LINEAR ALGEBRA 1

Homework 2

1. Calculate

a) $\begin{pmatrix} 3 & 2 & -5 \\ 1 & 1 & -2 \\ 0 & -2 & 3 \end{pmatrix}^{-1}$ b) $\begin{pmatrix} 2 & -1 \\ 3 & -7 \end{pmatrix}^{3}$

2. Let A, B be invertible square matrices of the same size. Simplify (using the properties of matrix operations)

a) $(I - B^T A^{-1})A + (A^T B)^T A^{-1}$,

b) $2B((AB)^{-1} - 3I)A + 7(A^T B^T)^T$.

3. Find a 3×3 matrix X such, that

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$$

4. Given

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 6 & 4 & 5 & 2 \end{pmatrix},$$

decompose p into cycles, then calculate it's inverse and p^{10}.

Solutions

1. a) Recall, that the algorithm for finding the inverse is using the Gauss-Jordan elimination on $(A|I)$ to obtain $(I|A^{-1})$. See below for a better explanation for why this works. This way we get that

$$\left(\begin{array}{ccc|ccc} 3 & 2 & -5 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 1 \\ 0 & 1 & 0 & -3 & 9 & 1 \\ 0 & 0 & 1 & -2 & 6 & 1 \end{array}\right).$$

b) we have to perform the matrix multiplication twice.

$$\begin{pmatrix} 2 & -1 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & -7 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 - 1\cdot 3 & 2\cdot(-1) - 1\cdot(-7) \\ 3\cdot 2 - 7\cdot 3 & 3\cdot(-1) - 7\cdot(-7) \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -15 & 46 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 \\ -15 & 46 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & -7 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 5\cdot 3 & 1\cdot(-1) + 5\cdot(-7) \\ -15\cdot 2 + 46\cdot 3 & -15\cdot(-1) + 46\cdot(-7) \end{pmatrix} = \begin{pmatrix} 17 & -36 \\ 108 & -307 \end{pmatrix}.$$

2. These are detailed solutions, more so than what I expect from you. Recall that by $-X$ we mean such Y that $X+Y = 0 = Y+X$ and by $X - Y$ we mean $X + (-Y)$. In each step I list all the properties used:

a)

$(I + (-B^T A^{-1}))A + (A^T B)^T A^{-1}$	Right-distributivity of $\cdot$ over $+$ i.e. $(X+Y)Z = XZ + YZ$
$IA + (-B^T A^{-1})A + (A^T B)^T A^{-1}$	$IX = X$, $(-X)Y = -(XY)$
$A - B^T A^{-1} A + (A^T B)^T A^{-1}$	$X^{-1}X = I$, $(XY)^T = Y^T X^T$
$A - B^T I + B^T (A^T)^T A^{-1}$	$XI = X$, $(X^T)^T = X$
$A - B^T + B^T A A^{-1}$	$XX^{-1} = I$ then $XI = X$
$A - B^T + B^T$	Associativity of $+$
$A + (-B^T + B^T)$	$(-X) + X = 0$
$A + 0 = A$	$X + 0 = X$.

Short version would be

$(I - B^T A^{-1})A + (A^T B)^T A^{-1} = A - B^T A^{-1} A + B^T A A^{-1} = A - B^T + B^T = A.$

b)

$2B((AB)^{-1} - 3I)A + 7(A^T B^T)^T$	$(XY)^T = Y^T X^T$ then $(X^T)^T = X$
$2B((AB)^{-1} - 3I)A + 7BA$	Right-distributivity of $\cdot$ over $+$ and $(cX)Y = c(XY)$
$(2B)((AB)^{-1}A + (-3I)A) + 7BA$	Left-distributivity of $\cdot$ over $+$ i.e. $X(Y+Z) = XY + XZ$
$2B(AB)^{-1}A + 2B(-3I)A + 7BA$	$(XY)^{-1} = Y^{-1}X^{-1}$, $-X = (-1)X$, then $c(rX) = (c \cdot r)X$
$2BB^{-1}A^{-1}A + 2B(-3)IA + 7BA$	$IX = X$, $XcY = cXY$ and $c(rX) = (c \cdot r)X$
$2BB^{-1}A^{-1}A + (-6)BA + 7BA$	Left-distributivity of scalar $\cdot$ over $+$ i.e. $cX + rX = (c + r)X$
$2BB^{-1}A^{-1}A + (-6 + 7)BA$	$XX^{-1} = I = X^{-1}X$
$2I \cdot I + 1BA$	$XI = X$, $1X = X$
$2I + BA$	

Short version would be

$$2B((AB)^{-1} - 3I)A + 7(A^T B^T)^T = 2B(AB)^{-1}A - 6BIA + 7(A^T B^T)^T =$$
$$= 2BB^{-1}A^{-1}A - 6BA + 7BA = 2I + BA.$$

3. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$.

Solution 1 (Inverse): A is invertible. We can see that $(AA^{-1})B = B$, so $A(A^{-1}B) = B$ and $X = A^{-1}B$ works. We calculate A^{-1} using the Gauss-Jordan elimination on $(A|I)$ to obtain $(I|A^{-1})$. We get

$$A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

and

$$X = A^{-1}B = \frac{1}{2}\begin{pmatrix} -17 & 17 \\ 7 & -5 \end{pmatrix} = \begin{pmatrix} -\frac{17}{2} & \frac{17}{2} \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix}.$$

Solution 2 (Straight Gauss-Jordan): Recall that finding A^{-1} is really solving the equation $AX = I$. We can extend that method. Starting from $(A|B)$ and performing Gauss-Jordan elimination we arrive at $(I|X)$. See below for the more detailed explanation of why this works. Performing the elimination yields $\begin{pmatrix} -\frac{17}{2} & \frac{17}{2} \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix}$.

Solution 3 (Equations): We can write $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, from the equality

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}.$$

we can derive a system of equations:

$$\begin{cases} 1x + 3z = 2 \\ 1y + 3w = 1 \\ 2x + 4z = -3 \\ 2y + 4w = 7 \end{cases}.$$

After solving this by standard methods, we should get X.

4. To find cycles we trace any element through the permutation (more precisely we look for $p(x), p(p(x))...$) up until we get back to it. Then we start from another element that was not in the first loop and repeat until we write all elements. We usually do not write down cycles of length 1 i.e. when $p(x) = x$. Let us start from 2

$$2 \mapsto 1 \mapsto 3 \mapsto 7 \mapsto 2$$

and then from 4

$$4 \mapsto 6 \mapsto 5 \mapsto 4,$$

so

$$p = (2,\ 1,\ 3,\ 7)(4,\ 6,\ 5).$$

Calculating p^{-1} is simple: we just trace back through p i.e if in p we have $2 \mapsto 1$ then in p^{-1} there will be $1 \mapsto 2$. hence

$$p^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 3 & 5 & 6 & 4 & 3 \end{pmatrix} = (7,\ 3,\ 1,\ 2)(5,\ 6,\ 4).$$

To calculate p^{10} we can perform 10 jumps through p for each element, but we can also notice, that for the elements in the first cycle, we have $p^4(x) = x$ (after 4 steps we end up where we started), so for each of them we have $p^{10}(x) = p^6(p^4(x)) = p^6(x) = p^2(p^4(x)) = p^2(x)$. Similarly for each element in the second cycle we have $p^3(x) = x$ and $p^{10}(x) = p(p^9(x)) = p(x)$. So

$$p^{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 6 & 4 & 5 & 1 \end{pmatrix} = (1,\ 7)(2,\ 3)(4,\ 6,\ 5).$$

Extra: What if we wanted to calculate p^{12}? p^{13}?

As above, notice that in the first cycle we have $p^4(x) = x$ and in the second cycle $p^3(x) = x$. So, in the first cycle we have

$$p^{12}(x) = p^8(p^4(x)) = p^8(x) = p^4(x) = x$$

and also in the second cycle $p^{12}(x) = x$. Overall we get that

$$p^{12}(x) = x \text{ and } p^{12} = id = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Now clearly $p^{13} = p \circ p^{12} = p$.

In General: If $c_1, c_2, ..., c_n$ are lengths of cycles of a permutation q, then $d = LCM(c_1, c_2, \cdots c_n)$ is the smallest (> 1) number such that $q^d = id$. Then $q^{d+1} = p$, $q^{d+2} = q^2$ and so on. Also, $(q^{-1})^d = q^{-d} = id$ (as q^{-1} has the same lengths of cycles as q). Furthermore since $q \circ q^{d-1} = id$, we have that $q^{d-1} = q^{-1}$.

For example if $q = (1,\ 2,\ 3,\ 4)(6,\ 8,\ 7)(5,\ 9,\ 10,\ 11,\ 12,\ 13)$, then $d = 12$ – the least common multiple of $4, 3$ and 6 – works, so we have $q^{12} = id = q^{24} = q^{36} = q^{-12}$ and $q^{-1} = q^{11}$.

Detailed explanation of why $(A|I) \sim (I|A^{-1})$ and $(A|B) \sim (I|X)$ whenever $AX = B$.

(note: both work only when A is invertible)
First notice that the question of finding A^{-1} is the same as finding such X that $AX = I$. To convince yourself that this is enough, see that (*first step below is to multiply both sides from the left by* A^{-1})

$$AX = I \implies A^{-1}AX = A^{-1}I \implies IX = A^{-1}I \implies X = A^{-1}.$$

Now we will solve the general form $AX = B$. Consider the following claim:

If $(A|B) \sim (A'|B')$*, then* $AX = B$ *for exactly those* X *for which* $A'X = B'$.

In other words we want to know, that the elementary row operations do not change the set of solutions. We know this works for vectors x, b i.e. that if $(A|b) \sim (A'|b')$ then $Ax = b \iff A'x = b'$. One way to convince ourselves that this is still true for matrices is to see that, if $X = (x_1, x_2, x_3, \cdots x_n)$ and $B = (b_1, b_2, \cdots b_n)$ where x_i, b_i denote column vectors, then from the definition of matrix multiplication

$$AX = B$$

is equivalent to

$$Ax_1 = b_1,\ Ax_2 = b_b,\ \cdots\ Ax_{n-1} = b_{n-1} \text{ and } Ax_n = b_n.$$

Now we use this claim. We perform the Gauss-Jordan elimination on $(A|B)$ with A invertible. We end up with $(I|C)$ for some matrix C and we have only used elementary row operations. So, we have that

$$(A|B) \sim (I|C)$$

and from our claim

$$AX = B \text{ whenever } IX = C.$$

But $IX = C$ means exactly that $X = C$, so $X = C$ is the only solution to the equation $AX = B$.