9. Bases and Dimension

Bases and coordinates

1. Find the bases of the following vector spaces over given fields and determine their dimension.

a) $\mathbb{R}^2$ over $\mathbb{R}$,
b) $\mathbb{C}^2$ over $\mathbb{C}$,
c) $\mathbb{C}^2$ over $\mathbb{R}$,
d) $\mathcal{P}^2$ – the space of all polynomials of degree 2 or less,
e) $\mathbb{R}^{2\times 2}$ over $\mathbb{R}$,
f) the space of all symmetric matrices in $\mathbb{R}^{2\times 2}$ over $\mathbb{R}$.

2. Determine whether $(-1, 5, 3) \in \operatorname{span}\{(1, 2, 2), (4, 1, 3)\}$. If yes, find it's coordinates in the basis given.

3. Find, in the space $\mathcal{P}^2$, the coordinates of x^2+2 in the basis $x^2+1, x-2, 2x^2+x-1$.

4. The coordinates of a vector v in the basis $B = (b_1, b_2, b_3, b_4)$ are $[v]_B = (a_1, a_2, a_3, a_4)$. Find the coordinates of v in:

a) $B_1 = (b_4, b_3, b_2, b_1)$,
b) $B_2 = (b_1 + b_4, b_2, b_3, b_4)$,
c) $B_3 = (b_1 + b_4, b_2 + b_3, b_4, b_2)$.

Dimension

5. Find all subspaces of the vector space $\mathbb{R}^2$.

6. Find the number of subspaces of $\mathbb{Z}_p^2$ over $\mathbb{Z}_p$.

7. Let U, V be subspaces of a vector space W and let $\dim U = 7$, $\dim V = 8$, $\dim W = 13$.

a) Find the lowest and highest possible value of $\dim(U+V)$ and give examples for both
b) Find the lowest and highest possible value of $\dim(U \cap V)$ and again show that the estimate is tight.

8. Let W be a direct sum of it's subspaces U, V i.e. $W = U+V = \{u+v : u \in u, v \in v\}$ and $U \cap V = \{0\}$. Show that if $u_1, \ldots, u_n$ is a basis of U and $v_1, \ldots, v_m$ is a basis of V, then $u_1, \ldots, u_n, v_1, \ldots, v_m$ is a basis of W.
Hint: look at the list 8.