8. Linear Independence

1. When is a system of one, two or three vectors independent?

2. Check if the following vectors are linearly independent in $\mathbb{R}^3$:
a) $(2,3,-5)^T, (1,-1,1)^T, (3,2,-2)^T$,
b) $(2,0,3)^T, (1,-1,1)^T, (0,2,1)^T$.

3. Let u, v, w be three linearly independent vectors in a space V over $\mathbb{R}$. Are the following sets linearly independent?
a) $\{u, v, 0\}$,
b) $\{w, u, v\}$,
c) $\{u, u+v, u+w\}$,
d) $\{u-v, u-w, v-w\}$.

4. Let V be a vector space over the field $\mathbb{F}$ and let $X \subset Y \subset V$ be two sets of vectors. Decide whether the following are true theorems
a) If X is independent, then Y is independent,
b) If X is dependent, then Y is dependent,
c) If Y is independent, then X is independent,
d) If Y is dependent, then X is dependent.

5. Decide whether the vectors $(0,1,1,1)^T, (1,0,1,1)^T, (1,1,0,1)^T, (1,1,1,0)^T$ are independent in $\mathbb{R}^4$ and in $\mathbb{Z}_3^4$.

6. Let U, V be subspaces of W. Show that $U \cap V = \{0\}$ if and only if each vector $x \in U + V$ can be uniquely represented as $x = u + v$, where $u \in U, v \in V$.

7. Decide whether the following sets are independent in the space of all functions $\mathbb{R} \to \mathbb{R}$ (over $\mathbb{R}$):
a) $\{2x-1, x-2, 3x\}$,
b) $\{x^2+2x+3, x+1, x-1\}$,
c) $\{\sin x, \cos x\}$.

8. Find 4 dependent vectors in $\mathbb{R}^4$ so that
a) exactly one vector is a linear combination of the remaining three,
b) each vector is a linear combination of the remaining three,
c) exactly two vectors are a linear combination of the remaining three,
d) exactly three vectors are a linear combination of the remaining three.