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Calculus of multilinear differential operators, operator L_∞-algebras and IBL_∞-algebras ☆

Denis Bashkirov, Martin Markl *

The Czech Academy of Sciences, Institute of Mathematics, Žitná 25, 115 67 Prague 1, Czech Republic

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ABSTRACT

We propose an operadic framework suitable for describing algebraic structures with operations being multilinear differential operators of varying orders or, more generally, formal series of such operators. The framework is built upon the notion of a multifiltration of a linear operad generalizing the concept of a filtration of an associative algebra. We describe a particular way of constructing and analyzing multifiltrations based on a presentation of a linear operad in terms of generators and relations. In particular, that allows us to observe a special role played in this context by Lie, Lie-admissible and $\mathtt{Lie}_\infty$-structures. As a main application, and the original motivation for the present work, we show how a certain generalization of the well-known big bracket construction of Lecomte–Roger and Kosmann-Schwarzbach encompassing the case of homotopy involutive Lie bialgebras can be obtained.

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* Corresponding author.
E-mail addresses: bashkirov@math.cas.cz (D. Bashkirov), markl@math.cas.cz (M. Markl).

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Introduction

Motivation. The deformation complex of a Lie bialgebra V admits a particularly simple description in terms of a certain graded Lie structure, known in the literature as the *big bracket*, supported on the shifted Grassmann algebra $\wedge^\bullet(V \oplus V^*)[2]$, cf. [20,21,24,27]. Geometrically, the latter can be identified with an odd Poisson bracket on the shifted cotangent bundle $T^*V[-1]$ and thus can be constructed from as little input data as a pairing between V and V^*. One of the points of our interest for the present work is an analogous construction for a particular class of Lie bialgebras, namely *involutive Lie bialgebras*, characterized by an additional property of the form $[-,-] \circ \delta = 0$, where $[-,-] : V \otimes V \to V$ and $\delta : V \to V \otimes V$ are the bracket and the cobracket of a Lie bialgebra V respectively. Some of the most notable examples of such algebras include the Goldman–Turaev Lie bialgebra on the vector space of non-trivial free homotopy classes of loops on an oriented surface [5] and, more generally, the Chas–Sullivan Lie bialgebra on the string homology of a compact oriented manifold [6]. Furthermore, as shown by K. Cieliebak and J. Latschev [7], the linearized homology of an augmented strongly homotopy Batalin–Vilkovisky algebra (or a BV_∞*-algebra*), which is free as a strictly commutative associative algebra, comes equipped with an involutive Lie bialgebra structure. As an application, this gives rise to an involutive Lie bialgebra structure on the linearized contact homology of a closed contact manifold with respect to an exact symplectic filling.

The problem of devising an appropriate homotopy counterpart of involutive Lie bialgebras arises in context of the string field theory. While the classical (genus zero) open-closed string field theory is encoded by a certain splice of A_∞ and L_∞-algebras, comprising what is commonly known as the open-closed homotopy algebra [16], an enhanced structure - that of a *homotopy involutive Lie bialgebra* (or IBL_∞-algebra) - is needed to set up the full BV master equation in the quantum case (arbitrary genus). As per general theory, constructing such a homotopy algebraic structure involves building a minimal resolution for the PROP of involutive Lie bialgebras. While that was accomplished by R. Campos, S. Merkulov and T. Willwacher [4], our motivation for the present work was to elucidate on a Lie-algebraic structure, akin to the big bracket, present on the corresponding deformation complex.

As it turns out, constructing such an analog of the big bracket would require bypassing a certain no-go result concerning differential-operator properties of Lie brackets. Namely, the well-known results of A. Kirillov [18] and J. Grabowski [13] state that if A is an algebra of smooth functions on a smooth manifold or, more generally, a reduced commutative ring A, and $[-,-] : A \otimes A \to A$ is a Lie bracket that happens to be a differential operator of order $n < \infty$ with respect to each of the arguments, then the order n cannot exceed 1, yielding in the case of $n = 1$ the classical cases of Poisson and Jacobi structures.

One may attempt to overcome this by introducing nilpotents (thus considering infinitesimal deformations of Lie algebras), by relaxing the antisymmetry condition on the bracket (thus arriving to Leibniz-type algebras, cf. Example 8.1) or, as we undertake in the present work, by considering Lie brackets comprised as formal infinite series of bilinear differential operators of ongoingly increasing order

$$[-,-] = [-,-]_1 + [-,-]_2 h + [-,-]_3 h^2 + \cdots .$$

Formalizing the latter concept has lead us to a more general notion of a formal multilinear differential operator algebra, which in turn required extrapolating some basic notions of the differential calculus and D-modules from the realm of associative algebras to the case of operads. In a sense, that may be regarded as an approach to the general notion of a deformation quantization of an algebra over a linear operad.

Multifiltrations and the differential calculus for operads. As one recalls, differential operators on a commutative associative $\Bbbk$-algebra A, where $\Bbbk$ is a field, are defined in terms of a filtration

$$\mathrm{Diff}^{-1}(A) := 0 \subset \mathrm{Diff}^0(A) \subset \mathrm{Diff}^1(A) \subset \cdots \subset End(A)$$

of the linear endomorphism algebra $End_\Bbbk(A)$ compatible with the standard commutator bracket in the sense that

$$[\mathrm{Diff}^k(A), \mathrm{Diff}^l(A)] \subset \mathrm{Diff}^{k+l-1}(A),\ k,l \geq 0.$$

A proposed notion of a *multifiltration* of a linear operad (cf. Definition 2.8) is meant to provide an n-ary analog of this concept. Specifically, it is defined in terms of a poset of $\Bbbk$-linear subspaces of a given $\Bbbk$-linear operad $\mathcal{P}$ controlled by the combinatorics of integer-valued multiindices reflecting, in our case, the differential-operator orders of the individual inputs of a $\Bbbk$-linear mapping $O : A \otimes \cdots \otimes A \to A$. The corresponding combinatorial data is encoded by a certain poset-valued operad $M\mathbb{Z}$, similarly to how $\mathbb{Z}$-graded filtrations of associative algebras are defined in terms of the ordered monoid $(\mathbb{Z}, +)$. It is worth noting that in this generalized n-ary setting the commutator bracket gets replaced by a double-indexed family of operadic commutators $[-,-]_{ij}$. A multilinear differential *operator algebra* is then defined, just as in case of an ordinary algebra over an operad, in terms of a structure morphism into the endomorphism operad $\mathcal{E}nd_A$ or $\mathcal{E}nd_{A[\![h]\!]}$, where h is a formal parameter, but this time both come equipped with some extra data in the form of differential operator multifiltrations.

Due to the specifics of the original problem, we pay a particular attention to the case of *operator* L_∞*-algebras*, the latter being L_∞-algebras supported on (graded) commutative associative algebras and with the structure operations representable as formal infinite series of differential operators of certain orders. The big bracket and the IBL_∞-algebras arise as particular examples of such algebras. We generalize the former by introducing the *superbig* bracket. The term is meant to indicate that it contains the big bracket while, as noticed by Y. Kosmann-Schwarzbach, the big bracket itself gives rise to several simpler brackets relevant for deformation theory. The interest is further reinforced by Theorem 3.10 that singles out Lie-related operads as the ones satisfying a certain minimality condition with respect to their multifiltrations. The operads satisfying this minimality condition, which we call *tight*, are characterized by the property that each of them admits a presentation $\mathcal{P} = \mathbb{F}(E)/(R)$, where the relations R with respect to the generators E stay within the multifiltration components of $\mathcal{P}$ of the lowest possible order. Here, the relevant multifiltration (the *standard D-multifiltration*) is canonically associated with a choice of generators E of $\mathcal{P}$. For instance, in the particular case of the Lie operad $\mathscr{L}ie$ the property of being tight shows up in the fact that the Jacobiator

$$\mathrm{Jac}(a,b,c) := [a,[b,c]] + [b,[c,a]] + [c,[a,b]]$$

is a differential operator of order 1 in each variable, provided that an antisymmetric operation $[-,-] : A \otimes A \to A$ on a graded commutative associative algebra A is a differential operator of order 1 with respect to each of the arguments. An analogous property does not hold e.g. for associative algebras. Given an operation $\star : A \otimes A \to A$ which is a first-order differential operator in each variable with respect to the graded commutative associative structure on A, its associator $\mathrm{Ass}(a,b,c) = (a \star b) \star c - a \star (b \star c)$ is, in general, a differential operator of order 2 with respect to its arguments. The property of being tight comes handy in our approach to deformation of the big bracket.

Layout of the paper

The paper is divided into two parts. In Section 1 of **Part 1** we collect some basic facts concerning differential operators on commutative associative algebras. That section claims no originality whatsoever, but we pay a particular attention to the non-unital setting, keeping in mind the case of algebras of smooth functions with finite support on non-compact manifolds.

Section 2 features a proposed operadic framework for working with multilinear differential operators, where the notion of a multifiltration of a $\Bbbk$-linear operad is introduced. Particular attention is paid to the case of D-multifiltrations that formalize the compositional properties of multilinear differential operators. As the main technical for constructing D-multifiltrations we generalize the notion of the standard filtration of an associative algebra to $\Bbbk$-linear operads. Explicit examples of standard D-multifiltrations are to be found in Section 3. Finally, in Section 4, operator and a formal operator algebras are introduced and the first examples are given.

Part 2 is devoted to some concrete examples of multilinear differential operator algebras. Specifically, Sections 5 and 6 cover the case of operator L_∞-algebras and their particular instances – IBL_∞-algebras, commutative BV_∞-algebras, operator Lie algebras and Poisson algebras. In particular, we state some results concerning operator Lie algebras whose underlying algebra is free as a graded commutative associative algebra, having in mind a certain deformation of the big bracket that we discuss in Section 7. In the latter section, in addition to the operadic machinery, the proof of Theorem 7.2 and the content of Remarks 7.1 and 7.3 require some technical results concerning PROPs and properads. Since these are not used elsewhere in the paper, we refer the reader to the works [48] and [41] for the necessary background material. The last section features a brief overview of operator algebras over some operads other than $\mathscr{L}ie$ and $\mathscr{L}_\infty$. Namely, we discuss an example of a formal associative operator algebra provided by Terilla's deformation formula and an example of a Leibniz operator algebra due to Kanatchikov.

Conventions. Throughout the text, $\Bbbk$ will denote a field of characteristic 0. The symmetric group on n elements will be denoted by Σ_n, with $\mathbb{1}_n \in \Sigma_n$ denoting its unit. All algebraic objects will be assumed to live in the symmetric monoidal category of graded $\Bbbk$-vector spaces with the symmetry

$$\tau_{V,U} : x \otimes y \longmapsto (-1)^{|x|\cdot|y|} y \otimes x$$

for any homogeneous $x \in V$, $y \in U$. In particular for $\sigma \in \Sigma_n$ and homogeneous $x_1, \ldots, x_n \in V$, the *Koszul sign* $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n) \in \{-1, +1\}$ is defined via

$$x_1 \otimes \cdots \otimes x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

We will also denote

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) := \mathrm{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

We will use the notation $\uparrow V$, resp. $\downarrow V$ for the suspension, resp. the desuspension, of a graded vector space V. The free graded commutative unital associative algebra on a graded $\Bbbk$-vector space X will be denoted by

$$\mathbb{S}(X) := \bigoplus_{n \geq 0} \mathbb{S}^n(X),$$

where $\mathbb{S}^n(X)$ is the n-th symmetric power of X. We also denote

$$\mathbb{S}^{\leq n}(X) := \bigoplus_{0\leq k\leq n} \mathbb{S}^k(X) = \Bbbk \oplus X \oplus \mathbb{S}^2(X) \oplus \cdots \oplus \mathbb{S}^n(X), \text{ and}$$
$$\mathbb{S}^{\leq n}_+(X) := \bigoplus_{1\leq k\leq n} \mathbb{S}^k(X) = X \oplus \mathbb{S}^2(X) \oplus \cdots \oplus \mathbb{S}^n(X).$$

All operads are assumed to be unital. The Jacobiator is usually defined as

$$\mathrm{Jac}(a,b,c) = [a,[b,c]] + [b,[c,a]] + [c,[a,b]]$$

while in the context of L_∞-algebras the form

$$\mathrm{Jac}(a,b,c) = [[a,b],c] + [[b,c],a] + [[c,a],b]$$

is preferred. Since the difference is only an overall sign which plays no rôle in our theory, we will freely use, depending on the context, both conventions.

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Part 1. Calculus of multilinear differential operators

1. Higher order derivations and differential operators

In this section we present some necessary terminology and results concerning higher order differential operators and derivations. While standard citations [22,44,14,33,35,39] assume the existence of a unit in the underlying algebra, we need to work in a nonunital setup. This requires particular care since some concepts of the unital case do not translate directly. The main results here are Propositions 1.3, 1.6 and 1.8. Their proofs are given at the end of this section.

Throughout this section, we suppose that A is a graded commutative associative, not necessary unital, algebra and $\nabla : A \to A$, possibly decorated with indices, a homogeneous linear map. As in [33], we define inductively, for each $n \geq 1$, the *deviations* $\Phi^n_\nabla : A^{\otimes n} \to A$ by

$$\begin{aligned}
\Phi^1_\nabla(a) &:= \nabla(a),\\
\Phi^2_\nabla(a,b) &:= \nabla(ab) - \nabla(a)b - (-1)^{|\nabla|\cdot|a|}a\nabla(b),\\
&\ \ \vdots\\
\Phi^{n+1}_\nabla(a_1,\ldots,a_{n+1}) &:= \Phi^n_\nabla(a_1,\ldots,a_na_{n+1}) - \Phi^n_\nabla(a_1,\ldots,a_n)a_{n+1}\\
&\quad -(-1)^{|a_n|\cdot|a_{n+1}|}\Phi^n_\nabla(a_1,\ldots,a_{n-1},a_{n+1})a_n.
\end{aligned} \tag{1}$$

A non-inductive formula for Φ^n_∇ can be found in [33, page 373]. In noncommutative probability theory, Φ^{n+1}_∇ is known as the nth *infinitesimal cumulant* of ∇ with respect to the multiplication of A.

Definition 1.1. A linear map $\nabla : A \to A$ is a *derivation of order* r if Φ^{r+1}_∇ is identically zero. We denote by $\mathrm{Der}^r(A)$, $r \geq 0$, the linear space of derivations of order r.

Notice that $\mathrm{Der}^0(A) = 0$ while $\mathrm{Der}^1(A)$ is the space of usual derivations of the algebra A. It follows from (1) that if Φ^{r+1}_∇ identically vanishes, then so does Φ^{r+2}_∇, thus $\mathrm{Der}^r(A) \subset \mathrm{Der}^{r+1}(A)$. For homogeneous linear maps $\nabla_1, \nabla_2 : A \to A$ we denote, as usual, by

$$[\nabla_1,\nabla_2] := \nabla_1 \circ \nabla_2 - (-1)^{|\nabla_1||\nabla_2|}\nabla_2 \circ \nabla_1$$

their graded commutator.

Definition 1.2. The space $\mathrm{Diff}^r(A)$, $r \geq 0$, of differential operators of order r is defined inductively as follows:

(i) $\mathrm{Diff}^0(A) := \{ L_a : A \to A \mid a \in A \}$, where $L_a : x \mapsto ax$ is the operator of left multiplication by $a \in A$, and
(ii) $\mathrm{Diff}^r(A) := \{ \nabla \mid [\nabla, L_a] \in \mathrm{Diff}^{r-1}(A) \text{ for all } a \in A \}$, $r \geq 1$.

Furthermore, as a convenient convention we set $\mathrm{Diff}^{-1}(A) := 0$. This is consistent with the above definition, since $[\nabla, L_a] = 0 \in \mathrm{Diff}^{-1}(A)$ for any $\nabla \in \mathrm{Diff}^0(A)$. To complete the picture, we recall still another definition of differential operators that can be found in the literature. It uses the derived $\Bbbk$-linear mappings

$$\Psi^n_\nabla(a_1, a_2 \ldots a_n) := [\ldots[[\nabla, L_{a_1}], L_{a_2}], \ldots L_{a_n}] : A \longrightarrow A,\ a_1, \ldots, a_n \in A.$$

As a definition, we set $\Psi^0_\nabla := \nabla$. In particular, the first few iterations read

$$\begin{aligned} \Psi^0_\nabla(x) &= \nabla(x), \\ \Psi^1_\nabla(a_1)(x) &= \nabla(a_1 x) - (-1)^{|a_1|\cdot|\nabla|} a_1 \nabla(x), \\ \Psi^2_\nabla(a_1, a_2)(x) &= \nabla(a_1 a_2 x) - (-1)^{|a_1|\cdot|\nabla|} a_1 \nabla(a_2 x) \\ &\quad - (-1)^{|a_2|\cdot|\nabla|} a_2 \nabla(a_1 x) + (-1)^{(|a_1|+|a_2|)\cdot|\nabla|} a_1 a_2 \nabla(x), \quad \&c. \end{aligned} \tag{2}$$

For $r \geq -1$ we define

$$\overline{\mathrm{Diff}}^r(A) := \left\{ \nabla \mid \Psi^{r+1}_\nabla(a_1, \ldots, a_{r+1}) = 0 \ \text{ for all } \ a_1, \ldots, a_{r+1} \in A \right\}. \tag{3}$$

Below we formulate the main results of this section, Propositions 1.3, 1.6 and 1.8. Their proofs are to be found at the end of this section. The first one specifies the relation between the above three definitions.

Proposition 1.3. *For arbitrary $r \geq 0$, $\mathrm{Der}^r(A) \subset \mathrm{Diff}^r(A) \subset \overline{\mathrm{Diff}}^r(A)$. If A has a unit $1 \in A$, then $\mathrm{Diff}^r(A) = \overline{\mathrm{Diff}}^r(A)$ and, moreover,*

$$\mathrm{Der}^r(A) = \left\{ \nabla \in \mathrm{Diff}^r(A) \mid \nabla(1) = 0 \right\}. \tag{4}$$

Corollary 1.4. *If A is unital, then there exists a canonical isomorphism*

$$\mathrm{Diff}^r(A) \cong \mathrm{Der}^r(A) \oplus A,\ r \geq 0.$$

Proof. Notice that the operator of left multiplication $L_a : A \to A$ belongs to $\mathrm{Diff}^r(A)$ for any $r \geq 0$. Thus $\nabla - L_{\nabla(1)} \in \mathrm{Diff}^r(A)$ and, since it clearly annihilates the unit, $\nabla - L_{\nabla(1)} \in \mathrm{Der}^r(A)$ by (4). Thus the correspondence

$$\nabla \longmapsto (\nabla - L_{\nabla(1)}) \oplus \nabla(1)$$

defines a map $\mathrm{Diff}^r(A) \to \mathrm{Der}^r(A) \oplus A$ whose inverse is given by $\theta \oplus a \mapsto \theta + L_a$. □

Lemma 1.5. *Derivations (resp. differential operators) of order n on $\mathbb{S}(X)$ are uniquely determined by their restriction to $\mathbb{S}^{\leq n}_+(X)$ (resp. to $\mathbb{S}^{\leq n}(X)$).*

Proof. The derivations part is [33, Proposition 3]. Now, given a differential operator ∇ of order n, we consider, using Corollary 1.4, its unique decomposition $\nabla = \theta + L_a$, where θ is a derivation of order n and L_a is the operator of left multiplication by $a \in \mathbb{S}(X)$. As mentioned, the former is determined by its restriction to $\mathbb{S}^{\leq n}_+(X)$, while $a = \nabla(1)$. □

For subspaces S_1, S_2 of the space $\mathrm{Lin}_\Bbbk(A, A)$ of $\Bbbk$-linear endomorphisms $A \to A$ denote

$$S_1 \circ S_2 := \left\{ \nabla_1 \circ \nabla_2 \mid \nabla_1 \in S_1, \nabla_2 \in S_2 \right\} \ \text{and} \ [S_1, S_2] := \left\{ [\nabla_1, \nabla_2] \mid \nabla_1 \in S_1, \nabla_2 \in S_2 \right\}.$$

One then has

Proposition 1.6. *Under the above notation, the following inclusions hold for arbitrary $m, n \geq 0$:*

(i) $\mathrm{Der}^m(A) \circ \mathrm{Der}^n(A) \subset \mathrm{Der}^{m+n}(A)$,
(ii) $\mathrm{Diff}^m(A) \circ \mathrm{Diff}^n(A) \subset \mathrm{Diff}^{m+n}(A)$, *and*
(iii) $\overline{\mathrm{Diff}}^m(A) \circ \overline{\mathrm{Diff}}^n(A) \subset \overline{\mathrm{Diff}}^{m+n}(A)$.

Likewise, for the graded commutators one has

(iv) $\left[\mathrm{Der}^m(A), \mathrm{Der}^n(A)\right] \subset \mathrm{Der}^{m+n-1}(A)$, *and*
(v) $\left[\mathrm{Diff}^m(A), \mathrm{Diff}^n(A)\right] \subset \mathrm{Diff}^{m+n-1}(A)$.

Remark 1.7. Notice that, for a general non-unital algebra A, the inclusion

$$\big[\overline{\mathrm{Diff}}^m(A), \overline{\mathrm{Diff}}^n(A)\big] \subset \overline{\mathrm{Diff}}^{m+n-1}(A)$$

analogous to (iv) and (v) above, need not hold. As an example, take A to be a d-dimensional $\mathbb{k}$-vector space placed in degree zero with trivial multiplication. Then $\Psi^1_\nabla = 0$ for arbitrary ∇, so $\overline{\mathrm{Diff}}^0(A)$ is, by definition, the space of all linear endomorphisms $A \to A$, i.e. the algebra of $d \times d$ matrices $M_d(\mathbb{k})$. If $d \geq 2$, $M_d(\mathbb{k})$ is non-commutative, therefore

$$0 \neq \big[M_d(\mathbb{k}), M_d(\mathbb{k})\big] = \big[\overline{\mathrm{Diff}}^0(A), \overline{\mathrm{Diff}}^0(A)\big] \not\subset \overline{\mathrm{Diff}}^{-1}(A) = 0.$$

The last of the main statements of this section is

Proposition 1.8. *For a arbitrary $r \geq 0$, both $\mathrm{Diff}^r(A)$ and $\overline{\mathrm{Diff}}^r(A)$ are sub-bimodules of the space $\mathrm{Lin}_{\mathbb{k}}(A, A)$ of linear maps $A \to A$ with its natural A-A-bimodule structure*

$$(b, \nabla) \mapsto L_b \circ \nabla, \quad (\nabla, b) \mapsto \nabla \circ L_b.$$

Moreover, $\mathrm{Der}^r(A)$ is a left submodule of $\mathrm{Lin}_{\mathbb{k}}(A, A)$ with respect to the action $(b, \nabla) \mapsto L_b \circ \nabla$.

Remark 1.9. The subspaces $\mathrm{Der}^r(A) \subset \mathrm{Lin}_{\mathbb{k}}(A, A)$ are not, in general, *right* submodules with respect to the action $(\nabla, a) \mapsto \nabla \circ L_a$, not even when A is unital. Assume, for instance, that $\nabla \in \mathrm{Der}^1(A)$, i.e. that $\Phi^2_\nabla(a_1, a_2) = 0$ for each $a_1, a_2 \in A$. It is easy to check that then

$$\Phi^2_{\nabla \circ L_a}(a_1, a_2) = -\nabla(a)a_1a_2.$$

Now take A to be the polynomial ring $\mathbb{k}[x]$, $\nabla := \frac{d}{dx}$ the standard derivation and $a \in \mathbb{k}[x]$ any non-constant polynomial. While $\nabla \in \mathrm{Der}^1(A)$, $\Phi^2_{\nabla \circ L_a} \neq 0$, so $\nabla \circ L_a \notin \mathrm{Der}^1(A)$.

The relations between the various subspaces of $\mathrm{Lin}_{\mathbb{k}}(A, A)$ introduced above are summarized in the diagram

$$\begin{array}{ccccccccccc}
 & & 0 = \mathrm{Der}^0(A) & \hookrightarrow & \mathrm{Der}^1(A) & \hookrightarrow & \mathrm{Der}^2(A) & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Lin}_{\mathbb{k}}(A, A) \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \| \\
0 = \mathrm{Diff}^{-1}(A) & \hookrightarrow & \mathrm{Diff}^0(A) & \hookrightarrow & \mathrm{Diff}^1(A) & \hookrightarrow & \mathrm{Diff}^2(A) & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Lin}_{\mathbb{k}}(A, A) \\
\| & & \downarrow & & \downarrow & & \downarrow & & & & \| \\
0 = \overline{\mathrm{Diff}}^{-1}(A) & \hookrightarrow & \overline{\mathrm{Diff}}^0(A) & \hookrightarrow & \overline{\mathrm{Diff}}^1(A) & \hookrightarrow & \overline{\mathrm{Diff}}^2(A) & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Lin}_{\mathbb{k}}(A, A)
\end{array}$$

in which the top row consists of inclusions of left A-modules and the remaining two rows of inclusions of A-A-bimodules. The vertical inclusions between the upper two rows are inclusions of left A-modules, the inclusions between the bottom ones are that of A-A-bimodules. If A possesses a unit, the bottom two rows are isomorphic. The rest of this section is devoted to the proofs of the above propositions and necessary auxiliary results, some of them being of independent interest. In what follows, to simplify the exposition, we assume that all objects are of degree 0. In the general graded case, the formulas can be properly adjusted by following the Koszul sign rule saying that whenever we commute two entities of degrees p and q, respectively, we multiply by $(-1)^{pq}$.

Let R be a (noncommutative) ring and let $[x, y]$ denote the commutator $xy - yx$ for all $x, y \in R$. Then

$$[xy, z] = xyz - zxy = xyz - xzy + xzy - zxy = x[y, z] + [x, z]y. \tag{5}$$

Applying this to $R = \mathrm{Lin}_{\mathbb{k}}(A, A)$ with the standard composition as the multiplication, $x = L_a$, $y = \nabla_1$ and $z = \nabla_2$, where ∇_1, ∇_2 are arbitrary $\mathbb{k}$-linear endomorphisms of A, we get

$$[\nabla_1 \circ \nabla_2, L_a] = \nabla_1 \circ [\nabla_2, L_a] + [\nabla_1, L_a] \circ \nabla_2. \tag{6a}$$

Similarly, for any $\nabla \in \mathrm{Lin}_{\mathbb{k}}(A, A)$, $a_1, a_2 \in A$,

$$[\nabla, L_{a_1} \circ L_{a_2}] = [\nabla, L_{a_1}] \circ L_{a_2} + L_{a_1} \circ [\nabla, L_{a_2}]. \tag{6b}$$

Furthermore, the Jacobi identity for the commutator reads

$$\big[[\nabla_1, \nabla_2], L_a\big] = \big[\nabla_1, [\nabla_2, L_a]\big] - \big[\nabla_2, [\nabla_1, L_a]\big].$$

A simple formula

$$L_{a_1} \circ L_{a_2} = L_{a_1 a_2} \tag{6c}$$

that holds for any $a_1, a_2 \in A$, will also be useful. We will also need the following simple

Lemma 1.10. *Assume that A is unital. Then a map $\Delta : A \to A$ commutes with the operator L_a of left multiplication for any $a \in A$ if and only if it is itself an operator of left multiplication.*

Proof. By definition, $[\Delta, L_a] = 0$ means that $\Delta(au) = a\Delta(u)$ for all $u \in A$. Taking $u = 1$ gives $\Delta(a) = a\Delta(1) = \Delta(1)a$, so Δ is the operator of left multiplication by $\Delta(1)$. $\square$

The following statement is also an easy observation.

Lemma 1.11. *Assume that $\nabla \in \mathrm{Lin}_{\Bbbk}(A, A)$ is such that $\nabla(1) = 0$. For each $n \geq 1$, $\Phi^n_\nabla(a_1, \dots, a_n) = 0$ if at least one of its variables equals* 1.

Proof. The claim is obvious for $n = 1$. For $n > 2$ it follows from defining formulas (1) by simple induction. $\square$

For a commutative associative (unital or nonunital) algebra A denote by $\widetilde{A}$ its 'unitalization,' i.e. the original algebra with an artificially added unit. Explicitly, $\widetilde{A} = A \oplus \Bbbk$ as $\Bbbk$-vector spaces, and the multiplication given by

$$(a' \oplus \gamma')(a'' \oplus \gamma'') = (a'a'' + \gamma'a'' + \gamma''a') \oplus (\gamma'\gamma'')$$

for $a', a'' \in A$, $\gamma', \gamma'' \in \Bbbk$. Notice that if A was unital, its unit does not coincide with the newly added unit $\tilde{1}$ of $\widetilde{A}$. Lemma 1.11 will be used in the proof of

Lemma 1.12. *Let $\nabla \in \mathrm{Lin}_{\Bbbk}(A, A)$ and $\widetilde{\nabla} \in \mathrm{Lin}_{\Bbbk}(\widetilde{A}, \widetilde{A})$ be its extension by $\widetilde{\nabla}(\tilde{1}) := 0$. Then $\nabla \in \mathrm{Der}^r(A)$ if and only if $\widetilde{\nabla} \in \mathrm{Der}^r(\widetilde{A})$.*

Proof. If $\nabla \in \mathrm{Der}^r(A)$, Φ^{r+1}_∇ is identically zero by definition, and the same is true also for $\Phi^{r+1}_{\widetilde{\nabla}}$. Indeed, if all $a_1, \dots, a_{r+1}$ belong to A, then

$$\Phi^{r+1}_{\widetilde{\nabla}}(a_1, \dots, a_{r+1}) = \Phi^{r+1}_\nabla(a_1, \dots, a_{r+1}) = 0$$

since $\nabla \in \mathrm{Der}^r(A)$. If at least one of $a_1, \dots, a_{r+1}$ equals the added unit $\tilde{1}$, then $\Phi^{r+1}_{\widetilde{\nabla}}(a_1, \dots, a_{r+1})$ vanishes by Lemma 1.11. The opposite implication is clear, since the restriction of a differential operator to a subalgebra is a differential operator again. $\square$

The unitalization $\widetilde{A}$ and the related Lemma 1.12 will be invoked again in the proof of Proposition 1.6. The next lemma provides an inductive formula for iterated left multiplications of the same spirit as (1).

Lemma 1.13. *For any $n \geq 1$ and any $a_1, a_2 \dots a_{n+1} \in A$,*

$$\Psi^{n+1}_\nabla(a_1, \dots, a_{n+1}) = \Psi^n_\nabla(a_1, \dots, a_n a_{n+1}) - a_n \Psi^n_\nabla(a_1, \dots, a_{n-1}, a_{n+1}) - a_{n+1}\Psi^n_\nabla(a_1, \dots, a_n).$$

Proof. First note that, by the noncommutative Leibniz identity (6b), we have

$$\begin{aligned}[\Psi^{n-1}_\nabla(a_1, \dots, a_{n-1}), L_{a_n} \circ L_{a_{n+1}}] &= [\Psi^{n-1}_\nabla(a_1, \dots, a_{n-1}), L_{a_n}] \circ L_{a_{n+1}} + L_{a_n} \circ [\Psi^{n-1}_\nabla(a_1, \dots, a_{n-1}), L_{a_{n+1}}] \\ &= \Psi^n_\nabla(a_1, \dots, a_n) \circ L_{a_{n+1}} + a_n \Psi^n_\nabla(a_1, \dots, a_{n-1}, a_{n+1}).\end{aligned}$$

Invoking (6c), we also have

$$[\Psi^{n-1}_\nabla(a_1, \dots, a_{n-1}), L_{a_n} \circ L_{a_{n+1}}] = [\Psi^{n-1}_\nabla(a_1, \dots, a_{n-1}), L_{a_n a_n+1}] = \Psi^n_\nabla(a_1, \dots, a_n a_{n+1})$$

which, combined with the previous display, gives

$$\Psi^n_\nabla(a_1, \dots, a_n) \circ L_{a_{n+1}} = \Psi^n_\nabla(a_1, \dots, a_n a_{n+1}) - a_n \Psi^n_\nabla(a_1, \dots, a_{n+1}).$$

Substituting this into

$$\Psi^{n+1}_\nabla(a_1, \dots, a_{n+1}) = [\Psi^n_\nabla(a_1, \dots, a_n), L_{a+1}] = \Psi^n_\nabla(a_1, \dots, a_n) \circ L_{a+1} - L_{a_{n+1}} \circ \Psi^n_\nabla(a_1, \dots, a_n)$$

yields the required result. $\square$

Corollary. *For any $n \geq 1$, $\Psi^n_\nabla(a_1, \dots, a_n)$ is symmetric as a function of $a_1, \dots, a_n \in A$.*

Proof. Induction on n, using Lemma 1.13. $\square$

An intriguing and important relation between Ψ^n_∇, Φ^n_∇ and Φ^{n+1}_∇ is given in

Proposition 1.14. *Let $\nabla : A \to A$ be a $\Bbbk$-linear mapping. Then, for $n \geq 1$ and any $x, a_1, \dots, a_n \in A$,*

$$\Psi^n_\nabla(a_1, \dots, a_n)(x) = \Phi^{n+1}_\nabla(a_1, \dots, a_n, x) + x\Phi^n_\nabla(a_1, \dots, a_n). \tag{7}$$

Proof. For the base case $n = 1$, we have

$$\Psi^1_\nabla(a)(x) = \nabla(ax) - a\nabla(x) = \nabla(ax) - a\nabla(x) - x\nabla(a) + x\nabla(a) = \Phi^2_\nabla(a, x) + x\Phi^1_\nabla(a).$$

For $n \geq 1$, we begin by noting that, by definition,

$$\begin{aligned}\Psi^{n+1}_\nabla(a_1, \dots, a_{n+1})(x) =\ & [\Psi^n_\nabla(a_1, \dots, a_n), L_{a_{n+1}}](x)\\ =\ & \Psi^n_\nabla(a_1, \dots, a_n)(a_{n+1}x) - a_{n+1}\Psi^n_\nabla(a_1, \dots, a_n)(x).\end{aligned}$$

By induction, the two terms in the right-hand side are equal to

$$\Phi^{n+1}_\nabla(a_1, \dots a_n, a_{n+1}x) + a_{n+1}x\Phi^n_\nabla(a_1, \dots, a_n)$$

and

$$-a_{n+1}\Phi^{n+1}_\nabla(a_1, \dots, a_n, x) - a_{n+1}x\Phi^n_\nabla(a_1, \dots, a_n),$$

respectively. Hence,

$$\begin{aligned}\Psi^{n+1}_\nabla(a_1, \dots, a_{n+1})(x) &= \Phi^{n+1}_\nabla(a_1, \dots, a_n, a_{n+1}x) - a_{n+1}\Phi^{n+1}_\nabla(a_1, \dots, a_n, x)\\ &= \Phi^{n+1}_\nabla(a_1, \dots, a_n, a_{n+1}x) - a_{n+1}\Phi^{n+1}_\nabla(a_1, \dots, a_n, x)\\ &\quad - x\Phi^{n+1}_\nabla(a_1, \dots, a_{n+1}) + x\Phi^{n+1}_\nabla(a_1, \dots, a_{n+1})\\ &\overset{\text{by (1)}}{=} \Phi^{n+2}_\nabla(a_1, \dots, a_{n+1}, x) + x\Phi^{n+1}_\nabla(a_1, \dots, a_{n+1})\end{aligned}$$

as desired. □

Notice that (7) with $x = 1$ combined with Lemma 1.11 gives the well-known equation

$$\Psi^n_\nabla(a_1, \dots, a_n)(1) = \Phi^n_\nabla(a_1 \dots, a_n). \tag{8}$$

Lemma 1.15. *Assume that $\nabla : A \to A$ is a $\Bbbk$-linear mapping. Then $\nabla \in \mathrm{Diff}^r(A)$ for some $r \geq 1$ if and only if, for arbitrary $a_1, \dots, a_r \in A$,*

$$\Psi^r_\nabla(a_1, \dots, a_r) \in \mathrm{Diff}^0(A). \tag{9a}$$

If A is unital, the above condition is equivalent to

$$\Psi^{r+1}_\nabla(a_1, \dots, a_{r+1}) = 0 \tag{9b}$$

for any $a_1, \dots, a_{r+1} \in A$.

Proof. Let $\nabla \in \mathrm{Diff}^r(A)$. Then, as it follows directly from the definition, for any $0 < k \leq r + 1$ and $a_1, a_2, \dots, a_k \in A$,

$$\Psi^k_\nabla(a_1, a_2, \dots, a_k) = [...[[\nabla, L_{a_1}], L_{a_2}], \dots L_{a_k}] \in \mathrm{Diff}^{r-k}(A).$$

This with $r = k$ gives (9a).

To get the converse direction, consider first the base case $r = 1$. Namely, let $\nabla : A \to A$ be such that for any $a \in A$, $\Psi^1_\nabla(a) = [\nabla, L_a] \in \mathrm{Diff}^0(A)$. Then ∇ is a differential operator of order one by its very definition.

Now, let $r > 1$ and ∇ be such that $\Psi^r_\nabla(a, a_1, \dots, a_{r-1}) \in \mathrm{Diff}^0(A)$ for any $a, a_1, \dots, a_{r-1} \in A$. We have

$$\Psi^{r-1}_{[\nabla, L_a]}(a_1, \dots a_{r-1}) = \Psi^r_\nabla(a_1, \dots, a_{r-1}, a) \in \mathrm{Diff}^0(A).$$

Then, by induction, $[\nabla, L_a] \in \mathrm{Diff}^{r-1}(A)$. Hence, $\nabla \in \mathrm{Diff}^r(A)$ which finishes the proof of the first part of the lemma.

Let us proceed to the second part assuming that A is unital. If $\nabla \in \mathrm{Diff}^r(A)$, we already know that $\Psi^r_\nabla(a_1, \dots, a_r) \in \mathrm{Diff}^0(A)$. Since $[\Delta, L_a] = 0$ for any $\Delta \in \mathrm{Diff}^0(A)$ and $a \in A$,

$$\Psi^{r+1}_\nabla(a_1,\dots,a_{r+1}) = [\Psi^r_\nabla(a_1,\dots,a_r), L_{a_{r+1}}] = 0,$$

which is (9b).

On the other hand, the vanishing (9b) means that $\Psi^r_\nabla(a_1,\dots,a_{r-1})$ commutes with the operator L_a for any $a \in A$ so it is, by Lemma 1.10, an operator of left multiplication, i.e. (9a) holds. Thus $\nabla \in \mathrm{Diff}^r(A)$ by the first part of the lemma. □

Remark 1.16. Note that, without the unitality assumption on A, the statement of the second part of Lemma 1.15 is false. Indeed, let A be a vector space with trivial multiplication. Since the operators of left multiplication are trivial as well, $\Psi^1_\nabla(a) = [\nabla, L_a] = 0$ for any $a \in A$ and $\nabla : A \to A$, yet ∇ is a differential operator in $\mathrm{Diff}^0(A)$ as per Definition 1.2 only if $\nabla = 0$.

Proof of Proposition 1.3. As in the proof of the first part of Lemma 1.15, we inductively establish that $\nabla \in \overline{\mathrm{Diff}}^r(A)$ if and only if $\Psi^{r+1}_\nabla(a_1,\dots,a_{r+1}) = 0$ for each $a_1,\dots,a_{r+1} \in A$. If $\nabla \in \mathrm{Diff}^r(A)$, $\Psi^r_\nabla(a_1,\dots,a_r) \in \mathrm{Diff}^0(A)$ by (9a), so

$$\Psi^{r+1}_\nabla(a_1,\dots,a_{r+1}) = [\Psi^r_\nabla(a_1,\dots,a_r), L_{a_{r+1}}] = 0,$$

thus $\nabla \in \overline{\mathrm{Diff}}^r(A)$. This proves the inclusion $\mathrm{Diff}^r(A) \subset \overline{\mathrm{Diff}}^r(A)$.

Assume that $\nabla \in \mathrm{Der}^r(A)$. By definition, $\Phi^{r+1}_\nabla(a_1,\dots,a_r,x) = 0$ for arbitrary $a_1,\dots,a_r,x$ so, by (7) with $n = r$,

$$\Psi^r_\nabla(a_1,\dots,a_r)(x) = \Phi^r_\nabla(a_1,\dots,a_r)x.$$

Thus $\Psi^r_\nabla(a_1,\dots,a_r)$ is the operator of left multiplication by $\Phi^r_\nabla(a_1,\dots,a_r) \in A$, meaning that

$$\Psi^r_\nabla(a_1,\dots,a_r) \in \mathrm{Diff}^0(A)$$

thus $\nabla \in \mathrm{Diff}^r(A)$ by Lemma 1.15. Therefore $\mathrm{Der}^r(A) \subset \mathrm{Diff}^r(A)$, finishing the proof of the first part of the proposition.

Assume that A is unital. By Lemma 1.10, $\Psi^r_\nabla(a_1,\dots,a_n)$ commutes with the operator L_a for any $a \in A$ if and only if it is an operator of left multiplication. This proves that $\mathrm{Diff}^r(A) = \overline{\mathrm{Diff}}^r(A)$.

Let $\nabla \in \mathrm{Diff}^r(A)$ be such that $\nabla(1) = 0$. By (8), if $\Psi^{r+1}_\nabla(a_1,\dots,a_{r+1})$ vanishes, and so does $\Phi^{r+1}_\nabla(a_1\dots,a_{r+1})$. Thus $\nabla \in \mathrm{Der}^r(A)$.

On the other hand, if $\nabla \in \mathrm{Der}^r(A)$, Φ^{r+1}_∇ vanishes by definition, and so does $\Psi^{r+1}_\nabla(a_1,\dots,a_{r+1})$ for all $a_1,\dots,a_{r+1}$ by (7) with $n = r+1$. Therefore $\nabla \in \mathrm{Diff}^r(A)$. It remains to prove that $\nabla(1) = 0$. Taking $a_1 = a_2 = \cdots = 1$ in (1) gives

$$\Phi^n_\nabla(1,\dots,1) = -\Phi^{n+1}_\nabla(1,\dots,1) \tag{10}$$

for any $n \geq 1$. Since ∇ is a derivation of order r, $\Phi^{r+1}_\nabla(1,\dots,1) = 0$ thus $\Phi^1_\nabla(1) = \nabla(1) = 0$ by iterating (10). This finishes the proof. □

Proof of Proposition 1.6. We start by proving item (ii), by induction on $m+n$. Let ∇_1 and ∇_2 belong to $\mathrm{Diff}^0(A)$ which by definition means that they are both operators of left multiplication. By (6c), their composite $\nabla_1 \circ \nabla_2$ is an operator of left multiplication as well, so $\nabla_1 \circ \nabla_2 \in \mathrm{Diff}^0(A)$.

Suppose that (ii) been established for all $m+n \leq k$. Consider the case $m+n = k+1$. For any $a \in A$, in equation (6a), $[\nabla_1, L_a]$ is of order $< m-1$ and $[\nabla_2, L_a]$ is of order $< n-1$. Then by the inductive assumption, each of the summands, and hence the left hand side of (6a), is a differential operator of order $< m+n-1$, thus $\nabla_1 \circ \nabla_2$ is of order $< m+n$, as desired.

The proof of (iii) differs from that of (ii) only in the first inductive step. If $\nabla_1, \nabla_2 \in \overline{\mathrm{Diff}}^0(A)$, by definition, for any $a \in A$, we have $[\nabla_1, L_a] = [\nabla_2, L_a] = 0$. Hence, by (6a), $[\nabla_1 \circ \nabla_2, L_a] = 0$, and thus $\nabla_1 \circ \nabla_2 \in \overline{\mathrm{Diff}}^0(A)$. We then proceed inductively as before.

It follows from (4) combined with already proven cases that (i) holds for derivations annihilating the unit of an unital algebra. Let A be an arbitrary, not necessarily unital, algebra, and ∇_1, ∇_2 derivations of orders $< m$ and $< n$, respectively. Their extensions $\widetilde{\nabla}_1$, $\widetilde{\nabla}_2$ to the unital algebra $\widetilde{A}$ are derivations of the same respective orders by Lemma 1.12, so $\widetilde{\nabla}_1 \circ \widetilde{\nabla}_2$ is a derivation of order $< m+n$ by the above reasoning related to the unital case. Notice that the composite $\widetilde{\nabla}_1 \circ \widetilde{\nabla}_2$ annihilates the unit $\tilde{1}$ of $\widetilde{A}$ and extends $\nabla_1 \circ \nabla_2$, so the later is a derivation of order $< m+n$ by Lemma 1.12 again.

The proof of the remaining items is similar except that instead of (6a) we use the Jacobi identity (6b). Let us prove (v) by induction on $m+n$. The base case is $m = n = 0$ when $\nabla_1 = L_a$, $\nabla_2 = L_b$ for some $a, b \in A$. Then $[\nabla_1, \nabla_2] = [L_a, L_b] = 0$ as expected.

Suppose that the statement has been established for $m+n \leq k$. Consider the case $m+n = k+1$. In (6b), $[L_a, \nabla_1]$ is of order $m-1$ and $[L_a, \nabla_2]$ is of order $n-1$. Then by the inductive assumption, each of the summands, and hence the left hand side of (6b), is a differential operator of order $m+n-2$. Thus $[\nabla_1, \nabla_2]$ is of order $m+n-1$, as desired. Item (iv) can be easily derived from (v) by the extension trick employed in the proof of (i). □

Proof of Proposition 1.8. The invariance of all spaces with respect to the left action follows from the obvious equations

$$\Phi^n_{L_b\circ\nabla} = b\cdot\Phi^n_\nabla,\ \Psi^n_{L_b\circ\nabla} = b\cdot\Psi^n_\nabla,\ b\in A,$$

and (6c) which proves the invariance of $\mathrm{Diff}^0(A)$. The invariance under the right action can be proved inductively, using the equation

$$[\nabla\circ L_b, L_a] = \nabla\circ[L_b, L_a] + [\nabla, L_a]\circ L_b = [\nabla, L_a]\circ L_b.$$

The details can be safely left to the reader. □

2. Multifiltrations and multilinear differential operators

The section is devoted to introducing an operadic framework for describing algebraic structures, whose operations are represented by multilinear differential operators. The standard references for operads include [37,34] and a more recent [29]. In what follows, all operads are assumed to be unital, $\Bbbk$-linear and connected, meaning that $\mathcal{P}(0) = 0$.

We ought to warn the reader that the term 'operator algebra' that is to be introduced in this section is an abbreviation for a 'multilinear differential operator algebra' and is not directly related to the homonymous, but more elaborate, functional analytic concept.

2.1. Filtrations of algebras

We proceed by recalling the basic terminology concerning filtrations of associative algebras. Recall that an (increasing) *filtration* $\{F_i\mathcal{A}\}_{i\in\mathbb{Z}}$ on an associative $\Bbbk$-linear algebra $\mathcal{A}$ is a collection of $\Bbbk$-subspaces

$$\cdots\subseteq F_{-1}\mathcal{A}\subseteq F_0\mathcal{A}\subseteq F_1\mathcal{A}\subseteq\cdots\subseteq\mathcal{A} \tag{11}$$

such that $F_i\mathcal{A}\cdot F_j\mathcal{A}\subseteq F_{i+j}\mathcal{A}$ for all $i,j\in\mathbb{Z}$. If $\mathcal{A}$ is unital with a unit $1_\mathcal{A}$, we assume in addition that $1_\mathcal{A}\in F_0\mathcal{A}$. All filtrations on $\mathcal{A}$ are naturally ordered by the componentwise inclusion and form a poset $\mathrm{Filt}(\mathcal{A})$ with the largest element being the trivial filtration $F_i\mathcal{A} := \mathcal{A}$ for all $i\in\mathbb{Z}$

One says that a filtration $FA = \{F_i\mathcal{A}\}_{i\in\mathbb{Z}}$ is *exhaustive* if $\bigcup_{i\in\mathbb{Z}} F_i\mathcal{A} = \mathcal{A}$ and is *bounded* if there exists $b\in\mathbb{Z}$ such that $F_i\mathcal{A} = 0$ for all $i < b$. Furthermore, a filtration FA such that

$$[F_i\mathcal{A}, F_j\mathcal{A}]\subseteq F_{i+j-1}\mathcal{A} \tag{12}$$

for all $i,j\in\mathbb{Z}$. is called a *D-filtration* [1]. Note, in particular, that in such a case F_1A is a Lie algebra with respect to the commutator bracket. An algebra $\mathcal{A}$ is said to be *almost commutative* if it admits a bounded exhaustive D-filtration. The following standard example is of particular relevance for us.

Example 2.1. Let C be a commutative associative $\Bbbk$-algebra and $\mathcal{A} := End(C)$ be the associative algebra of $\Bbbk$-linear endomorphisms of C with the multiplication given by the usual composition of linear maps. Proposition 1.6 implies that the collections of subspaces

$$\mathrm{Diff}^k(C) = \big\{O : C\to C \mid O \text{ is a differential operator of order } k\big\},\ k\in\mathbb{Z}$$

and

$$\mathrm{Der}^k(C) = \big\{O : C\to C \mid O \text{ is a derivation of order } k\big\},\ k\in\mathbb{Z}$$

comprise well-defined bounded D-filtrations of $\mathcal{A}$.

Given an algebra filtration $F\mathcal{A} = \{F_i\mathcal{A}\}_{i\in\mathbb{Z}}$, one readily notes that a $\Bbbk$-algebra homomorphism $f : \mathcal{A}\to\mathcal{B}$ induces a filtration on $\mathcal{B}$ by setting $G_i\mathcal{B} := f(F_i\mathcal{A})$ for all $i\in\mathbb{Z}$. Furthermore, if $F\mathcal{A}$ is exhaustive, then so is the induced filtration $G\mathcal{B} = \{G_i\mathcal{B}\}_{i\in\mathbb{Z}}$, provided that f is surjective.

Example 2.2. Let $\mathfrak{g}$ be a Lie algebra. An application of the previous observation to the evident filtration on the tensor algebra $T(\mathfrak{g})$ by the monomial degrees and the canonical surjection of $T(\mathfrak{g})$ onto $U(\mathfrak{g}) := T(\mathfrak{g})/I$, where I is the ideal generated by the elements of the form $x\otimes y - y\otimes x - [x,y]$ for all $x,y\in\mathfrak{g}$, yields a filtration on the universal enveloping algebra of $\mathfrak{g}$.

One may find it instructive to consider the special case of $\mathfrak{g}$ being the free Lie algebra $\mathbb{L}(E)$ on a generator space E, in which case $U(\mathbb{L}(E))\simeq T(E)$. The filtration $FT(E) = \{F_iT(E)\}_{i\in\mathbb{Z}}$ of $T(E)$ obtained in this way is different from the canonical filtration of $T(E)$ by the monomial degrees.

Example 2.3. More generally, given a unital algebra $\mathcal{A}$ with of a generating set S, the canonical surjection $T(E) \twoheadrightarrow \mathcal{A}$, where $E := \mathrm{Span}(S)$, gives rise to an exhaustive filtration on $\mathcal{A}$. Explicitly, such a filtration can be constructed inductively by setting $F_i\mathcal{A} = 0$ for all $i < 0$, $F_0\mathcal{A} := \Bbbk$, $F_1\mathcal{A} := \Bbbk + E$ and $F_k\mathcal{A} := F_1\mathcal{A} \cdot F_{k-1}\mathcal{A}$ for all $k > 1$.

The filtration obtained in this way is known as the *standard filtration* of an algebra $\mathcal{A}$ with respect to a generating set S [40, Section 1.6]. It is a D-filtration whenever the space of generators E is closed under the commutator. Its operadic analog, which is to be introduced in Section 3, will be particularly useful for us later on.

Example 2.4. Consider the tensor algebra $T(E)$ on the graded generator space $E = \mathrm{Span}(\Delta)$ with Δ being of an odd degree. If $\{F_pT(E)\}_{p\geq 0}$ is the filtration of $T(E)$ as per Example 2.2 and $\{T^q(E)\}_{q\geq 0}$ is the standard grading of the same algebra by the monomial degrees, then

$$F_pT(E) = \bigoplus_{k\leq p} T^{2k}(E).$$

Interpreting the associative unital algebra $\mathcal{D}g := T(E)/(\Delta^2)$ as an operad concentrated in arity one, $\mathcal{D}g$-algebras are differential graded (dg) vector spaces. The operad $\mathcal{D}g$ will serve in Section 3 as an example of a tight operad generated by an operation of arity one.

An index-free description of the filtration data can be obtained as follows. Given a graded vector space $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$, let $\mathtt{Sub}(\mathcal{A})$ denote the modular lattice of all graded linear subspaces $V \subset \mathcal{A}$, i.e. families $\{V_i \subset \mathcal{A}_i\}_{i\in\mathbb{Z}}$ of linear subspaces, with the meet and the join operations being the componentwise subspace intersection and the componentwise subspace sum, respectively. As a poset, $\mathtt{Sub}(\mathcal{A})$ carries a natural category structure.

The tensor product induces, for arbitrary graded vector spaces $\mathcal{A}'$ and $\mathcal{A}''$, a functor

$$\mathtt{Sub}(\mathcal{A}') \times \mathtt{Sub}(\mathcal{A}'') \to \mathtt{Sub}(\mathcal{A}' \otimes \mathcal{A}'')$$

defined by

$$\mathtt{Sub}(\mathcal{A}') \times \mathtt{Sub}(\mathcal{A}'') \ni \{V'_i\}_i \times \{V''_j\}_j \longmapsto \Big\{ \bigoplus_{i+j=s} V'_i \otimes V''_j \Big\}_s \in \mathtt{Sub}(\mathcal{A}' \otimes \mathcal{A}'').$$

Likewise, any homogeneous linear map $\mathcal{A} \to \mathcal{B}$ induces a functor $\mathtt{Sub}(\mathcal{A}) \to \mathtt{Sub}(\mathcal{B})$. As a consequence, any homogeneous $\Bbbk$-bilinear operation $\vartheta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defines a functor (denoted by the same symbol)

$$\vartheta : \mathtt{Sub}(\mathcal{A}) \times \mathtt{Sub}(\mathcal{A}) \to \mathtt{Sub}(\mathcal{A}) \tag{13}$$

given as the composition $\mathtt{Sub}(\mathcal{A}) \times \mathtt{Sub}(\mathcal{A}) \to \mathtt{Sub}(\mathcal{A} \otimes \mathcal{A}) \to \mathtt{Sub}(\mathcal{A})$ of the above functors. An obvious analog of the induced functor (13) holds also for arbitrary multilinear maps $\vartheta : \mathcal{A}^{\otimes n} \to \mathcal{A}$.

Example 2.5. Let $\mathcal{A}$ be a unital associative algebra. Its multiplication $\cdot : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ induces on $\mathtt{Sub}(\mathcal{A})$ a monoid structure in the cartesian category $\mathtt{Cat}$ of small categories, whose unit is $\mathrm{Span}(e)$, the linear span of the algebra unit $e \in \mathcal{A}$. Likewise, the graded commutator bracket $[-,-] : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ given by $[a', a''] := a' \cdot a'' - (-1)^{|a'||a''|} a'' \cdot a'$ induces a symmetric bifunctor (denoted by the same symbol) $[-,-] : \mathtt{Sub}(\mathcal{A}) \times \mathtt{Sub}(\mathcal{A}) \to \mathtt{Sub}(\mathcal{A})$.

Let $\mathbb{Z}$ be the integers, $+ : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ the standard addition and $[-,-] : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ the operation given by $[p,q] := p+q-1$ for $p, q \in \mathbb{Z}$. If we consider $\mathbb{Z}$ as a small category with the category structure given by its standard order, then $(\mathbb{Z}, +)$ and $(\mathbb{Z}, [-,-])$ are commutative monoids in $\mathtt{Cat}$, with the units being 0 and 1 respectively. The following proposition refers to the structures introduced in Example 2.5. By a magma we mean a binary operation with no specified axioms.

Proposition 2.6. *Let $\mathcal{A}$ be a unital associative algebra. The data of a D-filtration on $\mathcal{A}$ is equivalent to that of a functor $\mathscr{F} : \mathbb{Z} \to \mathtt{Sub}(\mathcal{A})$ which is a lax monoidal morphism $(\mathbb{Z}, +) \to (\mathtt{Sub}(\mathcal{A}), \cdot)$ and, simultaneously, a lax magma morphism $(\mathbb{Z}, [-,-]) \to (\mathtt{Sub}(\mathcal{A}), [-,-])$*

Proof. Given such a functor $\mathscr{F} : \mathbb{Z} \to \mathtt{Sub}(\mathcal{A})$, let use denote $F_p\mathcal{A} := \mathscr{F}(p)$, $p \in \mathbb{Z}$. The functoriality of $\mathscr{F}$ is equivalent to $F_p\mathcal{A} \subseteq F_q\mathcal{A}$ whenever $p \leq q$. Furthermore, for $\mathscr{F}$ to be a lax morphism $(\mathbb{Z}, +) \to (\mathtt{Sub}(\mathcal{A}), \cdot)$ means, by definition, the existence of natural morphisms

$$\mathscr{F}(p) \cdot \mathscr{F}(q) \longrightarrow \mathscr{F}(p+q),\ p, q \in \mathbb{Z} \ \text{and}\ \mathscr{F}(\mathrm{Span}(e)) \longrightarrow \mathscr{F}(0)$$

in $\mathtt{Sub}(\mathcal{A})$. In terms of the filtration determined by $\mathscr{F}$ it means that $F_p\mathcal{A} \cdot F_q\mathcal{A} \subseteq F_{p+q}\mathcal{A}$ and $e \in F_0\mathcal{A}$. By the same argument we verify that $\mathscr{F}$ is a lax morphism $(\mathbb{Z}, [-,-]) \to (\mathtt{Sub}(\mathcal{A}), [-,-])$ if and only if $[F_p\mathcal{A}, F_q\mathcal{A}] \subseteq F_{p+q-1}\mathcal{A}$. □

2.2. Multifiltrations of operads

We are going to generalize the notions recalled in the previous subsection to the realm of $\Bbbk$-linear operads. In essence, this amounts to introducing an n-ary analog of the $\mathbb{Z}$-indexed flag of subspaces (11) and modifying appropriately the corresponding compatibility condition for the algebra multiplication. Furthermore, extrapolating the notion of a D-filtration will require introducing an operadic analog of the standard commutator bracket.

While filtrations of associative algebras are defined in terms of the ordered monoid of integers $\mathbb{Z}$, filtrations of operads are to be controlled by an operad $M\mathbb{Z} = \{M\mathbb{Z}(n)\}_{n\geq 1}$ of posets of integer-valued multiindices. Specifically, for all $n \geq 1$, we set $M\mathbb{Z}(n) := \mathbb{Z}^n$ with the poset product order inherited from $\mathbb{Z}$. That is, the set $M\mathbb{Z}(n)$ is partially ordered via

$$(p'_1, \ldots, p'_n) \preceq (p''_1, \ldots, p''_n) \text{ if and only if } p'_i \leq p''_i \text{ for each } 1 \leq i \leq n.$$

With the obvious right permutation action of the symmetric groups Σ_n, the unit $(0) \in M\mathbb{Z}(1)$ and the structure operations

$$\circ_i : M\mathbb{Z}(m) \times M\mathbb{Z}(n) \to M\mathbb{Z}(m+n-1),\ n \geq 1,\ 1 \leq i \leq m,$$

given for $(a_1, \ldots, a_m) \in M\mathbb{Z}(m)$ and $(b_1, \ldots, b_n) \in M\mathbb{Z}(n)$ by

$$(a_1, \ldots, a_m) \circ_i (b_1, \ldots, b_m) := (a_1, \ldots, a_{i-1}, b_1 + a_i, \ldots, b_n + a_i, a_{i+1}, \ldots, a_m), \tag{14}$$

$M\mathbb{Z}$ forms an operad in the cartesian category `Cat` of small categories.

Furthermore, for each $n \geq 1$, $M\mathbb{Z}(n)$ is a lattice with the join and the meet operations being

$$(p'_1, \ldots, p'_n) \vee (p''_1, \ldots, p''_n) := (\max(p'_1, p''_1), \ldots, \max(p'_n, p''_n))$$

and

$$(p'_1, \ldots, p'_n) \wedge (p''_1, \ldots, p''_n) := (\min(p'_1, p''_1), \ldots, \min(p'_n, p''_n))$$

respectively. Finally, for $\vec{p} = (p_1, \ldots, p_n) \in M\mathbb{Z}(n)$, we denote the maximum of $p_1, \ldots, p_n$ by $\max \vec{p} \in \mathbb{Z}$.

In what follows, given $\vec{a} = (a_1, \ldots, a_m)$ and an index $1 \leq i \leq m$, we will use a shorthand notation $\vec{a}_L := (a_1, \ldots, a_{i-1})$, $\vec{a}_R := (a_{i+1}, \ldots, a_m)$ and write $\vec{a} = (\vec{a}_L, a_i, \vec{a}_R)$. Now, for $\vec{b} = (b_1, \ldots, b_n)$, equation (14) reads

$$\vec{a} \circ_i \vec{b} := (\vec{a}_L, \vec{b} + a_i, \vec{a}_R),$$

where $\vec{b} + a_i := (b_1 + a_i, \ldots, b_n + a_i)$.

Remark 2.7. The `Cat`-operad $M\mathbb{Z}$ introduced above is an ordered version of S. Giraudo's combinatorial operad TM [10, Eq. (3.1)] for M being the additive monoid of integers. More applications of that construction are presented in [8].

Definition 2.8. An (increasing) *multifiltration* $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p}\in M\mathbb{Z}(n), n\geq 1}$ of a $\Bbbk$-linear operad $\mathcal{P}$ is a collection of $\Bbbk$-subspaces of $F_{\vec{p}}\mathcal{P}(n) \subset \mathcal{P}(n)$ indexed by the elements $\vec{p} \in M\mathbb{Z}(n)$ for all $n \geq 1$ subject to the following conditions:

(i) Monotonicity: $F_{\vec{p}'}\mathcal{P}(n) \subseteq F_{\vec{p}''}\mathcal{P}(n)$ if $\vec{p}' \preceq \vec{p}''$.
(ii) Equivariance: $F_{\vec{p}}\mathcal{P}(n) \cdot \sigma = F_{\vec{p}\cdot\sigma}\mathcal{P}(n)$, where $\sigma \in \Sigma_n$ acts on $\vec{p} = (p_1, \ldots, p_n)$ via

$$\vec{p} \cdot \sigma = (p_{\sigma(1)}, \ldots, p_{\sigma(n)}).$$

(iii) Compositional compatibility: $F_{\vec{a}}\mathcal{P}(m) \circ_i F_{\vec{b}}\mathcal{P}(n) \subseteq F_{\vec{a}\circ_i\vec{b}}\mathcal{P}(m+n-1)$ for all $\vec{a} \in M\mathbb{Z}(m)$, $\vec{b} \in M\mathbb{Z}(n)$ and $1 \leq i \leq m$.
(iv) Unitality: If $e \in \mathcal{P}(1)$ is the operadic unit, then $e \in F_{(0)}\mathcal{P}(1)$.

We will routinely shorten $\{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p}\in M\mathbb{Z}(n), n\geq 1}$ to $\{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$, whenever there is no danger of confusion.

A multifiltration $F\mathcal{P}$ is said to be *exhaustive* if

$$\mathcal{P}(n) = \bigcup_{\vec{p}\in M\mathbb{Z}(n)} F_{\vec{p}}\mathcal{P}(n), \quad \text{for all } n \geq 1,$$

and is said to be *bounded* if for any $n \geq 1$, there exists $\vec{b}_n \in M\mathbb{Z}(n)$ such that $F_{\vec{p}}\mathcal{P}(n) = 0$ for all $\vec{p} < \vec{b}_n$.

Definition 2.9. Let $\mathcal{P}$ and $\mathcal{Q}$ be $\Bbbk$-linear operads equipped with multifiltrations $F\mathcal{P}$ and $G\mathcal{Q}$ respectively. Then a *multifiltered operad morphism* $\phi : (\mathcal{P}, F\mathcal{P}) \to (\mathcal{Q}, G\mathcal{Q})$ is an operad morphism $\phi : \mathcal{P} \to \mathcal{Q}$ such that $\phi(F_{\vec{p}}\mathcal{P}(n)) \subseteq G_{\vec{p}}\mathcal{Q}(n)$ for all $n \geq 1$ and $\vec{p} \in M\mathbb{Z}(n)$.

All multifiltrations of $\mathcal{P}$ form a poset MFilt($\mathcal{P}$), where $F'\mathcal{P} \preceq F''\mathcal{P}$ if and only if $F'_{\vec{p}}\mathcal{P}(n) \subseteq F''_{\vec{p}}\mathcal{P}(n)$ for all $\vec{p} \in M\mathbb{Z}(n)$ and $n \geq 1$. Furthermore, for any two multifiltrations $F'\mathcal{P}$, $F''\mathcal{P}$, their componentwise intersections

$$G_{\vec{p}}\mathcal{P}(n) := F'_{\vec{p}}\mathcal{P}(n) \cap F''_{\vec{p}}\mathcal{P}(n), \quad n \geq 1, \vec{p} \in M\mathbb{Z}(n) \tag{15}$$

and sums

$$K_{\vec{p}}\mathcal{P}(n) := F'_{\vec{p}}\mathcal{P}(n) + F''_{\vec{p}}\mathcal{P}(n), \quad n \geq 1, \vec{p} \in M\mathbb{Z}(n) \tag{16}$$

constitute well-defined multifiltrations. Both constructions admit obvious generalizations to the case of intersections and sums of arbitrarily large collections of multifiltrations and turn MFilt($\mathcal{P}$) into a lattice. Two trivial examples of multifiltrations are immediate.

Example 2.10. Any $\Bbbk$-linear operad $\mathcal{P}$ possesses the multifiltration with $F_{\vec{p}}\mathcal{P}(n) := \mathcal{P}(n)$ for all $\vec{p} \in M\mathbb{Z}(n)$ and $n \geq 1$. It is the largest element of MFilt($\mathcal{P}$).

Example 2.11. A filtration $F\mathcal{A} = \{F_i\mathcal{A}\}_{i\in\mathbb{Z}}$ of an associative algebra $\mathcal{A}$ is a multifiltration of $\mathcal{A}$ treated as an operad concentrated in arity 1. The qualities of being exhaustive and bounded transfer accordingly.

Example 2.12. Any $\Bbbk$-linear algebra A gives rise to a Giraudo's combinatorial operad MA, where $MA(n) := A^{\otimes n}$, for all $n \geq 1$, and

$$\circ_i : MA(m) \otimes MA(n) \to MA(m+n-1)$$
$$(a_1, \dots, a_m) \otimes (b_1, \dots, b_n) \mapsto (a_1, \dots, a_{i-1}, a_i \cdot b_1, \dots, a_i \cdot b_n, a_{i+1}, \dots, a_m)$$

for all $m, n \geq 1$ and $1 \leq i \leq m$. Any exhaustive (resp. bounded) filtration $FA = \{F_iA\}_{i\in\mathbb{Z}}$ of A induces an exhaustive (resp. bounded) multifiltration FMA of MA given by

$$F_{\vec{p}}MA(n) = F_{(p_1,p_2\dots p_n)}MA(n) := F_{p_1}A \otimes \cdots \otimes F_{p_n}A,\ \vec{p} \in M\mathbb{Z}(n),\ n \geq 1.$$

Example 2.13. Let A be a commutative associative $\Bbbk$-algebra and $\mathscr{E}nd_A$ be the endomorphism operad on the underlying vector space. Generalizing Example 2.1, we equip $\mathscr{E}nd_A$ with a multifiltration $F\mathscr{E}nd_A$ by taking $F_{(p_1,\dots,p_n)}\mathscr{E}nd_A(n)$ to be the space of $\Bbbk$-linear maps $O : A^{\otimes n} \to A$ that are differential operators of order p_i in the i-th variable, for each $1 \leq i \leq n$.

Example 2.14. Let $A[\![h]\!] := A \otimes \Bbbk[\![h]\!]$. A 'formally deformed' version of the previous example is the multifiltration $F\mathscr{E}nd_{A[\![h]\!]}$ of $\mathscr{E}nd_{A[\![h]\!]}$ whose $(p_1, \dots, p_n)$-th component is comprised of all $\Bbbk[\![h]\!]$-multilinear maps $O : A[\![h]\!]^{\otimes n} \to A[\![h]\!]$, $n \geq 1$, such that, for $a_1, \dots, a_n \in A$,

$$O(a_1, \dots, a_n) = O_0(a_1, \dots, a_n) + O_1(a_1, \dots, a_n) \cdot h + O_2(a_1, \dots, a_n) \cdot h^2 + \cdots, \tag{17}$$

where $O_s(a_1, \dots, a_n)$ is a differential operator of order $p_i + s$ with respect to the i-th argument, for each $1 \leq i \leq n$. The multifiltrations $F\mathscr{E}nd_A$ and $F\mathscr{E}nd_{A[\![h]\!]}$ of Examples 2.13 and 2.14 are bounded, but not exhaustive in general.

Remark 2.15. The relation between the orders of differential operators in (17) and the powers of h is motivated by the notion of IBL_∞-algebras, cf. Example 5.4. The latter fits into the scheme of *binary QFT algebras* of Park [45], which are, by definition, graded commutative associative algebras over $\Bbbk[\![h]\!]$ equipped with a differential D whose deviation Φ_D^{n+1} is divisible by h^n, for each $n \geq 0$.

Similarly to the case of filtered algebras, multifiltrations can be passed along morphisms.

Lemma 2.16. *Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a morphism of $\Bbbk$-linear operads and $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ be a multifiltration of $\mathcal{P}$. Then the system $\phi(F)\mathcal{Q} = \{\phi(F)_{\vec{p}}\mathcal{Q}(n)\}_{\vec{p},n}$ of subspaces of $\mathcal{Q} = \{\mathcal{Q}(n)\}_n$, where*

$$\phi(F)_{\vec{p}}\mathcal{Q}(n) := \phi(F_{\vec{p}}\mathcal{P}(n)),$$

is a multifiltration of $\mathcal{Q}$. Furthermore, if ϕ is surjective and $F\mathcal{P}$ is exhaustive, then $\phi(F)\mathcal{Q}$ is exhaustive as well.

Proof. The monotonicity and unitality of $\phi(F)\mathcal{Q}$ are automatic. The equivariance of $\phi(F)\mathcal{Q}$ follows from ϕ being an operad morphism and thus compatible with the symmetric group action. To check the compositional compatibility with the $\circ_i$-products, assume that $\alpha \in \phi(F)_{\vec{a}}\mathcal{Q}(m)$, $\beta \in \phi(F)_{\vec{b}}\mathcal{Q}(n)$. By the definition of $\phi(F)\mathcal{Q}$, there are $\alpha' \in F_{\vec{a}}\mathcal{P}(m)$ and $\beta' \in F_{\vec{b}}\mathcal{P}(n)$ such that $\phi(\alpha') = \alpha$ and $\phi(\beta') = \beta$, and thus

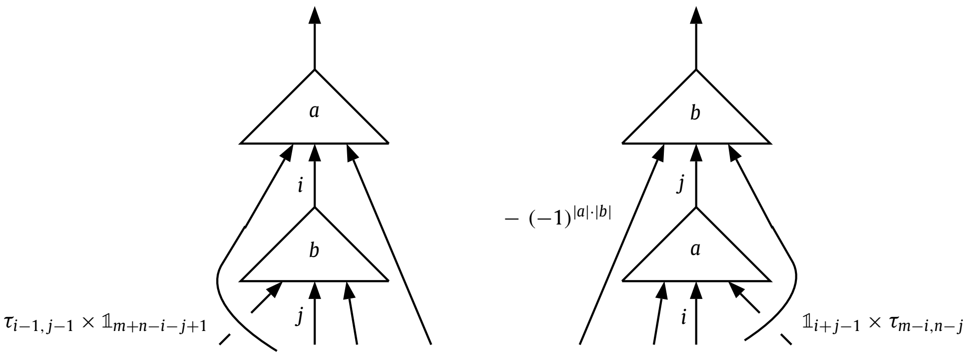


Fig. 1. The operadic commutator $[a,b]_{ij}$.

$$\alpha \circ_i \beta = \phi(\alpha') \circ_i \phi(\beta') = \phi(\alpha' \circ_i \beta') \in \phi(F_{\vec{a}\circ_i\vec{b}}\mathcal{P}(m+n-1)) = \phi(F)_{\vec{a}\circ_i\vec{b}}\mathcal{P}(m+n-1),$$

yielding the desired inclusion $\phi(F)_{\vec{a}}\mathcal{Q}(m) \circ_i \phi(F)_{\vec{b}}\mathcal{Q}(n) \subset \phi(F)_{\vec{a}\circ_i\vec{b}}\mathcal{Q}(m+n-1)$. Finally, it remains to observe that for a surjective ϕ

$$\bigcup_{\vec{p}\in M\mathbb{Z}(n)} \phi(F)_{\vec{p}}\mathcal{Q}(n) = \bigcup_{\vec{p}\in M\mathbb{Z}(n)} \phi(F_{\vec{p}}\mathcal{P}(n)) = \phi\left(\bigcup_{\vec{p}\in M\mathbb{Z}(n)} F_{\vec{p}}\mathcal{P}(n)\right) = \phi(\mathcal{P}(n)) = \mathcal{Q}(n)$$

for any $n \geq 1$. □

2.3. D-multifiltrations

An additional property that the multifiltrations of Examples 2.13–2.14 possess is implied by parts (iv)–(v) of Proposition 1.6 and concerns the behavior of multidifferential operators under commutation. To state it, for a $\Bbbk$-linear operad $\mathcal{P}$ with the operadic compositions

$$\circ_i : \mathcal{P}(m)\otimes\mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1),\ n \geq 1,\ 1 \leq i \leq m, \tag{18}$$

we introduce a family of commutators

$$[-,-]_{ij} : \mathcal{P}(m)\otimes\mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1),\ 1 \leq i \leq m,\ 1 \leq j \leq n, \tag{19}$$

generalizing the standard commutator bracket of an associative algebra. Namely, for $a \in \mathcal{P}(m)$ and $b \in \mathcal{P}(n)$, we set

$$[a,b]_{ij} := (a \circ_i b) \cdot (\tau_{i-1,j-1} \times \mathbb{1}_{m+n-i-j+1}) - (-1)^{|a||b|}(b \circ_j a) \cdot (\mathbb{1}_{i+j-1} \times \tau_{m-i,n-j}), \tag{20}$$

where $\tau_{i-1,j-1} \times \mathbb{1}_{m+n+i+j+1}) \in \Sigma_{m+n-1}$ is the permutation that exchanges the first $i-1$ symbols with the next $j-1$ symbols and leaves the remaining symbols unchanged. The action of $\mathbb{1}_{i+j-1} \times \tau_{m-j,n-i}$ is determined in a similar way. The flow diagram on Fig. 1 illustrates the idea.

As in (13), operations (20) induce bifunctors (denoted by the same symbol)

$$[-,-]_{ij} : \mathtt{Sub}(\mathcal{P}(m)) \times \mathtt{Sub}(\mathcal{P}(n)) \longrightarrow \mathtt{Sub}(\mathcal{P}(m+n-1)),\ 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{21}$$

The property we wish to formalize stipulates its behavior in terms of the indexing operad $M\mathbb{Z}$. To this end, we define

$$[-,-]_{ij} : M\mathbb{Z}(m) \times M\mathbb{Z}(n) \to M\mathbb{Z}(m+n-1),\ 1 \leq i \leq m,\ 1 \leq j \leq n, \tag{22}$$

by setting

$$[\vec{a},\vec{b}]_{ij} := (\vec{b}_L + a_i, \vec{a}_L + b_j, a_i + b_j - 1, \vec{b}_R + a_i, \vec{a}_R + b_j),$$

for $\vec{a} = (\vec{a}_L, a_i, \vec{a}_R) \in M\mathbb{Z}(m)$ and $\vec{b} = (\vec{b}_L, b_j, \vec{b}_R) \in M\mathbb{Z}(n)$. Note that $(M\mathbb{Z}(1), +, [-,-]_{11})$ is equivalent to $(\mathbb{Z}, +, [-,-])$ in the sense of Proposition 2.6.

Definition 2.17. We say that a multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ of a $\Bbbk$-linear operad $\mathcal{P}$ is a *D-multifiltration* if

$$[F_{\vec{p}}\mathcal{P}(m), F_{\vec{q}}\mathcal{P}(n)]_{ij} \subseteq F_{[\vec{p},\vec{q}]_{ij}}\mathcal{P}(m+n-1)$$

for all $\vec{p} \in M\mathbb{Z}(m)$, $\vec{q} \in M\mathbb{Z}(n)$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Example 2.18. Let FA be a D-filtration of A. Then, trivially, it is a D-multifiltration of A regarded as an operad concentrated in arity 1. More generally, the multifiltration of Giraudo's operad MA inherited from FA as per Example 2.12 is a D-multifiltration.

Example 2.19. The multifiltrations of Examples 2.13 and 2.14 are D-multifiltrations.

The property of being a D-multifiltration is preserved under taking sums and intersections of D-multifiltrations. Furthermore, as a corollary to Lemma 2.16, we get

Lemma 2.20. *For any $\Bbbk$-linear operad morphism $\phi : \mathcal{P} \to \mathcal{Q}$, the image $\phi(F)\mathcal{Q}$ of a D-multifiltration $F\mathcal{P}$ is a D-multifiltration.*

2.4. Saturated multifiltrations

In general, the benefit of having a non-trivial filtration on an algebra, or a multifiltration on a linear operad, lies in gaining means of better understanding its structure in terms of some smaller constituents and their mutual relation. For multifiltrations, the complexity of the indexing operad $M\mathbb{Z}$ encoding such a decomposition is noticeably higher than that of a linearly ordered monoid $\mathbb{Z}$ governing the case of ordinary associative algebra filtrations. With that in mind, we would like to single out a certain class of multifiltrations characterized by a relatively simple underlying combinatorics.

Definition 2.21. A multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ is *saturated* if

$$F_{\vec{p}'}\mathcal{P}(n) \cap F_{\vec{p}''}\mathcal{P}(n) \subseteq F_{\vec{p}' \wedge \vec{p}''}\mathcal{P}(n), \quad \text{for all } \vec{p}', \vec{p}'' \in M\mathbb{Z}(n). \tag{23}$$

Due to monotonicity (i) of Definition 2.8, inclusion (23) amounts to the equality

$$F_{\vec{p}'}\mathcal{P}(n) \cap F_{\vec{p}''}\mathcal{P}(n) = F_{\vec{p}' \wedge \vec{p}''}\mathcal{P}(n), \quad \text{for all } \vec{p}', \vec{p}'' \in M\mathbb{Z}(n).$$

For $n = 1$, the defining condition automatically holds true. In that regard, the notion of being saturated gets trivial in the realm of associative algebras.

Example 2.22. All multifiltrations in Examples 2.10–2.14 are saturated.

Example 2.23. As an example of a non-saturated multifiltration, let $\mathcal{P}$ be the $\Bbbk$-linear operad with $\mathcal{P}(2) = \mathrm{Span}(\alpha, \beta, \gamma)$ and $\mathcal{P}(n) = 0$ for any $n \neq 2$. That is, it is an operad generated by three binary operations with the vanishing pairwise- and self-compositions. Then the multifiltration defined by setting $F_{(0,1)}\mathcal{P}(2) := \mathrm{Span}(\alpha, \gamma)$, $F_{(1,0)}\mathcal{P}(2) := \mathrm{Span}(\beta, \gamma)$, $F_{\vec{p}}\mathcal{P}(2) := \mathcal{P}(2)$ for all $\vec{p} \geq (1, 1)$ and $F_{\vec{p}}\mathcal{P}(2) := 0$ for all $\vec{p} \leq (0, 0)$ is not saturated. More natural examples of non-saturated multifiltration are to appear in Section 3.

An elementary check shows that the property of being saturated is not preserved under taking sums and images of multifiltrations, but is preserved under taking arbitrary intersections. That, combined with the fact that for any $\Bbbk$-linear operad $\mathcal{P}$ the largest (trivial) multifiltration of $\mathcal{P}$ as per Example 2.10 is saturated, confirms the validity of the following

Definition 2.24. The *saturation* $\overline{F}\mathcal{P}$ of a multifiltration $F\mathcal{P}$ is the smallest saturated multifiltration containing $F\mathcal{P}$.

Specifically, $\overline{F}\mathcal{P}$ can be described as the intersection of all saturated multifiltrations containing $F\mathcal{P}$. In such a case, by [9, Theorem 2], the correspondence $F\mathcal{P} \mapsto \overline{F}\mathcal{P}$ makes up a *closure operator* on the lattice $\mathrm{MFilt}(\mathcal{P})$. That is, it enjoys the following formal properties

$$F\mathcal{P} \preceq \overline{F}\mathcal{P} \tag{24a}$$

$$\overline{\overline{F}}\mathcal{P} = \overline{F}\mathcal{P} \tag{24b}$$

and that $F'\mathcal{P} \preceq F''\mathcal{P}$ implies

$$\overline{F'}\mathcal{P} \preceq \overline{F''}\mathcal{P}. \tag{24c}$$

As it is the case for any closure operator, it is a functor $\mathrm{MFilt}(\mathcal{P}) \to \mathrm{SMFilt}(\mathcal{P})$ left adjoint to the inclusion $\mathrm{SMFilt}(\mathcal{P}) \hookrightarrow \mathrm{MFilt}(\mathcal{P})$, where $\mathrm{SMFilt}(\mathcal{P})$ is the sublattice of $\mathrm{MFilt}(\mathcal{P})$ consisting of all saturated multifiltrations of $\mathcal{P}$, and both lattices are regarded as categories with $x \to y$ if and only if $x \preceq y$.

Our next task is to identify the saturation $\overline{F}\mathcal{P}$ with the colimit of suitable intermediate steps. To this end, given a multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ of a $\Bbbk$-linear operad $\mathcal{P}$, we define its *presaturation* as the family of subspaces $\{F'_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ where

$$F'_{\vec{p}}\mathcal{P}(n) := \sum_{k\geq 1} \sum_{\substack{\vec{p}_1,\ldots,\vec{p}_k \\ \vec{p}_1\wedge\cdots\wedge\vec{p}_k=\vec{p}}} F_{\vec{p}_1}\mathcal{P}(n) \cap \cdots \cap F_{\vec{p}_k}\mathcal{P}(n), \quad \text{for } n \geq 1. \tag{25}$$

Lemma 2.25. *Let $F\mathcal{P}$ be a multifiltration of a $\Bbbk$-linear operad $\mathcal{P}$ and $F'\mathcal{P}$ be its presaturation.*

(i) The family $F'\mathcal{P} = \{F'_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ is a multifiltration of $\mathcal{P}$.
(ii) If $F\mathcal{P}$ is a D-multifiltration, then so is $F'\mathcal{P}$.

Proof. (i) Monotonicity, unitality and equivariance of $\{F'_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ are straightforward. It remains to prove that for any $\vec{p} \in M\mathbb{Z}(m), \vec{q} \in M\mathbb{Z}(n)$, $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$F'_{\vec{p}}\mathcal{P}(m) \circ_i F'_{\vec{q}}\mathcal{P}(n) \subset F'_{\vec{p}\circ_i\vec{q}}\mathcal{P}(m+n-1). \tag{26}$$

By $\Bbbk$-linearity of the $\circ_i$-compositions it suffices to show that

$$(F_{\vec{p}_1}\mathcal{P}(m) \cap \cdots \cap F_{\vec{p}_k}\mathcal{P}(m)) \circ_i (F_{\vec{q}_1}\mathcal{P}(n) \cap \cdots \cap F_{\vec{q}_l}\mathcal{P}(n)) \subset F'_{\vec{p}\circ_i\vec{q}}\mathcal{P}(m+n-1)$$

for all $\vec{p}_1, \ldots, \vec{p}_k \in M\mathbb{Z}(m)$, $\vec{q}_1, \ldots, \vec{q}_l \in M\mathbb{Z}(n)$, such that $\vec{p}_1 \wedge \cdots \wedge \vec{p}_k = \vec{p}$ and $\vec{q}_1 \wedge \cdots \wedge \vec{q}_l = \vec{q}$. To this end, observe first that

$$\vec{p} \circ_i \vec{q} = (\vec{p}_1 \wedge \cdots \wedge \vec{p}_k) \circ_i (\vec{q}_1 \wedge \cdots \wedge \vec{q}_l) = \bigwedge_{u,v} (\vec{p}_u \circ_i \vec{q}_v), \tag{27}$$

where in the rightmost term u and v run from 1 to k, and from 1 to l, respectively. Then

$$\begin{aligned}(F_{\vec{p}_1}\mathcal{P}(m) \cap \cdots \cap F_{\vec{p}_k}\mathcal{P}(m)) \circ_i (F_{\vec{q}_1}\mathcal{P}(n) \cap \cdots \cap F_{\vec{q}_l}\mathcal{P}(n)) &\subset \bigcap_{u,v} (F_{\vec{p}_u}\mathcal{P}(m) \circ_i F_{\vec{q}_v}\mathcal{P}(n)) \\ \subset \bigcap_{u,v} F_{\vec{p}_u\circ_i\vec{q}_v}\mathcal{P}(m+n-1) \subset F'_{\bigwedge_{u,v}(\vec{p}_u\circ_i\vec{q}_v)}\mathcal{P}(m+n-1) &= F'_{\vec{p}\circ_i\vec{q}}\mathcal{P}(m+n-1).\end{aligned}$$

(ii) Let $F\mathcal{P}$ be a D-multifiltration. Then an argument analogous to the one used in part (i) with the identity

$$[\vec{p}, \vec{q}]_{ij} = [\vec{p}_1 \wedge \cdots \wedge \vec{p}_k, \vec{q}_1 \wedge \cdots \wedge \vec{q}_l]_{ij} = \bigwedge_{u,v} [\vec{p}_u, \vec{q}_v]_{ij}$$

in place of (27) shows that

$$[F'_{\vec{p}}\mathcal{P}(m), F'_{\vec{q}}\mathcal{P}(n)]_{ij} \subset F'_{[\vec{p},\vec{q}]_{ij}}\mathcal{P}(m+n-1), \tag{28}$$

for all $n \geq 1$, $\vec{p}, \vec{q} \in M\mathbb{Z}(n)$. $\square$

Given a multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$, let $F^{(s)}\mathcal{P} = \{F^{(s)}_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ denote the multifiltration obtained by iterating the presaturation of $F\mathcal{P}$ s times.

Proposition 2.26. *The components of the saturation of $F\mathcal{P}$ are given by*

$$\overline{F}_{\vec{p}}\mathcal{P}(n) := \bigcup_{s\geq 1} F^{(s)}_{\vec{p}}\mathcal{P}(n), \ \vec{p} \in M\mathbb{Z}(n), \ n \geq 1. \tag{29}$$

In other words, $\overline{F}\mathcal{P} = \operatorname{colim} F^{(s)}\mathcal{P}$ in the poset MFilt($\mathcal{P}$).

Proof. The colimit of $F^{(s)}\mathcal{P}$ is a saturated multifiltration by Lemma 2.25. Each saturated multifiltration containing $F\mathcal{P}$ clearly contains also $F'\mathcal{P}$ and thus also $F^{(s)}\mathcal{P}$ for each $s \geq 1$. It therefore contains also the union of iterated presaturations, thus (29) indeed defines the smallest saturated multifiltration containing $F\mathcal{P}$. $\square$

Remark 2.27. Since the definition (25) of a component $F'_{\vec{p}}\mathcal{P}(n)$ of the presaturation $F'\mathcal{P}$ does not involve the operad structure of $\mathcal{P}$, a priori it might not be obvious that $F'\mathcal{P} = \{F'_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$, and consequently the saturation $\overline{F}\mathcal{P}$, would comprise a well-defined non-trivial multifiltration. As follows from the proof of Lemma 2.25, it does happen to be the case due to the distributive identity (27) noteworthy in that it connects the mutually independent lattice structures of the components $M\mathbb{Z}(n)$ for all $n \geq 1$ with the operad structure of $M\mathbb{Z}$.

Corollary 2.28. *Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a morphism of $\Bbbk$-linear operads and $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ be a multifiltration of $\mathcal{P}$. Then*

$$\phi(\overline{F})\mathcal{Q} \preceq \overline{\phi(F)}\mathcal{Q}. \tag{30}$$

The equality $\phi(\overline{F})\mathcal{Q} = \overline{\phi(F)}\mathcal{Q}$ holds if and only if $\phi(\overline{F})\mathcal{Q}$ is saturated.

Proof. From the definition of the presaturation (25), for any $n \geq 1$ and $\vec{p} \in M\mathbb{Z}(n)$,

$$\phi(F')_{\vec{p}}\mathcal{Q}(n) = \phi(F'_{\vec{p}}\mathcal{P}(n)) = \phi\left(\sum_{k\geq 1}\sum_{\substack{\vec{p}_1,\ldots,\vec{p}_k \\ \vec{p}_1\wedge\cdots\wedge\vec{p}_k=\vec{p}}} F_{\vec{p}_1}\mathcal{P}(n)\cap\cdots\cap F_{\vec{p}_k}\mathcal{P}(n)\right)$$
$$\subset \sum_{k\geq 1}\sum_{\substack{\vec{p}_1,\ldots,\vec{p}_k \\ \vec{p}_1\wedge\cdots\wedge\vec{p}_k=\vec{p}}} \phi(F_{\vec{p}_1}\mathcal{P}(n))\cap\cdots\cap\phi(F_{\vec{p}_k}\mathcal{P}(n)) = \phi(F)'_{\vec{p}}\mathcal{Q}(n).$$

The inclusion $\phi(\overline{F})\mathcal{Q} \preceq \overline{\phi(F)}\mathcal{Q}$ now follows from (29).

Now, let $F\mathcal{P}$ and ϕ be such that $\phi(\overline{F})\mathcal{Q}$ is saturated. We have $\phi(F)\mathcal{Q} \preceq \phi(\overline{F})\mathcal{Q}$. Upon taking the saturation on both sides, we get

$$\overline{\phi(F)}\mathcal{Q} \preceq \overline{\phi(\overline{F})}\mathcal{Q} = \phi(\overline{F})\mathcal{Q}$$

and the last claim follows. □

In general, the inclusion (30) could be proper as illustrated by Example 3.6.

2.5. Stabilized multifiltrations

The following notion is meant to single out multifiltrations whose components, in a given arity n, get eventually constant with respect to a natural order on $M\mathbb{Z}(n)$.

Definition 2.29. Let $n \geq 1$. We say that a multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ *stabilizes in arity n at N* for some integer N if, for each $\vec{p} \in M\mathbb{Z}(n)$,

$$F_{\vec{p}}\mathcal{P}(n) = F_{\vec{p}\wedge(N,\ldots,N)}\mathcal{P}(n). \tag{31}$$

In particular, $F_{\vec{p}}\mathcal{P}(n) = F_{(N,\ldots,N)}\mathcal{P}(n)$ for any $\vec{p} \geq (N,\ldots,N)$. The multifiltration $F\mathcal{P}$ is said to be *stabilized* if it stabilizes in each arity $n \geq 1$ at some $N_n \geq 1$.

Remark 2.30. The term above is not to be confused with the notion of a *stable filtration* (with respect to an ideal) of a module or an associative algebra.

By monotonicity, condition (31) implies that $F_{\vec{p}}\mathcal{P}(n) \subset F_{(N,\ldots,N)}\mathcal{P}(n)$ for any $\vec{p} \in M\mathbb{Z}(n)$. If $F\mathcal{P}$ is saturated, then the converse implication holds as well. The utility of this condition is to become more apparent in the next section, where a certain natural class of stabilized multifiltrations will be constructed. Otherwise, it is largely due to the following

Proposition 2.31. *Suppose that a multifiltration $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ stabilizes in arity n at N. Then,*

(i) the presaturation $F'\mathcal{P}$ stabilizes in arity n at N as well, and
(ii) for each $\vec{p} = (a_1,\ldots,a_n) \in M\mathbb{Z}(n)$, we have

$$F'_{\vec{p}}\mathcal{P}(n) = F_{(a_1,N,\ldots,N)}\mathcal{P}(n)\cap F_{(N,a_2,N,\ldots,N)}\mathcal{P}(n)\cap\cdots\cap F_{(N,\ldots,N,a_{n-1},N)}\mathcal{P}(n)\cap F_{(N,\ldots,N,a_n)}\mathcal{P}(n). \tag{32}$$

Proof. (i) Let $\vec{p} \in M\mathbb{Z}(n)$ and $\vec{p}_1,\ldots,\vec{p}_k$ be such that $\vec{p}_1\wedge\cdots\wedge\vec{p}_k = \vec{p}$. By stabilization of $F\mathcal{P}(n)$ at N, we have

$$F_{\vec{p}_1}\mathcal{P}(n)\cap\cdots\cap F_{\vec{p}_k}\mathcal{P}(n) = F_{\vec{p}_1\wedge(N,\ldots,N)}\mathcal{P}(n)\cap\cdots\cap F_{\vec{p}_k\wedge(N,\ldots,N)}\mathcal{P}(n)$$

The definition of presaturation (25) implies now that $F'_{\vec{p}}\mathcal{P}(n) = F'_{\vec{p}\wedge(N,\ldots,N)}\mathcal{P}(n)$.

(ii) We need to show that

$$F'_{\vec{p}}\mathcal{P}(n) \subseteq F_{(a_1,N,\ldots,N)}\mathcal{P}(n)\cap F_{(N,a_2,N,\ldots,N)}\mathcal{P}(n)\cap\cdots\cap F_{(N,\ldots,N,a_{n-1},N)}\mathcal{P}(n)\cap F_{(N,\ldots,N,a_n)}\mathcal{P}(n), \tag{33}$$

as the inclusion in the opposite direction follows readily from the definition of the presaturation $F'\mathcal{P}$. To this end, let $\vec{p}_i = (u^i_1, u^i_2, \dots, u^i_n) \in M\mathbb{Z}(n)$ for $i = 1, 2, \dots, k$ be such that $\vec{p}_1 \wedge \cdots \wedge \vec{p}_k = \vec{p}$. In particular, $a_j = \min_{i=1,\dots,k} u^i_j$ for each $j = 1, 2, \dots, n$. We have

$$F_{\vec{p}_1}\mathcal{P}(n) \cap \cdots \cap F_{\vec{p}_k}\mathcal{P}(n) = F_{\vec{p}_1 \wedge (N,\dots,N)}\mathcal{P}(n) \cap \cdots \cap F_{\vec{p}_k \wedge (N,\dots,N)}\mathcal{P}(n) \subseteq \bigcap_{i=1}^{k} F_{(u^i_1, N, \dots, N)}\mathcal{P}(n) = F_{(a_1, N, \dots, N)}\mathcal{P}(n),$$

where the first equality is due to stabilization and the subsequent inclusion follows by monotonicity. In a similar way, we obtain $F_{\vec{p}_1}\mathcal{P}(n) \cap \cdots \cap F_{\vec{p}_k}\mathcal{P}(n) \subseteq F_{(N, a_2, N, \dots, N)}$ and so on. The desired inclusion then follows from (25). □

Corollary 2.32. *Under the hypothesis of Proposition 2.31,* $\overline{F}\mathcal{P}(n) = F'\mathcal{P}(n)$.

Indeed, by (32), for any $j = 1, 2, \dots, n$, we have $F'_{(N,\dots,a_j,\dots,N)}\mathcal{P}(n) = F_{(N,\dots,a_j,\dots,N)}\mathcal{P}(n)$. Then by another application of the same equality, $F''_{\vec{p}}\mathcal{P}(n) = F'_{\vec{p}}\mathcal{P}(n)$ and the claim follows from Proposition 2.26.

2.6. Standard D-multifiltrations

We are about to introduce a construction of a multifiltration that can be associated with a $\Bbbk$-linear operad with a specified choice of generators. This can be thought of as an operadic analog of the standard filtration of an associative algebra as per Example 2.3 with two enhancements. First, the construction to be discussed here results in a D-multifiltration. Just as in the case of associative algebras, this would require the space of the chosen generators in arity 1 be closed under the commutator. At the same time, as we shall discuss below, the specifics of the combinatorics of $M\mathbb{Z}(n)$ for $n \geq 2$ may result in a non-trivial D-multifiltration on an operad without non-trivial unary operations. Second, the combinatorics of this D-multifiltration is simplified by taking its saturation. This does not have its direct analog in the realm of associative algebras, since all algebra filtrations are saturated in the sense of our Definition 25.

Let $\mathcal{P}$ be an $\Bbbk$-linear operad generated by a Σ-module $E = \{E(n)\}_{n\geq 1}$. For the rest of the section, we will assume that $\mathcal{P}(0) = 0$ and $E(1)$ is closed under the bracket $[-,-]_{11}$. In particular, this encompasses the case when $\mathcal{P}$ is simply-connected, i.e. when $\mathcal{P}(1)$ is one-dimensional and is spanned by the operad unit. The *prestandard D-multifiltration of $\mathcal{P}$ with respect to E* is defined as the smallest D-multifiltration $G\mathcal{P} = \{G_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ of $\mathcal{P}$ such that $E(n) \subset G_{(1,\dots,1)}\mathcal{P}(n)$ for all $n \geq 1$. Notice that such a multifiltration of $\mathcal{P}$ automatically satisfies $E(n) \cap G_{\vec{p}}\mathcal{P}(n) = 0$ for $\vec{p} \prec (1, \dots, 1)$. Explicitly, $G\mathcal{P}$ can be described as follows.

(i) For $n = 1$, we define $G_{(0)}\mathcal{P}(1) := \Bbbk \cdot e$, where $e \in \mathcal{P}(1)$ is the operad unit. Next, we set

$$G_{(1)}\mathcal{P}(1) := \Bbbk \cdot e + E(1).$$

For all $p > 1$, we inductively define

$$G_{(p)}\mathcal{P}(1) := G_{(p-1)}\mathcal{P}(1) + G_{(1)}\mathcal{P}(1) \circ_1 G_{(p-1)}\mathcal{P}(1).$$

(ii) For $n > 1$ and all $\vec{p} \in M\mathbb{Z}(n)$ such that $\vec{p} \not\succeq (1, \dots, 1)$, as well as for $n = 1$ and all $\vec{p} \in M\mathbb{Z}(1)$ such that $\vec{p} \prec (0)$, we put $G_{\vec{p}}\mathcal{P}(n) := 0$.

(iii) For $n \geq 2$ and $\vec{p} \succeq (1, \dots, 1)$ we proceed inductively by setting

$$G_{\vec{p}}\mathcal{P}(n) := E(n) + \sum_{i,\vec{p}',\vec{p}'',k,l,\sigma} (G_{\vec{p}'}\mathcal{P}(k) \circ_i G_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma + \sum_{i,j,\vec{p}',\vec{p}'',k,l,\sigma} [G_{\vec{p}'}\mathcal{P}(k), G_{\vec{p}''}\mathcal{P}(l)]_{ij} \cdot \sigma. \tag{34}$$

Here, both summations in (34) run over $k, l \geq 1$ such that $k + l = n + 1$. The first sum is taken over all $1 \leq i \leq k$, $\vec{p}' \in M\mathbb{Z}(k)$, $\vec{p}'' \in M\mathbb{Z}(l)$ and $\sigma \in \Sigma_n$ such that $(\vec{p}' \circ_i \vec{p}'')\sigma \preceq \vec{p}$. The second one goes over all $1 \leq i \leq k$, $1 \leq j \leq l$, $\vec{p}' \in M\mathbb{Z}(k)$, $\vec{p}'' \in M\mathbb{Z}(l)$ and $\sigma \in \Sigma_n$ such that $[\vec{p}', \vec{p}'']_{ij}\sigma \preceq \vec{p}$. We will routinely omit an explicit reference to the generating collection E, whenever there is no danger of confusion.

Proposition 2.33. *Let the family* $G\mathcal{P} = \{G_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ *be as described above. Then*

(i) $G\mathcal{P}$ is a D-multifiltration.
(ii) If $\mathcal{P}$ is simply-connected, then in arity n, it stabilizes at $N = n - 1$.
(iii) It is the smallest D-multifiltration such that $E(n) \subset G_{(1,\dots,1)}\mathcal{P}(n)$ for all $n \geq 1$.

Proof. (i) Monotonicity and unitality of $G\mathcal{P}$ are immediate. The equivariance follows from the compatibility of the operadic compositions with the symmetric group actions.

The property of being a D-multifiltration in arity $n=1$ follows by a simple inductive argument using (5). Now, given $n \geq 2$, $\vec{a} \in M\mathbb{Z}(x)$, $\vec{b} \in M\mathbb{Z}(y)$ with $x+y=n+1$ and $1 \leq s \leq k$, equation (34) for $\vec{p} := \vec{a} \circ_s \vec{b}$, yields

$$G_{\vec{a}\circ_s\vec{b}}\mathcal{P}(n) = E(n) + \sum_{i,\vec{p}',\vec{p}'',k,l,\sigma} (G_{\vec{p}'}\mathcal{P}(k) \circ_i G_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma + \sum_{i,j,\vec{p}',\vec{p}'',k,l,\sigma} [G_{\vec{p}'}\mathcal{P}(k), G_{\vec{p}''}\mathcal{P}(l)]_{ij} \cdot \sigma$$
$$\supset G_{\vec{a}}\mathcal{P}(x) \circ_s G_{\vec{b}}\mathcal{P}(y),$$

where the inclusion follows upon taking $k=x$, $l=y$, $\vec{p}'=\vec{a}$, $\vec{p}''=\vec{b}$, $i=s$ and $\sigma = \mathbb{1}_n$, the unit of Σ_n, in the first sum. In a similar way we get, using (34) again,

$$G_{[\vec{a},\vec{b}]_{st}}\mathcal{P}(n) \supset [G_{\vec{a}}\mathcal{P}(x), G_{\vec{b}}\mathcal{P}(y)]_{st}$$

for all $x, y, \vec{a}, \vec{b}, s, t$ for which the above expression makes sense.

(ii) Since, by definition, $G_{\vec{p}}\mathcal{P}(n)=0$ if $\vec{p} \not\succeq (1,\dots,1)$, we may assume that $\vec{p} \succeq (1,\dots,1)$. Since $\mathcal{P}$ is assumed to be simply-connected, the stabilization at arity $n=1$ is clear. For a given $n \geq 2$ and $\vec{p} \in M\mathbb{Z}(n)$ denote $\hat{p} := \vec{p} \wedge (n-1,\dots,n-1)$. As $n \geq 2$ by assumption, $\vec{p} \succeq (1,\dots,1)$ if and only if $\hat{p} \succeq (1,\dots,1)$. We must therefore prove that

$$G_{\vec{p}}\mathcal{P}(n) \subset G_{\hat{p}}\mathcal{P}(n) \text{ for all } n \geq 2 \text{ and } \vec{p} \in M\mathbb{Z}(n), \tag{35}$$

because the opposite inclusion and thus the equality would follow from the monotonicity. We proceed by the induction on the arity.

The base case $n=2$ is implied by $G_{\vec{p}}\mathcal{P}(2) = E(2) = G_{\hat{p}}\mathcal{P}(2)$. For the induction step consider formula (34) defining $G_{\vec{p}}\mathcal{P}(n)$ and compare it with the formula

$$G_{\hat{p}}\mathcal{P}(n) := E(n) + \sum_{i,\vec{q}',\vec{q}'',k,l,\sigma} (G_{\vec{q}'}\mathcal{P}(k) \circ_i G_{\vec{q}''}\mathcal{P}(l)) \cdot \sigma + \sum_{i,j,\vec{q}',\vec{q}'',k,l,\sigma} [G_{\vec{q}'}\mathcal{P}(k), G_{\vec{q}''}\mathcal{P}(l)]_{ij} \cdot \sigma \tag{36}$$

in which $(\vec{q}' \circ_i \vec{q}'')\sigma \preceq \hat{p}$ in the first sum, and $[\vec{q}', \vec{q}'']_{ij}\sigma \preceq \hat{p}$ in the second one. Since the term $E(n)$ occurs in both formulas, it remains to prove that all terms of the first and the second sum of (34) occur also in (36).

Consider an arbitrary term $(G_{\vec{p}'}\mathcal{P}(k) \circ_i G_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma$ of the first sum in (34) in which, of course, $(\vec{p}' \circ_i \vec{p}'')\sigma \preceq \vec{p}$. By the induction assumption,

$$(G_{\vec{p}'}\mathcal{P}(k) \circ_i G_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma = (G_{\hat{p}'}\mathcal{P}(k) \circ_i G_{\hat{p}''}\mathcal{P}(l)) \cdot \sigma . \tag{37}$$

At this point, one needs to verify that

$$\vec{p}' \circ_i \vec{p}'' \preceq \vec{p} \ \text{ implies } \ \hat{p}' \circ_i \hat{p}'' \preceq \hat{p}, \tag{38a}$$

thus the term in (37) indeed occurs in the first sum of (36), with $\vec{q}' := \hat{p}'$ and $\vec{q}'' := \hat{p}''$.

The second sum in (36) can be handled in a similar way using the property that

$$[\vec{p}', \vec{p}'']_{ij} \preceq \vec{p} \ \text{ implies } \ [\hat{p}', \hat{p}'']_{ij} \preceq \hat{p} \tag{38b}$$

Both (38a) and (38b) can be verified directly, using the elementary inequalities

$$\min(x,k) + \min(y,l) \leq \min(x+y, k+l), \ \text{ and}$$
$$\min(x-1,k) \leq \min(x,k) - 1$$

that hold for arbitrary $x, y, k, l \in \mathbb{Z}$.

(iii) Let $F\mathcal{P} = \{F_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ be a D-multifiltration of $\mathcal{P}$ such that $E(n) \subset F_{(1,\dots,1)}\mathcal{P}(n)$ for $n \geq 1$. We must prove that

$$G_{\vec{p}}\mathcal{P}(n) \subset F_{\vec{p}}\mathcal{P}(n), \ \text{ for each } n \geq 1, \vec{p} \in M\mathbb{Z}(n). \tag{39}$$

Since, by definition, $G_{\vec{p}}\mathcal{P}(n) = 0$ if $\vec{p} \not\succeq (1,\dots,1)$, we may assume that $\vec{p} \succeq (1,\dots,1)$.

Inclusion (39) is clear for $n=1$. For $n=2$ it follows from $G_{\vec{p}}\mathcal{P}(2) = E(2) \subset F_{(1,\dots,1)}\mathcal{P}(n) \subset F_{\vec{p}}\mathcal{P}(n)$. For $n>2$ we proceed by induction. Assuming that $G_{\vec{p}}\mathcal{P}(m) \subset F_{\vec{p}}\mathcal{P}(m)$ for all $m<n$, $\vec{p} \in M\mathbb{Z}(m)$, we get

$$G_{\vec{p}}\mathcal{P}(n) = E(n) + \sum_{i,\vec{p}',\vec{p}'',k,l,\sigma} (G_{\vec{p}'}\mathcal{P}(k) \circ_i G_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma + \sum_{i,j,\vec{p}',\vec{p}'',k,l,\sigma} [G_{\vec{p}'}\mathcal{P}(k), G_{\vec{p}''}\mathcal{P}(l)]_{ij} \cdot \sigma$$
$$\subset E(n) + \sum_{i,\vec{p}',\vec{p}'',k,l,\sigma} (F_{\vec{p}'}\mathcal{P}(k) \circ_i F_{\vec{p}''}\mathcal{P}(l)) \cdot \sigma + \sum_{i,j,\vec{p}',\vec{p}'',k,l,\sigma} [F_{\vec{p}'}\mathcal{P}(k), F_{\vec{p}''}\mathcal{P}(l)]_{ij} \cdot \sigma$$
$$\subset F_{\vec{p}}\mathcal{P}(n)$$

for all $\vec{p} \in M\mathbb{Z}(n)$, which proves the induction step. □

Remark 2.34. The stabilization bound estimate of Proposition 2.33 can be improved. Namely, if $k \geq 2$ is the smallest integer such that $E(k) \neq 0$. Then $\{G_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ stabilizes in arity $n \geq 2$ at $N = \left[\frac{n-1}{k-1}\right]$, the integral part of the fraction.

Remark 2.35. The above construction can be simplified to the one resulting in a multifiltration, rather than a D-multifiltration, by omitting the rightmost sum in (2.33) and dropping off the requirement of $E(1)$ being closed under the commutator $[-,-]_{11}$.

Definition 2.36. Let $\mathcal{P}$ be a $\Bbbk$-linear operad generated by a collection E. The *standard D-multifiltration with respect to E* is the smallest saturated D-multifiltration $F\mathcal{P}$ of $\mathcal{P}$ such that $E(n) \subset F_{(1,\ldots,1)}\mathcal{P}(n)$ for each $n \geq 1$.

More explicitly, we have the following

Lemma 2.37. *Let $\mathcal{P}$ be a $\Bbbk$-linear operad generated by a collection E. Then the standard D-multifiltration with respect to E is the saturation $\overline{G}\mathcal{P}$ of the prestandard D-multifiltration $G\mathcal{P}$.*

Proof. Let $F\mathcal{P}$ be the standard D-multifiltration of $\mathcal{P}$ with respect to E. By part (iii) of Proposition 2.33, we have $G\mathcal{P} \preceq F\mathcal{P}$. Upon taking the saturation on both sides, by (24c), we get $\overline{G}\mathcal{P} \preceq \overline{F}\mathcal{P} = F\mathcal{P}$. On the other hand, by the minimality of $F\mathcal{P}$ among saturated D-multifiltrations, we have $F\mathcal{P} \preceq \overline{G}\mathcal{P}$. □

In view of the above lemma, we will routinely use $\overline{G}\mathcal{P}$ as our notation for the standard D-multifiltration of an operad $\mathcal{P}$ with respect to a certain generating collection.

Remark 2.38. Since prestandard D-multifiltrations of simply-connected operads are stabilized, part (ii) of Proposition 2.31 along with Corollary 2.32 provides a recipe for computing the components of standard D-multifiltrations, which is to be made use of in the next section.

We conclude this section with a couple of results on the functoriality of (pre)standard D-multifiltrations that will be used later.

Lemma 2.39. *Let $G_E\mathcal{P} = \{G_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ be the prestandard D-multifiltration of $\mathcal{P}$ with respect to a generating collection $E = \{E(n)\}_{n\geq 1}$, $\phi : \mathcal{P} \twoheadrightarrow \mathcal{Q}$ be a surjective morphism and $F := \phi(E)$ be the subcollection of $\mathcal{Q}$ with components $F(n) := \phi(E(n))$, $n \geq 1$. Then the image $\phi(G_E)\mathcal{Q}$ of the prestandard D-multifiltration $G_E\mathcal{P}$ is the prestandard D-multifiltration $G_F\mathcal{Q}$ of $\mathcal{Q}$ with respect to the generators F, that is $G_F\mathcal{Q} = \phi(G_E)\mathcal{Q}$.*

Proof. It is easy to verify that, given a D-multifiltration $F\mathcal{Q}$ of $\mathcal{Q}$, the collection

$$\phi^{-1}(F)\mathcal{P} = \{\phi^{-1}(F)_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$$

with the components

$$\phi^{-1}(F)_{\vec{p}}\mathcal{P}(n) := \phi^{-1}(F_{\vec{p}}\mathcal{Q}(n)),\ \vec{p} \in M\mathbb{Z}(n),\ n \geq 1,$$

is a D-multifiltration of $\mathcal{P}$. Applying this to the prestandard D-multifiltration $G_F\mathcal{Q}$, we get a D-multifiltration $\phi^{-1}(G_F)\mathcal{P}$ of $\mathcal{P}$ with $E(n) \subset \phi^{-1}(G_F)_{(1,\ldots,1)}\mathcal{P}(n)$ for all $n \geq 1$. By the minimality property of $G_E\mathcal{P}$, we get $G_E\mathcal{P} \preceq \phi^{-1}(G_F)\mathcal{P}$. Hence,

$$\phi(G_E)\mathcal{P} \preceq \phi(\phi^{-1}(G_F))\mathcal{Q} = G_F\mathcal{Q}.$$

On the other hand, Lemma 2.20 implies that $\phi(G_E)\mathcal{Q}$ is a D-multifiltration of $\mathcal{Q}$ satisfying that $F(n) \subset \phi(G_E)_{(1,\ldots,1)}\mathcal{Q}(n)$ for all $n \geq 1$. Thus, by minimality of $G_F\mathcal{Q}$,

$$G_F\mathcal{Q} \preceq \phi(G_E)\mathcal{P}$$

and the desired equality follows. □

Proposition 2.40. *Assume that $\alpha : \mathcal{P} \to \mathcal{S}$ is a not necessarily surjective morphism of operads, $\overline{G}\mathcal{P} = \{\overline{G}_{\vec{p}}\mathcal{P}(n)\}_{\vec{p},n}$ is the standard D-multifiltration of $\mathcal{P}$ with respect to the generating collection $E = \{E(n)\}_{n\geq 1}$, and $F\mathcal{S} = \{F_{\vec{p}}\mathcal{S}(n)\}_{\vec{p},n}$ is a saturated D-multifiltration of $\mathcal{S}$ such that*

$$\alpha(E(n)) \subset F_{(1,\ldots,1)}\mathcal{S}(n),$$

for each $n \geq 2$. Then $\alpha(\overline{G})\mathcal{S} \preceq F\mathcal{S}$.

Proof. Denote by $\mathcal{Q} \subset \mathcal{S}$ the image of α so that α factorizes as $\mathcal{P} \overset{\phi}{\twoheadrightarrow} \mathcal{Q} \hookrightarrow \mathcal{S}$. Then $\mathcal{Q}$ is a suboperad of $\mathcal{S}$ carrying two D-multifiltrations. The first one is the restriction $F\mathcal{Q}$ of $F\mathcal{S}$ to $\mathcal{Q}$, and the second one is the image $\phi(G)\mathcal{Q}$ of the prestandard D-multifiltration of $\mathcal{P}$.

For any $n \geq 2$, we have $\alpha(E(n)) = \phi(E(n)) \subseteq F_{(1,\ldots,1)}\mathcal{Q}(n)$. By Lemma 2.39, $\phi(G)\mathcal{Q}$ is equal to the prestandard D-multifiltration $G\mathcal{Q}$ of $\mathcal{Q}$ with respect to the generators $\alpha(E)$. Then by the minimality property of prestandard D-multifiltrations, we have $\phi(G)\mathcal{Q} \preceq F\mathcal{Q}$. Passing to the saturations, by Corollary 2.28 and (24c), we get

$$\phi(\overline{G})\mathcal{Q} \preceq \overline{\phi(G)}\mathcal{Q} \preceq \overline{F}\mathcal{Q} = F\mathcal{Q} \subseteq F\mathcal{S},$$

where we use the fact that $F\mathcal{Q}$ is saturated as a restriction of a saturated multifiltration $F\mathcal{S}$ onto $\mathcal{Q}$. It remains to observe that $\alpha(\overline{G})\mathcal{S} = \phi(\overline{G})\mathcal{Q}$. □

3. Standard *D*-multifiltrations – examples and calculations

The section presents some explicit examples of standard D-multifiltrations on operads with a single generator. Much of the work done here amounts to analyzing the basic combinatorics of the standard D-multifiltration on a free operad and then passing to the saturation of its quotient. The corresponding results will be used later in the proof of Theorem 3.10.

The proposition below addresses the case of the standard D-multifiltration of the free operad $\mathbb{F}(E)$ when its generating collection E is spanned by a single fully symmetric or fully antisymmetric n-ary operation, $n \geq 2$, and degree of the same parity as n.

Proposition 3.1. *For the standard D-multifiltration $\overline{G}\mathbb{F}(E)$ of the free operad $\mathbb{F}(E)$, the following properties hold in arity $2n-1$:*

(i) $\overline{G}_{\vec{p}}\mathbb{F}(E)(2n-1) = \overline{G}_{\vec{p}\wedge(2,\ldots,2)}\mathbb{F}(E)(2n-1)$ *for each* $\vec{p} \in M\mathbb{Z}(2n-1)$,
(ii) $\overline{G}_{(2,2,\ldots,2)}\mathbb{F}(E)(2n-1) = \mathbb{F}(E)(2n-1)$, *thus* $\dim \overline{G}_{(2,2,\ldots,2)}\mathbb{F}(E)(2n-1) = \binom{2n-1}{n}$, *and*
(iii) $\dim \overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1) = \binom{2n-2}{n} + \frac{1}{2}\binom{2n-2}{n-1}$.
(iv) If E is spanned by a single n-ary antisymmetric operation $[-,\ldots,-]$, *then* $\overline{G}_{(1,1,\ldots,1)}\mathbb{F}(E)(n)$ *contains the Jacobiator*

$$\mathcal{J}ac_n := \sum_{\sigma \in Sh_{n,n-1}} \operatorname{sgn}(\sigma)\cdot\big[[\sigma(1),\ldots,\sigma(n)],\sigma(n+1),\ldots,\sigma(2n-1)\big] \in \mathbb{F}(E)(2n-1). \tag{40}$$

Here, the summation runs over all $(n,n-1)$-shuffles, i.e. permutations $\sigma \in \Sigma_{2n-1}$ such that

$$\sigma(1) < \cdots < \sigma(n) \ \text{ and } \ \sigma(n+1) < \cdots < \sigma(2n-1).$$

In the above formula, $\big[[\sigma(1),\ldots,\sigma(n)],\sigma(n+1),\ldots,\sigma(2n-1)\big]$ denotes the operation

$$\big[[-,\ldots,-]\circ_1[-,\ldots,-] \in \mathbb{F}(E)(2n-1)$$

acted on by $\sigma \in \Sigma_{2n-1}$. We will use the same type of notation for the action of the symmetric group also in the rest of the paper.

In can be shown by a tedious, but straightforward, argument that $\overline{G}_{(1,1,\ldots,1)}\mathbb{F}(E)(2n-1)$ in part (iv) is in fact one-dimensional and spanned by the Jacobiator (40). A particular case corresponding to $n=2$ is addressed in Example 3.2.

Proof of Proposition 3.1. Item (i) says that the standard D-multifiltration of $\mathbb{F}(E)$ stabilizes in arity $2n-1$ at 2. This follows from Proposition 2.33, resp. from its enhancement spelled out in Remark 2.34.

Since all the results of the proposition concern pieces of arity $2n-1$, we will not specify, in the rest of this proof, that arity explicitly where it is clear from the context. Let ω be a fully symmetric or fully antisymmetric operation of arity n spanning E. Then the expressions

$$\omega\big(\omega(\sigma(1),\ldots,\sigma(n)),\sigma(n+1),\ldots,\sigma(2n-1)\big),\ \sigma \in Sh_{n,n-1}, \tag{41}$$

form a basis of $\mathbb{F}(E)(2n-1)$. Since all terms above are obtained from $\omega \circ_1 \omega$ by the action of an element σ of Σ_{2n-1} such that $\big((1,1,\ldots,1)\circ_1(1,1,\ldots,1)\big)\cdot\sigma \preceq (2,2,\ldots,2)$, they all belong to $G_{(2,2,\ldots,2)}\mathbb{F}(E)$ by (34), and thus

$$G_{(2,2,\ldots,2)}\mathbb{F}(E)(2n-1) = \mathbb{F}(E)(2n-1).$$

The equality in (ii) then follows from the inclusions

$$G_{(2,2,\ldots,2)}\mathbb{F}(E)(2n-1) \subset \overline{G}_{(2,2,\ldots,2)}\mathbb{F}(E)(2n-1) \subset \mathbb{F}(E)(2n-1).$$

The second part of item (ii) expresses that there are exactly $\binom{2n-1}{n}$ shuffles in $Sh_{n,n-1}$.

Let us attend to (iii). It is straightforward to verify, using the (anti)symmetry of the generating operation, that in arity $2n-1$, equation (34) for $G_{(1,2,\ldots,2)}\mathbb{F}(E)$ reduces to

$$G_{(1,2,\ldots,2)}\mathbb{F}(E) := \sum_{\sigma}(E(n) \circ_2 E(n)) \cdot \sigma + \sum_{\sigma}[E(n), E(n)]_{11} \cdot \sigma. \tag{42}$$

The first sum in (42) generates the vectors

$$\omega\big(1, \omega(\sigma(2), \ldots, \sigma(n+1)), \sigma(n+2), \ldots, \sigma(2n-1)\big), \tag{43}$$

where σ is a permutation of the set $\{2, \ldots, 2n-1\}$ such that

$$\sigma(2) < \cdots < \sigma(n+1) \text{ and } \sigma(n+2) < \cdots < \sigma(2n-1). \tag{44}$$

The second sum in (42) generates the vectors

$$\omega\big(\omega(1, \sigma(2), \ldots, \sigma(n)), \sigma(n+1), \ldots, \sigma(2n-1)\big) \\ - (-1)^n \omega\big(\omega(1, \sigma(n+1), \ldots, \sigma(2n-1)), \sigma(2), \ldots, \sigma(n)\big), \tag{45}$$

where σ is a permutation of the set $\{2, \ldots, 2n-1\}$ such that

$$\sigma(2) < \cdots < \sigma(n),\ \sigma(n+1) < \cdots < \sigma(2n-1) \text{ and } \sigma(2) < \sigma(2n-1). \tag{46}$$

It is simple to show that the vectors in (43) and (45) are linearly independent, thus they form a basis of $G_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1)$. Moreover, by (32) with $N=2$,

$$\overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1) = G_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1).$$

The formula in (iii) then simply expresses the total number of the vectors in (43) and (45).

Let us finally turn our attention to (iv). By formula (32) with $N=2$,

$$\overline{G}_{(1,1,\ldots,1)}\mathbb{F}(E)(2n-1) = \bigcap G_{(2,\ldots,1,\ldots,2)}\mathbb{F}(E)(2n-1)$$

where the intersection runs over all positions of 1 in the multiindex. We thus need to show that $\mathcal{J}ac_n \in G_{(2,\ldots,1,\ldots,2)}\mathbb{F}(E) \times (2n-1)$ for each position of 1. Since $\mathcal{J}ac_n$ is stable under cyclic permutations, it suffices to establish that $\mathcal{J}ac_n \in G_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1)$. To this end we decompose $\mathcal{J}ac_n = A_n + B_n$, where

$$A_n := \sum_{\sigma} \operatorname{sgn}(\sigma)\Big\{\big[[1, \sigma(2), \ldots, \sigma(n)], \sigma(n+1), \ldots, \sigma(2n-1)\big] \\ - (-1)^n\big[[(1, \sigma(n+1), \ldots, \sigma(2n-1)], \sigma(2), \ldots, \sigma(n)\big]\Big\},$$

where σ runs over all permutations as in (46), and

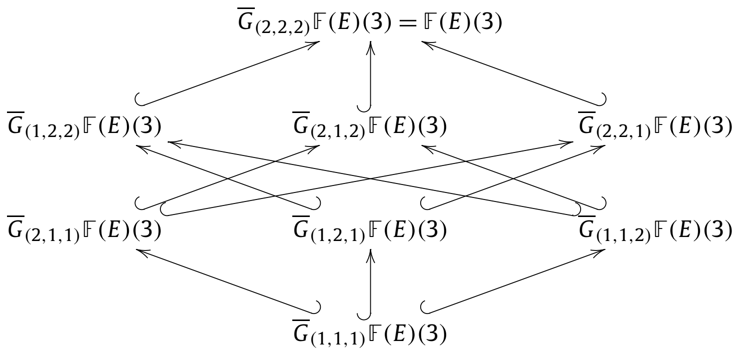
$$B_n := (-1)^n \sum_{\sigma} \operatorname{sgn}(\sigma)\big[[(\sigma(2), \ldots, \sigma(n+1)], 1, \sigma(n+2), \ldots, \sigma(2n-1)\big] \\ = (-1)^{n+1} \sum_{\sigma} \operatorname{sgn}(\sigma)\big[1, [(\sigma(2), \ldots, \sigma(n+1)], \sigma(n+2), \ldots, \sigma(2n-1)\big],$$

with σ running over permutations in (44). Now it suffices to notice that A_n is a linear combination of the vectors (45), while B_n is combination of the vectors (43), thus both A_n and B_n belong to $G_{(1,2,\ldots,2)}\mathbb{F}(E)(2n-1)$. The decomposition $\mathcal{J}ac_n = A_n + B_n$ is an abstract version of a similar trick used in the proof of Proposition 5.5. □

Let $\mathbb{F}(E)$ be the free operad on a collection of, possibly several, binary operations E. Then for the standard D-multifiltration $\{\overline{G}_{\vec{p}}\mathbb{F}(E)(n)\}_{\vec{p},n}$, we have

$$\overline{G}_{(p_1,p_2)}\mathbb{F}(E)(2) = \begin{cases} E(2) & \text{if } p_1, p_2 \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.33, in arity 3, $\overline{G}$ stabilizes at $N=2$ and its components form the lattice



(47)

By the saturation property, the above lattice is determined by the components $\overline{G}_{(1,2,2)}\mathbb{F}(E)(3)$, $\overline{G}_{(2,1,2)}\mathbb{F}(E)(3)$ and $\overline{G}_{(2,2,1)}\mathbb{F}(E)(3)$ as we illustrate in the following examples.

Example 3.2. Let $E = E(2)$ be spanned by a single antisymmetric operation $[-,-]$; its antisymmetry is expressed as $[1,2] = -[2,1]$. The pieces in the top two tiers of (47) are equal to

$$\overline{G}_{(2,2,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big([1,[2,3]],[2,[3,1]],[3,[1,2]]\big) = \mathbb{F}(E)(3),$$
$$\overline{G}_{(1,2,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big([1,[2,3]],[2,[3,1]]+[3,[1,2]]\big),$$
$$\overline{G}_{(2,1,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big([2,[3,1]],[3,[1,2]]+[1,[2,3]]\big), \text{ and}$$
$$\overline{G}_{(2,2,1)}\mathbb{F}(E)(3) = \mathrm{Span}\big([3,[1,2]],[1,[2,3]]+[2,[3,1]]\big).$$

To identify the remaining components, suppose that

$$\mu \in \overline{G}(1,1,2)\mathbb{F}(E)(3) = \mu \in \overline{G}(1,2,2)\mathbb{F}(E)(3) \cap \mu \in \overline{G}(2,1,2)\mathbb{F}(E)(3).$$

Then there exist $c_1, c_2, d_1, d_2 \in \Bbbk$ such that

$$\mu = c_1[1,[2,3]] + c_2\big([2,[3,1]]+[3,[1,2]]\big) = d_1[2,[3,1]] + d_2\big([3,[1,2]]+[1,[2,3]]\big).$$

Since $[1,[2,3]]$, $[2,[3,1]]$ and $[3,[1,2]]$ form a basis of $\overline{G}_{(2,2,2)}\mathbb{F}(E)(3)$, it follows that $c_1 = c_2 = d_1 = d_2$, and setting the common value of these coefficients to 1 produces $\mu = \mathcal{J}ac_3$, where

$$\mathcal{J}ac_3 := [1,[2,3]] + [2,[3,1]] + [3,[1,2]] \tag{48}$$

is the abstract Jacobian. Thus $\overline{G}_{(1,1,2)}\mathbb{F}(E)(3) = \mathrm{Span}(\mathcal{J}ac_3)$. Since we already know from part (iv) of Proposition 3.1 that $\mathcal{J}ac_3 \in \overline{G}_{(1,1,1)}\mathbb{F}(E)(3)$, we conclude that

$$\overline{G}_{(2,1,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,2,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,1,2)}\mathbb{F}(E)(3) = \overline{G}_{(1,1,1)}\mathbb{F}(E)(3) = \mathrm{Span}(\mathcal{J}ac_3). \tag{49}$$

Example 3.3. Let the generating collection of $\mathbb{F}(E)$ be spanned by one symmetric binary operation $(-,-)$. That is, $(1,2) = (2,1)$ in terms of the shorthand notation of Example 3.2. Then $\mathbb{F}(E)(3)$ has a basis consisting of $\big(1,(2,3)\big)$, $\big(2,(3,1)\big)$ and $\big(3,(1,2)\big)$. We easily verify that in (47)

$$\overline{G}_{(2,2,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big((1,(2,3)),(2,(3,1)),(3,(1,2))\big) = \mathbb{F}(E)(3),$$
$$\overline{G}_{(1,2,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big((1,(2,3)),(2,(3,1))-(3,(1,2))\big),$$
$$\overline{G}_{(2,1,2)}\mathbb{F}(E)(3) = \mathrm{Span}\big((2,(3,1)),(3,(1,2))-(1,(2,3))\big), \text{ and}$$
$$\overline{G}_{(2,2,1)}\mathbb{F}(E)(3) = \mathrm{Span}\big((3,(1,2)),(1,(2,3))-(2,(3,1))\big).$$

In contrast with the situation of Example 3.2, the remaining pieces of the poset (47) are trivial,

$$\overline{G}_{(2,1,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,2,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,1,2)}\mathbb{F}(E)(3) = \overline{G}_{(1,1,1)}\mathbb{F}(E)(3) = 0. \tag{50}$$

Indeed, by the saturation property,

$$\overline{G}_{(2,1,1)}\mathbb{F}(E)(3) = \overline{G}_{(2,1,2)}\mathbb{F}(E)(3) \cap \overline{G}_{(2,2,1)}\mathbb{F}(E)(3)$$
$$\overline{G}_{(1,2,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,2,2)}\mathbb{F}(E)(3) \cap \overline{G}_{(2,2,1)}\mathbb{F}(E)(3)$$
$$\overline{G}_{(1,1,2)}\mathbb{F}(E)(3) = \overline{G}_{(1,2,2)}\mathbb{F}(E)(3) \cap \overline{G}_{(2,1,2)}\mathbb{F}(E)(3)$$

whereas the intersections on the right hand sides are all trivial by a simple linear algebra.

Example 3.4. Consider finally the case when E is spanned by a single binary operation $(-,-)$ with no symmetry. Then $\mathbb{F}(E)(3)$ has a basis consisting of 12 vectors

$$\big\{ \big(\sigma(1), (\sigma(2), \sigma(3))\big),\ \big((\sigma(1), \sigma(2)), \sigma(3)\big) \big\}_{\sigma \in \Sigma_3}.$$

Just as before, we observe that $\overline{G}_{(1,2,2)}\mathbb{F}(E)(3)$, $\overline{G}_{(2,1,2)}\mathbb{F}(E)(3)$ and $\overline{G}_{(2,2,1)}\mathbb{F}(E)(3)$ have their respective bases

$$\left\{ \begin{matrix} \big(1,(2,3)\big),\ \big((2,3),1\big),\ \big(1,(3,2)\big),\ \big((3,2),1\big), \\ \big(2,(3,1)\big) - \big(3,(2,1)\big),\ \big((1,2),3\big) - \big((1,3),2\big),\ \big(3,(1,2)\big) - \big((3,1),2\big),\ \big(2,(1,3)\big) - \big((2,1),3\big) \end{matrix} \right\},$$

$$\left\{ \begin{matrix} \big(2,(3,1)\big),\ \big((3,1),2\big),\ \big(2,(1,3)\big),\ \big((1,3),2\big), \\ \big(3,(1,2)\big) - \big(1,(3,2)\big),\ \big((2,3),1\big) - \big((2,1),3\big),\ \big(1,(2,3)\big) - \big((1,2),3\big),\ \big(3,(2,1)\big) - \big((3,2),1\big) \end{matrix} \right\} \text{ and}$$

$$\left\{ \begin{matrix} \big(3,(1,2)\big),\ \big((1,2),3\big),\ \big(3,(2,1)\big),\ \big((2,1),3\big), \\ \big(1,(2,3)\big) - \big(2,(1,3)\big),\ \big((3,2),1\big) - \big((3,1),2\big),\ \big(1,(3,2)\big) - \big((1,3),2\big),\ \big(2,(3,1)\big) - \big((2,3),1\big) \end{matrix} \right\}.$$

Then $\overline{G}_{(1,1,2)}\mathbb{F}(E)(3) = \overline{G}_{(1,2,2)}\mathbb{F}(E)(3) \cap \overline{G}_{(2,1,2)}\mathbb{F}(E)(3)$ is 4-dimensional, spanned by

$$\begin{aligned} &\big(1,(2,3)\big) + \big((1,3),2\big) - \big((1,2),3\big),\ \ \big(2,(1,3)\big) + \big((2,3),1\big) - \big((2,1),3\big), \\ &\big(1,(3,2)\big) + \big((3,1),2\big) - \big(3,(1,2)\big),\ \ \big(2,(3,1)\big) + \big((3,2),1\big) - \big(3,(2,1)\big). \end{aligned}$$

Finally, $\overline{G}_{(1,1,1)}\mathbb{F}(E)(3) = \overline{G}_{(1,1,2)}\mathbb{F}(E)(3) \cap \overline{G}_{(2,2,1)}\mathbb{F}(E)(3)$ turns out to be one-dimensional, with a basis vector

$$\begin{aligned} &\big(1,(2,3)\big) + \big((1,3),2\big) - \big((1,2),3\big) - \big(2,(1,3)\big) - \big((2,3),1\big) + \big((2,1),3\big) \\ -&\big(1,(3,2)\big) - \big((3,1),2\big) + \big(3,(1,2)\big) + \big(2,(3,1)\big) + \big((3,2),1\big) - \big(3,(2,1)\big). \end{aligned}$$

The above vector can be conveniently rewritten using the associator

$$\mathrm{Ass}(1,2,3) := \big((1,2),3\big) - \big(1,(2,3)\big) \subset \mathbb{F}(E)(3)$$

of the operation $(-,-)$. Namely, it is

$$\mathrm{LieAdm}(1,2,3) := \sum_{\sigma \in \Sigma_3} \mathrm{sgn}(\sigma)\mathrm{Ass}\big(\sigma(1), \sigma(2), \sigma(3)\big)$$

making up a relation characterizing Lie admissible algebras [36, Example 6].

Example 3.5. Let $\mathscr{L}ie$ be the operad governing Lie algebras, presented as the quotient of the free operad $\mathbb{F}(E)$ in Example 3.2 modulo the Jacobian (48), and $\phi : \mathbb{F}(E) \twoheadrightarrow \mathscr{L}ie$ be the natural projection. We are going to describe the standard D-multifiltration of $\mathscr{L}ie$ with respect to the generator $\phi([-,-]) \in \mathscr{L}ie(2)$. Let us single out the following elements of $\mathscr{L}ie(3)$,

$$e := \phi\big([1,[2,3]]\big),\ f := \phi\big([2,[3,1]]\big),\ g := \phi\big([3,[1,2]]\big)$$

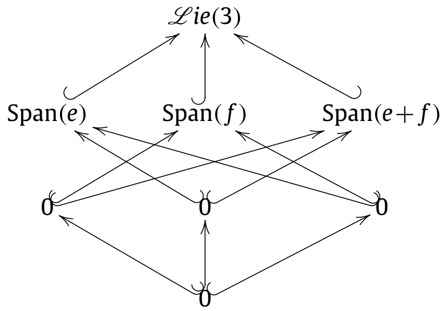
related by the Jacobi identity $e + f + g = 0$. We choose $\{e, f\}$ as a basis of $\mathscr{L}ie(3)$. According to Lemma 2.39, we may describe the relevant pieces of the prestandard D-multifiltration of $\mathscr{L}ie(3)$ as the image of the same pieces of the prestandard D-multifiltration of $\mathbb{F}(E)$. The result is

$$G_{(2,2,2)}\mathscr{L}ie(3) = \mathrm{Span}(e, f) = \mathscr{L}ie(3),$$
$$G_{(1,2,2)}\mathscr{L}ie(3) = \mathrm{Span}(e),\ G_{(2,1,2)}\mathscr{L}ie(3) = \mathrm{Span}(f) \text{ and } G_{(2,2,1)}\mathscr{L}ie(3) = \mathrm{Span}(e+f).$$

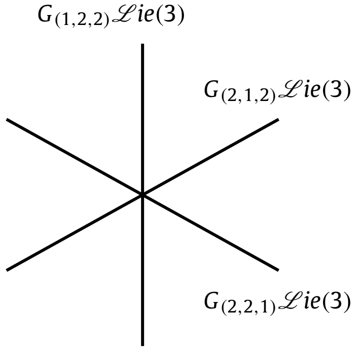
By part (i) of Proposition 2.31, the prestandard D-multifiltration $G\mathscr{L}ie$ stabilizes, so we may use formula (32) combined with Corollary 2.32 to describe the standard D-multifiltration of $\mathscr{L}ie(3)$. The result is

$$\overline{G}_{(2,2,2)}\mathscr{L}ie(3) = \mathrm{Span}(e, f) = \mathscr{L}ie(3),$$
$$\overline{G}_{(1,2,2)}\mathscr{L}ie(3) = \mathrm{Span}(e),\ \overline{G}_{(2,1,2)}\mathscr{L}ie(3) = \mathrm{Span}(f),\ \overline{G}_{(2,2,1)}\mathscr{L}ie(3) = \mathrm{Span}(e+f),$$
$$\overline{G}_{(1,1,2)}\mathscr{L}ie(3) = \overline{G}_{(2,1,1)}\mathscr{L}ie(3) = \overline{G}_{(1,2,1)}\mathscr{L}ie(3) = \overline{G}_{(1,1,1)}\mathscr{L}ie(3) = 0.$$

The lattice analogous to (47) for the standard D-multifiltration of $\mathscr{L}ie(3)$ thus looks as



The configuration of $\overline{G}_{(1,2,2)}\mathscr{L}ie(3)$, $\overline{G}_{(2,1,2)}\mathscr{L}ie(3)$ and $\overline{G}_{(2,2,1)}\mathscr{L}ie(3)$ in $\mathscr{L}ie(3)$ is portrayed in:



Notice that in this case the image $\phi(\overline{G}_{\vec{p}}\mathbb{F}(E)(3))$ equals $\overline{G}_{\vec{p}}\mathscr{L}ie(3)$ for each $\vec{p} \in M\mathbb{Z}(3)$. One may in fact prove that, more generally, if $\mathcal{P}$ is a binary quadratic operad with the quadratic presentation $\mathbb{F}(E)/(R)$ such that $R \subset \overline{G}_{(1,1,1)}\mathbb{F}(E)(3)$, then $\overline{G}_{(1,1,1)}\mathcal{P}(3)$ coincides with the image of $\overline{G}_{(1,1,1)}\mathbb{F}(E)(3)$ under the canonical projection $\mathbb{F}(E) \twoheadrightarrow \mathcal{P}$.

Example 3.6. Let $\mathscr{C}om$ be the operad for commutative associative algebras, presented as the quotient of the free operad $\mathbb{F}(E)$ of Example 3.3 modulo the associativity $\big(1, (2, 3)\big) = \big((1, 2), 3\big)$. Denoting by $\phi : \mathbb{F}(E) \twoheadrightarrow \mathscr{C}om$ the canonical projection and

$$a := \phi\big(1, (2, 3)\big),\ b := \phi\big(2, (3, 1)\big),\ c := \phi\big(3, (1, 2)\big),$$

the vector space $\mathscr{C}om(3)$ is isomorphic to Span(a), and $a = b = c$ in $\mathscr{C}om(3)$. Using the pattern of Example 3.5, we calculate

$$G_{(2,2,2)}\mathscr{C}om(3) = G_{(1,2,2)}\mathscr{C}om(3) = G_{(2,1,2)}\mathscr{C}om(3) = G_{(2,2,1)}\mathscr{C}om(3) = \mathrm{Span}(a) = \mathscr{C}om(3).$$

From this we conclude that $\overline{G}_{\vec{p}}\mathscr{C}om(3) = \mathscr{C}om(3)$ for each $\vec{p} \in M\mathbb{Z}(3)$. All entries of the lattice analogous to (47) equal $\mathscr{C}om(3)$. The standard D-multifiltration of $\mathscr{C}om$ is strictly bigger than the image of the standard D-multifiltration of $\mathbb{F}(E)$ under the projection $\phi : \mathbb{F}(E) \twoheadrightarrow \mathscr{C}om$.

Example 3.7. A 3-Lie algebra, aka Filipov algebra [11], is a vector space V together with a trilinear antisymmetric bracket $[-, -, -]$ satisfying

$$[1, 2, [3, 4, 5]] = [[1, 2, 3], 4, 5] + [3, [1, 2, 4], 5] + [3, 4, [1, 2, 5]].$$

One can verify that if $\mathbb{F}(E)$ is the free $\Bbbk$-linear operad generated by a single antisymmetric ternary operation, then the above identity determines an element in $\overline{G}_{(2,2,1,1,1)}\mathbb{F}(E)$.

Given a $\Bbbk$-linear operad $\mathcal{P}$ with a generating collection E, its relations $R = \{R(n)\}_{n\geq 1}$ are, in general, expected to be contained in the components $\overline{G}_{\vec{p}}\mathcal{P}(n)$ of the corresponding standard D-multifiltration with $\vec{p} \succ (1, \dots, 1)$. To single out the special case of R being confined to the lower-level components of $\overline{G}\mathcal{P}$, we introduce the following

Definition 3.8. A $\Bbbk$-linear operad $\mathcal{P}$ with a generating collection E is *tight* if it admits a *tight presentation*, that is, a presentation $\mathcal{P} = \mathbb{F}(E)/(R)$ such that the collection $R = \{R(n)\}_{n\geq 1}$ generating the operadic ideal (R) satisfies

$$R(n) \subset \overline{G}_{(1,\dots,1)}\mathbb{F}(E)(n), \quad \text{for each } n \geq 1. \tag{51}$$

Lemma 3.9. *Let $\mathcal{P}$ be a tight $\Bbbk$-linear operad with a tight presentation $\mathcal{P} = \mathbb{F}(E)/(R)$ and $\mathcal{S}$ be a $\Bbbk$-linear operad with a saturated D-multifiltration $F\mathcal{S}$. If $\tilde{\alpha} : \mathbb{F}(E) \to \mathcal{S}$ is an operad morphism such that*

$$\tilde{\alpha}(E(n)) \subset F_{(1,\dots,1)}\mathcal{S}(n) \text{ for all } n \geq 1,$$

then

$$\tilde{\alpha}(R(n)) \subset F_{(1,\dots,1)}\mathcal{S}(n) \text{ for all } n \geq 1.$$

Proof. We have $\tilde{\alpha}(R(n)) \subseteq \tilde{\alpha}(\overline{G}_{(1,\dots,1)}\mathbb{F}(E)(n)) \subseteq F_{(1,\dots,1)}\mathcal{S}$, where the second inclusion follows from Proposition 2.40. □

Theorem 3.10. *The only tight quadratic operads generated by a single binary operation are the free operad $\mathbb{F}(E)$, the operad $\mathscr{L}ie$ for Lie algebras, and the operad $\mathscr{L}ie\mathscr{A}dm$ for Lie admissible algebras.*

Proof. The result follows from the analysis carried out in Examples 3.2-3.4. If the generating collection of the quadratic operad $\mathcal{P} = \mathbb{F}(E)/(R)$ is spanned by one antisymmetric operation, then $\overline{G}_{(1,1,1)}\mathbb{F}(E)(3)$ is the one-dimensional span of the Jacobiator by (49). Thus either $R = 0$, in which case $\mathcal{P}$ is free, or $R = \mathrm{Span}(\mathcal{J}ac_3)$, in which case $\mathcal{P}$ is the operad for Lie algebras.

If the generating operation is commutative, then $\overline{G}_{(1,1,1)}\mathbb{F}(E)(3) = 0$ by (50), thus $\mathcal{P}$ must be free. If the generating operation has no symmetry, then either $\mathcal{P}$ is free or $\mathcal{P} = \mathscr{L}ie\mathscr{A}dm$ by the result of Example 3.4. □

Remark 3.11. It is evident that the coproduct $\mathcal{P}' \sqcup \mathcal{P}''$ of tight operads is tight again. As argued in [36, Example 6], the operad $\mathscr{L}ie\mathscr{A}dm$ is the coproduct

$$\mathscr{L}ie\mathscr{A}dm \cong \mathscr{L}ie \sqcup \mathbb{F}(\varpi)$$

of the operad for Lie algebras with the free operad on one commutative binary operation ϖ. Thus the tightness of $\mathscr{L}ie\mathscr{A}dm$ is corroborated by the tightness of the operads at the right hand side of the above display.

Theorem 3.10 indicates that in case of operads generated by a single binary operation tightness is a fairly restrictive property. Yet some meaningful examples of tight operads generated by multiple operations or operations of arities other than two are available. A rather simple example is the operad $\mathcal{D}g$ (cf. Example 2.4) whose algebras are differential graded vector spaces. A less trivial one is the operad $\mathscr{L}_\infty$ governing L_∞ (strongly homotopy Lie) algebras in the category of graded vector spaces. Recall the following

Definition 3.12 *([25, Definition 2.1]).* An *L_∞-algebra* consists of a $\Bbbk$-linear graded vector space L equipped with $\Bbbk$-linear maps $l_k : \bigotimes^k L \to L$, $k \geq 1$, of degree $k-2$ which are antisymmetric, i.e.

$$l_k(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)}) = \chi(\sigma) l_k(\lambda_1, \dots, \lambda_k) \tag{52}$$

for all permutations $\sigma \in \Sigma_k$ and homogeneous $\lambda_1, \dots, \lambda_k \in L$. Moreover, the following generalized form of the Jacobi identity is required to hold for any $k \geq 1$:

$$\mathrm{Jac}_k(\lambda_1, \dots, \lambda_k) := \sum \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(i)}), \lambda_{\sigma(i+1)}, \dots, \lambda_{\sigma(k)}) = 0, \tag{53}$$

with the summation running over all $i, j \geq 1$ with $i + j = k + 1$, and all $(i, k-i)$-shuffles $\sigma \in \Sigma_k$.

L_∞-algebras are governed by the operad $\mathscr{L}_\infty$ with a quadratic presentation $\mathbb{F}(E)/(R)$, where the generating collection $E = \{E(k)\}_{k\geq 1}$ is such that for each $k \geq 1$, $E(k)$ is the one-dimensional sign representation of the symmetric group Σ_k spanned by ℓ_k, and $R = \{R(k)\}_{k\geq 1}$ has its kth component $R(k)$ spanned by the following abstraction of (53):

$$\mathcal{J}ac_k(1, \dots, k) := \sum_{i+j=k+1} \sum_{\sigma} \mathrm{sgn}(\sigma)(-1)^{i(j-1)} \ell_j(\ell_i(\sigma(1), \dots, \sigma(i)), \sigma(i+1), \dots, \sigma(k)), \tag{54}$$

where the second summation runs over all $(i, k-i)$-shuffles $\sigma \in \Sigma_k$.

Proposition 3.13. *The operad $\mathscr{L}_\infty$ is tight quadratic.*

Proof. Consider the above presentation of $\mathscr{L}_\infty$ and let $\overline{G}\mathbb{F}(E)$ be the standard D-multifiltration of $\mathbb{F}(E)$. Since $E(k) \subset \overline{G}_{(1,\dots,1)}\mathbb{F}(E)(k)$, then $\mathcal{J}ac_k \in \overline{G}_{(2,\dots,2)}\mathbb{F}(E)(k)$. Since $\mathcal{J}ac_k$ is cyclically symmetric, it suffices to establish that

$$\mathcal{J}ac_k \in \overline{G}_{(1,2,\dots,2)}\mathbb{F}(E)(k). \tag{55}$$

Then, indeed, by the cyclic symmetry and equivariance

$$\mathcal{J}ac_k \in \overline{G}_{(2,\ldots,2,1,2,\ldots,2)}\mathbb{F}(E)(k)$$

for any position of 1 thus, by the saturation property, $\mathcal{J}ac_k \in \overline{G}_{(1,\ldots,1)}\mathbb{F}(E)(k)$ as required. To prove (55), we start by decomposing

$$\mathcal{J}ac_k(1,\ldots,k) = \mathcal{A}_k(1,\ldots,k) + \mathcal{B}_k(1,\ldots,k),$$

where $\mathcal{A}_k(1,\ldots,k)$ is the restriction of the second sum in the right hand side of (54) to $(i,k{-}i)$-shuffles with $\sigma(1)=1$, and $\mathcal{B}_k(1,\ldots,k)$ accounts for the remaining summands, i.e. those with $\sigma(i{+}1)=1$.

Notice first that all the summands of $\mathcal{B}_k$ belong to $\overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(k)$. Indeed, each summand

$$\mathrm{sgn}(\sigma)(-1)^{i(j-1)}\ell_j(\ell_i(\sigma(1),\ldots,\sigma(i)),\sigma(i{+}1),\ldots,\sigma(k))$$

of $\mathcal{B}_k(1,\ldots,k)$ can be written as

$$\mathrm{sgn}(\sigma)(-1)^{i(j-1)}(\ell_j \circ_1 \ell_i)\sigma,$$

using the operadic $\circ_1$-composition and the right action of the symmetric group Σ_k. By definition of the standard D-multifiltration, $\ell_j \in \overline{G}_{(1,\ldots,1)}\mathbb{F}(E)(j)$, $\ell_i \in \overline{G}_{(1,\ldots,1)}\mathbb{F}(E)(i)$ thus, by the compositional compatibility,

$$(\ell_j \circ_1 \ell_i) \in \overline{G}_{(2,\ldots,2,1,\ldots,1)}\mathbb{F}(E)(k)$$

where 2 occupies the first i positions of the multiindex and 1 the remaining ones, starting with position $i+1$. Since $\sigma(1)=i{+}1$, the first input of $(\ell_j \circ_1 \ell_i)\sigma$ is mapped to the $(i{+}1)$th input of $(\ell_j \circ_1 \ell_i)$, so

$$\mathrm{sgn}(\sigma)(-1)^{i(j-1)}(\ell_j \circ_1 \ell_i)\sigma \in \overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(k)$$

by the equivariance and monotonicity. Thus all the summands of $\mathcal{B}_k$ and thus also $\mathcal{B}_k$ itself belong to $\overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(k)$.

To prove that $\mathcal{A}_k \in \overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(k)$, we decompose it further as

$$\mathcal{A}_k(1,\ldots,k) = \mathcal{A}'_k(1,\ldots,k) + \mathcal{A}''_k(1,\ldots,k) + \mathcal{A}'''_k(1,\ldots,k),$$

where $\mathcal{A}'_k$ is the sum of terms of $\mathcal{A}_k$ with $i=j=1$, and $\mathcal{A}''_k$ is the sum of terms where either i or j, but not both, equals 1, and $\mathcal{A}'''_k$ is the sum of the remaining terms. Clearly, $\mathcal{A}'_k$ is nontrivial only for $k=1$ in which case

$$\mathcal{A}'_1 = \ell_1 \circ \ell_1 = \frac{1}{2}[\ell_1,\ell_1]_{11},$$

so $\mathcal{A}'_1 \in \overline{G}_{(1)}\mathbb{F}(E)(1)$ because $\overline{G}\mathbb{F}(E)$ is a D-multifiltration and $\ell_1 \in \overline{G}_{(1)}\mathbb{F}(E)(1)$. Furthermore,

$$\mathcal{A}''_k = \ell_1 \circ_1 \ell_k - (-1)^k \ell_k \circ_1 \ell_1 = [\ell_1,\ell_k]_{11},$$

thus $\mathcal{A}''_k \in \overline{G}_{(1,2,\ldots,2)}\mathbb{F}(E)(k)$ by the same argument. It remains to analyze $\mathcal{A}'''_k$. Notice that it decomposes as

$$\mathcal{A}'''_k(1,\ldots,k) = \mathcal{A}^<_k(1,\ldots,k) + \mathcal{A}^>_k(1,\ldots,k)$$

where

$$\mathcal{A}^<_k(1,\ldots,k) := \sum_{i+j=k+1}\sum_{\sigma} \mathrm{sgn}(\sigma)(-1)^{i(j-1)}\ell_j(\ell_i(1,\sigma(2),\ldots,\sigma(i)),\sigma(i{+}1),\ldots,\sigma(k))$$

with the second sum is restricted to the shuffles σ with $\sigma(2)<\sigma(i{+}1)$ and thus also $\sigma(1)=1$, while

$$\mathcal{A}^>_k(1,\ldots,k) := \sum_{i+j=k+1}\sum_{\sigma} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)(-1)^{j(i-1)}\ell_i(\ell_j(1,\sigma(i{+}1),\ldots,\sigma(k)),\sigma(2),\ldots,\sigma(i))$$

with the same range for σ and τ being the permutation

$$\sigma(2),\ldots,\sigma(i),\sigma(i{+}1),\ldots,\sigma(k) \longmapsto \sigma(i{+}1),\ldots,\sigma(k),\sigma(2),\ldots,\sigma(i).$$

Now, to any $(i,j{-}1)$-shuffle such that $\sigma(2)<\sigma(i{+}1)$ we associate a $(j,i{-}1)$-shuffle $\sigma^\dagger$ by

$$\sigma^\dagger := (\sigma(1),\sigma(i+1),\ldots,\sigma(k),\sigma(2),\ldots,\sigma(i)).$$

Then, with the same range for σ as before, we have

$$\mathcal{A}_k''' = \mathcal{A}_k^< + \mathcal{A}_k^> = \sum_{i+j=k+1} \sum_\sigma \operatorname{sgn}(\sigma)(-1)^{i(j-1)}(\ell_j \circ_1 \ell_i)\sigma + \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)(-1)^{j(i-1)}(\ell_i \circ_1 \ell_j)\sigma^\dagger$$

from which we conclude that

$$\mathcal{A}_k''' = \sum_{i+j=k+1} \sum_\sigma \operatorname{sgn}(\sigma)(-1)^{i(j-1)}[\ell_j, \ell_i]_{11}\sigma.$$

Since $\ell_i \in \overline{G}_{(1,\dots,1)}\mathbb{F}(E)(i)$ and $\ell_j \in \overline{G}_{(1,\dots,1)}\mathbb{F}(E)(j)$, and since $\overline{G}\mathbb{F}(E)$ is a D-multifiltration, each $[\ell_i, \ell_j]_{11}$ belongs to $\overline{G}_{(1,2,\dots,2)}\mathbb{F}(E)(k)$, and therefore $\mathcal{A}_k''' \in \overline{G}_{(1,2,\dots,2)}\mathbb{F}(E)(k)$. □

4. Operator algebras

Throughout this section, A will be a graded commutative associative $\Bbbk$-algebra. Let $\mathcal{P}$ be a $\Bbbk$-linear operad and $F\mathscr{E}nd_A = \{F_{\vec{p}}\mathscr{E}nd_A(n)\}_{\vec{p},n}$ the multifiltration of Example 2.13. Then the collection $\mathscr{D}iff_A := \{\mathscr{D}iff_A(n)\}_{n\geq 1}$, where

$$\mathscr{D}iff_A(n) := \bigcup_{\vec{p},n} F_{\vec{p}}\mathscr{E}nd_A(n),\ n \geq 1,$$

is, by Proposition 1.6, a multifiltered suboperad of the endomorphism operad $\mathscr{E}nd_A$. It has a multifiltered suboperad $\mathscr{D}er_A = \{\mathscr{D}er_A(n)\}_n$ such that $F_{(p_1,\dots,p_n)}\mathscr{D}er_A(n)$ consists of $\Bbbk$-linear maps $O : A^{\otimes n} \to A$ that are derivations of order p_i in the i-th variable, for each $1 \leq i \leq n$.

Definition 4.1. An *operator $\mathcal{P}$-algebra A* is an operad morphism $\alpha : \mathcal{P} \to \mathscr{D}iff_A$. We say that it is *of order $k \geq 0$*, if $\mathcal{P}$ is generated by a collection $E = \{E(n)\}_{n\geq 1}$ such that

$$\alpha(E(n)) \subset F_{(k,\dots,k)}\mathscr{D}iff_A(n),\ \text{for all } n \geq 1.$$

An operator $\mathcal{P}$-algebra on a graded commutative $\Bbbk$-algebra A is a $\mathcal{P}$-algebra in the ordinary operadic sense, via the composite $\mathcal{P} \to \mathscr{D}iff_A \hookrightarrow \mathscr{E}nd_A$. A *morphism* of operator $\mathcal{P}$-algebras with the underlying graded commutative associative algebras A' resp. A'' is an algebra morphism $f : A' \to A''$ which is simultaneously a morphism of $\mathcal{P}$-algebras in the usual sense [37, Definition II.1.21].

Operator $\mathcal{P}$-algebras can be regarded as algebras over the operad defined as the quotient of the coproduct $\mathcal{P} \sqcup \mathscr{C}om$, where $\mathscr{C}om$ is the operad of graded commutative associative $\Bbbk$-algebras, by the ideal $\mathscr{I}$ expressing that $\mathcal{P}$ acts via (higher order) differential operators with respect to the multiplicative structure encoded by $\mathscr{C}om$. Aside from some exceptional cases, such as Poisson algebras recalled in Example 4.4, the ideal $\mathscr{I}$ is not generated by a distributive law in the sense of [31]. Indeed, as proven in [3], the only operad tied to $\mathscr{C}om$ via a nontrivial distributive law is the operad $\mathscr{L}ie$. The relations expressing the higher derivation property do not even have the form resembling a distributive law.

Example 4.2. Commutative associative algebras are operator algebras of order 0.

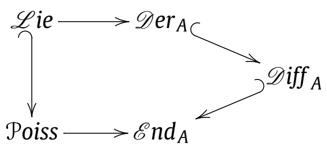
Example 4.3. A *Jacobi structure* on a manifold M is an operator Lie algebra on $C^\infty(M)$. A well-known [18,13] structure theorem gives a precise characterization of the only non-trivial bracket $[-,-]_1$. Namely,

$$[f, g]_1 = \Pi(df, dg) + \xi \lrcorner (f\,dg - g\,df)$$

for a 2-vector field Π and a 1-vector field ξ on M subject to the compatibility conditions

$$[\xi, \Pi] = 0, \quad [\Pi, \Pi] = 2\xi \wedge \Pi.$$

Example 4.4. A Poisson algebra A is, by definition, a Lie algebra whose underlying space is a commutative associative algebra such that the Lie bracket is a derivation of order 1 in each variable. It is thus the same as an operad map $\mathscr{L}ie \to \mathscr{D}er_A$ that sends the generator of $\mathscr{L}ie(2)$ into an antisymmetric operation $A^{\otimes 2} \to A$ which is an order 1 derivation in each variable. That is, A is operator $\mathscr{L}ie$-algebra of order 1. The situation is summarized by the commutative diagram



of operad morphisms in which $\mathcal{P}oiss$ is the operad governing Poisson algebras.

Example 4.5. Let $\mathcal{D}g$ be the operad treated in Example 2.4. A Batalin-Vilkovisky algebra with the underlying commutative associative algebra A is given by an operad morphism $\mathcal{D}g \to \mathscr{D}er_A$ that sends the generator of $\mathcal{D}g(1)$ to a derivation of order 2 and degree -1 with respect to the grading of A. Batalin-Vilkovisky algebras are therefore operator algebras of order 2. The situation is expressed by the diagram

$$
\begin{array}{ccccc}
\mathcal{D}g & \longrightarrow & \mathscr{D}er_A & & \\
\big\downarrow & & & \searrow & \\
 & & & & \mathscr{D}\!iff_A\ , \\
 & & & \swarrow & \\
\mathcal{BV} & \longrightarrow & \mathscr{E}nd_A & &
\end{array}
$$

where $\mathcal{BV}$ is the operad governing Batalin-Vilkovisky algebras.

On a practical side, given a $\Bbbk$-linear operad $\mathcal{P} = \mathbb{F}(E)/(R)$, defining an operator $\mathcal{P}$-algebra on a commutative associative $\Bbbk$-algebra A amounts to assigning a multilinear differential operator to each generator of $\mathcal{P}$ and verifying the defining relations within $\mathscr{D}\!iff_A$. The latter task simplifies if each of the relations gets mapped to a multilinear differential operator of order 1 in each variable, which is, in particular, the case for tight operads. More specifically, we have the following

Corollary 4.6 *(to Lemma 3.9). Let $\mathcal{P}$ be a $\Bbbk$-linear operad with a tight presentation $\mathcal{P} = \mathbb{F}(E)/(R)$, A be a commutative associative unital $\Bbbk$-algebra with a set of generators S and $\tilde{\alpha} : \mathbb{F}(E) \to \mathscr{D}\!iff_A$ be an operad morphism such that $\tilde{\alpha}(E(n)) \subseteq F_{(1,\dots,1)}\mathscr{D}\!iff_A$ for all $n \geq 1$. Then $\tilde{\alpha}$ factorizes through $\alpha : \mathbb{F}(E)/(R) \to \mathscr{D}\!iff_A$ if and only if $O(x_1, \dots, x_n) = 0$ for all $O \in \tilde{\alpha}(R(n))$, $x_i \in S \sqcup \{1\}$, $i = 1, 2, \dots, n$, and $n \geq 1$.*

Proof. By Lemma 3.9, $\tilde{\alpha}(R(n)) \subseteq F_{(1,\dots,1)}\mathscr{D}\!iff_A(n)$ for all $n \geq 1$. The claim follows now upon noting that by (9b) and by setting $x = 1$ in (2), a differential operator $\nabla : A \to A$ of order ≤ 1 is identically zero if and only if $\nabla(x) = 0$ for each $x \in S \sqcup \{1\}$. $\square$

If A is not necessarily unital, an analogous result holds upon replacing $\mathscr{D}\!iff_A$ in the statement of the above corollary by $\mathscr{D}er_A$ and restricting x_i's to the generators S.

We are going to introduce a deformed version of the multifiltered operads $\mathscr{D}\!iff_A$ and $\mathscr{D}er_A$ to account for the case of algebraic structures with operations representable as formal series of differential operators. To this end, let h be a formal parameter of an even degree and $\mathscr{E}nd_{A[\![h]\!]}$ be the endomorphism operad of the $\Bbbk[\![h]\!]$-module $A[\![h]\!]$ with the multifiltration $F\mathscr{E}nd_{A[\![h]\!]} = \{F_{\vec{p}}\mathscr{E}nd_{A[\![h]\!]}(n)\}_{\vec{p},n}$ of Example 2.14. The collection $\mathscr{D}\!iff_A[\![h]\!] := \{\mathscr{D}\!iff_A[\![h]\!](n)\}_{n\geq 1}$, where

$$\mathscr{D}\!iff_A[\![h]\!](n) := \bigcup_{\vec{p}\in M\mathbb{Z}(n)} F_{\vec{p}}\mathscr{E}nd_{A[\![h]\!]}(n)\ ,\ n \geq 1,$$

form a multifiltered suboperad of the endomorphism operad $\mathscr{E}nd_{A[\![h]\!]}$. Analogously we define a multifiltered suboperad $\mathscr{D}er_A[\![h]\!]$ of $\mathscr{D}\!iff_A[\![h]\!]$, requiring that the multilinear maps $O_0, O_1, O_2, \dots$ in (17) are derivations in each of its variables of the indicated degrees.

Definition 4.7. A *formal operator $\mathcal{P}$-algebra* is an operad morphism $\alpha : \mathcal{P} \to \mathscr{D}\!iff_A[\![h]\!]$. We say that such an algebra is *of order* $k \geq 0$, if $\mathcal{P}$ admits a generating collection $E = \{E(n)\}_{n\geq 1}$ such that

$$\alpha(E(n)) \subset F_{(k,\dots,k)}\mathscr{D}\!iff_A[\![h]\!](n),\ \text{for all } n \geq 1.$$

Expanding this definition, we note that a formal operator $\mathcal{P}$-algebra $\alpha : \mathcal{P} \to \mathscr{D}\!iff_A[\![h]\!]$ of order k is a $\mathcal{P}$-algebra supported on $A[\![h]\!]$ and such that its n-ary generating operations are of the form

$$O(a_1, \dots, a_n) = O_0(a_1, \dots, a_n) + O_1(a_1, \dots, a_n) \cdot h + O_2(a_1, \dots, a_n) \cdot h^2 + \dots$$

where each O_s is a multilinear differential operator of order $\leq k + s$ with respect to each of its arguments. In particular, any formal operator algebra of order k is automatically a formal operator algebra of order l for any $l > k$. The canonical inclusion $A \hookrightarrow A[\![h]\!]$ and the reduction $A[\![h]\!] \to A$ mod h determines an operad morphism $\mathscr{E}nd_{A[\![h]\!]} \to \mathscr{E}nd_A$, which restricts to a morphism of multifiltered suboperads $\pi : \mathscr{D}\!iff_A[\![h]\!] \to \mathscr{D}\!iff_A$ as per Definition 2.9.

A *morphism* of formal operator $\mathcal{P}$-algebras with underlying algebras A' resp. A'' is a $\Bbbk[\![h]\!]$-linear morphism $f : A'[\![h]\!] \to A''[\![h]\!]$ of graded commutative associative algebras which is also a standard morphism of $\mathcal{P}$-algebras [37, Definition II.1.21] with underlying graded vector spaces $A'[\![h]\!]$ resp. $A''[\![h]\!]$.

Definition 4.8. The *semiclassical limit* of a formal operator algebra $\mathcal{P} \to \mathscr{D}\!iff_A[\![h]\!]$ is an operator $\mathcal{P}$-algebra whose structure map is the composite

$$\mathcal{P} \to \mathscr{D}iff_A[\![h]\!] \xrightarrow{\pi} \mathscr{D}iff_A.$$

The semiclassical limit of a formal operator algebra of order k is an operator algebra of the same order.

Example 4.9. An example of a formal operator $\mathcal{P}$-algebra of order 0 for $\mathcal{P}$ the associative operad $\mathcal{A}ss$ is provided by Terilla's quantization [47] as recalled in Subsection 8.1. Its semiclassical limit is the commutative associative algebra $\widehat{\mathbb{S}}(V \oplus V^*)$. Another example of the same type is the celebrated Kontsevich deformation quantization of a Poisson manifold M [19]. Its semiclassical limit is the algebra $C^\infty(M)$ of smooth functions on M.

Part 2. Applications

5. Formal operator L_∞-algebras

This and the following section are devoted to a discussion of operator L_∞-algebras and some of their instances – IBL_∞-algebras, commutative BV_∞-algebras, operator Lie algebras and Poisson algebras.

5.1. Formal operator L_∞-algebras

Let A be a graded commutative associative algebra.

Definition 5.1. A *formal operator L_∞-algebra* is an L_∞-algebra whose underlying vector space L is $A[\![h]\!] := A \otimes \Bbbk[\![h]\!]$, the structure operations are $\Bbbk[\![h]\!]$-linear and such that, for each $a_1, \ldots, a_k \in A \subset A[\![h]\!]$,

$$l_k(a_1, \ldots, a_k) = \sum_{n \geq 1} l_{k,n}(a_1, \ldots, a_k) \cdot h^{n-1}, \tag{56}$$

where $l_{k,n} : A^{\otimes k} \to A$ is a differential operator of order n in each variable. In terms of Definition 4.7, such an algebra is a formal operator $\mathscr{L}_\infty$-algebra of order one.

Example 5.2. If the underlying algebra A is a graded vector space with the trivial multiplication, then the differential operators of order one on A are all $\Bbbk$-linear maps. Thus the usual L_∞-algebras can be thought of as a particular case of the formal operator L_∞-algebras with the structure operations $l_{k,n}$ vanishing for $n > 1$. In Example 5.7 we describe a less trivial embedding of the category of L_∞-algebras with weak morphisms into the category of formal operator L_∞-algebras.

Example 5.3. In a similar vein, L. Vitagliano's *multiderivation L_∞-algebras* [49] are a particular case of formal operator L_∞-algebras, where $l_{k,n} = 0$ for $n > 1$ and each $l_{k,1}$ is a derivation of order one in each of its variables for all $k \geq 1$.

Example 5.4. Assume that A is unital. A formal operator L_∞-algebra whose all structure operations except $l_1 : A[\![h]\!] \to A[\![h]\!]$ vanish and $l_1(1) = 0$, is the same as a *commutative* BV_∞*-algebra* [23, Definition 7]. In particular, if $A = \mathbb{S}(V)$ for a graded vector space V, we recover the definition of an IBL_∞-algebra [43, Subsection 4.2], cf. also [39, Example 9].

Due to the $\Bbbk[\![h]\!]$-linearity of the structure operations l_k and the decomposition (56), the condition $\mathrm{Jac}_k(\lambda_1, \ldots, \lambda_k) = 0$ for $\lambda_1, \ldots, \lambda_k \in A[\![h]\!]$ is equivalent to the system of equalities

$$\mathrm{Jac}_{k,n}(a_1, \ldots, a_k) := \sum \chi(\sigma)(-1)^{i(j-1)} l_{j,s}(l_{i,t}(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(k)}) = 0, \tag{57}$$

where i, j and σ run over the same ranges as in (53) and $s, t \geq 1$ run over all values such that $s + t = n + 1$, for each $n \geq 1$ and $a_1, \ldots, a_k \in A$.

Proposition 5.5. *The multilinear map $\mathrm{Jac}_{k,n} : A^{\otimes k} \to A$ defined in (57) is a differential operator of order n in each of its variables.*

Proof. We will refer to the notation introduced in the paragraph before Proposition 3.13. Let l_k, $k \geq 1$, be as in (56) and notice that $l_k \in \overline{G}_{(1,\ldots,1)}\mathscr{D}iff_A[\![h]\!](k)$. Let us define an operad map $\tilde{\alpha} : \mathbb{F}(E) \to \mathscr{D}iff_A[\![h]\!]$ by $\tilde{\alpha}(\ell_k) := l_k$, $k \geq 1$. Then clearly

$$\tilde{\alpha}(\mathcal{J}ac_k) = \sum_{n \geq 1} \mathrm{Jac}_{k,n} \cdot h^{n-1}.$$

Since $\mathscr{L}_\infty$ is tight by Proposition 3.13, then Lemma 3.9 implies that $\tilde{\alpha}(\mathcal{J}ac_k)$ and thus also the right-hand side of the above display belongs to $\overline{G}_{(1,\ldots,1)}\mathscr{D}iff_A[\![h]\!](k)$, which is equivalent to the statement of the proposition. □

Remark 5.6. It is known that the individual structure operations $l_k : \bigotimes^k L \to L$, $k \geq 1$, of an L_∞-algebra can be assembled into a degree -1 coderivation δ on the cofree conilpotent cocommutative coassociative coalgebra $\mathbb{S}^c(\uparrow L)$ cogenerated by the suspension of the underlying vector space L. Then the infinite system of relations (53) is equivalent to a single equation $\delta^2 = 0$, cf. [25, Theorem 2.3].

Likewise, the structure operations of a formal operator L_∞-algebra in Definition 5.1 assemble into a coderivation δ_h of $\mathbb{S}^c(\uparrow L)[\![h]\!]$ that squares to 0. We however do not know how to express conveniently the required decomposition (56) in terms of δ_h.

Example 5.7. Consider an L_∞-algebra whose structure operations $l_k : \bigotimes^k L \to L$, $k \geq 1$, are assembled into a coderivation δ of the coalgebra $\mathbb{S}^c(\uparrow L)$ as in Remark 5.6. Clearly

$$\delta = \delta_1 + \delta_2 + \delta_3 + \cdots ,$$

where δ_k, corresponding to l_k, takes $\mathbb{S}^k(\uparrow L)$ to $\uparrow L$. By inspecting explicit formulas [26, Eq. (3)] for the components δ_k's of the coderivation δ (denoted $\hat{l}_n$'s in loc. cit.), it is simple to check that, under the canonical isomorphism of graded vector spaces $\mathbb{S}^c(\uparrow L) \cong \mathbb{S}(\uparrow L)$, each δ_k is a differential operator of order $\leq k$ on the free graded commutative associative algebra $\mathbb{S}(\uparrow L)$. Taking, in Definition 5.1, $A := \mathbb{S}(\uparrow L)$, $l_k := 0$ for $k \geq 2$, and

$$l_1 := \delta_1 + \delta_2 h + \delta_3 h^2 + \cdots ,$$

one represents the initial classical L_∞-algebra as a formal operator L_∞-algebra with the underlying commutative associative algebra $\mathbb{S}(\uparrow L)$.

A (weak) *morphism* of L_∞-algebras that are represented in the language of Remark 5.6 by the dg coalgebras by $\big(\mathbb{S}^c(\uparrow L'), \delta'\big)$ resp. $\big(\mathbb{S}^c(\uparrow L''), \delta''\big)$ is, by definition, a dg coalgebra morphism

$$F : \big(\mathbb{S}^c(\uparrow L'), \delta'\big) \longrightarrow \big(\mathbb{S}^c(\uparrow L''), \delta''\big). \tag{58}$$

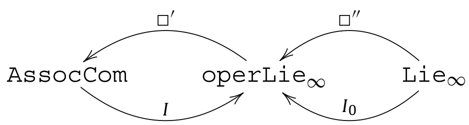
It turns out that (58) is the same as a system

$$f_k : \bigotimes^k L' \to L'',\ k \geq 1, \tag{59}$$

of degree $k-1$ graded antisymmetric linear maps that satisfy an infinite system of equations listed e.g. in [12, Section 7.3], whose explicit form is not needed in the present article.

Analogously, a (weak) *morphism* of formal operator L_∞-algebras with underlying graded commutative associative algebras A' resp. A'' is given by a system (59) with $L' := A'[\![h]\!]$, $L'' := A''[\![h]\!]$ satisfying equations in [12, Section 7.3], such that each f_k is $\Bbbk[\![h]\!]$-linear in each variable and, moreover, $f_1 : A'[\![h]\!] \to A''[\![h]\!]$ is a morphism of commutative associative algebras. With this definition, Poisson algebras and their morphisms form a subcategory of formal operator L_∞-algebras with $l_{2,0}$ a derivation in both variables, and all other $l_{k,n}$'s trivial.

Let us denote by $\mathtt{AssocCom}$, $\mathtt{Lie}_\infty$ and $\mathtt{operLie}_\infty$ the categories of commutative associative algebras, L_∞-algebras and formal operator L_∞-algebras, respectively. We clearly have a diagram of functors



in which $\square'$ forgets the L_∞-structure and remembers the underlying commutative associative algebra only, while the functor I equips a commutative associative algebra with the trivial L_∞-structure. The functor I_0 interprets an L_∞-algebra as a formal operator L_∞-algebra supported on a trivial graded commutative associative algebra.

Finally, $\square'' : \mathtt{operLie}_\infty \to \mathtt{Lie}_\infty$ forgets the underlying commutative associative algebra structure and remembers only the operations $l_{n,k}$ with $n = 1$, cf. (56) for the notation. One is tempted to define another functor $\mathtt{operLie}_\infty \to \mathtt{Lie}_\infty$ by putting $h = 1$ to (56), but the infinite sum thus obtained might not be well defined. Examples when this happens are easy to construct.

6. Formal operator Lie algebras

In this section we consider the case of formal operator algebras over the operad $\mathscr{L}ie$.

Definition 6.1. A *formal operator Lie algebra* is a formal operator $\mathscr{L}ie$-algebra of order one in the sense of Definition 4.7.

Explicitly, it is given by a graded commutative associative algebra A and a $\Bbbk[\![h]\!]$-linear Lie bracket $[-,-]$ on $A[\![h]\!]$ whose restriction onto $A \subset A[\![h]\!]$ decomposes as

$$[a', a''] = \sum_{n\geq 1} [a', a'']_n \cdot h^{n-1},\ a', a'' \in A, \tag{60}$$

where $[-,-]_n : A \otimes A \to A$ is a differential operator of order n in each variable. Such an algebra is said to be a *formal derivation Lie algebra* if $[-,-]_n$ is a derivation of order n in each variable.

Trivially, a formal operator Lie algebra A with $[-,-]_n$ vanishing for $n \geq 2$ is an operator $\mathscr{L}ie$ algebra of order one, which includes Poisson algebras as a special case, cf. Example 4.4. An example of a formal operator Lie algebra with non-trivial higher-order brackets is to be presented in Section 7.2. In that regard, the following result providing a way of constructing bilinear differential operators of arbitrary orders will be quite useful for us:

Lemma 6.2. *Let X be a graded vector space and*

$$\left\{\Upsilon(-,-)^{ij}_n : \mathbb{S}^i(X) \otimes \mathbb{S}^j(X) \longrightarrow \mathbb{S}(X) \mid 1 \leq i, j \leq n\right\} \tag{61}$$

be a family of $\Bbbk$-linear maps such that

$$\Upsilon(a', a'')^{ij}_n = (-1)^{|a'||a''|} \cdot \Upsilon(a'', a')^{ji}_n,\ \ \text{for } a' \otimes a'' \in \mathbb{S}^i(X) \otimes \mathbb{S}^j(X). \tag{62}$$

Then the formula

$$\begin{aligned} &[x'_1 x'_2 \cdots x'_s, x''_1 x''_2 \cdots x''_t]_n := \\ &\quad \sum_{1\leq i,j\leq n} \sum_{\sigma,\mu} \epsilon(\sigma)\epsilon(\mu) x'_{\sigma(1)} \cdots x'_{\sigma(s-i)} \Upsilon(x'_{\sigma(s-i+1)} \cdots x'_{\sigma(s)}, x''_{\mu(1)} \cdots x''_{\mu(j)})^{ij}_n x''_{\mu(j+1)} \cdots x''_{\mu(t)} \end{aligned} \tag{63}$$

where σ runs over $(s-i,i)$-shuffles, μ runs over $(j,t-j)$-shuffles, $x'_1 \cdots x'_s \in \mathbb{S}^s(X)$, $x''_1 \cdots x''_t \in \mathbb{S}^t(X)$, defines a unique graded antisymmetric operation $[-,-]_n : \mathbb{S}(X) \otimes \mathbb{S}(X) \to \mathbb{S}(X)$ which is a derivation of order n in each variable and such that $\Upsilon(-,-)^{ij}_n$ is equal to the restriction of $[-,-]_n$ onto $\mathbb{S}^i(X) \otimes \mathbb{S}^j(X)$.

Proof. Existence follows by a direct verification, cf. the formula at the top of page 374 of [33], whereas uniqueness is due to Lemma 1.5. □

Now, given a formal operator Lie algebra A, denote for $a, b, c \in A$ and $n \geq 1$

$$\mathrm{Jac}_n(a,b,c) := \sum_{s+t=n+1} \left\{(-1)^{|a||c|}[a,[b,c]_s]_t + (-1)^{|b||a|}[b,[c,a]_s]_t + (-1)^{|c||b|}[c,[a,b]_s]_t\right\}. \tag{64}$$

Note that the Jacobi identity for $[-,-]$ is equivalent to

$$\mathrm{Jac}_n(a,b,c) = 0 \tag{65}$$

holding for all $n \geq 1$ and $a, b, c \in A$. Since Jac_n can be identified, up to a sign, with $\mathrm{Jac}_{3,n}$ of equation (57), then by Proposition 5.5, it is a differential operator of order n in each of its three variables. This observation leads to the following

Corollary 6.3. *Let $A = \mathbb{S}(X)$ for a graded vector space X, and $[-,-] : A[\![h]\!] \otimes A[\![h]\!] \to A[\![h]\!]$ be an antisymmetric $\Bbbk[\![h]\!]$-linear mapping admitting a decomposition as in (60). Then for each $n \geq 1$, (65) is satisfied for all $a, b, c \in \mathbb{S}(X)$ if and only if it holds for all $a, b, c \in \mathbb{S}^{\leq n}(X)$. If each of the brackets in (60) is a derivation of the corresponding order with respect to each of the arguments, then it is enough to check this condition for $a, b, c \in \mathbb{S}^{\leq n}_+(X)$.*

Corollary 6.4. *A graded Poisson algebra with the underlying graded commutative associative algebra $\mathbb{S}(X)$ is uniquely determined by a graded antisymmetric bilinear map $\langle -,-\rangle : X \otimes X \to \mathbb{S}(X)$ whose extension $[-,-] : \mathbb{S}(X) \otimes \mathbb{S}(X) \to \mathbb{S}(X)$ into a derivation in each variable satisfies the Jacobi identity on $X \otimes X \otimes X$.*

Proof. The extension $[-,-]$ of $\langle -,-\rangle$ mentioned in the corollary is given by formula (63) with

$$\Upsilon(-,-)^{ij}_n := \begin{cases} \langle -,-\rangle & \text{for } i=j=n=1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The only nontrivial Jacobian (64) is $\mathrm{Jac}_1(a,b,c)$, which is a derivation of order 1 in each variable. Hence, by Corollary 6.3, it is enough to verify that it vanishes for any $a, b, c \in \mathbb{S}^{\leq 1}_+(X) = X$. □

Example 6.5. The basic example illustrating Corollary 6.4 is obtained by taking X to be a Lie algebra L with the Lie bracket $\langle -,-\rangle$. Note that assignment $L \mapsto \mathbb{S}(L)$ yields a left adjoint to the forgetful functor from the category of Poisson algebras to the category of Lie algebras. By imposing an additional degree shift on L and treating the bracket as a bilinear mapping $\uparrow L \otimes \uparrow L \to \uparrow L$ of degree (-1), the same construction returns the Schouten bracket on $\wedge(L) \simeq \mathbb{S}(\uparrow L)$.

Remark 6.6. Let us consider the complete topological algebra $\widehat{\mathbb{S}}(X) := \lim_k \mathbb{S}(X)/\mathbb{S}^{\geq k}(X)$ where,

$$\widehat{\mathbb{S}}^{\geq k}(X) := \bigoplus_{n\geq k} \mathbb{S}^n(X).$$

It is not difficult to prove that each differential operator ∇ of order $\leq r$ defined on $\mathbb{S}(X)$ uniquely extends into a continuous linear map $\widehat{\mathbb{S}}(X) \to \widehat{\mathbb{S}}(X)$. Moreover, any algebraic equation that ∇ satisfies is, by continuity, satisfied also for its extension. In particular, any Poisson algebra with the underlying commutative associative algebra $\mathbb{S}(X)$ determines a Poisson algebra with the underlying commutative associative algebra $\widehat{\mathbb{S}}(X)$.

7. L_∞-bialgebras, IBL_∞-algebras, the big and the superbig bracket

The section is devoted to a discussion of a certain formal operator Lie algebra with non-vanishing higher-order brackets arising in the context of the deformation theory of involutive Lie bialgebras. We precede the actual discussion of the subject with a brief recollection of some auxiliary facts concerning linear algebra in the graded setting, followed by a reminder on the big bracket algebra.

Let W be a graded vector space, $\uparrow W$ its suspension and $\uparrow : W \to \uparrow W$ the obvious degree $+1$ isomorphism. The exterior (aka Grassmann) algebra generated by W is the quotient

$$\wedge(W) := T(W)/\mathcal{I}$$

of the tensor algebra $T(W)$ modulo the ideal $\mathcal{I}$ generated by the expressions

$$w' \otimes w'' + (-1)^{|w'||w''|} w'' \otimes w'$$

with homogeneous $w', w'' \in W$. One has a sequence $\{f_n\}_{n\geq 0}$ of $\Bbbk$-linear degree n décalage isomorphisms

$$f_n : \wedge^n(W) \longrightarrow \mathbb{S}^n(\uparrow W),$$

between the components spanned by the products of generators of length n, inductively defined by $f_0 := \mathbb{1}_\Bbbk$, $f_1(w) := \uparrow w$ for $w \in W$ while

$$f_{a+b}(u \wedge v) := (-1)^{b|u|} f_a(u) \cdot f_b(v), \text{ for } u \in \wedge^a(W),\ v \in \wedge^b(W),\ a, b \geq 1. \tag{66}$$

The family $\{f_n\}_{n\geq 0}$ assembles into an isomorphism of non-graded algebras

$$f : \wedge(W) \cong \mathbb{S}(\uparrow W) \tag{67}$$

whose components satisfy (66). Likewise one defines a sequence $\{g_n\}_{n\geq 0}$ of linear degree $2n$ isomorphisms

$$g_n : \mathbb{S}^n(\downarrow W) \longrightarrow \mathbb{S}^n(\uparrow W),$$

inductively by $g_0 := \mathbb{1}_\Bbbk$, $g_1(\downarrow w) := \uparrow w$ for $w \in W$, while

$$g_{a+b}(u \cdot v) := g_a(u) \cdot g_b(v), \text{ for } u \in \mathbb{S}^a(\downarrow W),\ v \in \mathbb{S}^b(\downarrow W),\ a, b \geq 1.$$

The family $\{g_n\}_{n\geq 0}$ again gives rise to an isomorphism

$$g : \mathbb{S}(\downarrow W) \cong \mathbb{S}(\uparrow W).$$

7.1. The big bracket

Let V be a graded $\Bbbk$-vector space, $\uparrow V$ its suspension, and $\downarrow V^*$ the desuspension of its linear dual. In what follows, we will assume that V is finite-dimensional. This assumption can be relaxed using the compact-linear topology on dual spaces and completed tensor products [30, Chapter 1], but since this generalization brings nothing conceptually new, we will stick to the finite-dimensional case. Take, in Corollary 6.4, $X := \uparrow V \oplus \downarrow V^*$ and $\langle -,-\rangle : X \otimes X \to \mathbb{S}(X)$ given by

$$\langle \alpha, u\rangle = -(-1)^{|u||\alpha|}\langle u, \alpha\rangle := \alpha(u) \in \Bbbk = \mathbb{S}^0(X) \subset \mathbb{S}(X) \tag{68}$$

for $\alpha \in \downarrow V^*$, $u \in \uparrow V$, while $\langle \uparrow V, \uparrow V \rangle = \langle \downarrow V^*, \downarrow V^* \rangle := 0$. Notice that $|u|$ must equal $|\alpha|$ for $\langle \alpha, u \rangle$ above to be nonzero. The assumptions of Corollary 6.4 are easy to verify. Denote by $\big(\mathbb{S}(\uparrow V \oplus \downarrow V^*), [-,-]\big)$ the resulting Poisson algebra.

Take $B(V) := \downarrow^2 \mathbb{S}(\uparrow V \oplus \uparrow V^*)$ and equip $B(V)$ with the Lie bracket $\{-,-\}$ induced from the one on $\mathbb{S}(\uparrow V \oplus \downarrow V^*)$ by the vector space isomorphism

$$\mathbb{S}(\uparrow V \oplus \downarrow V^*) \cong \mathbb{S}(\uparrow V) \otimes \mathbb{S}(\downarrow V^*) \xrightarrow{\mathbb{1} \otimes g} \mathbb{S}(\uparrow V) \otimes \mathbb{S}(\uparrow V^*) \cong \mathbb{S}(\uparrow V \oplus \uparrow V^*) \xrightarrow{\downarrow^2} \downarrow^2 \mathbb{S}(\uparrow V \oplus \uparrow V^*) = B(V). \tag{69}$$

Since the above isomorphism involves even degree shifts only, the related signs issues are trivial. The bracket $\{-,-\}$ thus constructed is the *big bracket* of [27]. Applying (67) gives its standard presentation

$$\big(B(V)^*, \{-,-\}\big) \cong \big(\wedge^{*+2}(V \oplus V^*), \{-,-\}\big), \tag{70}$$

cf. [24, Equation (2)].

Let $B^p_q(V) \subset B(V)$ be the subspace of $B(V)$ spanned, in the presentation (70), by the exterior products of p elements of V and q elements of V^*. Maurer-Cartan elements in $B^1_2(V) \oplus B^2_1(V)$ describe Lie bialgebras, and those in $B^1_2(V) \oplus B^2_1(V) \oplus B^0_3(V)$ Lie quasi-bialgebras [23, Section 3].

To make room for L_∞-bialgebras, we define $\mathfrak{g}_{biLie}(V)$ to be the Poisson subalgebra of $B(V)$ with the underlying space

$$\mathfrak{g}_{biLie}(V) := \bigoplus_{p,q \geq 1,\ p+q \geq 3} B(V)^p_q. \tag{71}$$

Its closure $\widehat{\mathfrak{g}}_{biLie}(V)$ in $\downarrow^2 \widehat{\mathbb{S}}(\uparrow V \oplus \uparrow V^*)$ has an induced Poisson structure by Remark 6.6. Maurer-Cartan elements in $\widehat{\mathfrak{g}}_{biLie}(V)$ are known to describe L_∞-bialgebras [23, Subsection 4.4].

Remark 7.1. Recall that works [32,41] provide an explicit method of constructing L_∞-algebras controlling deformations of algebraic structures starting with a cofibrant or, if it exists, minimal model of the governing operad- or PROP-like object. For structures with quadratic relations, the resulting L_∞-algebra is actually a dg-Lie algebra [32, Proposition 2].

It turns out that $\widehat{\mathfrak{g}}_{biLie}(V)$ is precisely the Lie algebra capturing deformations of Lie bialgebras constructed using the explicit minimal model of the properad $\mathcal{L}ie\mathcal{B}$ for Lie bialgebras. Indeed, by definition, the degree n part of the completion $\widehat{\mathfrak{g}}_{biLie}(V)$ consists of infinite families

$$\big\{ f^p_q \in B^p_q(V) \mid p, q \geq 1,\ p+q \geq 3 \big\}, \tag{72}$$

where f^p_q, interpreted as a linear map $\wedge^q V \to \wedge^p V$, raises the degree by $(p+q) - n - 2$.

On the other hand, according to [38, Example 20], cf. also [42, Eq. (2)], the minimal model of the properad for Lie bialgebras is generated by the collection Υ spanned by the generators ξ^p_q, $p, q \geq 1$, $p+q \geq 3$, with q fully antisymmetric inputs and p fully antisymmetric outputs, placed in degree $p+q-3$. By [32, Eq. (9)], the degree n piece of the Lie algebra capturing deformations of Lie bialgebras is given by

$$C^n_{biLie}(V;V) := Lin^{1-n}_{\Sigma-\Sigma}(\Upsilon, \mathtt{End}_V).$$

Here $\mathtt{End}_V$ denotes the endomorphism properad of V and $Lin^{1-n}_{\Sigma-\Sigma}(\Upsilon, \mathtt{End}_V)$ the vector space of $\Sigma - \Sigma$-equivariant degree $1-n$ maps $\Upsilon \to \mathtt{End}_V$ of bicollections. The elements of $C^n_{biLie}(V;V)$ are precisely the families (72), the map f^p_q being the image of the generator ξ^p_q of Υ. This verifies that $\widehat{\mathfrak{g}}_{biLie}(V) \cong C_{biLie}(V;V)$ as graded vector spaces.

The Lie bracket on $C_{biLie}(V;V)$ is determined by the differential of the minimal model of $\mathcal{L}ie\mathcal{B}$ by formula (11) of [32]. Applying that formula to the differential described pictorially in [42, Eq. (3)], we conclude that the bracket $\{f'^p_q, f''^r_s\}$ of generators in (72) is the sum of all possible contractions of one output of f''^r_s with one input of f'^p_q, minus the sum of all contractions of one output of f'^p_q with one input of f''^r_s, with appropriate signs. This is precisely what the bracket in $\widehat{\mathfrak{g}}_{biLie}(V)$ does.

We are going to give, following [4, Subsection 3.1], an interesting alternative description of $\widehat{\mathfrak{g}}_{biLie}(V)$. Choose a basis $(e_1, e_2, \ldots)$ of V and its dual basis $(\alpha^1, \alpha^2, \ldots)$ of V^*. Let

$$(\psi_1, \psi_2, \ldots) := (\uparrow e_1, \uparrow e_2, \ldots) \text{ resp. } (\eta^1, \eta^2, \ldots) := (\uparrow \alpha^1, \uparrow \alpha^2, \ldots)$$

be the corresponding bases of $\uparrow V$ resp. $\uparrow V^*$. Elements of $\widehat{\mathbb{S}}(\uparrow V \oplus \uparrow V^*)$ then appear as power series $f \in \Bbbk[\![\psi, \eta]\!] := \Bbbk[\![\psi_1, \psi_2, \ldots, \eta^1, \eta^2, \ldots]\!]$. In this language, $\widehat{\mathfrak{g}}_{biLie}(V)$ consists of power series $f \in \Bbbk[\![\psi, \eta]\!]$ satisfying the boundary conditions

$$f(\psi, \eta) \in \mathfrak{m}^3,\ f(\psi, \eta)|_{\psi_i = 0} = 0,\ f(\psi, \eta)|_{\eta^j = 0} = 0,\ i, j = 1, 2, \ldots,$$

where $\mathfrak{m}$ is the maximal ideal of the complete local ring $\Bbbk[\![\psi,\eta]\!]$, assigned the degree two less than the degree of f in $\Bbbk[\![\psi,\eta]\!]$. With this convention, the big bracket is expressed as

$$\{f,g\} := \sum_{i=1,2,\ldots}\left(\frac{\partial f}{\partial \eta^i}\frac{\partial g}{\partial \psi_i} - (-1)^{|f|\cdot|g|}\frac{\partial g}{\partial \eta^i}\frac{\partial f}{\partial \psi_i}\right).$$

The big bracket turns out to be the semiclassical limit, in the sense of Definition 4.8, Part 1, of the superbig bracket constructed in the following subsection.

7.2. The superbig bracket

We will construct a formal operator Lie algebra $(B(V)[\![\hbar]\!], \{\!\{\!\{-,-\}\!\}\!\})$ deforming the big bracket algebra recalled earlier. Similarly to the big bracket, the only input required for constructing $\{\!\{\!\{-,-\}\!\}\!\}$ is the natural pairing between the graded vector space V and its linear dual V^*, extended to the corresponding exterior algebras. A Lie algebra $\mathfrak{g}_{IBL}(V)$, whose completion $\widehat{\mathfrak{g}}_{IBL}(V)$ constructed in [4] controls deformations of involutive Lie bialgebras, is going to be contained in $B(V)[\![\hbar]\!]$ as a subalgebra. Consequently, both $\mathfrak{g}_{IBL}(V)$ and $\widehat{\mathfrak{g}}_{IBL}(V)$ are intrinsic in the sense of [46]. The material below may also be regarded as an elementary version of the construction of $\widehat{\mathfrak{g}}_{IBL}(V)$ given in [4].

Let V be a graded $\Bbbk$-vector space and $X = \uparrow V \oplus \downarrow V^*$. Using the notation of Lemma 6.2, we define

$$\Upsilon(-,-)^{ij}_n : \mathbb{S}^i(X)\otimes\mathbb{S}^j(X) \longmapsto \mathbb{S}(X)$$

for all $n \geq 1$ and $1 \leq i, j \leq n$ as follows. First, for $n \geq 1$ and $i \neq n$ or $j \neq n$, $\Upsilon(-,-)^{ij}_n := 0$. Next, to define the terms for $n \geq 1$ and $i = j = n$, we invoke the canonical isomorphism

$$\mathbb{S}^n(X) \cong \bigoplus_{p+q=n}\mathbb{S}^p(\uparrow V)\otimes\mathbb{S}^q(\downarrow V^*) \cong \bigoplus_{p+q=n}\mathbb{S}^p(\uparrow V)\otimes\mathbb{S}^q(\downarrow V)^*$$

and set

$$\Upsilon\big(\mathbb{S}^p(\uparrow V)\otimes\mathbb{S}^q(\downarrow V^*),\ \mathbb{S}^s(\uparrow V)\otimes\mathbb{S}^t(\downarrow V^*)\big)^{nn}_n := 0$$

if $(p,q;s,t) \notin \big\{(n,0;0,n),(0,n;n,0)\big\}$, while

$$\Upsilon(1\otimes\alpha, u\otimes 1)^{nn}_n = -(-1)^{|\alpha|\cdot|u|}\Upsilon(u\otimes 1, 1\otimes\alpha)^{nn}_n := \alpha(u) \in \Bbbk = \mathbb{S}^0(X) \subset \mathbb{S}(X),$$

for $\alpha \in \mathbb{S}^n(\downarrow V)^*$, $u \in \mathbb{S}^n(\uparrow V)$. Notice that $\Upsilon(-,-)^{11}_1$ is the map $\langle -,-\rangle : X\otimes X \to \mathbb{S}(X)$ in (68). Formula (63) then defines a map

$$[-,-]_n : \mathbb{S}(X)\otimes\mathbb{S}(X) \to \mathbb{S}(X),\ \ X = \uparrow V \oplus \downarrow V^*, \tag{73}$$

which is a differential operator of order n in each variable. The main result of this subsection is

Theorem 7.2. *Under the above notation, the formula*

$$[a',a''] := [a',a'']_1 + [a',a'']_2\cdot h + [a',a'']_3\cdot h^2 + \cdots,\ \ a',a'' \in \mathbb{S}(X), \tag{74}$$

equips the graded commutative associative algebra $S(X)[\![h]\!] = \mathbb{S}(\uparrow V \oplus \downarrow V^*)[\![h]\!]$ with the structure of a formal operator Lie algebra.

Proof. The only property which is not obvious is the Jacobi identity for the bracket. Rather that verifying it directly, we identify $\big(\mathbb{S}(X)[\![h]\!],[-,-]\big)$ with a slight generalization of a construction in [41].

For a pair of elements a' and a'' of a properad $\mathcal{P}$, Merkulov and Vallette denoted, in [41, Subsection 2.2], by $a' \circ a''$ the sum of all possible composites of a' by a'' in $\mathcal{P}$ along any 2-leveled graph with two vertices. They proved that the commutator

$$[a',a''] := a'\circ a'' - (-1)^{|a'|\cdot|a''|}a''\circ a'$$

is a Lie bracket on the total space $\bigoplus\mathcal{P} := \bigoplus_{m,n\geq 0}\mathcal{P}(m,n)$, and that it induces a Lie bracket on the total space $\bigoplus\mathcal{P}^\Sigma$ of invariants. Let us modify their definition of the $\circ$-operation into

$$a'\circ_h a'' = a'\circ_1 a'' + (a'\circ_2 a'')h + (a'\circ_3 a'')h^2 + \cdots \tag{75}$$

where $a'\circ_k a''$, $k \geq 1$, is the sum of all possible composites of a' by a'' along 2-leveled graphs with two vertices connected by k edges. The proof of [41, Theorem 8] can be easily modified to show that also the commutator $[-,-]_h$ of the $\circ_h$-operation is a Lie bracket on the $\Bbbk[\![h]\!]$-linear extension $\bigoplus\mathcal{P}[\![h]\!]$ of the total space of $\mathcal{P}$, which in turn induces a Lie algebra structure on the $\Bbbk[\![h]\!]$-linear extension $\bigoplus\mathcal{P}^\Sigma[\![h]\!]$ of the space of invariants.

Let us apply the above constructions to the endomorphism properad $\mathcal{P} := \mathcal{E}nd_{\uparrow V}$ of the suspension of V. Recall that

$$\mathcal{E}nd_{\uparrow V}(m,n) = \mathrm{Lin}_{\Bbbk}\left(\bigotimes^m \uparrow V, \bigotimes^n \uparrow V\right),\ m,n \geq 0,$$

the space of $\Bbbk$-linear maps $\bigotimes^m \uparrow V \to \bigotimes^n \uparrow V$. One has the canonical isomorphism

$$\mathrm{Lin}_{\Bbbk}\left(\bigotimes^m \uparrow V, \bigotimes^n \uparrow V\right) \cong \bigotimes^m(\uparrow V) \otimes \bigotimes^n(\downarrow V^*)$$

which translates the properadic composition of $\mathcal{E}nd_{\uparrow V}$ into the contraction via the canonical pairing between $\uparrow V$ and $\downarrow V^*$. For the space of invariants one gets

$$\mathrm{Lin}_{\Bbbk}\left(\bigotimes^m \uparrow V, \bigotimes^n \uparrow V\right)^{\Sigma_m \times \Sigma_n} \cong \mathrm{Lin}_{\Bbbk}\left(\mathbb{S}^m(\uparrow V), \mathbb{S}^n(\uparrow V)\right) \cong \mathbb{S}^m(\uparrow V) \otimes \mathbb{S}^n(\downarrow V^*)$$

therefore

$$\bigoplus \left(\mathcal{E}nd_{\uparrow V}\right)^{\Sigma} \cong \mathbb{S}(X).$$

It is easy to check that, under the canonical isomorphism above, the commutator of the $\circ_h$-product in (75) becomes the bracket in Theorem 7.2. □

Let $n \geq 1$ and, as before, $B(V) = \downarrow^2 \mathbb{S}(\uparrow V \oplus \uparrow V^*)$. Denote by $\{\!\{-,-\}\!\}_n : B(V) \otimes B(V) \to B(V)$ the linear map induced from the operation $[-,-]_n$ in (73) via isomorphism (69). Since $[-,-]_n$ is a differential operator of order n in each variable, so is $\{\!\{-,-\}\!\}_n$. A simple degree count shows that $\{\!\{-,-\}\!\}_n$ is of degree $2-2n$. Furthermore, $\{\!\{-,-\}\!\}_1$ matches the big bracket $\{-,-\}$ recalled in Subsection 7.1.

We want to assemble all $\{\!\{-,-\}\!\}_n$'s into a degree 0 operation. To this end we introduce a formal variable $\hbar$ of degree $+2$. Denoting $B(V)[\![\hbar]\!] := B(V) \otimes \Bbbk[\![\hbar]\!]$, the formula

$$\{\!\{-,-\}\!\} := \{-,-\} + \{\!\{-,-\}\!\}_2 \hbar + \{\!\{-,-\}\!\}_3 \hbar^2 + \cdots$$

indeed defines a degree 0 $\Bbbk$-linear map $\{\!\{-,-\}\!\} : B(V) \otimes B(V) \to B(V)[\![\hbar]\!]$. We will denote its $\Bbbk[\![\hbar]\!]$-bilinear extension by the same symbol and call it the *superbig bracket*.

Notice that $\{\!\{-,-\}\!\}$ defined this way corresponds to the Lie bracket (74) under the obvious isomorphism $\mathbb{S}(\uparrow V \oplus \downarrow V^*) \times [\![h]\!] \cong B(V)[\![\hbar]\!]$ of vector spaces induced by (69). Since this isomorphism involves even degree shifts only, it does not influence the signs involved, thus $(B(V)[\![\hbar]\!], \{\!\{-,-\}\!\})$ is a Lie algebra as well. By definition, for $a', a'' \in B(V)$, $\{\!\{a', a''\}\!\}_n = 0$ if n is big enough. The superbig bracket is therefore a *global deformation* of the big bracket.

Let us denote by $\mathfrak{g}_{IBL}(V)$ the Poisson subalgebra of $\left(B(V)[\![\hbar]\!], \{\!\{-,-\}\!\}\right)$ given by

$$\mathfrak{g}_{IBL}(V) := \bigoplus_{p,q \geq 1,\ p+q \geq 3} B(V)^p_q[\![\hbar]\!],$$

where $B(V)^p_q$ has the same meaning as in (71). Its closure $\widehat{\mathfrak{g}}_{IBL}(V)$ in the completed tensor product $\downarrow^2 \widehat{\mathbb{S}}(\uparrow V \oplus \uparrow V^*) \widehat{\otimes} \Bbbk[\![\hbar]\!]$ bears an induced Poisson structure by Remark 6.6.

The following description of $\widehat{\mathfrak{g}}_{IBL}(V)$ is taken almost verbatim from [4, Subsection 3.1]; the notation is the same as in the second half of Subsection 7.1. The authors of [4] interpreted the elements of $\widehat{\mathfrak{g}}_{IBL}(V)$ as power series $f \in \Bbbk[\![\psi,\eta,\hbar]\!] := \Bbbk[\![\psi_1,\psi_2,\ldots,\eta^1,\eta^2,\ldots,\hbar]\!]$ subject to the conditions

$$f(\psi,\eta,\hbar)|_{\hbar=0} \in \mathfrak{m}^3,\ f(\psi,\eta,\hbar)|_{\psi_i=0} = 0 \text{ and } f(\psi,\eta,\hbar)|_{\eta^j=0} = 0,\ i,j = 1,2,\ldots,$$

where $\mathfrak{m}$ is the maximal ideal in $\Bbbk[\![\psi,\eta,\hbar]\!]$. As in Subsection 7.1, such an f is taken with degree two less than its actual degree in $\Bbbk[\![\psi,\eta,\hbar]\!]$. The induced superbig bracket can then be written as

$$\{\!\{f,g\}\!\} = f *_\hbar g - (-1)^{|f|\cdot|g|} g *_\hbar f$$

where

$$f *_\hbar g := \sum_{n=1}^{\infty} \frac{\hbar^{n-1}}{n!} \sum_{i_1,\ldots,i_n} \epsilon \cdot \frac{\partial^n f}{\partial \eta^{i_1} \cdots \partial \eta^{i_n}} \frac{\partial^n g}{\partial \psi_{i_1} \cdots \partial \psi_{i_n}}, \tag{76}$$

with ϵ the Koszul sign of the permutation

$$\eta^{i_1},\ldots,\eta^{i_n},\psi_{i_1},\ldots,\psi_{i_n} \longmapsto \eta^{i_1},\psi_{i_1},\ldots,\eta^{i_n},\psi_{i_n}.$$

Notice that, unlike the authors of [4], we did not allow $n=0$ in the above sum. The operation $*_\hbar$ in (76) decomposes as

$$f *_\hbar g = f *_1 g + f *_2 g \cdot \hbar + f *_3 g \cdot \hbar^2 + \cdots,$$

where

$$*_n := \sum_{i_1,\ldots,i_n} \epsilon \cdot \frac{\partial^n f}{\partial \eta^{i_1} \cdots \partial \eta^{i_n}} \frac{\partial^n g}{\partial \psi_{i_1} \cdots \partial \psi_{i_n}}, \ n \geq 1,$$

is a differential operator of order n in each variable, $k \geq 1$. The $*_\hbar$-product can thus serve as an example of a *formal operator Lie-admissible algebra.*

Remark 7.3. The Lie algebra $\widehat{\mathfrak{g}}_{IBL}(V)$ is isomorphic to the Lie algebra controlling deformations of *IBL*-algebras (i.e. involutive Lie bialgebras), constructed using the recipe of [32,41], from the explicit minimal model for the properad $\mathcal{L}ie^\diamond\mathcal{B}$ for *IBL*-algebras, which is described in [4, Subsection 2.3]. The scheme of verification is the same as in Remark 7.1; the useful pictorial presentation of the differential of the minimal model of $\mathcal{L}ie^\diamond\mathcal{B}$ is given in [4, Eq. (7)], cf. also [42, Eq. (5)]. If V has a differential, then $\widehat{\mathfrak{g}}_{IBL}(V)$ receives an induced differential δ, and the Maurer-Cartan elements in the dg-Lie algebra $(\widehat{\mathfrak{g}}_{IBL}(V), \delta)$ are IBL_∞-algebras.

8. Miscellany

We take a brief look on some known constructions from the point of view of (formal) operator algebras.

8.1. Terilla's quantization

Let V be a graded vector space and $P := \mathrm{Lin}_{\Bbbk}\big(\mathbb{S}(V), \widehat{\mathbb{S}}(V)\big)$. In [47] Terilla studied a deformation quantization of a graded commutative associative algebra structure on P, where the algebra product is given by the symmetrization and completion of the simple associative product

$$\circ_0 : \mathrm{Lin}_{\Bbbk}(V^{\otimes i}, V^{\otimes j}) \otimes \mathrm{Lin}_{\Bbbk}(V^{\otimes m}, V^{\otimes n}) \to \mathrm{Lin}_{\Bbbk}(V^{\otimes (i+m)}, V^{\otimes (j+n)})$$

defined for all $i, j, m, n \geq 0$ by setting

$$(f \circ_0 g)(v_1, \ldots, v_{i+m}) := f(v_1, \ldots, v_i) \otimes g(v_{i+1}, \ldots, v_{j+m}). \tag{77}$$

Let $\{e_1, e_2 \ldots\}$ be a basis of V and $\{\alpha_1, \alpha_2 \ldots\}$ be the corresponding dual basis of V^*. Upon identifying the completed symmetric algebra $\widehat{\mathbb{S}}(V \oplus V^*)$ with a subalgebra of P, the product $f \circ_0 g$ for any $f, g \in \widehat{\mathbb{S}}(V \oplus V^*)$ becomes the standard product of the corresponding power series. Terilla's deformation of $\widehat{\mathbb{S}}(V \oplus V^*)$ is then defined by setting

$$f \star g := \sum_{k=0}^{\infty} \frac{h^k}{k!} \sum_{i_1,\ldots,i_k} \epsilon \cdot \frac{\partial^k f}{\partial \alpha^{i_1} \cdots \partial \alpha^{i_k}} \frac{\partial^k g}{\partial e_{i_1} \cdots \partial e_{i_k}},$$

for any $f, g \in \widehat{\mathbb{S}}(V \oplus V^*) = \Bbbk[\![e_1, e_2 \ldots \alpha_1, \alpha_2 \ldots]\!]$, and extending it further to $\widehat{\mathbb{S}}(V \oplus V^*)[\![h]\!]$ by $\Bbbk[\![h]\!]$-bilinearity. In the above formula, ϵ is the Koszul sign of the permutation

$$\alpha^{i_1}, \ldots, \alpha^{i_k}, e_{i_1}, \ldots, e_{i_k} \longmapsto \alpha^{i_1}, e_{i_1}, \ldots, \alpha^{i_k}, e_{i_k}.$$

Notice that $f \star g$ is of the form

$$f \star g = f \circ_0 g + f \circ_1 g \cdot h + f \circ_2 g \cdot h^2 + \cdots, \tag{78}$$

where each $\circ_i$ is a $\Bbbk$-bilinear differential operator of order i in each variable.

The above construction provides an example of a formal operator $\mathcal{A}ss$-algebra of order 0, in the sense of Definition 4.7, Part 1, with $\mathcal{A}ss$ being the operad governing associative algebras. As illustrated above, such a structure consists of a graded commutative associative algebra A carrying a $\Bbbk[\![h]\!]$-bilinear associative multiplication $\star$ on $A[\![h]\!]$ whose restriction to $A \subset A[\![h]\!]$ decomposes as

$$a' \star a'' = \sum_{n \geq 0} a' \star_n a'' \cdot h^n, \ a', a'' \in A, \tag{79}$$

where $\star_n : A \otimes A \to A$ is a differential operator of order n in both variables.

It is worth pointing out that the relation between the orders of $\star_n$ and the powers of h differs from the one in (60) where the power series starts with the differential operator of degree ≤ 1. The associator $\mathrm{Ass}(a, b, c) := a \star (b \star c) - (a \star b) \star c$ decomposes as

$$\mathrm{Ass}(a, b, c) = \sum_{n \geq 0} \mathrm{Ass}_n(a, b, c) \cdot h^n,$$

where each

$$\mathrm{Ass}_n(a,b,c) := \sum_{s+t=n} a \star_s (b \star_t c) - (a \star_s b) \star_t c \tag{80}$$

is a differential operator of order n by [33, Proposition 1].

8.2. Operator Leibniz algebras

Recall that a (left) *Leibniz algebra* structure on a graded $\Bbbk$-vector space V, also known as a *Loday algebra*, is a $\Bbbk$-bilinear mapping $[-,-] : V \otimes V \to V$ subject to the (left) Leibniz rule

$$[[a,b],c] = [a,[b,c]] + (-1)^{|a|\cdot|b|}[b,[a,c]]$$

for all $a, b, c \in V$. In particular, any Lie algebra is trivially a Leibniz algebra, where the above bracket happens to be graded skew-symmetric. The concept was originally conceived as a noncommutative analogue of a Lie algebra in [2] and was further elaborated in [28].

Example 8.1. The following construction, originally due to Kanatchikov [17], arises in the context of the De Donder–Weyl covariant Hamiltonian formulation of classical field theory. It might serve as an example of an operator Leibniz algebra. Let E be a smooth manifold, regarded as a *polymomentum phase space*, equipped with a multisymplectic form Ω, that is, a closed $(k+1)$-form subject to the non-degeneracy condition

$$X \lrcorner \Omega = 0 \Rightarrow X = 0$$

for any vector field X on E. The special case of $k = 1$ corresponds to the ordinary symplectic setting.

Here, E is to be thought of as the total space of a fiber bundle $E \to M$ over an n-dimensional space-time manifold M. Locally, Ω provides a splitting of a sufficiently small coordinate chart on E into the *horizontal* (the space-time coordinates) and the *vertical* (the field variables and the corresponding polymomenta) directions. In particular, that enables one to single out the vertical component d^V of the de Rham differential d on E. The crucial feature of this set-up is that for any p-form F on E, there is a vertical multivector-valued horizontal 1-form X_F satisfying $X_F \lrcorner \Omega = d^V F$. Now, taking A to be the de Rham algebra $A^*_{dR}(E)$ with the exterior product and setting

$$[F,G] = (-1)^{n-|F|} X_F \lrcorner d^V G, \ \text{ for } G \in A^*_{dR}(E),$$

gives rise to an operator Leibniz algebra. A peculiar asymmetry of this bracket is manifest. Namely, the bracket satisfies the Leibniz rule with respect to the first argument, and it is a differential operator $[F,-]$ of order $n-p$ with respect to the second argument whenever F is a p-form. In that regard, the above construction presents an example of an operator algebra over the corresponding operad of Leibniz algebras.

8.3. F-manifolds

A generalization of Poisson manifolds is provided by the following algebraic abstraction of *F-manifolds* of Hertling and Manin [15]. What we mean is a commutative associative algebra A equipped with a Lie algebra product, but the standard derivation property

$$0 = [a_1'a_2', a''] - a_1'[a_2', a''] - (-1)^{|a_1'|\cdot|a_2'|} a_2'[a_1', a'']$$

of Poisson algebras replaced by[1]

$$\begin{aligned} 0 = {} & [a_1'a_2', a_1''a_2''] - a_1'[a_2', a_1''a_2''] - (-1)^{|a_1'|\cdot|a_2'|} a_2'[a_1', a_1''a_2''] - [a_1'a_2', a_1'']a_2'' \\ & - (-1)^{|a_1''|\cdot|a_2''|}[a_1'a_2', a_2'']a_1'' + a_1'[a_2', a_1'']a_2'' + (-1)^{|a_1'|\cdot|a_2'|} a_2'[a_1', a_1'']a_2'' \\ & + (-1)^{|a_1''|\cdot|a_2''|} a_1'[a_2', a_2'']a_1'' + (-1)^{|a_1'|\cdot|a_2'| + |a_1''|\cdot|a_2''|} a_2'[a_1', a_2'']a_1''. \end{aligned} \tag{81}$$

The above condition says that the linear map $[-,-] : A \otimes A \to A$ is a bidifferential operator in the sense we introduce below.

Let A be a commutative associative algebra and $\nabla : A \otimes A \to A$ a linear map. To shorten the formulas, we will write $\nabla(a', a'')$ for $\nabla(a' \otimes a'')$. Inspired by the scheme (1), we define the *bideviations* $\Psi^n_\nabla : A^{\otimes n} \otimes A^{\otimes n} \to A$, $n \geq 1$ inductively as

[1] Here and below, the possibly decorated symbol a will denote an element of A.

$$\Psi^1_\nabla(a', a'') := \nabla(a', a''),$$

while, for $n \geq 2$,

$$\begin{aligned}
&\Psi^{n+1}_\nabla(a'_1 a'_2, \ldots, a'_{n+1}, a''_1, \ldots, a''_n a''_{n+1}) := \\
&\quad + a'_1 \Psi^n_\nabla(a'_2, \ldots, a'_n, a''_1, \ldots, a''_n a''_{n+1}) + (-1)^{|a'_1|\cdot|a'_2|} a'_2 \Psi^n_\nabla(a'_1, a'_3, \ldots, a'_n, a''_1, \ldots, a''_n a''_{n+1}) \\
&\quad + \Psi^n_\nabla(a'_1 a'_2, \ldots, a'_{n+1}, a''_1, \ldots, a''_n) a''_{n+1} + (-1)^{|a''_n|\cdot|a''_{n+1}|} \Psi^n_\nabla(a'_1 a'_2, \ldots, a'_{n+1}, a''_1, \ldots, a''_{n-1}, a''_{n+1}) a''_n \\
&\quad - a'_1 \Psi^n_\nabla(a'_2, \ldots, a'_{n+1}, a''_1, \ldots, a''_n) a''_{n+1} - (-1)^{|a'_1|\cdot|a'_2|} a'_2 \Psi^n_\nabla(a'_1, a'_3, \ldots, a'_{n+1}, a''_1, \ldots, a''_n) a''_{n+1} \\
&\quad - (-1)^{|a''_n|\cdot|a''_{n+1}|} a'_1 \Psi^n_\nabla(a'_2, \ldots, a'_{n+1}, a''_1, \ldots, a''_{n-1}, a''_{n+1}) a''_n \\
&\quad - (-1)^{|a'_1|\cdot|a'_2| + |a''_n|\cdot|a''_{n+1}|} a'_2 \Psi^n_\nabla(a'_1, a'_3, \ldots, a'_{n+1}, a''_1, \ldots, a''_{n-1} a''_{n+1}) a''_n.
\end{aligned}$$

Notice that the right hand side of (81) equals $\Psi^2_\nabla(a'_1 a'_2, a''_1 a''_2)$ with $\nabla = [-,-]$. We say that ∇ is a *bidifferential operator of order r* if Ψ^{r+1}_∇ is identically zero. Thus (81) says that the bracket of an F-manifold is a bidifferential operator of order 1.

Proposition 8.2. *If a linear map $\nabla : A \otimes A \to A$ is a differential operator of order r in both variables, then it is a bidifferential operator of order r.*

Proof. For permutations $\sigma, \mu \in \Sigma_n$, elements $a'_1, \ldots, a'_n, a''_1, \ldots, a''_n \in A$ and $0 \leq i, j \leq n$ we define the linear maps $L_i(\sigma), R_j(\sigma) : A \to A$ by the formulas

$$\begin{aligned}
&L_i(\sigma)(a) := \epsilon(\sigma) \cdot a'_{\sigma(1)} \cdots a'_{\sigma(i)} \nabla(a'_{\sigma(i+1)} \cdots a'_{\sigma(n)}, a), \ \text{ and} \\
&R_j(\mu)(a) := \epsilon(\mu) \cdot \nabla(a, a''_{\mu(1)} \cdots a''_{\mu(j)}) a''_{\mu(j+1)} \cdots a''_{\mu(n)}.
\end{aligned}$$

Notice that when ∇ is a differential operator of order r in both variables, then both $L_i(\sigma)$ and $R_j(\sigma)$ are differential operators of order r.

The proposition follows from the fact that the bideviation Ψ^n_∇ is, for each $n \geq 1$, a linear combinations of the deviations $\Phi^r_{L_i(\sigma)}$ and $\Phi^r_{R_j(\sigma)}$. Namely, one can verify that

$$2\Psi^n_\nabla(a'_1, \ldots, a'_n, a''_1, \ldots, a''_n) = \sum_{0 \leq i < n} (-1)^i \sum_\sigma \Phi^n_{L_i(\sigma)}(a''_1, \ldots, a''_n) + \sum_{0 < j \leq n} (-1)^{j+n} \sum_\mu \Phi^n_{R_j(\sigma)}(a'_1, \ldots, a'_n),$$

where σ runs over all $(i, r-i)$-shuffles and μ over all $(j, r-j)$-shuffles. □

An immediate consequence of Proposition 8.2 is the fact that the bracket of a Poisson algebra satisfies (81). One obviously can likewise introduce polydifferential operators $\nabla : A^{\otimes k} \to A$ for arbitrary k and modify e.g. Definition 5.1 by requiring that the structure maps l_k's are polydifferential operators. On the other hand, polydifferential operators on the free commutative associative algebra $\mathbb{S}(X)$ are not necessarily determined by their values on $\mathbb{S}^{\leq n}(X)$ for a finite n, so one cannot expect results as e.g. Corollary 6.3.

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