7. Vector Spaces

1. Decide whether the following are vector spaces over a given field:

a) $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$,

b) $\mathbb{R}^n$ over $\mathbb{Q}$,

c) $\mathbb{Q}^n$ over $\mathbb{R}$,

d) $\mathbb{R}^n$ with operations $x \oplus y = x + y, \alpha \odot x = -a \cdot x$ over $\mathbb{R}$,

e) $\mathbb{R}^n$ with operations $x \oplus y = x + y, \alpha \odot x = |a| \cdot x$ over $\mathbb{R}$,

f) $U \times V$ over $\mathbb{F}$ where U, V are vector spaces over $\mathbb{F}$ and the operations are defined coordinate-wise,

g) the set of all functions $f : M \to V$ (with point-wise operations) over $\mathbb{F}$, where M is any set, V is a vector space over $\mathbb{F}$.

2. Find a non-trivial subset of $\mathbb{R}^2$, that is

a) closed under addition and subtraction, but not scalar multiplication,

b) closed under scalar multiplication, but not addition.

3. Decide if the following sets form a subspace of $\mathbb{R}^2$:

a) $\{(s, 5s)^T : s \in \mathbb{R}\}$,

b) $\{(s + t, 1)^T : s, t \in \mathbb{R}\}$,

c) $\{(s, s^2)^T : s \in \mathbb{R}\}$,

d) $\{(s - t, 2t)^T : s, t \in \mathbb{R}\}$.

4. Let $A \in \mathbb{R}^{m \times n}$. Show that $\{x \in \mathbb{R}^n : Ax = 0\}$ forms a subspace of $\mathbb{R}^n$.

5. Decide if the following form a subspace of the space of all real-valued sequences $\mathbb{R}^\infty = \{(x_1, x_2, \cdots) : x_i \in \mathbb{R}, i \in \mathbb{N}\}$:

a) all sequences with infinitely many zeros,

b) all sequences with finitely many non-zero elements,

c) monotone sequences (non-increasing and non-decreasing)

d) all Fibonacci sequences (st. $x_{n+2} = x_{n+1} + x_n$ for all n).

6. Let V be any vector space and $M, N \subset V$ be any sets of vectors. Decide whether the following hold:

a) $\text{span}(\text{span}(M)) = \text{span}(M)$,

b) $M \subseteq N \implies \text{span}(M)$ is a subspace of $\text{span}(N)$,

c) $\text{span}(M)$ is a subspace of $\text{span}(N) \implies M \subseteq N$,

d) $\text{span}(M \cap N) = \text{span}(M) \cap \text{span}(N)$

7. Do vectors $(1, 2)^T$, $(3, 4)^T$ generate $\mathbb{R}^2$?

8. Decide if $x = (1, 2, 3)^T$ is a linear combination (over $\mathbb{R}$) of given vectors. If yes, find that combination.

a) $(1, 1, 1)^T, (2, 1, 3)^T, (3, 1, 5)^T$,

b) $(2, 1, 3)^T, (3, 1, 2)^T, (1, 1, 1)^T$.