (T^mB, S^nC)$, $g \in (T^iA, S^jB)$:

$$
\begin{array}{ll}
f \circ \mu S \circ g = f \circ \mu \circ g & f \circ \delta T \circ g = f \circ \delta \circ g \\
f \circ \mu T^{i+m} = \mu T^i S^n \circ f & \delta S^{i+n} \circ f = f \circ \delta S^i T^m
\end{array}
$$

6 Examples

6.1 Example: The first useful example of a distributive law was, of course, the distributive law of multiplication over addition in a ring. In this setting rings are viewed as sets which have two algebraic structures, multiplication and addition, given by the semigroup triple $\mathbf{T}_1$ on *Sets* and the abelian group triple $\mathbf{T}_2$, with their natural distributive law $T_1T_2 \longrightarrow T_2T_1$. That the cohomology of rings viewed as $\mathbf{T}_{21}$-algebras over sets is Shukla cohomology is the main result of [2].

It is instructive to formulate diagram (9) for this simple example using the more usual presentation of multiplication and addition. First write

$$\begin{aligned} T_1X &= X + X \times X + X \times X \times X + \cdots \\ T_2X &= 1 + X + X * X + X * X * X + \cdots \end{aligned}$$

where we have used $*$ to denote the symmetrized product. One usually thinks of the multiplication $T_1X \longrightarrow X$ as being defined by its quadratic piece $\cdot : X \times X \longrightarrow X$ (we hope that the reader understands that the dot '$\cdot$' denotes the operation). Similarly the addition $T_2X \longrightarrow X$ is defined by $+ : X * X \longrightarrow X$. Then (9) becomes

$$\begin{array}{ccc} (X * X) \times (X * X) & \xrightarrow{+ \times +} & X \times X \\ \downarrow{\scriptstyle \lambda} & & \\ (X \times X) * (X \times X) & & \downarrow{\scriptstyle \cdot} \\ \downarrow{\scriptstyle \cdot * \cdot} & & \\ X * X & \xrightarrow[+]{} & X \end{array} \tag{13}$$

where λ is given by

$$(w * x, y * z) \mapsto (w, y) * (w, z) + (x, y) * (x, z) \tag{14}$$

Of course $X \times X$ and $X * X$ are only pieces of T_1 and T_2 respectively, but the diagram above gives the idea of how to define λ in general. Note that λ defines a monoid structure on $X * X$ with respect to which $+$ becomes a map of monoids. Note also that (14) is a bit of a cheat, since we have used the addition in $(X \times X) * (X \times X)$. Clearly the distributive law should be viewed as a map

$$\lambda : (X * X) \times (X * X) \longrightarrow (X \times X) * (X \times X) * (X \times X) * (X \times X) \tag{15}$$

This may seem like a trivial point, but it shows that one should not expect the distributive law to be defined between quadratic terms only.

6.2 Example: As mentioned above a Poisson algebra is a module equipped with an associative commutative product and a Lie bracket connected by the equation

$$[x, yz] = [x, y]z + y[x, z]. \tag{16}$$

If $\mathbf{T}_1$ is the Lie algebra triple on a category of modules and $\mathbf{T}_2$ is the commutative algebra triple, then (16) above yields a distributive law

$$\lambda : T_1T_2 \longrightarrow T_2T_1 \tag{17}$$

taking a bracket of products to a product of brackets. Regard T_1A as a quotient of the exterior algebra $A + A \wedge A + A \wedge A \wedge A \cdots$ and write

$$T_2A = k + A + A * A + A * A * A + \cdots$$

Then (16) shows how to define a map $A \wedge (A * A) \longrightarrow (A \wedge A) * A + A * (A \wedge A)$ which is a piece of λ. In this case the diagram corresponding to (13) becomes

$$\begin{array}{ccc} (A * A) \wedge (A * A) & \xrightarrow{\ \cdot \wedge \cdot\ } & A \wedge A \\ \downarrow{\scriptstyle \lambda'} & & \\ A * A * (A \wedge A) & & \downarrow \\ \downarrow & & \\ A * A & \xrightarrow[\ \cdot\ ]{} & A \end{array}$$

The map λ' above is the piece of λ coming from the equation $[wx, yz] = wy[x,z] + wz[x,y] + xy[w,z] + xz[w,y]$ which we must check is well-defined by (16). As in the previous example we have cheated a bit by using the algebra of $A * A$ to define the left edge of the diagram above. The lower left corner should be $A * A * A$.

As mentioned in section 3, another way of looking at this problem is as follows: Given a Lie algebra $\alpha : T_1 A \longrightarrow A$, the map $T_2\alpha\lambda$ should define a Lie algebra structure on $T_2 A$. (More traditionally speaking, given a Lie algebra $[-,-] : A \wedge A \longrightarrow A$, the map $(A * A) \wedge (A * A) \longrightarrow (A * A)$ should define a Lie algebra structure on $(A * A)$.) We must check that (16) gives a well-defined map, and that it satisfies the requisite identities, including the Jacobi equation. A simple induction shows we need only check products of the forms $[wx, yz]$ and $[w, xyz]$. The commutativity of the product then gives well-definedness, and the fact that α already defines a Lie algebra takes care of the rest. It has been shown by the second author that this example is typical of all "quadratic" algebras (see section 9 below or [24]).

6.3 Example: Before turning to bialgebras, we mention an odd-looking example that seems to be an extreme case. Consider a module A having two associative multiplications $\cdot : A \otimes A \longrightarrow A$ and $\langle -,- \rangle : A \otimes A \longrightarrow A$ connected by the equations

$$x \cdot \langle y, z \rangle = \langle x \cdot y, z \rangle \qquad \langle x, y \rangle \cdot z = \langle x, y \cdot z \rangle \tag{18}$$

These algebras were introduced by the second author in [24] and called there *non-symmetric Poisson algebras*. Letting TA denote the tensor algebra generated by A, the multiplications $\cdot$ and $\langle -,- \rangle$ yield maps α_1 and $\alpha_2 : TA \longrightarrow A$, and we have a distributive law $\lambda : TTA \longrightarrow TTA$ defined by (18). To understand this let both AA and $A \cdot A$ denote $A \otimes A$. Then

$$TTA = A + AA + A \cdot A + AAA + A \cdot A \cdot A + A \cdot (AA) + (AA) \cdot A + \cdots$$

The distributive law is determined by $A \cdot (AA) + (AA) \cdot A \mapsto (A \cdot A)A + A(A \cdot A)$ using (18). This is an example of a distributive law between "two" triples whose inverse is again a distributive law.

6.4 Example: As an example of the bialgebra situation, we consider classical associative/coassociative bialgebras. If $\mathbf{T}$ is the tensor algebra triple and $\mathbf{S}$ is the coassociative coalgebra cotriple, a bialgebra B has a multiplication $\beta : TB \longrightarrow B$ and a comultiplication $b : B \longrightarrow SB$ defined by $\cdot : B \otimes B \longrightarrow B$ and $\Delta : B \longrightarrow B \otimes B$. The multiplication and diagonal are classically connected by the equation

$$\Delta(x \cdot y) = (\Delta x) \cdot (\Delta y)$$

The mixed distributive law $\lambda B : TSB \longrightarrow STB$ is then generated by the usual middle interchange $(B \otimes B) \otimes (B \otimes B) \longrightarrow (B \otimes B) \otimes (B \otimes B)$, which is exactly what is needed to put an algebra or coalgebra structure on $B \otimes B$. To write this completely in terms of the triple and cotriple would necessitate a discussion of the cofree coalgebra construction, which we do not want to go into here (see [14] for details). Suffice to say that SB may be realized as $B \times (B \star B) \times (B \star B \star B) \times \cdots$ where this is a submodule of $B \times (B \otimes B) \times (B \otimes B \otimes B) \times \cdots$. The map $b : B \longrightarrow SB$ sends x to $(x, \Delta x, (\Delta \otimes 1)\Delta x, \ldots)$, which is well-defined since we are looking at coassociative coalgebras, and the distributive law λ then looks like

$$(19) \qquad \begin{aligned} &(B \times B \star B \cdots) + (B \times B \star B \cdots) \otimes (B \times B \star B \cdots) \longrightarrow \\ &\quad (B + B \otimes B \cdots) \times ((B + B \otimes B \cdots) \star (B + B \otimes B \cdots)) \cdots \end{aligned}$$

The middle interchange defines λ on the quadratic pieces of TSB, i.e. it can be thought of as a map $(B \star B) \otimes (B \star B) \longrightarrow (B \otimes B) \star (B \otimes B)$ and (12) becomes

$$\begin{array}{ccc} (B \star B) \otimes (B \star B) & \xleftarrow{\Delta \otimes \Delta} & B \otimes B \\ \lambda \downarrow & & \\ (B \otimes B) \star (B \otimes B) & & \downarrow \cdot \\ \cdot \star \cdot \downarrow & & \\ B \star B & \xleftarrow[\Delta]{} & B \end{array}$$

6.5 Example: We may also look at Lie bialgebras, i.e. Lie algebras B which also have a Lie-diagonal $\Delta : B \longrightarrow B \otimes B$ [27]. In this case, the triple $\mathbf{T}$ is the Lie algebra triple, while the cotriple $\mathbf{S}$ defines Lie coalgebras. The usual way of presenting the connection between the diagonal and bracket product is $\Delta[x, y] = [\Delta x, y] + [x, \Delta y]$ or

$$(20)\ \Delta[x, y] = \sum [x_{(1)}, y] \otimes x_{(2)} + [x, y_{(1)}] \otimes y_{(2)} + x_{(1)} \otimes [x_{(2)}, y] + y_{(1)} \otimes [x, y_{(2)}]$$

where $\Delta x = \sum x_{(1)} \otimes x_{(2)}$ and $\Delta y = \sum y_{(1)} \otimes y_{(2)}$. This serves as a guide to the definition of $\lambda : TSB \longrightarrow STB$, which may be written as a map

$$(21) \qquad \begin{aligned} &(B \times B \star B \cdots) + (B \times B \star B \cdots) \wedge (B \times B \star B \cdots) \longrightarrow \\ &\quad (B + B \wedge B \cdots) \times ((B + B \wedge B \cdots) \star (B + B \wedge B \cdots)) \ldots \end{aligned}$$

Equation (20) shows the second coordinate (the diagonal) of λ acting on the quadratic piece $(B \star B) \wedge (B \star B)$ of TSB:

$$(B \star B) \wedge (B \star B) \longrightarrow (B \wedge B) \star B + B \star (B \wedge B)$$

In this case (12) becomes

$$
\begin{array}{ccc}
(B \star B) \wedge (B \star B) & \longleftarrow & B \wedge B \\
\downarrow{\scriptstyle \lambda'} & & \\
(B \wedge B) \star B + B \star (B \wedge B) & & \downarrow \\
\downarrow & & \\
B \star B & \xleftarrow[\Delta]{} & B
\end{array}
$$

6.6 Example: We end these examples by considering trialgebras, i.e. modules equipped with three operations. We will look at the situation involving two triples $\mathbf{T}_1$ and $\mathbf{T}_2$ and a cotriple $\mathbf{S}$. An $\mathbf{ST}_2\mathbf{T}_1$-*trialgebra* A will have three structure maps $\alpha_1 : T_1A \longrightarrow A$, $\alpha_2 : T_2A \longrightarrow A$ and $a : A \longrightarrow SA$. The question is what relationship must there be between the three functors for this to determine a reasonable class of algebras. First of all we must assume there is a distributive law $\lambda : T_1T_2 \longrightarrow T_2T_1$, so that we may look at the category of $\mathbf{T}_2\mathbf{T}_1$-*alg* of $\mathbf{T}_2\mathbf{T}_1$-algebras. Then we look for $\mathbf{S}$-coalgebras in $\mathbf{T}_2\mathbf{T}_1$-*alg* so we must be able to lift the cotriple $\mathbf{S}$.

The easiest situation occurs when there are two more mixed distributive laws $\lambda_1 : T_1S \longrightarrow ST_1$ and $\lambda_2 : T_2S \longrightarrow ST_2$. We may then sensibly look at the category $(\mathbf{T}_2\mathbf{S})\mathbf{T}_1$-*trialg* of $\mathbf{T}_2\mathbf{S}$-bialgebras in $\mathbf{T}_1$-*alg*, the category $\mathbf{T}_2(\mathbf{ST}_1)$-*trialg* of $\mathbf{T}_2$ algebras in $\mathbf{ST}_1$-*bialg*, the category $\mathbf{S}(\mathbf{T}_2\mathbf{T}_1)$-*trialg* of $\mathbf{S}$ coalgebras in $\mathbf{T}_2\mathbf{T}_1$-*alg* , or the category $\mathbf{ST}_{21}$-*trialg* of $\mathbf{S}$ coalgebras in $\mathbf{T}_{21}$-*alg* . These four categories are, of course, identical, and the last makes sense because we have a distributive law

$$T_2\lambda_1 \cdot \lambda_2 T_1 : T_{21}S \longrightarrow ST_{21} \tag{22}$$

The cohomology of such trialgebras is straightforward, being given by a tricomplex which the reader can easily construct. Unfortunately, the only natural examples of trialgebras we know do not fit this neat scheme. In particular, a distributive law $T_{21}S \longrightarrow ST_{21}$ need not decompose as in (22).

Recall that a Poisson bialgebra B is a Poisson algebra equipped with a coassociative diagonal $\Delta : B \longrightarrow B \otimes B$ satisfying

$$\Delta(x \cdot y) = (\Delta x) \cdot (\Delta y) \tag{23}$$

$$\Delta[x, y] = [x_{(1)}, y_{(1)}] \otimes x_{(2)}y_{(2)} + x_{(1)}y_{(1)} \otimes [x_{(2)}, y_{(2)}] \tag{24}$$

One usually thinks of B as a classical commutative/coassociative bialgebra having an added Lie multiplication, but this is a bit deceptive (see [31]). Let $\mathbf{T}_1$, $\mathbf{T}_2$, and λ be as in (17) above, and let $\mathbf{S}$ be the cofree coalgebra cotriple. Then equation (23) determines a distributive law $\lambda_2 : T_2S \longrightarrow ST_2$ defined by the usual middle interchange (c.f. (19)), but there is no distributive law of the form $\lambda_1 : T_1S \longrightarrow ST_1$ (this explains why there is no category of bialgebras having a Lie multiplication and coassociative comultiplication). Rather, equation (24) determines a distributive law $T_{21}S \longrightarrow ST_{21}$, so Poisson bialgebras should be thought of as coassociative coalgebras in the category of Poisson algebras. Their cohomology groups are determined by a double complex of the form (T_{21}^*, S^*), each row (T_{21}^*, S^n) is homotopic to a double complex $(T_1^*T_2^*, S^n)$, and there are well-defined bicomplexes $(T_1^mT_2^*, S^*)$, but the whole mess does not determine a tricomplex since the double complexes $(T_1^* T_2^m, S^*)$ remain undefined. The implication for deformation theory (quantization) is that one may not look for deformations of a Poisson bialgebra that leave *one* of the multiplications fixed.

Finally we must point out that quasi-whatever bialgebras do not fit into the scenario of triples and distributive laws, since free and cofree objects do not exist unless one fixes the inner automorphisms involved in the definitions of such objects.

7 Algebras defined by operads

We now turn our attention to an alternative description of algebras and their cohomology. As in the first part of the paper, we work over the category $\mathcal{M}$ of graded $\mathbf{k}$-modules, but here the graded structure of $\mathcal{M}$ becomes more manifest because we must immediately address the question of signs. In the triple case the differential of an n-cochain was always a summation of $(n+1)$ pieces (see formula (4)) and the sign convention was the obvious one. This is no longer true for the operadic cohomology which we are going to introduce now, where the signs are a much more delicate matter (for example the Chevalley-Eilenberg cohomology , which is the operadic cohomology of Lie algebras). A solution of this problem is the systematic use of the Koszul sign convention (recalled below) which give us a proper sign convention almost for free. *From now on, we will always assume that the ground field* $\mathbf{k}$ *is of characteristic zero.*

For two graded vector spaces, $V, W \in \mathcal{M}$, let $\mathcal{M}^p(V,W)$ denote the set of linear homogeneous maps $f : V \longrightarrow W$ of degree p, and denote by $\mathcal{M}(V,W)$ the graded vector space $\bigoplus_p \mathcal{M}^p(V,W)$. For $V \in \mathcal{M}$, let $\uparrow V$ (resp. $\downarrow V$) be the suspension (resp. the desuspension) of V, i.e. the graded vector space defined by $(\uparrow V)_p = V_{p-1}$ (resp. $(\downarrow V)_p = V_{p+1}$). By $\#V$ we denote the graded dual of V, i.e. the graded vector space $\bigoplus_p (\#V)_p$ with $(\#V)_p = \mathcal{M}^p(V,\mathbf{k}) = \mathcal{M}(V_{-p},\mathbf{k})$, the space of linear maps from V_{-p} to $\mathbf{k}$. For a graded vector space V we have the natural maps $\uparrow: V \longrightarrow \uparrow V$ and $\downarrow: V \longrightarrow \downarrow V$. We will consider $\mathcal{M}$ as a *symmetric monoidal category* with the symmetry isomorphism given by $x \otimes y \mapsto (-1)^{|x|\cdot|y|} y \otimes x$. The symmetry isomorphism is based on the *Koszul sign convention*, meaning that whenever we commute two "things" of degrees p and q, respectively, we multiply the sign by $(-1)^{pq}$. We will systematically use this convention throughout the paper. Let us recall some basic definitions of [26].

7.1 Definition: By an *operad* we mean an operad in the symmetric monoidal category $\mathcal{M}$, i.e. a sequence $\mathcal{P} = \{\mathcal{P}(n); n \geq 1\}$ of graded vector spaces such that:

(i) Each $\mathcal{P}(n)$ is equipped with a $\mathbf{k}$-linear (left) action of the symmetric group Σ_n on n elements, $n \geq 1$.

(ii) For any $m_1, \ldots, m_l \geq 1$ we have degree zero linear maps (called the *composition maps*)

$$\gamma = \gamma_{m_1,\ldots,m_l} : \mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_l) \longrightarrow \mathcal{P}(m_1 + \cdots + m_l).$$

These data have to satisfy the usual axioms including the existence of a unit $1 \in \mathcal{P}(1)$, for which we refer again to [26]. We sometimes write $\mu(\nu_1, \cdots, \nu_l)$ or $\mu(\nu_1 \otimes \cdots \otimes \nu_l)$ instead of $\gamma(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_l)$.

Cooperads are defined in the dual manner. Hence a *cooperad* is a sequence $\mathcal{Q} = \{\mathcal{Q}(n); n \geq 1\}$ of graded vector spaces such that each $\mathcal{Q}(n)$ has an action of the symmetric group Σ_n, $n \geq 1$, and, for any $n \geq 1$, the map

$$\nu(n) : \mathcal{Q}(n) \longrightarrow \bigoplus \mathcal{Q}(l) \otimes \mathcal{Q}(m_1) \otimes \cdots \otimes \mathcal{Q}(m_l)$$

is given, where the summation is taken over all $l, m_1, \ldots, m_l \geq 1$ with $m_1 + \cdots + m_l = n$. These maps have to satisfy some axioms which are exactly the duals of the axioms of an operad. As we will need cooperads only marginally, we just say that, under an obvious finite-dimension assumption, the dual of an operad is a cooperad and vice versa.

By a *map* $\chi : \mathcal{P} \longrightarrow \mathcal{Q}$ of operads we mean a sequence $\chi = \{\chi(n) : \mathcal{P}(n) \longrightarrow \mathcal{Q}(n);\ n \geq 1\}$ of degree zero Σ_n-invariant linear maps, such that $\chi(1)(1) = 1$ and the sequence χ commutes in the obvious sense with the composition maps. We denote by *Oper* the category of operads and their maps in the above sense. As a matter of fact, we will tacitly assume that all operads $\mathcal{P}$ considered in our paper are such that $\mathcal{P}(n)$ is of finite type for any $n \geq 1$, and that the unitary ring $\mathcal{P}(1)$ is isomorphic to the ground field $\mathbf{k}$, with two very important exceptions – the endomorphism operad $\mathcal{E}_V$ and the dual endomorphism operad $\mathcal{G}_W$ defined below.

For a graded vector space V, define the operad $\mathcal{E}_V$ as follows: Let $\mathcal{E}_V(n) = \mathcal{M}(V^{\otimes n}, V)$ and define the composition maps by the usual composition, i.e.

$$\gamma_{m_1,\ldots,m_l}(f \otimes \varphi_1 \otimes \cdots \otimes \varphi_l) = f(\varphi_1 \otimes \cdots \otimes \varphi_l),$$

for $f \in \mathcal{E}_V(l)$ and $\varphi_i \in \mathcal{E}_V(m_i)$, $l \geq 1$ and $m_1, \ldots, m_l \geq 1$. The action of the symmetric group is given, for $f \in \mathcal{E}_V(n)$ and $\sigma \in \Sigma_n$, by

$$(\sigma f)(x_1, \ldots, x_n) = \epsilon(\sigma; x_1, \ldots, x_n) \cdot f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \tag{25}$$

where $\epsilon(\sigma; x_1, \ldots, x_n)$ (the *Koszul sign*) is determined by the equation

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \cdots, x_n) \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}, \tag{26}$$

which has to be satisfied in the free graded commutative algebra $\bigwedge(x_1, \ldots, x_n)$. The operad $\mathcal{E}_V$ will be called the *operad of endomorphisms* on V.

Dually, for a (graded) vector space W, let $\mathcal{G}_W$ be the operad defined as follows: Put $\mathcal{G}(n) = \mathcal{M}(W, W^{\otimes n})$ and define the composition maps as

$$\gamma_{m_1,\ldots,m_l}(f \otimes \varphi_1 \otimes \cdots \otimes \varphi_l) = (-1)^{|f|\cdot(|\varphi_1|+\cdots+|\varphi_l|)} \cdot (\varphi_1 \otimes \cdots \otimes \varphi_l) \circ f,$$

for $f \in \mathcal{G}_W(l)$ and $\varphi_i \in \mathcal{G}_W(m_i)$, $l \geq 1$ and $m_1, \ldots, m_l \geq 1$. The action of the symmetric group is defined as in (25). We call $\mathcal{G}_W$ the *dual endomorphism operad* on the space W.

By an *algebra* over an operad $\mathcal{P}$ (or by a $\mathcal{P}$-*algebra*), we mean a graded vector space A together with a map $a : \mathcal{P} \longrightarrow \mathcal{E}_A$ of operads. We will write $A = (A, a)$. If not necessary, we make no distinction between the algebra and its underlying space. A $\mathcal{P}$-algebra is thus given by a sequence of degree zero Σ_n-invariant linear maps $a(n) : \mathcal{P}(n) \longrightarrow \mathcal{M}^*(A^{\otimes n}, A)$ satisfying certain axioms, hence the elements of $\mathcal{P}$ act as n-multilinear operations on A. A *homomorphism* of two $\mathcal{P}$-algebras $A = (A, a)$ and $B = (B, b)$ is a homogeneous degree zero map $g : A \longrightarrow B$ which commutes in the obvious sense with the $\mathcal{P}$-algebra structures a and b. We denote by $\mathcal{P}$-*alg* the category of $\mathcal{P}$-algebras and their homomorphisms.

Dually, by a $\mathcal{P}$-*coalgebra* we mean a (graded) vector space C together with an operad map $c : \mathcal{P} \longrightarrow \mathcal{G}_C$; we write $C = (C, c)$. We say that a $\mathcal{P}$-coalgebra C is *connected* if for any $x \in C$ there exists $n \geq 2$ such that $c(n)(\mu)(x) = 0$ for all $\mu \in \mathcal{P}(n)$.

Let us recall the following well-known classical facts. If A is an "ordinary" algebra (such as associative, commutative, Lie, &c.), then an A-module structure on a vector space M is the same as an algebra structure on the direct sum $A \oplus M$ such that both the projection $A \oplus M \longrightarrow A$ and the inclusion $A \longrightarrow A \oplus M$ are algebra homomorphisms and that the product of $(0 \oplus m)$ and $(0 \oplus m')$ in $A \oplus M$ is trivial for all $m, m' \in M$. This motivates the following definition.

Let $A = (A, a)$ be a $\mathcal{P}$-algebra. An A-*module* is a (graded) vector space M together with a $\mathcal{P}$-algebra structure $m : \mathcal{P} \longrightarrow \mathcal{E}_{A \oplus M}$ on $A \oplus M$ such that both the projection $A \oplus M \longrightarrow A$ and the inclusion $A \longrightarrow A \oplus M$ are algebra homomorphisms and that for all $n \geq 1$ and $\mu \in \mathcal{P}$ we have $m(\mu)(x_1, \ldots, x_n) = 0$ if $x_i \in M \subset A \oplus M$ for at least two i, $1 \leq i \leq n$. It is possible to show that an A-module is an abelian group object in the slice category $\mathcal{P}$-*alg*$/A$, but we will not need this result (see the remarks in 2.3).

8 Operadic cohomology

We define, for an algebra A over a quadratic (see below for the definition) operad $\mathcal{P}$ and for an A-module M, a cohomology theory $H_{\mathcal{P}}(A; M)$ based on a very small chain complex. This cohomology is a natural generalization of such classical constructions as Hochschild cohomology, Harrison cohomology and Chevalley-Eilenberg cohomology. The ideas of the construction are implicit in the papers [17, 16] and it is quite possible that they circulate among people as folklore, but, as Murray Gerstenhaber pointed out to us, the problem with folklore is that we are not all of the same folk, so we decided to give an explicit construction here.

The construction is motivated by the following observation, which is certainly well known to people working in rational homotopy theory, though we are not able to find a suitable reference (for related ideas see [22, 23, 29, 30]). Our observation is that the Hochschild cohomology can be computed as the cohomology of the algebra of coderivations of a certain coassociative coalgebra. Similarly, the Chevalley-Eilenberg cohomology of a Lie algebra can be computed as the cohomology of the algebra of coderivations of a certain cocommutative coalgebra and, finally, the Harrison cohomology of a commutative algebra can be computed as the cohomology of the algebra of coderivations of a certain Lie coalgebra. The plan of our construction will be based on a similar scheme.

The basic ingredient of our definition is the notion of Koszul duality for operads, introduced in [17]. This construction gives, for any quadratic operad $\mathcal{P}$, another operad, denoted by $\mathcal{P}^!$ and called the *Koszul dual* of $\mathcal{P}$. Our cohomology will be then defined as the cohomology of the algebra of coderivations of a certain (almost) cofree $\mathcal{P}^!$-coalgebra (see below for the definition).

Let us recall first some notions from [17]. By a *collection* we mean a sequence $E = \{E(n);\ n \geq 2\}$ such that each $E(n)$ is a $\mathbf{k}$-linear Σ_n-space. The obvious forgetful functor from the category of operads into the category of collections has a left adjoint $\mathcal{F}$, and we call the operad $\mathcal{F}(E)$ the *free operad* generated by the collection E. We will assume that all collections considered in this paper have the property that the graded vector space $E(n)$ is of finite type, $n \geq 2$.

Let $\mathcal{P}$ be an operad. A sequence $I = \{I(n);\ n \geq 1\}$ of Σ_n-invariant linear subspaces $I(n) \subset \mathcal{P}(n)$ is called an *ideal* if, for $\lambda \in \mathcal{P}(n)$ and $\mu_i \in \mathcal{P}(m_i)$, $1 \leq i \leq l$, the composition $\lambda(\mu_1 \otimes \cdots \otimes \mu_l)$ belongs to $I(m_1 + \cdots + m_l)$ if either $\lambda \in I(l)$ or if for at least one i, $1 \leq i \leq l$, we have $\mu_i \in I(m_i)$. For an ideal I in $\mathcal{P}$, it makes sense to speak about the quotient operad $\mathcal{P}/I$. For a given sequence $R = \{R(n);\ n \geq 2\}$ of linear invariant subspaces $R(n) \subset \mathcal{P}(n)$, denote by (R) the ideal generated by R.

Every Σ_2-invariant linear space E defines a collection $\{E(2) = E,\ E(n) = 0 \text{ for } n \geq 3\}$ (denoted also by E), and we can consider the free operad $\mathcal{F}(E)$ on E. Choose a Σ_3-invariant linear subspace $R \subset \mathcal{F}(E)(3)$ and form the operad $\langle E; R\rangle = \mathcal{F}(E)/(R)$ We say that an operad $\mathcal{P}$ is *quadratic* if $\mathcal{P} = \langle E; R\rangle$ for some E and R as above.

Let V be a $\mathbf{k}$-linear Σ_n-space of finite type. We equip its dual $\#V$ with the Σ_n-action given by $\sigma\varphi(v) = \mathrm{sgn}(\sigma) \cdot \varphi(\sigma^{-1}v)$. If E is a linear Σ_2 space of finite type, then there exists, for each $n \geq 2$, a natural Σ_n-invariant identification $\#\mathcal{F}(E)(n) \cong F(\#E)(n)$, given by the pairing $\langle -|-\rangle_n : F(E)(n) \otimes F(\#E)(n) \longrightarrow \mathbf{k}$, which is characterized by the following conditions.

(i) $\langle -|-\rangle_2$ is the evaluation between $F(E)(2) = E$ and $F(\#E)(2) = \#E$.

(ii) For $\mu \in F(E)(k)$, $\nu \in F(E)(l)$, $\varphi \in F(\#E)(k)$ and $\psi \in F(\#E)(l)$ we have

$$\langle \mu(1^{\otimes(i-1)} \otimes \nu \otimes 1^{\otimes(k-i)}) | \varphi(1^{\otimes(j-1)} \otimes \psi \otimes 1^{\otimes(k-j)})\rangle_{k+l-1} = \\ = \begin{cases} (-1)^{(l+1)(i+1)} \cdot \langle \mu|\varphi\rangle_k \cdot \langle \nu|\psi\rangle_l, & \text{for } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) $\langle \sigma\mu|\sigma\varphi\rangle_l = \mathrm{sgn}(\sigma) \cdot \langle \mu|\varphi\rangle_l$, for $\mu \in F(E)(l)$, $\varphi \in F(\#E)(l)$ and $\sigma \in \Sigma_l$.

Let $\mathcal{P} = \langle E; R\rangle$ be a quadratic operad. Using the above identification, we can view the subspace $R \subset \mathcal{F}(E)(3)$ as a subspace (denoted by the same symbol) of $\#F(\#E)(3)$ and we can take its annihilator $R^\perp \subset F(\#E)(3)$. The operad $\mathcal{P}^! = \langle \#E; R^\perp\rangle$ was introduced in [17] as the *Koszul dual* of the operad $\mathcal{P}$.

For a graded vector space W, let $C_{\mathcal{P}}(W) = \bigoplus_{n\geq 1} C^n_{\mathcal{P}}(W)$, where

$$C^n_{\mathcal{P}}(W) = (\#\mathcal{P}(n) \otimes W^{\otimes n})^{\Sigma_n}$$

Here Σ_n acts on $W^{\otimes n}$ by permuting the factors (but taking into account the Koszul convention as in (25)) and $(\#\mathcal{P}(n) \otimes W^{\otimes n})^{\Sigma_n}$ is the space of invariants of the diagonal action of Σ_n on $\#\mathcal{P}(n) \otimes W^{\otimes n}$. We equip $C_{\mathcal{P}}(W)$ with the obvious $\mathcal{P}$-coalgebra structure. Denote by $\pi : C_{\mathcal{P}}(W) \longrightarrow W$ the canonical projection. The coalgebra $C_{\mathcal{P}}(W)$ is connected and it has the following universal property (which we state without proof, which is a standard one). For any connected $\mathcal{P}$-coalgebra C and for any homogeneous degree zero linear map $\psi : C \longrightarrow W$, there exists exactly one coalgebra homomorphism $g : C \longrightarrow C_{\mathcal{P}}(W)$ making the following diagram commutative.

$$\begin{array}{ccc} C & \xrightarrow{\ g\ } & C_{\mathcal{P}}(W) \\ & {}_{\psi}\searrow & \downarrow \pi \\ & & W \end{array}$$

Let us point out, however, that the coalgebra $C_{\mathcal{P}}(W)$ is the cofree *connected* coalgebra on W, not an honest cofree coalgebra; see also the remarks in Example 6.4.

Dually, for any graded vector space V we can define, following [17], the *free $\mathcal{P}$-algebra* on V by $A_{\mathcal{P}}(V) = \bigoplus_{n\geq 1} A^n_{\mathcal{P}}(V)$, with

$$A^n_{\mathcal{P}}(V) = (\mathcal{P}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

where $(-)_{\Sigma_n}$ denote the space of coinvariants. This algebra has the classical and absolutely obvious universal property. Moreover, any $\mathcal{P}$-algebra structure $a : \mathcal{P} \longrightarrow \mathcal{E}_V$ on V defines, by dualization, the *canonical map* (which we denote again by a) $a : A_{\mathcal{P}}(V) \longrightarrow V$. One would expect that there exists also a dual analog of this map for coalgebras. More precisely, let $c : \mathcal{P} \longrightarrow \mathcal{G}_W$ be a $\mathcal{P}$-coalgebra structure on W. We would like to dualize this map to get a map (again denoted by the same symbol) $c : W \longrightarrow C_{\mathcal{P}}(W)$. An immediate observation shows that such a dualization hits, in general, infinitely many components of $C_{\mathcal{P}}(W)$, so the target space is $\prod_{n\geq 1} C^n_{\mathcal{P}}(W)$ rather than $C_{\mathcal{P}}(W) = \bigoplus_{n\geq 1} C^n_{\mathcal{P}}(W)$. This is related with the fact that $C_{\mathcal{P}}(W)$ is not the honest cofree coalgebra; see also the comments above. But it still makes sense to take, for any $n \geq 1$, the component $c^n : W \longrightarrow C^n_{\mathcal{P}}(W)$ of this dual, and by the *canonical map* in the coalgebra case we mean the *sequence* $\{c^n : W \longrightarrow C^n_{\mathcal{P}}(W)\}_{n\geq 1}$ of these maps. The functor $A_{\mathcal{P}}(-)$ gives rise to a triple on the category $\mathcal{M}$ and the algebras over this triple are exactly $\mathcal{P}$-algebras, see [16]. This connection gives formal meaning to the analogy between the triple and operadic definition of algebras.

Let $C = (C, c)$ be a $\mathcal{P}$-coalgebra and $\theta : C \longrightarrow C$ a homogeneous degree p linear map. We say that θ is a degree p *coderivation* of C (into itself) if

$$c(n)(\mu) \circ \theta = (-1)^{|\mu|\cdot|\theta|} \sum_{0\leq i\leq n-1} (\mathbb{1}^{\otimes i} \otimes \theta \otimes \mathbb{1}^{\otimes(n-i-1)}) \circ c(n)(\mu),$$

for any $n \geq 2$ and $\mu \in \mathcal{P}(n)$. We denote by $\mathrm{Coder}^p(C)$ the linear space of all degree p coderivations of C.

Consider again the coalgebra $C_{\mathcal{P}}(W)$ introduced above. The grading $C_{\mathcal{P}}(W) = C^n_{\mathcal{P}}(W)$ induces on $\mathrm{Coder}^p(C_{\mathcal{P}}(W))$ a second grading, namely $\mathrm{Coder}^p(C_{\mathcal{P}}(W)) = \bigoplus_n \mathrm{Coder}^{p,n}(C_{\mathcal{P}}(W))$ with

$$\mathrm{Coder}^{p,n}(C_{\mathcal{P}}(W)) = \{\theta \in \mathrm{Coder}^p(C_{\mathcal{P}}(W));\ (\pi \circ \theta)(C^q_{\mathcal{P}}(W)) = 0 \text{ for } q \neq n+1\}.$$

8.1 Lemma: *The map $\omega : \mathrm{Coder}^{p,n}(C_{\mathcal{P}}(W)) \longrightarrow \mathcal{M}^p(C^{n+1}_{\mathcal{P}}(W), W)$, given by $\omega(\theta) = \pi \circ \theta$, is an isomorphism for all p, n.*

Proof: The lemma is a consequence of the universal property of $C_{\mathcal{P}}(W)$ mentioned above and we leave the proof to the reader. □

In the rest of the paper, all operads will be assumed to be quadratic.

8.2 Theorem: *Let V be a graded vector space and let $W = \downarrow V$. Then there is a natural one-to-one correspondence between $\mathcal{P}$-algebra structures $a : \mathcal{P} \longrightarrow \mathcal{E}_V$ on V and coderivations $d \in \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W))$ of the $\mathcal{P}^!$-coalgebra $C_{\mathcal{P}^!}(W)$ with $d^2 = 0$.*

Proof: To fix the notation, let $\mathcal{P} = \langle E; R\rangle$. First we show that there is a one-to-one natural map

$$\Omega : \mathit{Oper}(\mathcal{F}(E), \mathcal{E}_V) \longrightarrow \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W)).$$

To this end, observe that, since $\mathcal{F}(E)$ is free, $\mathit{Oper}(\mathcal{F}(E), \mathcal{E}_V)$ is naturally isomorphic to $\mathcal{M}_{\Sigma_2}(E, \mathcal{M}^*(V^{\otimes 2}, V))$, while, by Lemma 8.1, we have

$$\begin{aligned}\mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W)) \cong \mathcal{M}^1(C^2_{\mathcal{P}^!}(W), W) &\cong \mathcal{M}^1((\#\mathcal{P}^!(2) \otimes W^{\otimes 2})^{\Sigma_2}, W)\\ &\cong \mathcal{M}^1((E \otimes W^{\otimes 2})^{\Sigma_2}, W)\end{aligned}$$

Let now $\Xi : \mathcal{M}^0_{\Sigma_2}(E, \mathcal{M}(V^{\otimes 2}, V)) \to \mathcal{M}^1((E \otimes W^{\otimes 2})^{\Sigma_2}, W)$ be the isomorphism defined by

$$\Xi(f)(e \otimes w_1 \otimes w_2) = (-1)^{|w_1|} \downarrow f(e)(\uparrow w_1 \otimes \uparrow w_2).$$

We define Ω by the commutativity of the diagram

$$\begin{array}{ccc} \mathit{Oper}(\mathcal{F}(E), \mathcal{E}_V) & \xrightarrow{\ \Omega\ } & \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W)) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{M}^0_{\Sigma_2}(E, \mathcal{M}(V^{\otimes 2}, V)) & \xrightarrow[\ \Xi\ ]{} & \mathcal{M}^1((E \otimes W^{\otimes 2})^{\Sigma_2}, W) \end{array}$$

It is easy to verify that $a \in \mathit{Oper}(\mathcal{F}(E), \mathcal{E}_V)$ defines an algebra structure on V, i.e. factors through $\mathcal{F}(E)/(R)$, if and only if the coderivation $d = \Omega(a)$ satisfies $d^2 = 0$, which finishes the proof. □

Suppose now that $A = (A, a)$ is a $\mathcal{P}$-algebra and $M = (M, m)$ an A-module. Let $U = \downarrow A$, $X = \downarrow M$ and $W = U \oplus X$. Notice that the decomposition $W^{\otimes n} = \bigoplus_{i+j=n} (W^{\otimes n})^{i,j}$, where $(W^{\otimes n})^{i,j}$ is the subspace formed by i-tuple products of elements of U and j-tuple products of elements of X, induces the decomposition $C^n_{\mathcal{P}^!}(W) = \bigoplus_{i+j=n} C^{i,j}_{\mathcal{P}^!}(W)$ for any $n \geq 1$. Let $C^{p,n}_{\mathcal{P}}(A; M)$ be the subspace of $\mathrm{Coder}^{p,n}(C_{\mathcal{P}^!}(W))$ consisting of those coderivations θ for which $(\pi \circ \theta)(C^{i,j}_{\mathcal{P}^!}(W)) = 0$ whenever $j \geq 1$, and $(\pi \circ \theta)(C^{n+1}_{\mathcal{P}^!}(W)) \subset X \subset W$.

Suppose now that $d \in \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W))$ corresponds to the algebra structure m on $A \oplus M$ as in Proposition 8.2. Let us denote, for $\theta \in \mathrm{Coder}^{p,n}(C_{\mathcal{P}^!}(W))$, by $\nabla(\theta) = d \circ \theta - (-1)^p \theta \circ d$ the (graded) commutator. It is easy to verify that ∇ maps $C^{p,n}_{\mathcal{P}}(A; M)$ to $C^{p+1,n+1}_{\mathcal{P}}(A; M)$ and that $\nabla^2 = 0$. We can then formulate the following definition.

8.3 Definition: Let A be an algebra over a quadratic operad $\mathcal{P}$ and let M be an A-module. Define the cohomology $H^{p,n}_{\mathcal{P}}(A; M)$ of A with coefficients in M as

$$H^{p,n}_{\mathcal{P}}(A; M) := H^{p,n}(C^{*,*}_{\mathcal{P}}(A; M), \nabla).$$

We call p and n the *inner* and the *simplicial* degrees, respectively.

8.4 Remark: The map ω of Lemma 8.1 induces an isomorphism

$$\begin{aligned} (27) \quad C^{p,n}_{\mathcal{P}}(A; M) &\cong \mathcal{M}^p(C^{n+1}_{\mathcal{P}^!}(U), X) = \\ &= \mathcal{M}^p(C^{n+1}_{\mathcal{P}^!}(\downarrow A), \downarrow M) \cong \mathcal{M}^p_{\Sigma_{n+1}}(\#\mathcal{P}^!(n+1) \otimes (\downarrow A)^{\otimes(n+1)}, \downarrow M). \end{aligned}$$

We could have *defined* $C^{p,n}_{\mathcal{P}}(A; M)$ by the equation above, without having referred to coderivations of some coalgebra. The problem is that we do not know an easy way to describe the differential in terms of $\mathcal{M}^p_{\Sigma_{n+1}}(\#\mathcal{P}(n+1) \otimes (\downarrow A)^{\otimes(n+1)}, \downarrow M)$, the only way we know is the one based on the somewhat explicit description of the Koszul dual operad $\mathcal{P}^!$ using the language of trees as in [16, 17].

8.5 Examples: If A is an associative algebra, i.e. an algebra over the associative operad Ass, we know ([17]) that the Koszul dual $\mathit{Ass}^!$ is again Ass and the description above gives us the usual definition of the Hochschild cohomology. For a Lie algebra, i.e. for an algebra over the Lie operad Lie, we have $\mathit{Lie}^! = \mathit{Comm}$, the commutative operad, and the description above gives the Chevalley-Eilenberg cohomology. Finally, for a commutative algebra, i.e. for an algebra over the commutative operad *Comm*, we have $Comm^! = Lie$ and the description above gives us the Harrison cohomology.

It can be shown, using the computation of [17], that for a so-called *Koszul operad* (see again [17] for the definition) we have $H^{p,\geq 1}_{\mathcal{P}}(A;M) = 0$ whenever the algebra A is free. This is, by [5], enough to infer the following proposition.

8.6 Proposition: *For an algebra A over a Koszul operad $\mathcal{P}$ and for an A-module M, the cohomology $H^{p,n}_{\mathcal{P}}(A;M)$ coincides with the Barr-Beck triple cohomology of A with coefficients in M.*

This proposition relates the operadic cohomology with the "triple cohomology" introduced in Definition 2.2.

9 Distributive laws and operads

In this paragraph we discuss distributive laws from the operadic point of view. The definitions and results quoted here were taken from the paper [24] of the second author. We formulate them for quadratic operads only, since the range of our applications is limited by our definition of the cohomology. Similar definitions can be made for more general types of operads as well.

Suppose that a Σ_2-space E has an invariant decomposition $E = E_1 \oplus E_2$. This decomposition induces the decomposition

$$\mathcal{F}(E)(3) = \mathcal{F}(E)(3)_{11} \oplus \mathcal{F}(E)(3)_{12} \oplus \mathcal{F}(E)(3)_{21} \oplus \mathcal{F}(E)(3)_{22},$$

where $\mathcal{F}(E)(3)_{ij}$ is the Σ_3-invariant subspace of $\mathcal{F}(E)(3)$ generated by the compositions of the form $\mu(1,\nu)$ (or $\mu(\nu,1)$) with $\mu \in E_i$ and $\nu \in E_j$, for $i,j = 1$ or 2. Notice that $\mathcal{F}(E)(3)_{ii}$ can be identified with the image of the map $F(E_i)(3) \longrightarrow \mathcal{F}(E)(3)$ induced by the inclusion $E_i \subset E$, $i = 1,2$. Let us consider a Σ_3-invariant map $\mathcal{D} : \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}$. Every such a map defines an invariant subspace $R_{\mathcal{D}} \subset \mathcal{F}(E)(3)$ generated by elements of the form $x - \mathcal{D}(x)$, for $x \in \mathcal{F}(E)(3)_{12}$.

Let $\mathcal{P} = \langle E; R\rangle$ be a quadratic operad for which there exists a Σ_2-invariant decomposition $E = E_1 \oplus E_2$, an invariant linear map $\mathcal{D} : \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}$ and Σ_3-invariant subsets $R_i \subset \mathcal{F}(E)(3)_{ii}$, $i = 1,2$, such that $R = R_1 \oplus R_{\mathcal{D}} \oplus R_2$. In this case we write

$$\mathcal{P} = \langle E_1, E_2; R_1, \mathcal{D}, R_2\rangle.$$

We can clearly form the suboperads $\mathcal{P}_i = \langle E_i; R_i\rangle \subset \mathcal{P}$, for $i = 1,2$. Denote, for any $n \geq 1$ and $l \leq n$, by $\mathcal{P}(n)_l$ the invariant subspace of $\mathcal{P}(n)$ generated by the elements of the form $\mu(\nu_1,\ldots,\nu_l)$, for $\mu \in \mathcal{P}_2(l)$ and $\nu_s \in \mathcal{P}_1(m_s)$, for a sequence $m_1,\ldots,m_l \geq 1$, $m_1 + \cdots + m_l = n$, and $1 \leq s \leq l$. The inclusions $\mathcal{P}_i \subset \mathcal{P}$, $i = 1,2$ induce, for any $n \geq 2$, a map

$$\xi(n) : \bigoplus_{1\leq l\leq n} \mathcal{P}(n)_l \to \mathcal{P}(n). \tag{28}$$

9.1 Definition: We say that $\mathcal{D}$ is a distributive law (or sometimes also that $\mathcal{P} = \langle E_1, E_2; R_1, \mathcal{D}, R_2\rangle$ is an operad with a distributive law) if the map $\xi(n)$ is an isomorphism for each $n \geq 2$.

We refer to [24] for how the distributive law in the sense of the above definition induces the transformation λ from the triple definition (Definition 3.1) of the distributive law. Observe that $\xi(n)$ is always an isomorphism for $n = 2, 3$. The second author proved in [24] the following coherence theorem.

9.2 Theorem: *The map $\xi(n)$ is an isomorphism for any $n \geq 2$ if and only if it is an isomorphism for $n = 4$.*

The following lemma can be verified directly.

9.3 Lemma: *Let $\mathcal{P} = \langle E_1, E_2; R_1, \mathcal{D}, R_2\rangle$ be a quadratic operad with a distributive law $\mathcal{D}$. Then its Koszul dual operad $\mathcal{P}^!$ is again a (quadratic) operad with a distributive law, namely*

$$\mathcal{P}^! = \langle \# E_2, \# E_1; R_2^\perp, \#\mathcal{D}, R_1^\perp\rangle,$$

where $\#\mathcal{D} : F(\# E)(3)_{12} \longrightarrow F(\# E)(3)_{21}$ is the dual of $\mathcal{D} : \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}$ under the natural identification $F(\# E)(3)_{ij} \cong \#\mathcal{F}(E)(3)_{ji}$.

9.4 Example: Below we give, following some ideas of [16], an innocuous generalization of Poisson algebras already discussed in Example 6.2 and describe the Koszul dual of the corresponding operad (Proposition 9.5). This will also be an example of a nontrivially graded operad. By an (m, n)-*algebra* we mean a (graded) vector space P together with two bilinear maps, $- \cup - : P \otimes P \longrightarrow P$ of degree m, and $[-, -] : P \otimes P \longrightarrow P$ of degree n (m and n are natural numbers), such that, for any homogeneous $a, b, c \in P$,

(i) $a \cup b = (-1)^{|a|\cdot|b|+m} \cdot b \cup a$,

(ii) $[a, b] = -(-1)^{|a|\cdot|b|+n} \cdot [b, a]$,

(iii) $- \cup -$ is associative in the sense that

$$a \cup (b \cup c) = (-1)^{m\cdot(|a|+1)} \cdot (a \cup b) \cup c,$$

(iv) $[-, -]$ satisfies the following form of the Jacobi identity:

$$(-1)^{|a|\cdot(|c|+n)} \cdot [a, [b, c]] + (-1)^{|b|\cdot(|a|+n)} \cdot [b, [c, a]] + (-1)^{|c|\cdot(|b|+n)} \cdot [c, [a, b]] = 0,$$

(v) the operations $- \cup -$ and $[-, -]$ are compatible in the sense that

$$(-1)^{m\cdot|a|}[a, b \cup c] = [a, b] \cup c + (-1)^{(|b|\cdot|c|+m)}[a, c] \cup b.$$

Obviously $(0, 0)$-algebras are exactly (graded) Poisson algebras, $(0, -1)$-algebras are Gerstenhaber algebras introduced under this name in [15], while $(0, n-1)$-algebras are n-algebras of [16]. For the relation between the cohomology of configuration spaces and n-algebras we refer to [11]. We may think of an (m, n)-structure on P as a Lie algebra structure on $\uparrow^n P$ together with an associative commutative algebra structure on $\uparrow^m P$ such that both structures are related via the compatibility axiom (v).

Let us give an operadic description of (m, n)-algebras. To simplify the notation, let $\chi = (-1)^n$ and $\lambda = (-1)^m$. Let E be the space spanned by two elements, μ (for

$-\cup-$) of degree m and ℓ (for $[-,-]$) of degree n, with the action of Σ_2 defined by $s\mu = \lambda\mu$ and $s\ell = -\chi\ell$, s being the generator of Σ_2. We can easily see that

$$\mathcal{F}(E)(3) = \mathrm{Span}(X_{ij}, Y_{ij}, Z_{ij};\ i,j = 1 \text{ or } 2)$$

with

(29) $$X_{ij} = e_i(e_j \otimes 1),\ Y_{ij} = e_i(1 \otimes e_j) \text{ and } Z_{ij} = e_i(e_j \otimes 1)(\mathbb{1} \otimes s),$$

where $e_i = \ell$ for $i = 1$ and $e_i = \mu$ for $i = 2$. The action of the group Σ_3 on $\mathcal{F}(E)(3)$ is described by the following table (which is, unforunately, broken into two pieces, due to the limited width of the page), whose meaning is clear, we hope.

	X_{11}	Y_{11}	Z_{11}	X_{12}	Y_{12}	Z_{12}
(123)	X_{11}	Y_{11}	Z_{11}	X_{12}	Y_{12}	Z_{12}
(312)	$-\chi Z_{11}$	$-\chi X_{11}$	Y_{11}	λZ_{12}	$-\chi X_{12}$	$-\chi\lambda Y_{12}$
(231)	$-\chi Y_{11}$	Z_{11}	$-\chi X_{11}$	$-\chi Y_{12}$	$-\chi\lambda Z_{12}$	λX_{12}
(213)	$-\chi X_{11}$	$-\chi Z_{11}$	$-\chi Y_{11}$	λX_{12}	$-\chi Z_{12}$	$-\chi Y_{12}$
(321)	Y_{11}	X_{11}	$-\chi Z_{11}$	$-\chi\lambda Y_{12}$	$-\chi\lambda X_{12}$	λZ_{12}
(132)	Z_{11}	$-\chi Y_{11}$	X_{11}	Z_{12}	λY_{12}	X_{12}

	X_{21}	Y_{21}	Z_{21}	X_{22}	Y_{22}	Z_{22}
(123)	X_{21}	Y_{21}	Z_{21}	X_{22}	Y_{22}	Z_{22}
(312)	$-\chi Z_{21}$	λX_{21}	$-\lambda\chi Y_{21}$	λZ_{22}	λX_{22}	Y_{22}
(231)	λY_{21}	$-\lambda\chi Z_{21}$	$-\chi X_{21}$	λY_{22}	Z_{22}	λX_{22}
(213)	$-\chi X_{21}$	λZ_{21}	λY_{21}	λX_{22}	λZ_{22}	λY_{22}
(321)	$-\lambda\chi Y_{21}$	$-\lambda\chi X_{21}$	$-\chi Z_{21}$	Y_{22}	X_{22}	λZ_{22}
(132)	Z_{21}	$-\chi Y_{21}$	X_{21}	Z_{22}	λY_{22}	X_{22}

The space of relations R is the Σ_3-invariant subspace of $\mathcal{F}(E)(3)$ generated by the elements

$$X_{11} - \chi Y_{11} - \chi Z_{11} \qquad \text{(for the Jacobi identity)}$$
$$X_{22} - \lambda Y_{22} \qquad \text{(for the associativity)}$$
$$Y_{12} - \lambda Z_{21} - X_{21} \qquad \text{(for the compatibility).}$$

From this we get easily that the operad $\mathcal{P}(m,n)$ for the category of (m,n)-algebras can be described as

$$\mathcal{P}(m,n) = \langle E_1, E_2; R_1, \mathcal{D}, R_2 \rangle$$

with

$$E_1 = \mathrm{Span}(\ell),\ E_2 = \mathrm{Span}(\mu),\ R_1 = \mathrm{Span}(X_{11} - \chi Y_{11} - \chi Z_{11}),$$
$$R_2 = \mathrm{Span}(X_{22} - \lambda Y_{22}, Y_{22} - Z_{22})$$

and the distributive law $\mathcal{D} : \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}$ defined by

$$\mathcal{D}(X_{12}) = Y_{21} + Z_{21},\ \mathcal{D}(Y_{12}) = X_{21} + \lambda Z_{21} \text{ and } \mathcal{D}(Z_{12}) = X_{21} - \chi Y_{21}.$$

We leave it to the reader to verify that the condition of Theorem 9.2 is satisfied.

Let us describe the Koszul dual of the operad $\mathcal{P}(m,n)$. For $\bar{E} = \#E$, $\bar{E} = \bar{E}_1 \oplus \bar{E}_2$ with $\bar{E}_1 = \mathrm{Span}(\bar{\ell})$, $\deg(\bar{\ell}) = -n$, and $\bar{E}_2 = \mathrm{Span}(\bar{\mu})$, $\deg(\bar{\mu}) = -m$, with the action of Σ_2 given by $s\bar{\mu} = \bar{\mu}$ and $s\bar{\ell} = -\bar{\ell}$. The evaluation between E and $\bar{E} = \#E$ is given by

(30) $$\langle \mu | \bar{\ell} \rangle = \langle \ell | \bar{\mu} \rangle = 1 \text{ and } \langle \mu | \bar{\mu} \rangle = \langle \ell | \bar{\ell} \rangle = 0.$$

As above, $\mathcal{F}(\bar{E})(3)$ has the basis

(31) $$\bar{X}^{ij} = \bar{e}^i(\bar{e}^j \otimes 1),\ \bar{Y}^{ij} = \bar{e}^i(1 \otimes \bar{e}^j) \text{ and } \bar{Z}^{ij} = \bar{e}^i(\bar{e}^j \otimes 1)(1\!\!1 \otimes s),$$

where $\bar{e}^i = \bar{\ell}$ for $i = 1$ and $\bar{e}^i = \bar{\mu}$ for $i = 2$. The pairing between $\mathcal{F}(E)(3)$ and $\mathcal{F}(\bar{E})(3)$ is given by

(32) $$\langle X_{ij}|\bar{X}^{kl}\rangle = \delta_i^k\delta_j^l,\ \langle Y_{ij}|\bar{Y}^{kl}\rangle = -\delta_i^k\delta_j^l \text{ and } \langle Z_{ij}|\bar{Z}^{kl}\rangle = -\delta_i^k\delta_j^l$$

while the pairing is trivial on other combinations of the basis elements and δ_*^* denotes the Kronecker delta. From this we get immediately that

$$R_2^\perp = \mathrm{Span}(\bar{X}^{11} - \lambda\bar{Y}^{11} - \lambda\bar{Z}^{11}),\ R_1^\perp = \mathrm{Span}(\bar{X}^{22} - \chi\bar{Y}^{22}, \bar{Y}^{22} - \bar{Z}^{22})$$

and that $\#\mathcal{D}$ is given by

$$\#\mathcal{D}(\bar{X}^{12}) = \bar{Y}^{12} + \bar{Z}^{21},\ \#\mathcal{D}(\bar{Y}^{12}) = \bar{X}^{21} + \chi\bar{Z}^{21} \text{ and } \#\mathcal{D}(\bar{Z}^{12}) = \bar{X}^{21} - \lambda\bar{Y}^{21}.$$

We have proved the following proposition.

9.5 Proposition: *For any natural numbers m and n we have*

$$\mathcal{P}(m,n)^! = \mathcal{P}(-n,-m).$$

In particular, the category of $(m,-m)$-algebras is Koszul self-dual, $\mathcal{P}(m,-m)^! = \mathcal{P}(m,-m)$, for any m.

9.6 Example: Recall from Example 6.3 that a nonsymmetric Poisson algebra consists of a vector space P and two associative multiplications, $\cdot\,, \langle -,- \rangle : P \otimes P \longrightarrow P$ such that:

$$\langle a \cdot b, c\rangle = a \cdot \langle b, c\rangle \text{ and } \langle a, b \cdot c\rangle = \langle a, b\rangle \cdot c,$$

for any $a, b, c \in P$. The operadic description appears as follows: Let E_1 (resp. E_2) be the free Σ_2-space generated by a symbol ℓ (resp. μ), ℓ corresponding to $\langle -,- \rangle$ and μ corresponding to $\cdot$. If $E = E_1 \oplus E_2$. Then $\mathcal{F}(E)(3)$ is generated, as a Σ_3-module, by $X_{ij} = e_i(e_j \otimes 1)$ and $Y_{ij} = e_i(1 \otimes e_j)$, where $e_i = \ell$ for $i = 1$ and $e_i = \mu$ for $i = 2$. Define $\mathcal{D} : \mathcal{F}(E)(3)_{12} \longrightarrow \mathcal{F}(E)(3)_{21}$ by

$$\mathcal{D}(X_{12}) = Y_{21} \text{ and } \mathcal{D}(Y_{12}) = X_{21}.$$

It was verified in [24] that this map defines a distributivity law and that nonsymmetric Poisson algebras are algebras over the operad $\mathcal{P} = \langle E_1, E_2; R_1, \mathcal{D}, R_2\rangle$, where R_1 (resp. R_2) is the associativity axiom for ℓ (resp. μ). It is also almost immediate to see that the category of these algebras is Koszul self-dual.

10 Effect on cohomology (operad case)

Let $\mathcal{P} = \langle E_1, E_2; R_1, \mathcal{D}, R_2\rangle$ be an operad with a distributive law. Recall that this means, by definition, that the map

$$\xi(n) : \bigoplus_{1 \le l \le n} \mathcal{P}(n)_l \longrightarrow \mathcal{P}(n)$$

of (28) is an isomorphism for each $n \geq 2$. We use this isomorphism to *identify* $\mathcal{P}(n)$ with $\bigoplus_{1\leq l\leq n}\mathcal{P}(n)_l$. This identification induces, for any (graded) vector space W, the decomposition

$$C^n_{\mathcal{P}}(W) = \bigoplus C^{\alpha,\beta}_{\mathcal{P}}(W) \tag{33}$$

where the summation runs over all $\alpha+\beta = n$, $1 \leq \alpha \leq n$, and

$$C^{\alpha,\beta}_{\mathcal{P}}(W) = (\#\mathcal{P}(\alpha{+}\beta)_\alpha \otimes W^{\otimes(\alpha+\beta)})^{\Sigma_{(\alpha+\beta)}}. \tag{34}$$

On the other hand, take the $\mathcal{P}_1$-coalgebra $C_{\mathcal{P}_1}(W)$, forget the coalgebra structure, and form the $\mathcal{P}_2$-coalgebra $C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W))$. This coalgebra again decomposes as

$$C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W)) = \bigoplus C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W))^{\alpha,\beta},$$

where the summation runs again over all $\alpha+\beta = n$, $1 \leq \alpha \leq n$, with

$$C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W))^{\alpha,\beta} = \bigoplus \left\{\#\mathcal{P}_2(\alpha)\otimes(\#\mathcal{P}_1(m_1)\otimes W^{\otimes m_1})^{\Sigma_{m_1}}\otimes\cdots\otimes(\#\mathcal{P}_\alpha(m_\alpha)\otimes W^{\otimes m_\alpha})^{\Sigma_{m_\alpha}}\right\}^{\Sigma_\alpha},$$

where the summation runs over all $m_1,\ldots,m_\alpha \geq 1$ with $m_1+\cdots+m_\alpha = \alpha+\beta$. We can show that the map

$$\#\mathcal{P}(\alpha{+}\beta)_\alpha \otimes W^{\otimes(\alpha+\beta)} \longrightarrow \#(\mathcal{P}_2(\alpha)\otimes\mathcal{P}_1(m_1)\otimes\cdots\otimes\mathcal{P}_1(m_\alpha))\otimes W^{\otimes(\alpha+\beta)}$$

induced by the composition map $\mathcal{P}_2(\alpha)\otimes\mathcal{P}_1(m_1)\otimes\cdots\otimes\mathcal{P}_1(m_\alpha) \longrightarrow \mathcal{P}(\alpha{+}\beta)$ gives an identification $C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W))^{\alpha,\beta} \cong C^{\alpha,\beta}_{\mathcal{P}}(W)$, which in turn induces an isomorphism

$$C_{\mathcal{P}}(W) \cong C_{\mathcal{P}_2}(C_{\mathcal{P}_1}(W)) \tag{35}$$

of $\mathcal{P}_2$-coalgebras. The decomposition (33) then induces the decomposition

$$\mathrm{Coder}^{p,n}(C_{\mathcal{P}}(W)) = \bigoplus_{i+j=n}\mathrm{Coder}^{p,i,j}(C_{\mathcal{P}}(W))$$

with

$$\mathrm{Coder}^{p,i,j}(C_{\mathcal{P}}(W)) = \{\theta\in\mathrm{Coder}^p(C_{\mathcal{P}}(W));\ \pi\circ\theta(C^{\alpha,\beta}_{\mathcal{P}}(W)) = 0 \text{ for } (\alpha,\beta)\neq(i+1,j)\}.$$

Notice that the two most extreme pieces of the decomposition above have a particularly easy description:

$$\begin{aligned}\mathrm{Coder}^{p,n,0}(C_{\mathcal{P}}(W)) &= \mathrm{Coder}^{p,n}(C_{\mathcal{P}_2}(W)) \text{ and}\\ \mathrm{Coder}^{p,0,n}(C_{\mathcal{P}}(W)) &= \mathrm{Coder}^{p,n}(C_{\mathcal{P}_1}(W)).\end{aligned} \tag{36}$$

All the decompositions above apply, by Lemma 9.3, also to $\mathcal{P}^!$ in place of $\mathcal{P}$, with $\mathcal{P}^!_1 = (\mathcal{P}_2)^!$ and $\mathcal{P}^!_2 = (\mathcal{P}_1)^!$. In particular, (36) gives

$$\begin{aligned}\mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W)) &= \mathrm{Coder}^{1,1,0}(C_{\mathcal{P}^!}(W))\oplus\mathrm{Coder}^{1,0,1}(C_{\mathcal{P}^!}(W))\\ &= \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!_1}(W))\oplus\mathrm{Coder}^{1,1}(C_{\mathcal{P}^!_2}(W)).\end{aligned} \tag{37}$$

Let $a : \mathcal{P} \longrightarrow \mathcal{E}_V$ be a $\mathcal{P}$-algebra structure on a (graded) vector space V. The composition of a and the inclusion $\mathcal{P}_i \hookrightarrow \mathcal{P}$ induces on V a $\mathcal{P}_i$-algebra structure a_i, for $i = 1, 2$. Let $d \in \mathrm{Coder}^{1,1}(C_{\mathcal{P}^!}(W))$ (resp. $d_i \in \mathrm{Coder}^{1,1}(C_{\mathcal{P}_i^!}(W))$) correspond to a (resp. to a_i) as in Proposition 8.2. We immediately have the following lemma.

10.1 Lemma: *Under the identification of (37), $d = d_1 + d_2$.*

Let A be a $\mathcal{P}$-algebra over an operad with a distributive law as above, and let M be an A-module. Then

$$C^{p,n}_{\mathcal{P}}(A; M) = \bigoplus_{i+j=n} C^{p,i,j}_{\mathcal{P}}(A; M)$$

with $C^{p,i,j}_{\mathcal{P}}(A; M) = C^{p,i+j}_{\mathcal{P}}(A; M) \cap \mathrm{Coder}^{p,i,j}(C_{\mathcal{P}^!}(W))$. Notice that we have, much as in (36),

$$C^{p,n,0}_{\mathcal{P}}(A; M) = C^{p,n}_{\mathcal{P}_1}(A; M) \text{ and } C^{p,0,n}_{\mathcal{P}}(A; M) = C^{p,n}_{\mathcal{P}_2}(A; M).$$

We may easily verify that

$$d_1(C^{p,i,j}_{\mathcal{P}}(A; M)) \subset C^{p,i+1,j}_{\mathcal{P}}(A; M) \text{ and } d_2(C^{p,i,j+1}_{\mathcal{P}}(A; M)) \subset C^{p,i,j}_{\mathcal{P}}(A; M),$$

which immediately gives the following statement.

10.2 Theorem: *The cohomology $H^{*,*}_{\mathcal{P}}(A; M)$ of a $\mathcal{P}$-algebra A with coefficients in an A-module M, where $\mathcal{P} = \mathcal{P}(E_1, E_2; R_1, \mathcal{D}, R_2)$ is a quadratic operad with a distributivity law, can be computed as the cohomology of the total complex of the bicomplex below*

$$\begin{array}{ccccccccc}
\uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
C^{*,2}_{\mathcal{P}_2}(A; M) & \xrightarrow{d_1} & C^{*,1,2}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & C^{*,2,2}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & C^{*,3,2}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & \\
\uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
C^{*,1}_{\mathcal{P}_2}(A; M) & \xrightarrow{d_1} & C^{*,1,1}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & C^{*,2,1}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & C^{*,3,1}_{\mathcal{P}}(A; M) & \xrightarrow{d_1} & \\
\uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \\
\mathcal{M}^*(A, M) & \xrightarrow[d_1]{} & C^{*,1}_{\mathcal{P}_1}(A; M) & \xrightarrow[d_1]{} & C^{*,2}_{\mathcal{P}_1}(A; M) & \xrightarrow[d_1]{} & C^{*,3}_{\mathcal{P}_1}(A; M) & \xrightarrow[d_1]{} &
\end{array}$$

Notice that the bottom row (resp. the extreme left column) of the above bicomplex computes the cohomology $H^{*,*}_{\mathcal{P}_1}(A; M)$ (resp. $H^{*,*}_{\mathcal{P}_2}(A; M)$).

The bicomplex in the above theorem is the operadic analog of the bicomplex in (10). Having in mind Theorem 8.6, it is useful to quote the following result of the second author [24].

10.3 Theorem: *If the operads $\mathcal{P}_1$, $\mathcal{P}_2$ are Koszul, then the operad $\mathcal{P}$ is Koszul as well.*

10.4 Example: In this example we construct the operadic cohomology of non-symmetric Poisson algebras from examples 6.3 and 9.6. Let $P = (P, \cdot\,, \langle -,-\rangle)$ be such an algebra. A P-module is a graded vector space M equipped with two actions, $\nu, \lambda : P \otimes M \oplus M \otimes P \longrightarrow M$ such that, for all $a, b \in P$ and $x \in M$,

(i) $\nu(a, \nu(b, x)) = \nu(a \cdot b, x)$, $\nu(a, \nu(x, b)) = \nu(\nu(a, x), b)$ and $\nu(\nu(x, a), b) = \nu(x, a \cdot b)$,

(ii) $\lambda(a, \lambda(b, x)) = \lambda(\langle a, b\rangle, x)$, $\lambda(a, \lambda(x, b)) = \lambda(\lambda(a, x), b)$ and $\lambda(\lambda(x, a), b) = \lambda(x, \langle a, b\rangle)$,

(iii) $\lambda(a, \nu(b, x)) = \nu(\langle a, b\rangle, x)$, $\nu(a, \lambda(b, x)) = \lambda(a \cdot b, x)$, $\lambda(a, \nu(x, b)) = \nu(\lambda(a, x), b)$, $\nu(a, \lambda(x, b)) = \lambda(\nu(a, x), b)$, $\lambda(x, a \cdot b) = \nu(\lambda(x, a), b))$ and $\nu(x, \langle a, b\rangle) = \lambda(\nu(x, a), b))$.

As we already observed in Example 9.6, the category of such algebras is Koszul self-dual, so $\mathcal{P}^! = \mathcal{P}$ for the corresponding operad. The distributive law gives the isomorphism $C_{\mathcal{P}^!}(W) \cong C_{\mathcal{P}_1^!}(C_{\mathcal{P}_2^!}(W))$ of (35) and, because $\mathcal{P}_i = \mathcal{P}_i^! = Ass$, for $i = 1, 2$, $C_{\mathcal{P}_1^!}(C_{\mathcal{P}_2^!}(W)) \cong T(T(W))$, the "double" tensor coalgebra. Notice the perfect symmetry $C_{\mathcal{P}_1^!}(C_{\mathcal{P}_2^!}(W)) \cong C_{\mathcal{P}_2^!}(C_{\mathcal{P}_1^!}(W))$ (see also the comments in Example 6.3).

The observation above makes it clear that a typical element of the space $C^n_{\mathcal{P}^!}(W)$ is of the form $w_1 \circ_1 w_2 \circ_2 \cdots \circ_{n-1} w_n$, where $\circ_i \in \{*, \star\}$ with $*$ corresponding, via the Koszul duality, to $\langle -,-\rangle$ and $\star$ corresponding to $\cdot$. The space $C^{\alpha,\beta}_{\mathcal{P}^!}(W)$ of (34) then consists to those elements of $C^{\alpha+\beta}_{\mathcal{P}^!}(W)$ for which the cardinality of the set $\{i;\ \circ_i = *\}$ is $\alpha - 1$.

To make life easier, we suppose P to be trivially graded, so we may neglect the internal degrees and identify $C^{n+1}_{\mathcal{P}}(P, M)$ with $\mathcal{M}(C^n_{\mathcal{P}^!}(P), M)$ using (27). For $f \in C^{n-1}_{\mathcal{P}}(P, M)$ the differential d is then given by the formula

$$df(a_0 \circ_0 \cdots \circ_{n-1} a_n) = \sum_{j=0}^{n+1} (-1)^j d^j f(a_0 \circ_0 \cdots \circ_{n-1} a_n)$$

where

$$d^0 f(a_0 \circ_0 \cdots \circ_{n-1} a_n) = \begin{cases} \lambda(a_0, f(a_1 \circ_1 \cdots \circ_{n-1} a_n)), & \text{if } \circ_0 = * \\ \nu(a_0, f(a_1 \circ_1 \cdots \circ_{n-1} a_n)), & \text{if } \circ_0 = \star \end{cases}$$

$$d^j f(a_0 \circ_0 \cdots \circ_{n-1} a_n) = \begin{cases} f(a_0 \circ_0 \cdots \langle a_{j-1}, a_j\rangle \cdots \circ_{n-1} a_n), & \text{if } \circ_j = * \\ f(a_0 \circ_0 \cdots a_{j-1} \cdot a_j \cdots \circ_{n-1} a_n), & \text{if } \circ_j = \star \end{cases}$$

$$d^{n+1} f(a_0 \circ_0 \cdots \circ_{n-1} a_n) = \begin{cases} \lambda(f(a_0 \circ_0 \cdots \circ_{n-2} a_{n-1}), a_n), & \text{if } \circ_{n-1} = * \\ \nu(f(a_0 \circ_0 \cdots \circ_{n-2} a_{n-1}), a_n), & \text{if } \circ_{n-1} = \star \end{cases}$$

with $1 \leq j \leq n$. The differential d thus defined obviously decomposes as $d = d_1 + d_2$ where d_1 (resp. d_2) corresponds to the first (resp. second) cases of the formulas above. This decomposition reflects the bicomplex description of the cohomology as predicted by Theorem 10.2.

In the first draft of the paper we did some computations related to the cohomology of Poisson algebras (Example 9.4), but we found them rather technical and not very stimulating, hence we decided not to include them in this revision of the paper. The computations are available on request from the second author.

11 Bialgebras (operadic approach)

In this section we are going to develop an operadic definition of various types of bialgebras and prove a technical statement about induced structures (Theorem 11.11). The conceptual trouble here is related to the fact that bialgebras are not algebras over some operad (the presence of co-algebraic operations makes this impossible), but rather algebras over PROPs [19] or, more precisely, over their **k**-linear versions called *theories* in [21]. A theory is a system $\{\mathbf{A}(m,n)\}_{m,n\geq 1}$ of **k**-linear spaces together with some operations of composition and an action of the symmetric group. We may view a $\mathbf{A}(m,n)$ as a space of operations with m "inputs" and n "outputs." Roughly speaking, an operad is then a theory generated by $\{\mathbf{A}(m,1)\}_{m\geq 1}$ while a cooperad is a theory generated by $\{\mathbf{A}(1,m)\}_{n\geq 1}$.

Suppose now that $\mathcal{P}$ and $\mathcal{Q}$ are two operads. Temporarily denote by $\mathbf{F} = \{\mathbf{F}(m,n)\}_{m,n\geq 1}$ the free theory generated by $\mathcal{P}$ and by the cooperad $\#\mathcal{Q}$. A *mixed distributive law* is a relation M in $\mathbf{F}$ of a special type. A bialgebra is then an algebra over the quotient-theory $\mathbf{F}/(M)$. We now give precise meanings for these notions without referring to theories.

Let $N, s_1, \ldots, s_n, t_1, \ldots, t_m$ be natural numbers, $s_1 + \cdots s_n = t_1 + \cdots + t_m = N$. Then $\Sigma_{s_1} \times \cdots \times \Sigma_{s_n}$ and $\Sigma_{t_1} \times \cdots \times \Sigma_{t_m}$ are subgroups of Σ_N, hence we may consider $\mathbf{k}[\Sigma_N]$ as a $\Sigma_{s_1} \times \cdots \times \Sigma_{s_n}$- $\Sigma_{t_1} \times \cdots \times \Sigma_{t_m}$-bimodule. Also $\mathcal{P}(t_1) \otimes \cdots \otimes \mathcal{P}(t_m)$ (resp. $\mathcal{Q}(s_1) \otimes \cdots \otimes \mathcal{Q}(s_n)$) is a natural left $\Sigma_{t_1} \times \cdots \times \Sigma_{t_m}$ (resp. right $\Sigma_{s_1} \times \cdots \times \Sigma_{s_n}$) module and it makes sense to form the product

$$\mathcal{Q}(s_1) \otimes \cdots \otimes \mathcal{Q}(s_n) \otimes_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_n}} \mathbf{k}[\Sigma_N] \otimes_{\Sigma_{t_1} \times \cdots \times \Sigma_{t_m}} \mathcal{P}(t_1) \otimes \cdots \otimes \mathcal{P}(t_m).$$

We simplify the notation by writing

$$- \mathrel{\overline{\otimes}} \mathbf{k}[\Sigma_N] \mathrel{\overline{\otimes}} - \ \text{ instead of } \ - \otimes_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_n}} \mathbf{k}[\Sigma_N] \otimes_{\Sigma_{t_1} \times \cdots \times \Sigma_{t_m}} -.$$

By a *mixed distributive law* (between $\mathcal{P}$ and $\mathcal{Q}$) we mean a sequence

$$M = \{M(m,n)\}_{m,n\geq 1}$$

of maps

$$\begin{aligned} (38) \qquad M = M(m,n) : \mathcal{P}(m) \otimes \mathcal{Q}(n) \longrightarrow \\ \bigoplus \{\mathcal{Q}(t_1) \otimes \cdots \otimes \mathcal{Q}(t_m) \mathrel{\overline{\otimes}} \mathbf{k}[\Sigma_N] \mathrel{\overline{\otimes}} \mathcal{P}(s_1) \otimes \cdots \otimes \mathcal{P}(s_n)\}, \end{aligned}$$

where the summation is taken over all $N \geq 1$ and $s_1 + \cdots + s_n = t_1 + \cdots + t_m = N$. The spaces above are in fact subspaces of $\mathbf{F}(m,n)$, so we may understand the latter as things having m "inputs" and n "outputs." To make this even more mnemonic, we write the target space on the right-hand side as

$$(39) \qquad \bigoplus \begin{pmatrix} \mathcal{Q}(t_1) \\ \vdots \\ \mathcal{Q}(t_m) \end{pmatrix} \mathrel{\overline{\otimes}} \mathbf{k}[\Sigma_N] \mathrel{\overline{\otimes}} \begin{pmatrix} \mathcal{P}(s_1) \\ \vdots \\ \mathcal{P}(s_n) \end{pmatrix}.$$

The map M, considered as a map of theories, must satisfy some conditions which mean that it is compatible with the structure maps of the theory. We formulate these conditions in operadic terms.

First, we have an obvious action of the symmetric group Σ_m "on inputs" of $\mathcal{P}(m) \otimes \mathcal{Q}(n)$ given by $(\sigma, p \otimes q) \mapsto \sigma p \otimes q$, where the action of Σ_m on $\mathcal{P}(m)$ is the one given by the operad structure. Similarly we have an action of Σ_N on the "inputs" of the space in (39) given as follows: For $\sigma \in \Sigma_m$, let $\sigma^{-1}_{t_1,\dots,t_n} \in \Sigma_N$ be the permutation sending $(1,\dots,N)$ to

$$(t_1+\cdots+t_{\sigma(1)-1}+1,\dots,t_1+\cdots+t_{\sigma(1)},\dots,t_1+\cdots+t_{\sigma(m)-1}+1,\dots,t_1+\cdots+t_{\sigma(m)}).$$

We may think of $\sigma^{-1}_{t_1,\dots,t_n}$ as of the permutation permuting the blocks

$$(1,\dots,t_1)(t_1+1,\dots,t_1+t_2)\dots(t_1+\cdots+t_{m-1}+1,\dots,N)$$

via σ^{-1}. The action of σ is then given by

$$(\sigma,\begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix} \mathbin{\overline{\otimes}} \mu \mathbin{\overline{\otimes}} \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}) \longmapsto$$

$$\epsilon(\sigma; q_1,\dots,q_m)\cdot \begin{pmatrix} q_{\sigma(1)} \\ \vdots \\ q_{\sigma(m)} \end{pmatrix} \mathbin{\overline{\otimes}} \sigma^{-1}_{t_1,\dots,t_n}\mu \mathbin{\overline{\otimes}} \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

The action of Σ_n "on outputs" is defined in a similar way. The first condition we require is that the map M is invariant with respect to both actions.

The second conditions is the compatibility with composition maps. We formulate it again first "for inputs". Suppose that $p \in \mathcal{P}(m)$ is of the form $p = \mu(\nu_1,\dots,\nu_k)$, with $\mu \in \mathcal{P}(k)$, $\nu_i \in \mathcal{P}(l_i)$ and $\sum l_i = m$. Let us write, for $q \in \mathcal{Q}(n)$

$$M(\mu \otimes q) = \sum_i \begin{pmatrix} q^i_1 \\ \vdots \\ q^i_k \end{pmatrix} \mathbin{\overline{\otimes}} \sigma^i \mathbin{\overline{\otimes}} \begin{pmatrix} p^i_1 \\ \vdots \\ p^i_n \end{pmatrix}.$$

The second condition then says that

$$M(p\otimes q) = \sum_i \xi^i \cdot \begin{pmatrix} M(\nu_1, q^i_1) \\ \vdots \\ M(\nu_k, q^i_k) \end{pmatrix} \otimes \sigma^i \otimes \begin{pmatrix} p^i_1 \\ \vdots \\ p^i_n \end{pmatrix}, \tag{40}$$

with $\xi^i = (-1)^{(|\nu_2|\cdot|q^i_1|+|\nu_3|\cdot(|q^i_1|+|q^i_2|)+\cdots+|\nu_k|\cdot(|q^i_1|+\cdots+|q^i_{k-1}|))}$, and similarly for "outputs." A little caution is needed to interpret formula (40) properly, because the right-hand side is not of the form required in (38). If $1 \le j \le k$ and $q^i_j = \mathcal{Q}(s^i_j)$ then each $M(\nu_j, q^i_j)$ is a sum of elements of the form

$$\begin{pmatrix} q^{i,j}_1 \\ \vdots \\ q^{i,j}_{l_j} \end{pmatrix} \mathbin{\overline{\otimes}} \sigma^{i,j} \mathbin{\overline{\otimes}} \begin{pmatrix} p^{i,j}_1 \\ \vdots \\ p^{i,j}_{s^i_j} \end{pmatrix}.$$

To interpret formula (40) we must compose the "outputs"

$$\begin{pmatrix} p^{i,j}_1 \\ \vdots \\ p^{i,j}_{s_j} \end{pmatrix}$$

of $M(\nu_j, q_j^i)$ with the elements of the "output" column

$$\begin{pmatrix} p_1^i \\ \vdots \\ p_n^i \end{pmatrix}$$

of $M(\mu \otimes q)$ via the permutation σ^i using the structure maps of the operad $\mathcal{Q}$. We suggest that the reader look at formula (43) for an explicit and easy example of this kind of manipulation. Let us sum up the above remarks in a compact definition.

11.1 Definition: We say that a system $\{M = M(m,n)\}_{m,n\geq 1}$ of maps as in (38) is a mixed distributivity law if:

(i) it is compatible with the symmetric group actions both on the "input" and "output" sides

(ii) it is compatible with the compositions both on the "input" and the "output" sides.

Suppose now that our operads $\mathcal{P}$ and $\mathcal{Q}$ are quadratic, $\mathcal{P} = \langle E; R\rangle$, $\mathcal{Q} = \langle F; S\rangle$.

11.2 Proposition: *A mixed distributive law for quadratic operads is uniquely determined by its component $M(2,2)$ (which we denote $\mathcal{R}$),*

$$\mathcal{R} : E \otimes F \longmapsto \bigoplus \begin{pmatrix} \mathcal{Q}(t_1) \\ \mathcal{Q}(t_2) \end{pmatrix} \overline{\otimes}\, \mathbf{k}[\Sigma_N]\, \overline{\otimes} \begin{pmatrix} \mathcal{P}(s_1) \\ \mathcal{P}(s_2) \end{pmatrix}. \tag{41}$$

On the other hand, a map $\mathcal{R}$ as above determines a mixed distributive law if and only if it is compatible with the Σ_2-actions and if the obvious extensions of $\mathcal{R}$ to $R \otimes F$ and $E \otimes S$ are zero.

Proof: The compatibility condition (40) enables us to express the mixed distributivity law M on $\mathcal{P}(m) \otimes \mathcal{Q}(n)$ via its values on the space of generators, i.e. via $\mathcal{R} = M(2,2)$.

On the other hand, given $\mathcal{R}$, we may extend it inductively to a mixed distributive law between the free operads $\mathcal{F}(E)$ and $\mathcal{F}(F)$. It is not hard to see that this extension induces a mixed distributive law between $\mathcal{P}$ and $\mathcal{Q}$ if and only if it sends $R \otimes F$ and $E \otimes S$ to zero. □

11.3 Definition: We call a map $\mathcal{R}$ having the properties stated in the previous proposition a *replacement rule*.

Let $\mathcal{P}$ and $\mathcal{Q}$ be quadratic operads and $\mathcal{R}$ a replacement rule between $\mathcal{P}$ and $\mathcal{Q}$. Suppose we have a graded vector space B, a $\mathcal{P}$-algebra structure $a : \mathcal{P} \longrightarrow \mathcal{E}_B$ and a $\mathcal{Q}$-coalgebra structure $c : \mathcal{Q} \longrightarrow \mathcal{G}_B$ on B. These two structures give rise to a natural map $J_l : E \otimes F \longrightarrow \mathcal{M}(B^{\otimes 2}, B^{\otimes 2})$ given by $e \otimes f \mapsto c(f) \circ a(e)$. Similarly, they define another natural map

$$J_r : \begin{pmatrix} \mathcal{Q}(t_1) \\ \mathcal{Q}(t_2) \end{pmatrix} \overline{\otimes}\, \mathbf{k}[\Sigma_N]\, \overline{\otimes} \begin{pmatrix} \mathcal{P}(s_1) \\ \mathcal{P}(s_2) \end{pmatrix} \longrightarrow \mathcal{M}^*(B^{\otimes 2}, B^{\otimes 2})$$

given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \overline{\otimes}\, \sigma\, \overline{\otimes} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longmapsto (c(q_1) \otimes c(q_2)) \circ \sigma \circ (a(p_1) \otimes a(p_2)),$$

where we interpret $\sigma \in \mathbf{k}[\Sigma_N]$ as a map from $B^{\otimes n}$ onto itself given by

$$(b_1, \dots, b_n) \mapsto \epsilon(\sigma; b_1, \cdots, b_n) \cdot (b_{\sigma(1)}, \dots, b_{\sigma(n)}).$$

11.4 Definition: Let $\mathcal{P}$ and $\mathcal{Q}$ be quadratic operads and let $\mathcal{R}$ be a replacement rule between $\mathcal{P}$ and $\mathcal{Q}$. We say that $B = (B, a, c)$ is a $\mathcal{P}$-$\mathcal{Q}$-bialgebra if

(i) $a : \mathcal{P} \longrightarrow \mathcal{E}_B$ is a $\mathcal{P}$-algebra structure on B.

(ii) $c : \mathcal{Q} \longrightarrow \mathcal{G}_B$ is a $\mathcal{Q}$-coalgebra structure on B.

(iii) $J_l(e \otimes f) = J_r(\mathcal{R}(e \otimes f))$ in $\mathcal{M}(B^{\otimes 2}, B^{\otimes 2})$.

11.5 Example: In this example we discuss associative/coassociative bialgebras of Example 6.4 from the operadic point of view. Let $Ass = \langle E; R\rangle$ be the associative algebra operad, i.e. E is the free Σ_2-space generated by one symbol μ and R is the ideal generated by the associativity relation $\mu(\mu, 1) - \mu(1, \mu) = 0$. Define

$$\mathcal{R} : E \otimes E \longmapsto \begin{pmatrix} E \\ E \end{pmatrix} \overline{\otimes} \mathbf{k}[\Sigma_4] \overline{\otimes} \begin{pmatrix} E \\ E \end{pmatrix}$$

by

$$\mathcal{R}(\mu \otimes \mu) = \begin{pmatrix} \mu \\ \mu \end{pmatrix} \overline{\otimes} T_{1324} \overline{\otimes} \begin{pmatrix} \mu \\ \mu \end{pmatrix}. \tag{42}$$

Here T_{1324} denotes the map $B^{\otimes 4} \longrightarrow B^{\otimes 4}$ induced by the permutation $(1234) \mapsto (1324)$. Let us verify that it satisfies the conditions of Proposition 11.2. The Σ_2-Σ_2-equivariance of $\mathcal{R}$ is clear. For the extension M of R we have

$$\begin{aligned} M(\mu(\mu,1) \otimes \mu) &= \begin{pmatrix} M(\mu \otimes \mu) \\ \mu \end{pmatrix} \overline{\otimes} T_{1324} \overline{\otimes} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \\ &= \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \overline{\otimes}\, T_{135246}\, \overline{\otimes} \begin{pmatrix} \mu(\mu,1) \\ \mu(\mu,1) \end{pmatrix}, \end{aligned} \tag{43}$$

because $M(\mu \otimes \mu) = \mathcal{R}(\mu \otimes \mu)$ and we expanded $\mathcal{R}(\mu \otimes \mu)$ as in (42). Similarly,

$$M(\mu(1,\mu) \otimes \mu) = \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \overline{\otimes}\, T_{135246}\, \overline{\otimes} \begin{pmatrix} \mu(1,\mu) \\ \mu(1,\mu) \end{pmatrix}.$$

This computation shows that M takes $R \otimes E$ to zero, and similarly for $E \otimes R$. The corresponding *Ass*-*Ass*-bialgebras are bialgebras in the usual sense. We leave to the reader to verify that the same replacement rule defines *Comm*-*Ass*-bialgebras, *Ass*-*Comm*-bialgebras and even *Comm*-*Comm*-bialgebras (the latter are rather boring, due to the Structure Theorem).

11.6 Example: Here we discuss Lie bialgebras of Example 6.5 as bialgebras given by a replacement rule. Let $Lie = \langle E; R\rangle$ be the Lie algebra operad; let ℓ be an antisymmetric element generating E. Then R is generated by the Jacobi identity. Let us define the replacement rule

$$\mathcal{R} : E \otimes E \longmapsto$$
$$\begin{pmatrix} E \\ \mathbf{k} \end{pmatrix} \overline{\otimes} \mathbf{k}[\Sigma_3] \overline{\otimes} \begin{pmatrix} \mathbf{k} \\ E \end{pmatrix} \oplus \begin{pmatrix} E \\ \mathbf{k} \end{pmatrix} \overline{\otimes} \mathbf{k}[\Sigma_3] \overline{\otimes} \begin{pmatrix} E \\ \mathbf{k} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{k} \\ E \end{pmatrix} \overline{\otimes} \mathbf{k}[\Sigma_3] \overline{\otimes} \begin{pmatrix} E \\ \mathbf{k} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{k} \\ E \end{pmatrix} \overline{\otimes} \mathbf{k}[\Sigma_3] \overline{\otimes} \begin{pmatrix} \mathbf{k} \\ E \end{pmatrix}$$

between *Lie* and *Lie* by

$$\mathcal{R}(\ell,\ell) = \binom{\ell}{1}\overline{\otimes}1\overline{\otimes}\binom{1}{\ell} \oplus \binom{\ell}{1}\overline{\otimes}T_{132}\overline{\otimes}\binom{\ell}{1} \oplus \binom{1}{\ell}\overline{\otimes}1\overline{\otimes}\binom{\ell}{1} \oplus \binom{1}{\ell}\overline{\otimes}T_{213}\overline{\otimes}\binom{1}{\ell}.$$

We leave it to the reader to verify that this really does define a mixed distributive law (the computation is rather long, involving many terms).

11.7 Example: In this example we describe a nonsymmetric analog of Lie bialgebras. Let $Ass = \langle E; R\rangle$ be the associative algebra operad as in Example 11.5 and define the replacement rule

$$\mathcal{R} : E\otimes E \longmapsto \left(\begin{array}{c} E \\ \mathbf{k} \end{array}\right)\overline{\otimes}\mathbf{k}[\Sigma_3]\overline{\otimes}\left(\begin{array}{c} \mathbf{k} \\ E \end{array}\right) \oplus \left(\begin{array}{c} \mathbf{k} \\ E \end{array}\right)\overline{\otimes}\mathbf{k}[\Sigma_3]\overline{\otimes}\left(\begin{array}{c} E \\ \mathbf{k} \end{array}\right)$$

between *Ass* and *Ass* by

$$\mathcal{R}(\mu,\mu) = \left(\begin{array}{c} \mu \\ 1 \end{array}\right)\overline{\otimes}1\overline{\otimes}\left(\begin{array}{c} 1 \\ \mu \end{array}\right) \oplus \left(\begin{array}{c} 1 \\ \mu \end{array}\right)\overline{\otimes}1\overline{\otimes}\left(\begin{array}{c} \mu \\ 1 \end{array}\right).$$

We call these *Ass*-*Ass*-bialgebras *mock bialgebras.* So, a mock bialgebra is a vector space B with an associative multiplication $\cdot$ and a coassociative comultiplication Δ such that

$$\Delta(x\cdot y) = \sum x_{(1)}\otimes x_{(2)}\cdot y + x\cdot y_{(1)}\otimes y_{(2)}.$$

Mock bilagebras will illustrate the necessity of the homogeneity assumption in Theorem 11.11.

11.8 Example: In this example we discuss Poisson bialgebras already introduced in Example 6.6. Let $Poiss = \mathcal{P}(0,0) = \langle E; R\rangle$ be the operad for Poisson algebras, let ℓ (resp. μ) denote the antisymmetric (resp. symmetric) generator of E, see Example 9.4. Let $Ass = \langle F; S\rangle$ be the operad for associative algebras, let ν be the generator of F. Define the replacement rule

$$\mathcal{R} : E\otimes F \longmapsto \left(\begin{array}{c} F \\ F \end{array}\right)\overline{\otimes}\mathbf{k}[\Sigma_4]\overline{\otimes}\left(\begin{array}{c} E \\ E \end{array}\right)$$

between *Poiss* and *Ass* by

$$\mathcal{R}(\ell\otimes\nu) = \left(\begin{array}{c} \nu \\ \nu \end{array}\right)\overline{\otimes}T_{1324}\overline{\otimes}\left(\begin{array}{c} \ell \\ \mu \end{array}\right) \oplus \left(\begin{array}{c} \nu \\ \nu \end{array}\right)\overline{\otimes}T_{1324}\overline{\otimes}\left(\begin{array}{c} \mu \\ \ell \end{array}\right)$$

while $\mathcal{R}(\mu\otimes\nu)$ is defined by the same formula as in the bialgebra case, see Example 11.5.

11.9 Definition: We say that a replacement rule $\mathcal{R}$ is *homogeneous* if, on the right-hand side of (41), $t_1 = t_2$ and $s_1 = s_2$.

Observe that $\mathcal{R}$ is homogeneous if and only if on the right-hand side of (38) we always have $t_1 = t_2 = \cdots = t_m$ and $s_1 = s_2 = \cdots = s_n$. The replacement rules in examples 11.5 and 11.8 are homogeneous, the replacement rules in examples 11.6 and 11.7 are not.

11.10 Theorem: *Let B be a $\mathcal{P}$-$\mathcal{Q}$-bialgebra. Then* *(i) The $\mathcal{Q}$-coalgebra $C_{\mathcal{Q}}(B)$ has a natural $\mathcal{P}$-algebra structure. Moreover, in the homogeneous case this $\mathcal{P}$-algebra structure is such that each $C^n_{\mathcal{Q}}(B)$ forms a subalgebra and the canonical map $c^n : B \mapsto C^n_{\mathcal{Q}}(B)$ is a $\mathcal{P}$-algebra homomorphism for any $n \geq 1$.*

(ii) The $\mathcal{P}$-algebra $A_{\mathcal{P}}(B)$ has a natural $\mathcal{Q}$-coalgebra structure such that the canonical map $a : A_{\mathcal{P}}(B) \to B$ is a $\mathcal{Q}$-coalgebra homomorphism.

Proof: Let us prove (i). To define a $\mathcal{P}$-algebra structure on $C_{\mathcal{Q}}(B)$ means to define, for each $m, n \geq 1$, a map

$$(44) \quad \mathcal{P}(m)\otimes(\bigoplus \#\mathcal{Q}(t_1)\otimes B^{\otimes t_1})^{\Sigma_{t_1}}\otimes\cdots\otimes(\bigoplus \#\mathcal{Q}(t_m)\otimes B^{\otimes t_m})^{\Sigma_{t_m}}) \longrightarrow \\ \longrightarrow (\bigoplus \#\mathcal{Q}(n) \otimes B^{\otimes n})^{\Sigma_n}.$$

The mixed distributive law gives, after a proper dualization, the map

$$\mathcal{P}(m) \otimes (\bigoplus \#\mathcal{Q}(t_1) \otimes B^{\otimes t_1})^{\Sigma_{t_1}} \otimes \cdots \otimes (\bigoplus \#\mathcal{Q}(t_m) \otimes B^{\otimes t_m})^{\Sigma_{t_m}} \longrightarrow \\ \longrightarrow \bigoplus \#\mathcal{Q}(n) \otimes (\bigoplus \mathcal{P}(s_1) \otimes B^{\otimes s_1})_{\Sigma_{s_1}} \otimes \cdots \otimes (\bigoplus \mathcal{Q}(s_n) \otimes B^{\otimes s_n})_{\Sigma_{s_n}},$$

while the $\mathcal{P}$-algebra structure on B gives, for any $1 \leq i \leq n$, a map $(\mathcal{P}(s_i)\otimes B^{\otimes s_i})_{\Sigma_{s_i}} \to B$. The composition of these two maps give the requisite map of (44). We leave the verification of the desired properties to the reader. The proof of (ii) is the verbatim dual of the arguments above. □

The theorem we have just proven relates the operadic notion of a replacement rule with the triple definition of a mixed distributive law; see Definition 5.1.

11.11 Theorem: *Let B be a $\mathcal{P}$-$\mathcal{Q}$-bialgebra.*

(i) If the replacement rule is homogeneous, then both $C_{\mathcal{P}}(B)$ and $A_{\mathcal{Q}^!}(B)$ have natural (B, a)-module structures.

(ii) Both $A_{\mathcal{P}}(B)$ and $A_{\mathcal{Q}^!}(B)$ have natural (B, c)-comodule structures.

Proof: Let us prove (i). Observe first that the $\mathcal{P}$-algebra structure on $C^n_{\mathcal{P}}(B)$ of Theorem 11.10 induces, via the canonical map $c^n : B \to C^n_{\mathcal{Q}}(B)$, a (B, a)-module structure on $C^n_{\mathcal{P}}(B)$ for any $n \geq 1$. This gives rise to an obvious (B, a)-module structure on the direct sum $C_{\mathcal{P}}(B) = \bigoplus_{n\geq 1} C^n_{\mathcal{P}}(B)$. The homogeneity assumption was necessary to build up the action from the partial actions on the homogeneous parts, otherwise there is no way to guarantee the convergence of such an action, see Example 11.14.

The existence of a (B, a)-module structure on $A_{\mathcal{Q}^!}(B)$ is a much more delicate matter. To fix the notation, let $\mathcal{Q} = \langle F; S\rangle$. The first observation is that a replacement rule $\mathcal{R}$ between $\mathcal{P}$ and $\mathcal{Q}$ induces a mixed distributive law between $\mathcal{P}$ and $\mathcal{F}(F) = \langle F; 0\rangle$, the free operad on F, which is homogeneous if $\mathcal{R}$ is. We thus have a (B, a)-module structure on $C_{\mathcal{F}(F)}(B)$. The second important observation is that there is a canonical isomorphism of graded spaces $C^*_{\mathcal{F}(F)}(B) \cong A^*_{\mathcal{F}(\#F)}(B)$, therefore we have a natural (B, a)-module structure also on $A_{\mathcal{F}(\#F)}(B)$. The third observation is that the algebra $A_{\mathcal{Q}^!}(B)$ is a quotient of $A_{\mathcal{F}(\#F)}(B)$. So it remains to prove that the (B, a)-module structure on $A_{\mathcal{F}(\#F)}(B)$ induces a (B, a)-module structure on the quotient $A_{\mathcal{Q}^!}(B)$. We leave this rather technical verification to the reader. The proof of the second half of the theorem is an exact dual. The homogeneity assumption is not needed here, because the action always 'converges', see Example 11.14. □

In the following examples we focus our attention on the module structure on $A_{\mathcal{Q}^!}(B)$ which, we think, has not yet been observed except in the bialgebra case where it is rather obvious, as we will see in the next example.

11.12 Example: Let us begin with the classical bialgebras (see examples 6.4 and 11.5). Here $\mathcal{P} = \mathcal{Q} = Ass$, therefore $\mathcal{Q}^!$ is again Ass and the free $\mathcal{Q}^!$-algebra $A_{\mathcal{Q}^!}(B)$ on a vector space B is the usual tensor algebra, $A_{\mathcal{Q}^!}(B) = T(B) = \bigoplus_{k\geq 1} \bigotimes^k B$. Let $B = (B, \mu, \Delta)$ be a bialgebra. We introduce the *iterated diagonal* $\Delta^{[n]} : B \longrightarrow B^{\otimes n}$, $n \geq 1$, by $\Delta^{[1]} = 1\!\!1$ and $\Delta^{[n+1]} = (\Delta \otimes 1\!\!1^{\otimes n-1}) \circ \Delta^{[n]}$. Using the Sweedler notation $\Delta^{[n]}(b) = \sum b_{(1)} \otimes \cdots \otimes b_{(n)}$ we define a left action $\bullet : B \otimes T(B) \longrightarrow T(B)$ by

$$b \bullet (x_1 \otimes \cdots \otimes x_n) = \sum b_{(1)} \cdot x_1 \otimes \cdots \otimes b_{(n)} \cdot x_n,$$

where $\cdot$ denotes the multiplication μ. The right action is defined in a similar way and these together give a (B, μ)-module structure on $T(B)$.

11.13 Example: A bit less trivial is the case of *Ass*-*Comm*-bialgebras. We have $\mathcal{Q}^! = Lie$ and $A_{\mathcal{Q}^!}(B)$ is the free Lie algebra $L(B)$ on B. Let $B = (B, \mu, \Delta)$ be an *Ass*-*Comm*-bialgebra (i.e. a cocommutative bialgebra). Let us define a left action $\bullet$ of (B, a) on $L(B)$ inductively by saying that $b \bullet u = b \cdot u$ for $b, u \in B$ while

$$b \bullet [u, v] = \sum [b_{(1)} \bullet u, b_{(2)} \bullet v]$$

for $u, v \in L(B)$. The right action is defined by a similar rule. The well-definedness of the action means that it preserves the antisymmetry (this is almost obvious) and the ideal generated by the Jacobi identity. Let us inspect how $b \in B$ acts on it! We have, for $u, v, w \in B$,

$$\begin{aligned}
&b \bullet ([u, [v, w]] + [v, [w, u]] + [w, [u, v]]) = \\
&\quad = [b_{(1)} \cdot u, [b_{(2)} \cdot v, b_{(3)} \cdot w]] + [b_{(1)} \cdot v, [b_{(2)} \cdot w, b_{(3)} \cdot u]] + [b_{(1)} \cdot w, [b_{(2)} \cdot u, b_{(3)} \cdot v]] \\
&\quad = [b_{(1)} \cdot u, [b_{(2)} \cdot v, b_{(3)} \cdot w]] + [b_{(2)} \cdot v, [b_{(3)} \cdot w, b_{(1)} \cdot u]] + [b_{(3)} \cdot w, [b_{(1)} \cdot u, b_{(2)} \cdot v]] = 0.
\end{aligned}$$

Here the cocommutativity of Δ which implies that

$$\sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)} = \sum b_{(\sigma(1))} \otimes b_{(\sigma(2))} \otimes b_{(\sigma(3))}$$

for any $\sigma \in \Sigma_3$ is absolutely crucial.

11.14 Example: Let (B, μ, Δ) be the mock bialgebra introduced in Example 11.7. As in Example 11.5, $\mathcal{P} = \mathcal{Q} = \mathcal{Q}^!$ and $A_{\mathcal{Q}^!}(B) = T(B)$, the tensor algebra. Because of the nonhomogeneity of the replacement rule, this example is really weird, and we describe first the $\mathcal{P} =$ associative algebra structure on $C_{\mathcal{Q}}(B) = T(B)$. It is given by the multiplication $\star : T(B) \otimes T(B) \longrightarrow T(B)$ defined by

$$(x_1 \otimes \cdots \otimes x_n) \star (y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots \otimes x_n \cdot y_1 \otimes \cdots \otimes y_m,$$

for $(x_1 \otimes \cdots \otimes x_n) \in T^n(B)$ and $(y_1 \otimes \cdots \otimes y_m) \in T^m(B)$. If we try to blindly repeat the construction of a (B, a)-module structure on $A_{\mathcal{Q}^!}(B) = T(B)$ as it is given in the proof of Theorem 11.11, we arrive at the formula

$$b \bullet (x_1 \otimes \cdots \otimes x_n) = b \cdot x_1 \otimes \cdots \otimes x_n + \\ + \sum b_{(1)} \otimes b_{(2)} \cdot x_1 \otimes \cdots \otimes x_n + \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)} \cdot x_1 \otimes \cdots \otimes x_n + \cdots,$$

which is not an element of $C_{\mathcal{P}}(B) = \bigoplus_{n \geq 1} C^n_{\mathcal{P}}(B)$ but rather of $\prod_{n \geq 1} C^n_{\mathcal{P}}(B)$. We see that the homogeneity is a necessary assumption in Theorem 11.11 (i).

On the other hand, there are no problems with the existence of the structures of Theorem 11.10 (ii). The coalgebra structure, say $\delta : T(B) \longrightarrow T(B) \otimes T(B)$, is given by

$$\delta(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i \leq n} x_1 \otimes (x_i)_{(1)} \otimes (x_i)_{(2)} \otimes \cdots \otimes x_n$$

where we interpret $x_1 \otimes (x_i)_{(1)} \otimes (x_i)_{(2)} \otimes \cdots \otimes x_n$ as an element of $T^i(B) \otimes T^{n-i+1}(B)$. The left (B, c)-comodule structure on $T(B)$, say $\nu_l : T(B) \longrightarrow B \otimes T(B)$, is given by

$$\nu_l(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i \leq n} (x_1 \cdot \cdots \cdot x_i)_{(1)} \otimes (x_1 \cdot \cdots \cdot x_i)_{(2)} \otimes x_{i+1} \cdots \otimes x_n,$$

where we interpret $(x_1 \cdot \cdots \cdot x_i)_{(1)} \otimes (x_1 \cdot \cdots \cdot x_i)_{(2)} \otimes x_{i+1} \cdots \otimes x_n$ as an element of $B \otimes T^{n-i+1}(B)$. The right action is defined in a similar way.

12 Operadic cohomology of bialgebras

In this section we define a cohomology of a $\mathcal{P}$-$\mathcal{Q}$-bialgebra $B = (B, a, c)$, where $\mathcal{P}$ and $\mathcal{Q}$ are quadratic operads with a homogeneous replacement rule $\mathcal{R}$. The homogeneity assumption is not absolutely necessary, but in the general case we do not have a nice definition based on a bicomplex. There are also some problems with the grading because the resulting cohomology is not a direct sum of the homogeneous components but rather the direct product, this phenomenon being already observed on the cohomology of A(m)-algebras [20].

We need first the notion of the operadic cohomology of a coalgebra $C = (C, c)$ with coefficients in a C-comodule N. The definition is exactly the dual of the definition of the operadic cohomology of an algebra with coefficients in a module (Definition 8.3). We thus merely sketch the definition.

Let $A = (A, a)$ be a $\mathcal{P}$-algebra and $\theta : A \longrightarrow A$ a homogeneous degree p linear map. We say that θ is a *degree p derivation* if, for any $n \geq 2$ and $\mu \in \mathcal{P}(n)$,

$$\theta \circ a(n)(\mu) = (-1)^{|\mu| \cdot |\theta|} \cdot \sum_{0 \leq i \leq n-1} a(n)(\mu) \circ (\mathbb{1}^{\otimes i} \otimes \theta \otimes \mathbb{1}^{\otimes (n-i-1)}).$$

Let us denote by $\mathrm{Der}^p(A)$ the space of all degree p derivations of the algebra A. Recall that, for a graded vector space W, $A_{\mathcal{P}}(W)$ denotes the free $\mathcal{P}$-algebra on W. As in the case of coderivations, $\mathrm{Der}^p(A_{\mathcal{P}}(W))$ has a second grading, namely

$$\mathrm{Der}^{p,n}(A_{\mathcal{P}}(W)) = \{\theta \in \mathrm{Der}^p(A_{\mathcal{P}}(W));\ \theta(W) \subset A^{n+1}_{\mathcal{P}}(W)\}.$$

We have the following analog of Theorem 8.2.

12.1 Theorem: *Let V be a graded vector space and let $W = \uparrow V$. Then there exists a one-to -one correspondence between $\mathcal{P}$-coalgebra structures $c : \mathcal{P} \longrightarrow \mathcal{G}_V$ on*

V and degree one derivations $\delta \in \mathrm{Der}^{1,1}(A_{\mathcal{P}^!}(W))$ of the free $\mathcal{P}^!$-coalgebra $A_{\mathcal{P}^!}(W)$ with $\delta^2 = 0$.

Let $C = (C, c)$ be a $\mathcal{P}$-coalgebra. By a *C-comodule* we mean a graded vector space N together with a $\mathcal{P}$-coalgebra structure $\nu : \mathcal{P} \longrightarrow \mathcal{G}_{C\oplus N}$ on $C \oplus N$ such that both the projection $C \oplus N \longrightarrow C$ and the inclusion $C \oplus C \oplus N$ are $\mathcal{P}$-algebra homomorphisms and $\nu(n)(p) : C \oplus N \longrightarrow (C \oplus N)^{\otimes n}$, for each $n \geq 1$ and $p \in \mathcal{P}(n)$, maps N to the subspace of $(C \oplus N)^{\otimes n}$ spanned by monomials which contain exactly one element of N.

Finally, let $C = (C, c)$ be a $\mathcal{P}$-coalgebra and N a C-comodule. Let $V = \uparrow C$, $Y = \uparrow N$ and $Z = V \oplus X$. The decomposition $Z^{\otimes n} = \bigoplus_{i+j=n} (Z^{\otimes n})^{i,j}$, where $(Z^{\otimes n})^{i,j}$ is the subspace spanned by monomials containing i elements of V and j elements of Y, induces the decomposition $A^n_{\mathcal{P}^!}(Z) = \bigoplus_{i+j=n} A^{i,j}_{\mathcal{P}^!}(Z)$ for any $n \geq 1$. Let $D^{p,n}_{\mathcal{P}}(N, C)$ be the subspace of $\mathrm{Der}^{p,n}(A_{\mathcal{P}^!}(Z))$ consisting of those derivations θ for which $\theta(Y) \subset A^{i,0}_{\mathcal{P}^!}(Z)$ and $\theta(V) = 0$. The graded commutator with the derivation $\delta \in \mathrm{Der}^{1,1}(C_{\mathcal{P}^!}(W))$ which corresponds to the coalgebra structure on $C \oplus N$ via the correspondence of Theorem 12.1 induces a differential $\Delta : D^{p,n}_{\mathcal{P}}(N, C) \longrightarrow D^{p+1,n+1}_{\mathcal{P}}(N, C)$. We have the following analog of Definition 8.3.

12.2 Definition: Let C be a coalgebra over a quadratic operad $\mathcal{P}$ and let N be an C-comodule. The cohomology $H^{p,n}_{\mathcal{P}}(N; C)$ of C with coefficients in N is defined as

$$H^{p,n}_{\mathcal{P}}(N; C) := H^{p,n}(D^{*,*}_{\mathcal{P}}(N, C), \Delta).$$

We call p and n the *inner* and *simplicial* degrees, respectively. As in (27) we observe that

$$D^{p,n}_{\mathcal{P}}(N, C) \cong \mathcal{M}^p(\uparrow N, A^{n+1}_{\mathcal{P}^!}(\uparrow C)). \tag{45}$$

Now we proceed to the definition of the operadic bialgebra cohomology of a $\mathcal{P}$-$\mathcal{Q}$-coalgebra $B = (B, a, c)$ with a homogeneous distributive law $\mathcal{R}$. Put

$$C^{p,m,n}_{\mathcal{P},\mathcal{Q}}(B, B) = \mathcal{M}^{p-m-n}(C^{m+1}_{\mathcal{P}^!}(B), A^{n+1}_{\mathcal{Q}^!}(B)).$$

By Theorem 11.10 (i) there exists a (B, a)-module structure on $A^{n+1}_{\mathcal{Q}^!}(B)$ (recall that $\mathcal{R}$ is supposed to be homogeneous) and (27) gives the identification

$$C^{p,m,n}_{\mathcal{P},\mathcal{Q}}(B, B) \cong \mathcal{M}^{p-n}(C^{m+1}_{\mathcal{P}^!}(\downarrow B), \downarrow A^{n+1}_{\mathcal{Q}^!}(B)) = C^{p-n,m}_{\mathcal{P}}(B, A^{n+1}_{\mathcal{Q}^!}(B)). \tag{46}$$

Similarly, by Theorem 11.10 (ii) there exists a (B, c)-comodule structure on the space $C^{m+1}_{\mathcal{P}^!}(B)$ and we have the identification

$$C^{p,m,n}_{\mathcal{P},\mathcal{Q}}(B, B) \cong \mathcal{M}^{p-m}(\uparrow C^{m+1}_{\mathcal{P}^!}(B), A^{n+1}_{\mathcal{Q}^!}(\uparrow B)) = D^{p-m,n}_{\mathcal{Q}}(C^{m+1}_{\mathcal{P}^!}(B), B). \tag{47}$$

We may thus consider the following bicomplex.

$$
\begin{array}{ccccccccc}
\Delta\Big\uparrow & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \\
D^{*,2}_{\mathcal{Q}}(B;B) & \xrightarrow{\nabla} & C^{*,1,2}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & C^{*,2,2}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & C^{*,3,2}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & \\
\Delta\Big\uparrow & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \\
D^{*,1}_{\mathcal{Q}}(B;B) & \xrightarrow{\nabla} & C^{*,1,1}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & C^{*,2,1}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & C^{*,3,1}_{\mathcal{P},\mathcal{Q}}(B,B) & \xrightarrow{\nabla} & \\
\Delta\Big\uparrow & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \Big\uparrow\Delta & & \\
\mathcal{M}^*(B,B) & \xrightarrow[\nabla]{} & C^{*,1}_{\mathcal{P}}(B;B) & \xrightarrow[\nabla]{} & C^{*,2}_{\mathcal{P}}(B;B) & \xrightarrow[\nabla]{} & C^{*,3}_{\mathcal{P}}(B;B) & \xrightarrow[\nabla]{} &
\end{array}
$$

The n-th row of the bicomplex is the complex defining the operadic cohomology of (B,a) with coefficients in $A^{n+1}_{\mathcal{Q}^!}(B)$ (46) and the m-th column is the complex from the definition of the operadic cohomology of the coalgebra (B,c) with coefficients in $C^{m+1}_{\mathcal{P}^!}(B)$ (47). It is not hard to verify that the differentials in the above bicomplex commute, so the following definition makes sense.

12.3 Definition: The cohomology $H^{*,*}_{\mathcal{P},\mathcal{Q}}(B;B)$ of a $\mathcal{P}$-$\mathcal{Q}$-bialgebra $B=(B,a,c)$ is the cohomology of the bicomplex above,

$$H^{p,*}_{\mathcal{P},\mathcal{Q}}(B;B) = H^{p,*}\Big(\bigoplus_{m+n=*} C^{p,m,n}_{\mathcal{P},\mathcal{Q}}(B,B), \nabla+\Delta\Big).$$

12.4 Example: In this example we discuss the cohomology of bialgebras introduced in examples 6.4 and 11.5. We again consider the nongraded case only, so we may neglect the inner grading. The cohomology $H^*_{Ass,Ass}(B,B)$ of associative coassociative bialgebras coincides with the bialgebra cohomology introduced by Gerstenhaber and Schack in [15]. At the (m,n)-position of the bicomplex is then the space $\mathcal{M}(\bigotimes^{m+1}(B),\bigotimes^{n+1}(B))$.

The cohomology of *Ass*-*Comm*-bialgebras, i.e. associative coassociative bialgebras with the cocommutative comultiplication looks very similar except that the space at the (m,n)-position of the bicomplex is $\mathcal{M}(\bigotimes^{m+1}(B), L^{n+1}(B))$, where $L^{n+1}(B)$ is the subspace of the free Lie algebra $L(B)$ on B consisting of elements of length n.

In the dual case of the cohomology of *Comm*-*Ass*-bialgebras the element at the (m,n)-position is $\mathcal{M}(C^{m+1}_{Lie}(B),\bigotimes^{n+1}(B)) \cong \mathcal{M}_{\mathrm{Harr}}(\bigotimes^{m+1}(B),\bigotimes^{n+1}(B))$ where $\mathcal{M}_{\mathrm{Harr}}(\bigotimes^{m+1}(B),\bigotimes^{n+1}(B))$ is the subspace of $\mathcal{M}^*(\bigotimes^{m+1}(B),\bigotimes^{n+1}(B))$ of linear maps which are zero on decomposables of the shuffle product, see [28]. The differential in the last description is given by the restriction of the differential from the Gerstenhaber-Schack complex. A similar bicomplex was used in [25] to describe restricted deformations of triangular quasi-Hopf algebras.

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