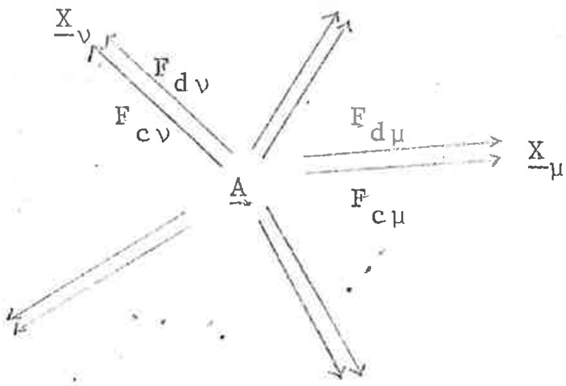
§ 3 Bialgebras in locally presentable categories

In this chapter some questions of "Universal bialgebra" in a locally presentable category $\underline{A}$ are investigated. Our definition of bialgebras in a category $\underline{A}$ is fairly broad and includes universal algebras and coalgebras in the sense of Birkhoff [2] or Lawvere [21], Σ-continuous and Σ-cocontinuous functors in the sense of Gabriel-Ulmer [13] 8.1, algebras, coalgebras, Hopfalgebras and bialgebras in the usual sense over a commutative ring Λ , or more generally bialgebras with respect to some tensor product and an arbitrary Prop [24], coalgebras over a cotriple with rank in $\underline{A}$ (e.g. comodules over a Λ-coalgebra), algebras over a triple with rank in $\underline{A}$, the category of descent data with respect to fibrations and more generally sections and cartesian closed sections with respect to a fibration or cofibration, etc. Roughly speaking a bialgebra (A,M,R) consists of an object A in $\underline{A}$ together with a set M of operations which satisfy certain relations R . An operation is represented by a morphism $\mu(A) : FA \longrightarrow F'A$ for some pair of functors $F,F' : \underline{A} \rightrightarrows \underline{X}$, and a relation by a morphism pair $r(A,M) : HA \rightrightarrows H'A$ for some pair of functors $H,H' : \underline{A} \rightrightarrows \underline{Y}$ (for details see 3.1). Given a bialgebra (A,M,R) and a subobject $U \subset A$ in the underlying category $\underline{A}$, which we assume to be locally noetherian for the moment, we are concerned with sub-bialgebras (U',M,R) of (A,M,R) containing U such that U' is not much bigger than U . We give in 3.22 a construction and size estimates for U' which in many cases are the best possible. For instance, if Λ is a commutative noetherian ring, then it follows that any countably generated submodule of a Hopfalgebra (resp. coalgebra, bialgebra, comodule over a fixed Λ-coalgebra) is contained in a sub-Hopfalgebra (resp. sub-coalgebra,...) whose underlying Λ-module is again countably generated regardless of the size of Λ . If the category $\underline{A}$ is not locally noetherian, the situation is different and the question should be put instead as follows: Given a bialgebra (A,M,R) and a morphism

$f : U \longrightarrow A$ in the underlying category $\underline{A}$ with U being a γ - presentable object, does then f factor through a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is γ' - presentable and γ' is not much bigger than γ ? In 3.8 we give a construction and size estimates for U' similar to the noetherian case which in particular implies the existence of dense generators in the above mentioned examples. If Λ is any commutative ring and (A,M,R) is a Λ-Hopfalgebra (resp. coalgebra, bialgebra, comodule over a fixed Λ-coalgebra), then by 3.8 any homomorphism $f : U \longrightarrow A$ with U being countably presentable factors through a Hopfalgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ (resp. coalgebra morphism ...) such that U' is again countably presentable. Also 3.8 implies that every descent data is effective provided every descent data on "small" objects is effective. For modules "small" means countably presentable. More generally for a fibration with countably presentable fibres "small" means countably presentable provided either the inverse image functors have right adjoints which preserve countably filtered direct limits or the inverse image functors take countably presentable objects into countably presentable objects and preserve filtered direct limits. The main results of this chapter are 3.8, 3.9, 3.22, 3.24 and 3.28. The last two concern conditions which guarantee that the category Bialg($\underline{A}$) of bialgebras in a locally γ-presentable category $\underline{A}$ is locally γ'-presentable and that γ' is not much bigger than γ . For instance, if Λ is any commutative ring, then they imply that the categories of commutative Λ-Hopfalgebras, cocommutative Λ-Hopfalgebras, Λ-coalgebras, Λ-bialgebras, comodules over a fixed Λ-coalgebra etc. are locally countably presentable, regardless of the size of Λ . Also if $\mathbb{G}$ is a cotriple with rank α in a locally γ-presentable category $\underline{A}$, then the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras in $\underline{A}$ is locally γ'-presentable, where $\gamma' = \sup(\aleph_1, \gamma, \alpha)$.

3.1 In this paragraph we give the basic definitions. Let $\underline{A}$ be a category. Let M be a set (or class) and assume that with each $\mu \in M$ there is associated an ordered pair of functors $F_{d\mu} : \underline{A} \longrightarrow \underline{X}_\mu$ and $F_{c\mu} : \underline{A} \longrightarrow \underline{X}_\mu$. Note that the domain is always $\underline{A}$ and that each pair has the same codomain (which can vary from pair to pair)



Also note that the assignment $\mu \rightsquigarrow (F_{d\mu}, F_{c\mu})$ need not be injective. A pre-bialgebra $(A, \mu(A))_{\mu \in M}$ in $\underline{A}$ with respect to M is an object $A \in \underline{A}$ together with a morphism $\mu(A) : F_{d\mu}A \longrightarrow F_{c\mu}A$ for every $\mu \in M$. We say that an element $\mu \in M$ is an operation and $\mu(A)$ is the structure morphism on A associated with μ . A morphism $(A, \mu(A))_{\mu \in M} \longrightarrow (A', \mu(A'))_{\mu \in M}$ between pre-bialgebras is a morphism $f : A \longrightarrow A'$ in $\underline{A}$ which is compatible with the structure morphisms, i.e. for every $\mu \in M$ the diagram

$$\begin{array}{ccc} F_{d\mu}A & \xrightarrow{\mu(A)} & F_{c\mu}A \\ {\scriptstyle F_{d\mu}f}\downarrow & & \downarrow{\scriptstyle F_{c\mu}f} \\ F_{d\mu}A' & \xrightarrow[\mu(A')]{} & F_{c\mu}A' \end{array}$$

commutes. The category of pre-bialgebras is denoted with $\text{P-Bialg}_M(\underline{A})$. Let $V : \text{P-Bialg}_M(\underline{A}) \longrightarrow \underline{A}$ denote the (faithful) forgetful functor $(A, \mu(A))_{\mu \in M} \rightsquigarrow A$. If it is clear which M we are referring to we write $\text{P-Bialg}(\underline{A})$ instead of $\text{P-Bialg}_M(\underline{A})$. Further we abbreviate $(A, \mu(A))_{\mu \in M}$ to (A, M) in order to avoid expressions of extreme complexity in the following. This notation does no longer distinguish between pre-bialgebras with the same underlying object. The reader should keep this in mind.

Clearly in practice one is not interested in all pre-bialgebras but only in those which satisfy certain given relations. The relations are normally expressed in terms of diagrams which have to commute. The diagrams are constructed from the structure morphisms and other canonical morphisms. However there is a great deal of variety and a scheme of sufficient generality to cover the above mentioned examples becomes hopelessly involved. Surprisingly it turned out – after many attempts – that the explicit description of the relations in terms of structure and canonical morphisms is not needed to establish the main results of this chapter. Instead the following common features suffice : 1) for every pre-bialgebra the diagrams expressing the relations are given in some way 2) the diagrams are natural with respect to pre-bialgebra morphisms.

More precisely by a <u>relation</u> r <u>on</u> $P\text{-Bialg}_M(\underline{A})$ we mean a pair of functors $F_{dr} : \underline{A} \to \underline{X}_r$ and $F_{cr} : \underline{A} \to \underline{X}_r$ together with a pair of natural transformations $F_{dr} \circ V \rightrightarrows F_{cr} \circ V$. Explicitely with every pre-bialgebra (A,M) there is associated a pair of morphisms $r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ in such a way that for every pre-bialgebra morphism $f: (A,M) \to (A',M)$ the diagram

$$\begin{array}{ccc} F_{dr}A & \overset{r(A,M)}{\rightrightarrows} & F_{cr}A \\ {\scriptstyle F_{dr}f}\downarrow & & \downarrow{\scriptstyle F_{cr}f} \\ F_{dr}A' & \underset{r(A',M)}{\rightrightarrows} & F_{cr}A' \end{array}$$

commutes in the obvious sense (i.e. with respect to both components of r).

Let R be a set (or class) of relations on $P\text{-Bialg}_M(\underline{A})$.

A <u>bialgebra</u> (A,M,R) in $\underline{A}$ with respect to M and R is a pre-bialgebra (A,M) such that for every $r \in R$ the morphisms $r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ coincide. In other words a bialgebra is a pre-bialgebra satisfying the relations of R. A morphism between bialgebras is a morphism between the underlying pre-bialgebras. The category of bialgebras is denoted with $\mathrm{Bialg}_{M,R}(\underline{A})$. If there is no ambiguity we drop the indices M and R. Clearly $\mathrm{Bialg}(\underline{A})$ is a full subcategory of $\text{P-Bialg}(\underline{A})$. The forgetful functor $\mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$, $(A,M,R) \rightsquigarrow A$, is also denoted with V.

By the support of the operations M and the relations R we mean the set (or class) $\mathbb{F}$ of all functors $F_{d\mu}$, $F_{c\mu}$, F_{dr} and F_{cr}, where μ and r are running through M and R respectively. The subclass of all functors of $\mathbb{F}$ which are the domain of either an operation or a relation is denoted with $\mathbb{F}_d$. Likewise $\mathbb{F}_c$ denotes the subclass of all functors appearing as the codomain of either an operation or a relation. In the following the hypothesis are often stated in terms of $\mathbb{F}$, $\mathbb{F}_d$, and $\mathbb{F}_c$ instead of M and R. It is therefore essential to keep their meanings in mind (d = domain , c = codomain).

3.2 Remarks I) It is easy to express that for some specified operation $\mu \in M$ the structure morphism $\mu(A) : F_{d\mu}A \longrightarrow F_{c\mu}A$ should be an isomorphism for every pre-bialgebra (A,M) : One has to add an operation $\bar{\mu}$ to M and two relations to R expressing $\bar{\mu}(A)\mu(A) = id_{F_{d\mu}A}$ and $\mu(A)\bar{\mu}(A) = id_{F_{c\mu}A}$ (cf. 3.2 IIId).

II) One can call an operation μ algebraic (resp. coalgebraic) if $F_{d\mu}$ and $F_{c\mu}$ are endofunctors of $\underline{A}$ and $F_{c\mu}$ is the identity of $\underline{A}$ (resp. $F_{d\mu} = id_{\underline{A}}$); and likewise for relations. Typical examples are functors $\underline{A} \longrightarrow \underline{A}$ which assign to an object A its n-fold product, coproduct or tensor product etc.

III) For examples of bialgebras see § 4 and § 6. It should however be clear at this point how to express some of the examples given in the introduction to § 3 as bialgebras, i.e. how to choose the underlying category $\underline{A}$, the operations M and the relations R such that $\mathrm{Bialg}(\underline{A})$ is

a) the category of groups, rings, ..., cogroups, ... in $\underline{A}$.

b) the category of algebras, coalgebras, bialgebras, Hopfalgebras ... over a commutative ring Λ.

c) the category of $\mathbb{T}$-algebras (resp. $\mathbb{G}$-coalgebras) for a triple $\mathbb{T}$ (resp. cotriple $\mathbb{G}$) in $\underline{A}$.

d) the category of descent data (or données de recollement) in the standard situation

$$\mathcal{F}_S \xrightarrow{\alpha^*} \mathcal{F}_{S'} \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{F}_{S'\times_S S'} \quad (\Delta^*: \mathcal{F}_{S'\times_S S'} \to \mathcal{F}_{S'}) \quad \overset{p_{31}^*,\ p_{32}^*,\ p_{21}^*}{\Rrightarrow} \mathcal{F}_{S'\times_S S'\times_S S'}$$

given by $\alpha : S' \to S$ (cf. Grothendieck [16] Def. 1.4 - Def 1.7).

The reader should be familiar with these examples, in particular know what the functors $F_{d\mu}$, $F_{c\mu}$, F_{dr}, F_{cr} and the natural transformations $F_{dr}\circ V \Longrightarrow F_{cr}\circ V$ look like for every operation $\mu \in M$ and relation $r \in R$. If not, he is advised to first have a look at § 4 because the following is often motivated by these examples.

3.3 We start with some elementary properties of the underlying functor $V: \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ concerning the preservation of limits and colimits.

Lemma Let $H : \underline{D} \longrightarrow \mathrm{Bialg}(\underline{A})$ be a functor such that the limit (resp. colimit) of the composite $V\circ H : \underline{D} \longrightarrow \underline{A}$ exists. Then the following hold:

a) If every $F \in \mathbb{F}_c$ preserves $\varprojlim V\circ H$, then $\varprojlim H$ exists in $\mathrm{Bialg}(\underline{A})$ and $\varprojlim H = (\varprojlim V\circ H, M, R)$.

b) <u>If every</u> $F \in \mathbb{F}_d$ <u>preserves</u> $\varinjlim V \circ H$, <u>then</u> $\varinjlim H$ <u>exists in</u> $\mathrm{Bialg}(\underline{A})$ <u>and</u> $\varinjlim H = (\varinjlim V \circ H, M, R)$.

<u>Proof</u> It suffices to consider a) because b) is dual.

By assumption for every operation $\mu \in M$ there is an unique morphism $\mu(\varprojlim V\circ H) : F_{d\mu}(\varprojlim V\circ H) \longrightarrow F_{c\mu}(\varprojlim V\circ H)$ such that for every $D \in \underline{D}$ the diagram

$$\begin{array}{ccccc}
F_{d\mu}(\varprojlim V\circ H) & \xrightarrow{\mu(\varprojlim V\circ H)} & F_{c\mu}(\varprojlim V\circ H) & \xrightarrow{\cong} & \varprojlim (F_{c\mu} \circ V\circ H) \\
\Big\downarrow {\scriptstyle F_{d\mu}(p_D)} & & \Big\downarrow {\scriptstyle F_{c\mu}(p_D)} & & \\
F_{d\mu}((V\circ H)D) & \xrightarrow{\mu((V\circ H)D)} & F_{c\mu}((V\circ H)D) & &
\end{array}$$

commutes, where $p_D : \varprojlim(V\circ H) \longrightarrow (V\circ H)D$ denotes the canonical morphism. Thus $\varprojlim (V\circ H)$ together with $\mu(\varprojlim V\circ H)$, $\mu \in M$, is a pre-bialgebra and p_D is a pre-bialgebra morphism for every $D \in \underline{D}$. Hence for every relation $r \in R$ and every $D \in \underline{D}$ the morphism pair $r(\varprojlim V\circ H, M)$ gives rise to a commutative diagram (with respect to both components of r)

$$\begin{array}{ccccc}
F_{dr}(\varprojlim V\circ H) & \overset{r(\varprojlim V\circ H, M)}{\rightrightarrows} & F_{cr}(\varprojlim V\circ H) & \xrightarrow{\cong} & \varprojlim F_{cr}\circ V\circ H \\
\Big\downarrow {\scriptstyle F_{dr}(p_D)} & & \Big\downarrow {\scriptstyle F_{cr}(p_D)} & & \\
F_{dr}((V\circ H)D) & \overset{r(HD)}{\rightrightarrows} & F_{cr}((V\circ H)D) & &
\end{array}$$

This shows that $r(\varprojlim V\circ H, M)$ is the inverse limit over all "pairs" $r(HD)$, $D \in \underline{D}$. Since the two components of $r(HD)$, $D \in \underline{D}$, coincide, the same holds for $r(\varprojlim V\circ H, M)$ and thus $(\varprojlim V\circ H, M)$ is a bialgebra. One readily checks that the latter together with the bialgebra morphisms $p_D : (\varprojlim V\circ H, M, R) \longrightarrow HD$ is the limit of $H : \underline{D} \longrightarrow \mathrm{Bialg}(\underline{A})$.

3.4 Corollary a) If A is complete and every $F \in \mathbb{F}_c$ preserves limits, then Bialg(A) is complete and the forgetful functor V : Bialg(A) $\longrightarrow$ A preserves (and creates) limits. Moreover V is tripleable provided it has a left adjoint.

b) Likewise if A is cocomplete and every $F \in \mathbb{F}_d$ preserves colimits, then Bialg(A) is cocomplete and the forgetful functor V preserves (and creates) colimits. Moreover V is cotripleable provided it has a right adjoint.

c) If A has α-filtered colimits and every $F \in \mathbb{F}_d$ preserves them, then Bialg(A) has α-filtered colimits and V preserves (and creates) them.

As for the tripleability and cotripleability note that by 3.3 a), b) the underlying V always preserves (and creates) both V-contractible kernels and cokernels. The condition c) holds in most examples for an appropriate α . This is not so for a) and b) . However a) holds when all operations and relations are algebraic (3.2 II), while condition b) holds when all operations and relations are coalgebraic (3.2 II).

3.5 In order to study the category Bialg(A) from the point of view of locally presentable categories the first question to answer is whether there exist α-presentable objects for sufficiently large α and how they look like. The following and 3.6, 3.7 give a partial answer.

Lemma Let A be a category with α-filtered colimits and let M and R be a data for bialgebras (3.1). Assume that card(M) $< \alpha$ and that every $F \in \mathbb{F}$ preserves α-filtered colimits. Then a bialgebra (U,M,R) is α-presentable in $\mathrm{Bialg}(\underline{A})$ provided $U \in \underline{A}$ and FU are α-presentable *) for every $F \in \mathbb{F}_d$.

Remark If the underlying functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ and the functors $F \in \mathbb{F}_c$ preserve monomorphisms - e.g. in the situation 3.4 a) - then there is an analoguous assertion for α-generated objects: A bialgebra (U,M,R) is α-generated provided 1) $\mathrm{card}(M) < \alpha$ and every $F \in \mathbb{F}$ preserves α-filtered colimits 2) U and FU are α-generated for every $F \in \mathbb{F}_d$. The proof is the same as for 3.5.

Proof First note that by 3.4 c) $\mathrm{Bialg}(\underline{A})$ has α-filtered colimits and $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ preserves (and creates) them. Let $(X,M,R) = \varinjlim_{\nu} (X_\nu, M, R)$ be an α-filtered colimit in $\mathrm{Bialg}(\underline{A})$ and let $f : (U,M,R) \longrightarrow \varinjlim_{\nu}(X_\nu, M, R)$ be a bialgebra morphism with $\pi(U) \leq \alpha$ and $\pi(FU) \leq \alpha$ for every $F \in \mathbb{F}_d$. Since $\varinjlim_{\nu} (X_\nu, M, R) = (\varinjlim_{\nu} X_\nu, M, R)$ and U is α-presentable, the underlying morphism $U \longrightarrow \varinjlim_{\nu} X_\nu$ of f admits a factorization $U \xrightarrow{f_\nu} X_\nu \xrightarrow{u_\nu} \varinjlim_{\nu} X_\nu$ for some ν, where u_ν denotes the underlying canonical morphism. In general f_ν is not a bialgebra morphism because for an operation $\mu \in M$ the morphisms $F_{c\mu} f_\nu \circ \mu(U)$ and $\mu(X_\nu) \circ F_{d\mu} f_\nu$ need not coincide. However they become equal when composed with $F_{c\mu} u_\nu : F_{c\mu} X_\nu \longrightarrow F_{c\mu} \varinjlim X_\nu$ because $f = u_\nu f_\nu$ is a bialgebra morphism. Since $F_{d\mu} U$ is α-presentable and $F_{c\mu} \varinjlim_{\nu} X_\nu \cong \varinjlim_{\nu} F_{c\mu} X_\nu$ is an α-filtered colimit, this implies that there is a transition morphism $u : X_\nu \longrightarrow X_{\nu'}$ - depending on μ - such that the diagram

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*) It is not assumed that the codomain of F, $F \in \mathbb{F}_d$, is locally presentable, but merely that $[FU,-]$ preserves all existing α-filtered colimits.

$$\begin{array}{ccc}
F_{c\mu}U & \xrightarrow{F_{c\mu}(u\circ f_\nu)} & F_{c\mu}X_{\nu'} \\
\uparrow{\scriptstyle \mu(U)} & & \uparrow{\scriptstyle \mu(X_{\nu'})} \\
F_{d\mu}U & \xrightarrow[F_{d\mu}(u\circ f_\nu)]{} & F_{d\mu}X_{\nu'}
\end{array}$$

commutes. Since card(M) < α one can find a transition morphism $X_\nu \longrightarrow X_{\nu''}$ which has this property for every $\mu \in M$. This shows that $f : (U,M,R) \longrightarrow \varinjlim_{\nu'} (X_\nu, M, R)$ admits a factorization into bialgebra morphisms $(U,M,R) \longrightarrow (X_{\nu''},M,R) \longrightarrow \varinjlim_{\nu'} (X_\nu,M,R)$, i.e. the canonical map

$$\varinjlim_{\nu} [(U,M,R),(X_\nu,M,R)] \longrightarrow [(U,M,R), \varinjlim_{\nu} (X_\nu,M,R)]$$

is surjective. In the same way one can show it is also injective. Hence (U,M,R) is α-presentable in Bialg($\underline{A}$). (The former follows also directly from the fact that V : Bialg($\underline{A}$) $\longrightarrow$ $\underline{A}$ is faithful and preserves α-filtered colimits and that U is α-presentable in $\underline{A}$).

3.6 Remark One would like to conclude from 3.5 that a bialgebra (U,M,R) is α-presentable in Bialg($\underline{A}$) provided its underlying object $U \in \underline{A}$ is. We will show below in 3.7 that this is true provided α is sufficiently large and $\underline{A}$ is locally presentable. If $\underline{A}$ is locally finitely presentable, then 3.7 provides the smallest α for which this is true. In general this is not so and resorting to 3.7 can give poor estimates. However in examples one often knows enough about the functors $F \in \mathbb{F}_d$ (e.g. in 3.2 III) to find out directly what the smallest α is such that $\pi(U) \leq \alpha$ implies $\pi(FU) \leq \alpha$ for every $F \in \mathbb{F}_d$. A particular situation is the following. Assume that the codomain of every $F \in \mathbb{F}_d$ is locally presentable and that every F has a right adjoint G_F. Then by 2.9 there is a (smallest) regular cardinal ß such that every G_F preserves β-filtered colimits. Hence $\pi(U) \leq \beta$ implies $\pi(FU) \leq \beta$ for every $F \in \mathbb{F}_d$ because $[FU,-] \cong [U,G_F -]$. Likewise $\varepsilon(U) \leq \beta$ implies $\varepsilon(FU) \leq \beta$ for every $F \in \mathbb{F}_d$ and $U \in \underline{A}$.

3.7 Lemma Let $\underline{A}$ be a locally α-presentable category and let $\mathbb{F}_d$ be a set of functors with domain $\underline{A}$ which preserve α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be a regular cardinal such that

1) if $X \in \underline{A}$ and $\pi(X) \leq \alpha$, then $\pi(FX) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$,
2) if $\rho < \alpha$ and $\beta < \bar{\alpha}$, then $\beta^{\rho} < \bar{\alpha}$.

Then $\pi(U) \leq \bar{\alpha}$ implies $\pi(FU) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$ and $U \in \underline{A}$.

Corollary Let $\underline{A}$ be a locally α-presentable category with a data M and R for bialgebras (3.1). Assume that card(M) < α and that every $F \in \mathbb{F}$ preserves α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be a cardinal with the above properties 1) and 2). Then a bialgebra (U,M,R) is $\bar{\alpha}$-presentable in Bialg($\underline{A}$) provided U is $\bar{\alpha}$-presentable in $\underline{A}$.

Remarks a) Note that condition 2) is trivially satisfied if either $\alpha = \aleph_0$ or $\bar{\alpha}$ is of the form $(2^{\gamma})^{+}$ for some $\gamma^{+} \geq \alpha$

b) Since the α-presentable objects in $\underline{A}$ form a small subcategory there exists always a cardinal $\bar{\alpha}$ with the properties 1) and 2).

c) Using 5.1 one can prove an assertion analoguous to 3.7 for locally α-generated categories (cf. remark 3.5).

Proof of 3.7 The case $\bar{\alpha} = \alpha$ is trivial and we assume $\bar{\alpha} > \alpha$. Given $U \in \underline{A}$ with $\pi(U) \leq \bar{\alpha}$ we are looking for an α-filtered colimit presentation $U = \varinjlim_K X_K$ such that $\pi(X_K) \leq \bar{\alpha}$ for every K and the cardinality of the index system is strictly smaller than $\bar{\alpha}$. Since $F \in \mathbb{F}_d$ preserves α-filtered colimits, it then follows easily that $\pi(FU) = \pi(\varinjlim_K FX_K) \leq \bar{\alpha}$.

We need some preparation. Let D be a partially ordered set which is α-filtered and let D' be a subset of cardinality $< \bar{\alpha}$. Then D' is contained in an α-filtered subset D" whose cardinality is also $< \bar{\alpha}$. One constructs D" by transfinite induction as follows. Let $D'_0 = D'$. If $\lambda < \alpha$ is a successor ordinal then let D'_λ be the subset consisting of $D'_{\lambda-1}$ and an upper bound in D for every subset $I \subset D'_{\lambda-1}$ with $\mathrm{card}(I) < \alpha$. If $\lambda < \alpha$ is a limit ordinal let $D'_{\lambda} = \bigcup_{\rho<\lambda} D_{\rho}$. In either case it follows from assumption 2) that $\mathrm{card}(D'_{\lambda}) < \bar{\alpha}$. Clearly $D'' = \bigcup_{\lambda<\alpha} D'_{\lambda}$ is α-filtered and $\mathrm{card}(D'') < \bar{\alpha}$ because $\alpha < \bar{\alpha}$.

Let $U \in \underline{A}$ be $\bar{\alpha}$-presentable. Then by 1.7 there is a cokernal diagram

$$\coprod_{i\in I} X_i \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{j\in J} X_j \longrightarrow Y$$

such that 1) $\mathrm{card}(I) < \bar{\alpha} > \mathrm{card}(J)$ 2) X_i and X_j are α-presentable for every $i \in I$ and $j \in J$ and 3) U is a retract of Y. We will show that $\pi(FY) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$. Since this implies $\pi(FU) \leq \bar{\alpha}$, we can assume without loss of generality that $Y = U$.

Let D be the partially ordered set consisting of quadruples (I_K, J_K, f_K, g_K), where $I_K \subset I$, $J_K \subset J$ and $f_K, g_K : \coprod_{i\in I} X_i \rightrightarrows \coprod_{j\in J_K} X_j$ are morphisms such that $\mathrm{card}(I_K) < \alpha > \mathrm{card}(J_K)$ and the canonical diagram

$$\begin{array}{ccc}
\coprod_{i\in I_K} X_i & \underset{g_K}{\overset{f_K}{\rightrightarrows}} & \coprod_{j\in J_K} X_j \\
\downarrow & & \downarrow \\
\coprod_{i\in I} X_i & \underset{g}{\overset{f}{\rightrightarrows}} & \coprod_{j\in J} X_j
\end{array}$$

commutes. The ordering is given by inclusion, i.e. $K < K'$ provided $I_K \subset I_{K'}$, $J_K \subset J_{K'}$, and the induced diagram

$$\begin{array}{ccc}
\coprod_{i\in I_K} X_i & \underset{g_K}{\overset{f_K}{\rightrightarrows}} & \coprod_{j\in J_K} X_j \\
\downarrow & & \downarrow \\
\coprod_{i\in I_{K'}} X_i & \underset{g_{K'}}{\overset{f_{K'}}{\rightrightarrows}} & \coprod_{j\in J_{K'}} X_j
\end{array}$$

commutes. Note that in both diagrams the vertical morphisms need not be monomorphic! (This complicates the proof considerably). Using that $\coprod_{i \in I_K} X_i$ is α-presentable and $\coprod_{j \in J} X_j$ is the α-filtered colimit of its subcoproducts with less than α summands, it is routine to verify that D is α-filtered. Let D' be the subset of D obtained in the following way: For every pair $I' \subset I$ and $J' \subset J$ with $\mathrm{card}(I') < \alpha > \mathrm{card}(J')$ pick <u>one</u> element (I_K, J_K, f_K, g_K) of D with the property $I' = I_K$ and $J' = J_K$ (provided there is such an element, there may be many or none with this property). Clearly condition 2) and $\mathrm{card}(I) < \bar{\alpha} > \mathrm{card}(J)$ imply $\mathrm{card}(D') < \bar{\alpha}$. Given $\coprod_{i \in I'} X_i$ with $\mathrm{card}(I') < \alpha$ there is an element (I_K, J_K, f_K, g_K) with $I' = I_K$ because $\coprod_{i \in I'} X_i$ is α-presentable and $\coprod_{j \in J} X_j$ is the α-filtered colimit of its subcoproducts with less than α summands. Likewise given $\coprod_{j \in J'} X_j$ with $\mathrm{card}(J') < \alpha$ one can find an element (I_K, J_K, f_K, g_K) such that $J' \subset J_K$. From this it follows that D' is not empty and that the colimits of $D'' \longrightarrow \underline{A}$, $K \rightsquigarrow \coprod_{i \in I_K} X_i$ and $D'' \longrightarrow \underline{A}$, $K \rightsquigarrow \coprod_{j \in J_K} X_j$, are $\coprod_{i \in I} X_i$ and $\coprod_{j \in J} X_j$ respectively (for D'' see above). Whence the colimit of $D'' \longrightarrow \underline{A}$, $K \rightsquigarrow X_K = \mathrm{coker}(f_K, g_K)$ is U. Note that D'' is α-filtered and $\mathrm{card}(D'') < \bar{\alpha}$. Since $X_K = \mathrm{coker}(f_K, g_K)$ is α-presentable, by condition 1) FX_K is $\bar{\alpha}$-presentable for every $F \in \mathbb{F}_d$. Summarizing we obtain

$$\pi(FU) = \pi(F \varinjlim_{K \in D''} X_K) = \pi(\varinjlim_{K \in D''} FX_K) \leq \bar{\alpha}$$

because an $\bar{\alpha}$-colimit of $\bar{\alpha}$-presentable objects is again $\bar{\alpha}$-presentable. This completes the proof.

3.8 <u>Theorem Let $\underline{A}$ be a locally presentable category and let</u> M, R <u>and</u> $\mathbb{F}$ <u>be a data for bialgebras</u> (cf. 3.1). <u>Assume there is a regular cardinal</u> β <u>such that every</u> $F \in \mathbb{F}$ <u>preserves β-filtered colimits.</u> <u>Let</u> $\gamma > \beta$ <u>be any regular cardinal such that</u>

a) $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ <u>and</u> $\underline{A}$ <u>is locally γ-presentable.</u>

b) if $U \in \underline{A}$ is γ-presentable, then FU is γ-presentable for every $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Let (A,M,R) be a bialgebra and let $U \in \underline{A}$ be a γ-presentable object. Then every morphism $f : U \longrightarrow A$ admits a factorization into a morphism $U \longrightarrow U'$ and a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that $U' \in \underline{A}$ is again γ-presentable. Moreover a bialgebra (X,M,R) is γ-presentable in $\mathrm{Bialg}(\underline{A})$ iff X is γ-presentable in $\underline{A}$.

Remark Note that γ has to be strictly bigger than β ; hence $\gamma \geq \aleph_1$. If the codomain of every $F \in \mathbb{F}_d$ is locally presentable, then by 3.7 there is always a cardinal $\gamma > \beta$ such that the above conditions a) and b) hold. The point is of course to choose γ as small as possible. The moste useful situation seems $\gamma = \aleph_1$ and $\beta = \aleph_0$. This happens in any of the following cases

I card(M) $\leq \aleph_0 \geq$ card(R) , $\pi(\underline{A}) = \aleph_0$, every $F \in \mathbb{F}$ preserves filtered colimits, and every $F \in \mathbb{F}_d$ takes finitely presentable objects into countably presentable objects (cf. 3.7).

II card(M) $\leq \aleph_0 \geq$ card(R), $\pi(\underline{A}) \leq \aleph_1$, every $F \in \mathbb{F}$ preserves filtered colimits, and every $F \in \mathbb{F}_d$ takes countably presentable objects into countably presentable objects.

III card(M) $\leq \aleph_0 \geq$ card(R), $\pi(\underline{A}) \leq \aleph_1$, every $F \in \mathbb{F}$ preserves filtered colimits and every $F \in \mathbb{F}_d$ has a right adjoint G_F which preserves countably filtered colimits (cf. 3.6).

3.9 Corollary Let $\underline{Y}(\gamma)$ be the full subcategory of $\underline{Y} = \mathrm{Bialg}(\underline{A})$ consisting of all γ-presentable objects. Then for every $Y \in \underline{Y}$ the category $\underline{Y}(\gamma)/Y$ is γ-filtered and the colimit of the forgetful functor $\underline{Y}(\gamma)/Y \longrightarrow \underline{Y}$ is Y ; i.e. the inclusion $\underline{Y}(\gamma) \hookrightarrow \underline{Y}$ is dense (cf.[13]3.1).

Definition A set valued functor on a small category is called γ-flat if it is a γ-filtered colimit of representable functors.

3.1o Corollary Let $\mathrm{Flat}_\gamma[\underline{Y}(\gamma)^o, \underline{\mathrm{Sets}}]$ denote the full subcategory of $[\underline{Y}(\gamma)^o, \underline{Sets}]$ consisting of all γ-flat functors. Then the functor

$$\underline{Y} \longrightarrow \mathrm{Flat}_\gamma[\underline{Y}(\gamma)^o, \underline{Sets}] \ , \ Y \rightsquigarrow [-, Y]$$

is an equivalence.

3.11 Remarks I One can view 3.1o as a "generalization" of [13] 7.9. The latter asserts that a locally γ-presentable category $\underline{X}$ is of the form $\underline{X} \xrightarrow{\simeq} \mathrm{St}_\gamma[\underline{X}(\gamma)^o, \underline{Me}]$. Thus, if $\underline{Y}(\gamma)$ is γ-cocomplete, then by [13] 5.4 a functor $\underline{Y}(\gamma)^o \longrightarrow \underline{Sets}$ is γ-flat iff it is γ-continuous, i.e. $\mathrm{Flat}_\gamma[\underline{Y}(\gamma)^o, \underline{Sets}] = \mathrm{St}_\gamma[\underline{Y}(\gamma)^o, \underline{Me}]$ (cf. [13] 7.9).

II It will be apparent from the proofs of 3.8 - 3.1o that the hypotheses have not been fully used; in particular the existence of arbitrary colimits in $\underline{A}$. Besides b) and $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ only the following properties are used

a) $\underline{A}$ has β-filtered colimits for some $\beta < \gamma$ and every $F \in \mathbb{F}$ preserves them,

b) for every $A \in \underline{A}$ the category $\underline{A}(\gamma)/A$ of γ-presentable objects over A is γ-filtered and A is the colimit of $\underline{A}(\gamma)/A \longrightarrow \underline{A}$, $(U \longrightarrow A) \rightsquigarrow U$ (cf. 2.8).

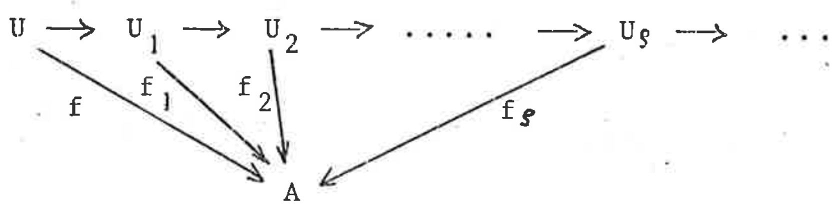
In general $\mathrm{Bialg}(\underline{A})$ is not locally presentable but it has again β-filtered colimits and by 3.9 it inherits property b). For instance the category of flat left Λ-modules over a ring Λ need not be locally presentable, but has filtered colimits and satisfies property b) for every γ. An important class of categories which are not locally presentable but for which 3.8 - 3.1o applies are the "catégories localisables" recently introduced by Y. Diers [5].

Proof of 3.1o. By 3.4 $\underline{Y}$ has γ-filtered colimits. The functor $\underline{Y} \longrightarrow [\underline{Y}(\gamma)^o, \underline{Sets}]$, $Y \rightsquigarrow [-, Y]$ is full and faithful because the inclusion $\underline{Y}(\gamma) \xrightarrow{\subset} \underline{Y}$ is dense, cf [ ] 3.4. Moreover it preserves and reflects γ-filtered colimits because the objects of $\underline{Y}(\gamma)$ are γ-presentable in $\underline{Y}$. Also its values are in $\mathrm{Flat}[\underline{Y}(\gamma)^o, \underline{Sets}]$

because by 3.9 $\underline{Y}(\gamma)/Y$ is γ-filtered for every $Y \in \underline{Y}$. Hence the factorization $\underline{Y} \longrightarrow \mathrm{Flat}_{\gamma}\,[\underline{Y}(\gamma)^{o}, \underline{\mathrm{Sets}}]$, $Y \rightsquigarrow [-,Y]$ is a full embedding which preserves γ-filtered colimits. If $F : \underline{Y}(\gamma)^{o} \longrightarrow \underline{\mathrm{Sets}}$ is γ-flat, let $\underline{D}$ be a γ-filtered category together with a functor $\underline{D} \longrightarrow \underline{Y}$, $D \rightsquigarrow Y_D$ such that $F = \varinjlim_{D \in \underline{D}} [-,Y_D]$. By the above $\varinjlim_{D \in \underline{D}} [-,Y_D] \cong [-, \varinjlim_{D \in \underline{D}} Y_D]$, whence $F \cong [-,Y]$ for $Y = \varinjlim_{D \in \underline{D}} Y_D$. This completes the proof.

Proof of 3.8 and 3.9 Since the proof is fairly involved and technical we first give a sketch.

In a first step (3.12 - 3.17) we construct factorizations of $f: U \longrightarrow V(A,M,R)$

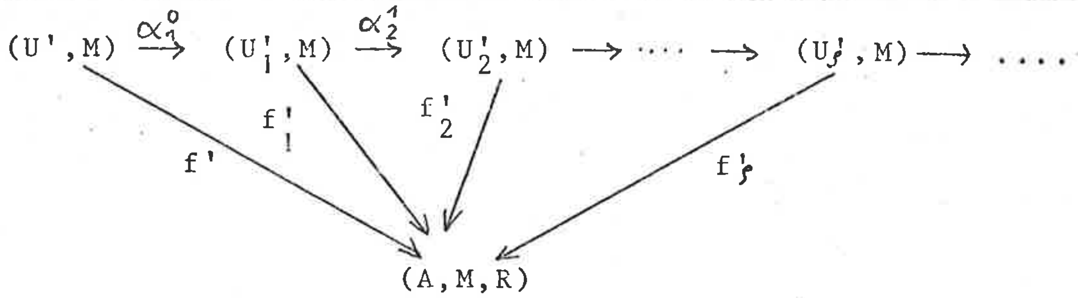


for every ordinal $\rho < \beta$ such that $\pi(U_\rho) \leq \gamma$ and for every operation $\mu \in M$ there is a morphism $\mu(\rho,\rho+1) : F_{d\mu}U_\rho \longrightarrow F_{c\mu}U_{\rho+1}$ making the diagram

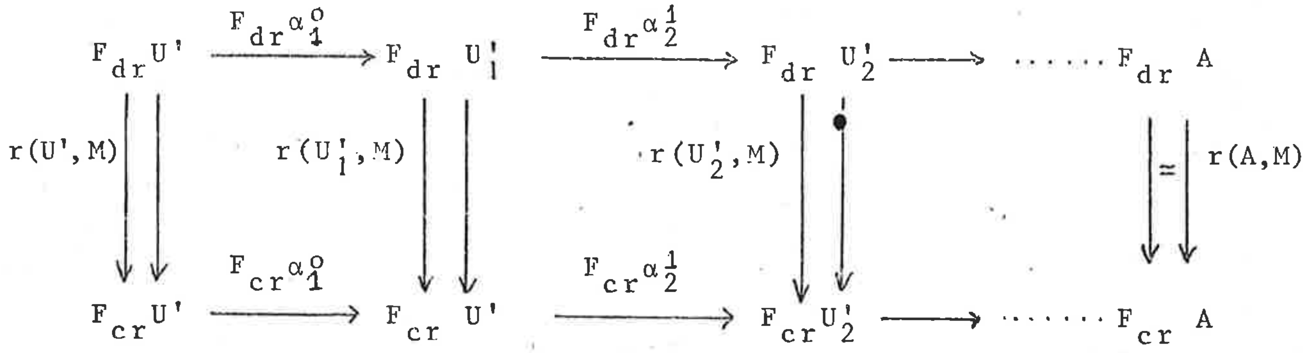
$$\begin{array}{ccc} F_{d\mu}U_\rho & \xrightarrow{\ \mu(\rho,\rho+1)\ } & F_{c\mu}U_{\rho+1} \\ \Big\downarrow{\scriptstyle F_{d\mu}f_\rho} & & \Big\downarrow{\scriptstyle F_{c\mu}f_{\rho+1}} \\ F_{d\mu}A & \xrightarrow{\ \mu(A)\ } & F_{c\mu}A \end{array}$$

commutative. Using that $F_{d\mu}$ and $F_{c\mu}$ preserve β-filtered colimits, we obtain in the limit a morphism $F_{d\mu}(\varinjlim_{\rho<\beta} U_\rho) \longrightarrow F_{c\mu}(\varinjlim_{\rho<\beta} U_{\rho+1})$ for every $\mu \in M$. The latter make $\varinjlim_{\rho<\beta} U_\rho$ into a pre-bialgebra and $\varinjlim_{\rho<\beta} f_\rho : \varinjlim_{\rho<\beta} U \longrightarrow A$ into a pre-bialgebra morphism. Since $\beta < \gamma$ the colimit $U' = \varinjlim_{\rho<\beta} U_\rho$ is again γ-presentable. In this way one obtains a factorization of $f : U \longrightarrow A$ into a morphism $U \longrightarrow U'$ and a pre-bialgebra morphism $f' = \varinjlim_{\rho<\beta} f_\rho : (U',M) \longrightarrow (A,M)$ with (U',M) being γ-presentable in P-Bialg($\underline{A}$) (cf. 3.5).

In a second step (3.18-3.2o) we construct factorizations



in the category P-Bialg($\underline{A}$) for every $\rho < \beta$ such that $\pi(U'_\rho) \le \gamma$ and for every relation $r \in R$ the vertical morphism pairs in the diagram



become equal when composed with the adjacent horizontal morphism (Note that the two components of r(A,M) coincide). Since relations on P-Bialg($\underline{A}$) commute with β-filtered colimits passing to the limit yields a pre-bialgebra $(U'',M) = \varinjlim_{\rho<\beta} (U'_\rho, M)$ which satisfies the relations. Since $\beta < \gamma$, one has also $\pi(U'') \le \gamma$. Thus the induced factorization $(U',M) \longrightarrow (U'',M) \longrightarrow (A,M,R)$ of $f' : (U',M) \longrightarrow (A,M,R)$ together with the one from the first step yields the desired decomposition of $f : U \longrightarrow V(A,M,R)$.

Finally to show that a γ-presentable bialgebra (X,M,R) has a γ-presentable underlying object X we study the category of those bialgebras (U,M,R) over (X,M,R) whose underlying object U is γ-presentable. We show that (X,M,R) is the colimit in Bialg($\underline{A}$) of these bialgebras and that this (comma) category is γ-filtered. Thus the identity of (X,M,R) admits a factorization
$(X,M,R) \longrightarrow (U,M,R) \longrightarrow (X,M,R)$ with $\pi(U) \le \gamma$. Hence X is a retract of U and thus also γ-presentable. Conversely, if X is γ-presentable,

then by 3.5 (X,M,R) is likewise in $\mathrm{Bialg}(\underline{A})$.

3.12 Let $R = \emptyset$ and let (A,M) be a bialgebra and $f : U \longrightarrow A$ a morphism with $\pi(U) \leq \gamma$. Let $(U_t,M)_{t \in T}$ be a family of bialgebras with $\mathrm{card}(T) < \gamma$ and let $(h_t : U_t \longrightarrow U)_{t \in T}$ be a family of morphisms such that U_t is γ-presentable and $f \circ h_t : (U_t,M) \longrightarrow (A,M)$ is a bialgebra morphism for every $t \in T$. We will show that f admits a factorization into a morphism $g' : U \longrightarrow U'$ and a bialgebra morphism $f' : (U',M) \longrightarrow (A,M)$ such that U' is γ-presentable and $g' \circ h_t : (U_t,M) \longrightarrow (U',M)$ is a bialgebra morphism for every $t \in T$.

3.13 Let $\underline{D}_{A,f}$ be the category whose objects are factorizations $i = (U \xrightarrow{g_i} U_i \xrightarrow{f_i} A)$ of f with $\pi(U_i) \leq \gamma$ and whose morphisms $i \longrightarrow j$ are morphisms $\alpha^i_j : U_i \longrightarrow U_j$ in $\underline{A}$ with $\alpha^i_j g_i = g_j$ and $f_j \alpha^i_j = f_i$. Since $\underline{D}_A = \underline{A}(\gamma)/A$ is γ-filtered (cf. 2.8), it easily follows that the functor

$$\underline{D}_{A,f} \longrightarrow \underline{D}_A, \quad (U \xrightarrow{g_i} U_i \xrightarrow{f_i} A) \rightsquigarrow (U_i \xrightarrow{f_i} A)$$

is cofinal and that $\underline{D}_{A,f}$ is also γ-filtered. Since $\beta < \gamma$ the category $\underline{D}_A$ has β-wellordered colimits which are computed pointwise. Hence the same holds for $\underline{D}_{A,f}$. For an ordinal $\lambda \leq \beta$ let $\underline{I}^-_\lambda$ (resp. $\underline{I}_\lambda$) denote the wellordered set of all ordinals $\rho < \lambda$ (resp. $\rho \leq \lambda$). By transfinite induction we will construct a functor

$$\Phi : \underline{I}^-_\beta \longrightarrow \underline{D}_{A,f}, \quad \lambda \rightsquigarrow (U \xrightarrow{g_\lambda} U_\lambda \xrightarrow{f_\lambda} A)$$

with $$\Phi(0) = (U \xrightarrow{id} U \xrightarrow{f} A) \quad \text{such that}$$

$$\varinjlim \Phi = (U \longrightarrow \varinjlim_{\lambda<\beta} U_\lambda \xrightarrow{\varinjlim f_\lambda} A)$$

is a factorization of $f : U \longrightarrow A$ with the properties stated in 3.12. For $\rho < \tau$ in $\underline{I}^-_\beta$ the transition morphism $\Phi(\rho) \longrightarrow \Phi(\tau)$ is denoted with α^ρ_τ.

3.14 The induction hypothesis for an ordinal λ is as follows.

There is a functor

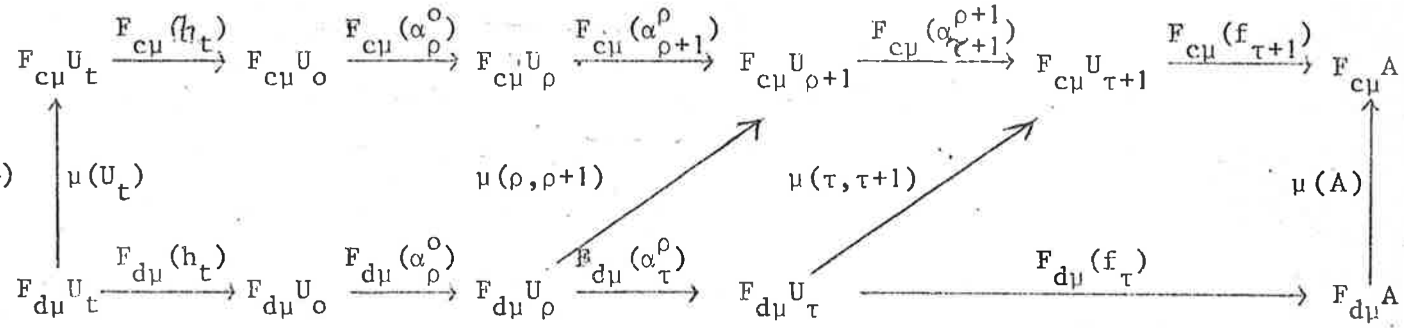
(*) $\quad \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}, \ \rho \rightsquigarrow (U \xrightarrow{g_\rho} U_\rho \xrightarrow{f_\rho} A)$

whose value at 0 is $(U \xrightarrow{id} U \xrightarrow{f} A)$ together with a morphism

(**) $\quad \mu(\rho,\rho+1) : F_{d\mu}U_\rho \longrightarrow F_{c\mu}U_{\rho+1}$

for every $\mu \in M$ and every ρ such that $\rho+1 < \lambda$ subject to the following condition: For every $t \in T$ and for every pair ρ,τ with $\rho+1 < \tau+1 < \lambda$ the diagrams

(***)

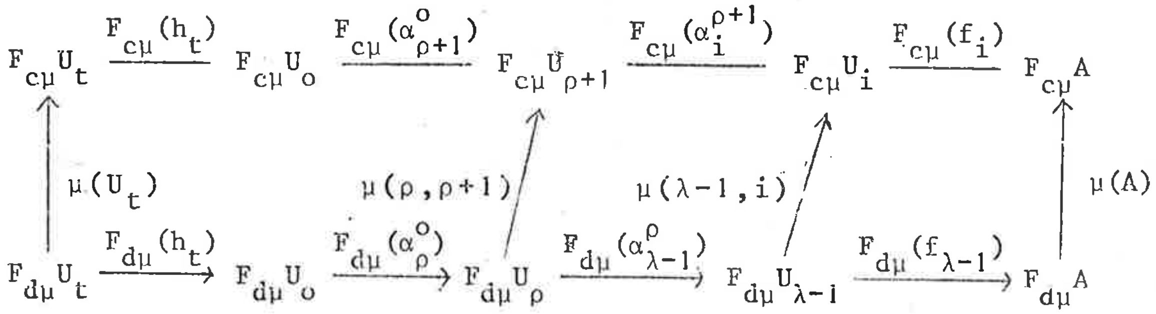


commute.

3.15 For $\lambda=1$ we put $\bar{\Phi}(0) = (id_U, f)$ and (*) trivially holds whereas (**) and (***) are vacuous. If λ is a limit ordinal $< \beta$, the functor $\bar{\Phi} : \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}$ given by induction hypothesis has to be extended to $\underline{I}_\lambda$. Since $\underline{D}_{A,f}$ is γ-filtered, the image of $\bar{\Phi} : \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}$ has an upper bound in $\underline{D}_{A,f}$, i.e. there is an object $(f_{\lambda'}, g_{\lambda'}) \in \underline{D}_{A,f}$ together with a morphism $\alpha_{\lambda'}^\tau : (f_\tau, g_\tau) \longrightarrow (f_{\lambda'}, g_{\lambda'})$ for every $\tau < \lambda$. Since $\underline{D}_{A,f}$ is γ-filtered, there is moreover an object $(f_\lambda, g_\lambda) \in \underline{D}_{A,f}$ together with a morphism $\alpha_\lambda^{\lambda'} : (f_{\lambda'}, g_{\lambda'}) \longrightarrow (f_\lambda, g_\lambda)$ such that for every pair $\rho < \tau$ in $\underline{I}_\lambda^-$ the equation $\alpha_\lambda^\rho = \alpha_\lambda^\tau \circ \alpha_\tau^\rho$ holds, where $\alpha_\lambda^\tau = \alpha_\lambda^{\lambda'} \circ \alpha_{\lambda'}^\tau$ and $\alpha_\lambda^\rho = \alpha_\lambda^{\lambda'} \circ \alpha_{\lambda'}^\rho$. Therefore we can define $\bar{\Phi}(\lambda) = (f_\lambda, g_\lambda)$ and $\bar{\Phi}(\rho < \alpha) = \alpha_\lambda^\rho$ and obtain an extension $\bar{\Phi} : \underline{I}_\lambda \longrightarrow \underline{D}_{A,f}$. Note that (**) and (***) hold trivially for every ρ with $\rho+1 \leq \lambda$ and every pair ρ,τ with

$\rho+1 < \tau+1 \leq \lambda$ because λ is a limit ordinal.

If λ is not a limit ordinal, then by assumption there is a functor $\Phi : \underline{I}_{\lambda-1} \to \underline{D}_{A,f}$ together with a morphism $\mu(\rho,\rho+1) : F_{d\mu}U_\rho \to F_{c\mu}U_{\rho+1}$ satisfying (***) for every $\mu \in M$ and every ρ with $\rho+1 < \lambda$. It suffices to construct $\alpha_\lambda^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_\lambda, g_\lambda)$ and $\mu(\lambda-1,\lambda) : F_{d\mu}U_{\lambda-1} \to F_{c\mu}U_\lambda$ such that the conditions (*), (**) and (***) hold. For an operation $\mu \in M$ it follows from $F_{c\mu}A = \varinjlim_{i \in \underline{D}_{A,f}} F_{c\mu}U_i$ and $\pi(F_{d\mu}U_{\lambda-1}) \leq \gamma \geq \pi(F_{d\mu}U_\rho)$ and $\lambda < \gamma$ that there is an object $i = i(\mu)$ in $\underline{D}_{A,f}$ together with morphisms $\alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_i, g_i)$ and $\mu(\lambda-1,i) : F_{d\mu}U_{\lambda-1} \to F_{c\mu}U_i$ such that for every $t \in T$ and every $\rho < \lambda-1$ the two squares on the right in the diagram

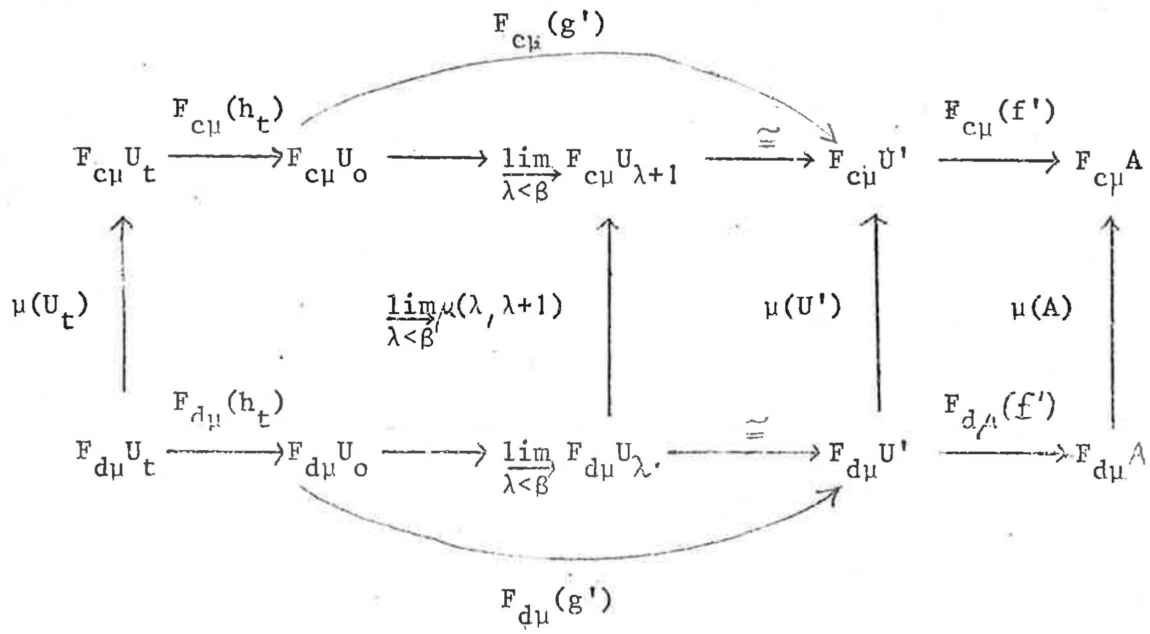


commute, where $\alpha_i^{\rho+1} = \alpha_i^{\lambda-1}\alpha_{\lambda-1}^{\rho+1}$. Note that for $\lambda-1 > 0$ the left side of the diagram commutes by induction hypothesis whereas for $\lambda = 1$ this can be established using $\pi(F_{d\mu}U_t) \leq \gamma > \text{card}(T)$ in the same way as for the middle square. Since $\text{card}(M) < \gamma$ and $\underline{D}_{A,f}$ is γ-filtered, there is an object $(U \xrightarrow{g_\lambda} U_\lambda \xrightarrow{f_\lambda} A)$ in $\underline{D}_{A,f}$ together with a morphism $\alpha_\lambda^i : (f_i, g_i) \to (f_\lambda, g_\lambda)$ for every $\mu \in M$ such that $\alpha_\lambda^i \circ \alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_\lambda, g_\lambda)$ is independant of $i = i(\mu)$. Hence we can define $\Phi(\lambda) = (f_\lambda, g_\lambda)$, $\Phi(\lambda-1<\lambda) = \alpha_\lambda^i \circ \alpha_i^{\lambda-1}$ and $\mu(\lambda-1,\lambda) = F_{c\mu}(\alpha_\lambda^i) \circ \mu(\lambda-1,i)$ for $\mu \in M$. With this one easily sees that $\Phi : \underline{I}_\lambda \to \underline{D}_{A,f}$ is an extension onto $\underline{I}_\lambda$ and that $\mu(\lambda-1,\lambda)$ satisfies (***).

3.16 We now construct a factorization of $f : U \to A$ into a morphism

$g' : U \to U'$ and a bialgebra morphism $f' : U' \to A$ such that the properties stated in 3.12 hold. Let $(U \xrightarrow{g'} U \xrightarrow{f'} A)$ be the colimit of $\Phi : \underline{I}_\beta \to \underline{D}_{A,f}$. Then, as mentioned in the middle of 3.13, we have $U' = \varinjlim_{\lambda<\beta} U_\lambda$ and $f' = \varinjlim_{\lambda<\beta} f_\lambda$ whereas $g' : U \to U'$ is the canonical morphism into the colimit.

For every $\mu \in M$ the functors $F_{d\mu}$ and $F_{c\mu}$ preserve β-filtered colimits, in particular $\varinjlim_{\lambda<\beta} F_{d\mu} U_\lambda \xrightarrow{\cong} F_{d\mu} \varinjlim_{\lambda<\beta} U_\lambda$ and $\varinjlim_{\lambda<\beta} F_{c\mu} U_\lambda \xrightarrow{\cong} F_{c\mu} \varinjlim_{\lambda<\beta} U_\lambda$. Passing to the colimit with 3.14 (**) and (***)yields an unique morphism $\mu(U') : F_{d\mu} U' \to F_{c\mu} U'$ such that the diagram



commutes. This shows that U' together with the morphisms $\mu(U')$, $\mu \in M$, is a bilagebra and that $f' : U' \to A$ and $g'h_t : U_t \to U'$ are bialgebra morphisms for every $t \in T$. This completes the proof of the assertion in 3.12.

3.17 For a bialgebra (A,M) let $\underline{D}_{(A,M)}$ be the category of bialgebras over (A,M) whose underlying object in $\underline{A}$ is γ-presentable. Recall that for every $A \in \underline{A}$ the category $\underline{D}_A = \underline{A}(\gamma)/A$ of γ-presentable objects over A is γ-filtered (even γ-cocomplete) and that the colimit of $\underline{D}_A \to \underline{A}, (U \xrightarrow{f} A) \rightsquigarrow U$, is A (cf. 2.8). From this and 3.12 it readily follows for a bialgebra (A,M) that the forgetful functor

(*) $\underline{D}_{(A,M)} \longrightarrow \underline{D}_A$, $\{(U,M) \xrightarrow{f} (A,M)\} \rightsquigarrow (U \xrightarrow{f} A)$

is cofinal and that $\underline{D}_{(A,M)}$ is γ-filtered (but in general not γ-cocomplete). In particular (A,M) is the colimit of

(**) $\underline{D}_{(A,M)} \longrightarrow \text{Bialg}(\underline{A})$, $\{(U,M) \xrightarrow{f} (A,M)\} \rightsquigarrow (U,M)$

and $\underline{D}_{(A,M)}$ has β-wellordered colimits which are preserved by the functors (*) and (**).

3.18 We now return to the general case and drop the assumption $R = \emptyset$ which was made at the beginning of the proof in 3.12. For a bialgebra (A,M,R) let $\underline{D}_{(A,M,R)}$ be the category of bialgebras over (A,M,R) whose underlying object in $\underline{A}$ is γ-presentable. Clearly the forgetful functor

(3.19) $\underline{D}_{(A,M,R)} \longrightarrow \underline{D}_{(A,M)}$, $\{(U,M,R) \xrightarrow{f} (A,M,R)\} \rightsquigarrow \{(U,M) \xrightarrow{f} (A,M)\}$

is a full embedding. We will show below in 3.2o that it is cofinal. From this and 3.17 it follows that $\underline{D}_{(A,M,R)}$ is also γ-filtered and that (A,M,R) is the colimit of

$$\underline{D}_{(A,M,R)} \longrightarrow \text{Bialg}(\underline{A}),\ \{(U,M,R) \xrightarrow{f} (A,M,R)\} \rightsquigarrow (U,M,R)$$

If (X,M,R) is γ-presentable in Bialg($\underline{A}$), then this implies that the identity of (X,M,R) admits a factorization $(X,M,R) \rightarrow (U,M,R) \xrightarrow{f} (X,M,R)$ with $f \in \underline{D}_{(X,M,R)}$. Hence X is a retract of U, in particular X is also γ-presentable. Conversely, if X is γ-presentable in $\underline{A}$, then by 3.5 (X,M,R) is γ-presentable in Bialg($\underline{A}$). This proves the second assertion of 3.8. Moreover this shows that the category $\underline{D}_{(A,M,R)}$ is the category of γ-presentable objects over (A,M,R) in Bialg($\underline{A}$) which completes the proof of 3.9.

3.2o For the cofinality of the functor 3.19 it suffices to show that a pre-bialgebra morphism $f : (U,M) \to (A,M,R)$ with $\pi(U) \leq \gamma$ factors through a bialgebra morphism $f' : (U',M,R) \to (A,M,R)$ such that $\pi(U') \leq \gamma$. The construction of f' is similar to 3.12 - 3.16. Let $\underline{D}_{(A,M),f}$ be the category whose objects are factorizations $i = \{(U,M) \xrightarrow{g_i} (U_i,M) \xrightarrow{f_i} (A,M)\}$ of f with $\pi(U_i) \leq \gamma$ and whose morphisms $i \to j$ are morphisms $\alpha^i_j : (U_i,M) \to (U_j,M)$ in P-Bialg($\underline{A}$) with the properties $f_i = f_j \alpha^i_j$ and $g_j = \alpha^i_j g_i$. Since $\underline{D}_{(A,M)}$ is γ-filtered and has β-wellordered colimits, it easily follows that the functor

$$\underline{D}_{(A,M),f} \longrightarrow \underline{D}_{(A,M)}, \quad \{g_i, f_i\} \rightsquigarrow \{f_i : (U_i,M) \to (A,M)\}$$

is cofinal and that $\underline{D}_{(A,M),f}$ is also γ-filtered and has β-well=ordered colimits.

Recall that $\underline{I}^-_\lambda$ (resp. $\underline{I}_\lambda$) denotes the wellordered set of all ordinals $\rho < \alpha$ (resp. $\rho \leq \lambda$). By means of transfinite inductions we construct a functor

$$\Omega : \underline{I}^-_\beta \longrightarrow \underline{D}_{(A,M),f},$$

with

$$\Omega(0) = \{(U,M) \xrightarrow{id} (U,M) \xrightarrow{f} (A,M)\}$$

such that the factorization $\varinjlim \Omega = \{(U,M) \xrightarrow{g'} (U',m) \xrightarrow{f'} (A,M)\}$ of f has the required properties. We write $\Omega(\rho) = \{(U,M) \xrightarrow{g_\rho} (U_\rho,M) \xrightarrow{f_\rho} (A,M)\}$ for $\rho \in \underline{I}^-_\beta$ and $\Omega(\rho<\tau) = \alpha^\rho_\tau$ for $\rho < \tau$ in $\underline{I}^-_\beta$.

We define $\Omega(0) = \{id_U, f\}$. Assume Ω has been constructed for all $\rho < \lambda$, i.e. there is a functor $\Omega : \underline{I}^-_\lambda \to \underline{D}_{(A,M),f}$ with $\Omega(0) = \{id_U, f\}$. $\Omega(0) = \{id_U, f\}$. If λ is a limit ordinal we extend Ω to $\underline{I}_\lambda$ by defining $\Omega(\lambda)$ as an appropriate upper bound of the image of $\Omega : \underline{I}^-_\lambda \to \underline{D}_{(A,M),f}$ (the details are as above in 3.15). Now let λ be a successor ordinal. For every relation $r \in R$ we have $\pi(F_{dr} U_{\lambda-1}) \leq \gamma$. Since $F_{cr}A = \varinjlim_{i \in \underline{D}_{(A,M),f}} F_{cr} U_i$ is a γ-filtered colimit and the morphisms

$r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ coincide, there is a factorization $\{(U,M) \xrightarrow{g_i} (U_i,M) \xrightarrow{f_i} (A,M)\}$ and a morphism

$$\alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \longrightarrow (f_i, g_i)$$

in $\underline{D}_{(A,M),f}$ depending on r such that the morphisms

$$r(U_{\lambda-1},M) : F_{dr}U_{\lambda-1} \rightrightarrows F_{cr}U_{\lambda-1}$$

become equal when composed with $F_{cr}\alpha_i^{\lambda-1} : F_{cr}U_{\lambda-1} \longrightarrow F_{cr}U_i$. Since card(R) $< \gamma$ and $\underline{D}_{(A,M),f}$ is γ-filtered there is a factorization

$$\{(U,M) \xrightarrow{g_\lambda} (U_\lambda,M) \xrightarrow{f_\lambda} (A,M)\}$$

together with morphisms $\alpha_\lambda^i : (f_i,g_i) \longrightarrow (f_\lambda,g_\lambda)$ in $\underline{D}_{(A,M),f}$ such that $\alpha_\lambda^{\lambda-1} = \alpha_\lambda^i \circ \alpha_i^{\lambda-1} : (f_{\lambda-1},g_{\lambda-1}) \longrightarrow (f_\lambda,g_\lambda)$ is independant of r . Thus we can define $\Omega(\lambda) = (g_\lambda,f_\lambda)$ and $\Omega(\lambda-1 < \lambda) = \alpha_\lambda^{\lambda-1}$ and it is clear that $\Omega : \underline{I}_\lambda \longrightarrow \underline{D}_{(A,M),f}$ is a functor. This shows that there is a functor $\Omega : \underline{I}_\beta^- \to \underline{D}_{(A,M),f}$ with $\Omega(0) = \{id_U, f\}$. By 3.17 and the cofinality of the forgetful functor

$$\underline{D}_{(A,M),f} \longrightarrow \underline{D}_{(A,M)}, \quad (f_i,g_i) \rightsquigarrow f_i$$

the colimit of Ω exists and can be computed pointwise, i.e.

$$\varinjlim \Omega = \{(U,M) \xrightarrow{g'} (\varinjlim_{\lambda<\beta} U_\lambda, M) \xrightarrow{\varinjlim f_\lambda} (A,M)\}$$

where g' denotes the canonical morphism into the colimit. From the construction of $\alpha_{\lambda+1}^\lambda : (f_\lambda,g_\lambda) \to (f_{\lambda+1},g_{\lambda+1})$ and the diagram

$$\begin{array}{ccccccc}
F_{cr}U_\lambda & \xrightarrow{F_{cr}(\alpha_{\lambda+1}^\lambda)} & F_{cr}U_{\lambda+1} & \xrightarrow{\text{can.}} & \varinjlim_{\lambda<\beta} F_{cr}U_\lambda & \xrightarrow{\cong} & F_{cr} \varinjlim_{\lambda<\beta} U_\lambda \\
{\scriptstyle r(U_\lambda,M)}\upuparrows & & \upuparrows{\scriptstyle r(U_{\lambda+1},M)} & & \upuparrows{\scriptstyle \varinjlim_{\lambda<\beta} r(U_\lambda,M)} & & \upuparrows{\scriptstyle r(\varinjlim_{\lambda<\beta} U_\lambda,M)} \\
F_{dr}U_\lambda & \xrightarrow{F_{dr}(\alpha_{\lambda+1}^\lambda)} & F_{dr}U_{\lambda+1} & \xrightarrow{\text{can.}} & \varinjlim_{\lambda<\beta} F_{dr}U_\lambda & \xrightarrow{\cong} & F_{dr} \varinjlim_{\lambda<\beta} U_\lambda
\end{array}$$

it follows for every $r \in R$ that the two components of the morphism pair $r(\varinjlim_{\lambda<\beta} U_\lambda, M)$ coincide. Hence $(\varinjlim_{\lambda<\beta} U_\lambda, M)$ is a bialgebra.

3.21 Definition A sub-bialgebra of a bialgebra (A,M,R) is a bialgebra (U,M,R) together with a bialgebra morphism $f : (U,M,R) \to (A,M,R)$ whose underlying morphism in $\underline{A}$ is a monomorphism.
Clearly $f : (U,M,R) \longrightarrow (A,M,R)$ is then also a monomorphism in Bialg($\underline{A}$). However the forgetful functor $V : \text{Bialg}(\underline{A}) \longrightarrow \underline{A}$ does not preserve monomorphisms in general (for an exception see 3.4 a)).
The question arises whether there is an assertion analogous to 3.8 for α-generated objects. This is not so. The reason for this asymmetry lies in the fact that the underlying functor $V : \text{Bialg}(\underline{A}) \longrightarrow \underline{A}$ and the functors $F \in \mathbb{F}_c$ need not preserve monomorphisms. We give below in 3.22 a version of 3.8 for γ-generated objects correcting this deficiency by additional assumptions. From the point of view of applications 3.22 is useful in either of the following situations:
1) $\underline{A}$ is locally γ-noetherian, i.e. every γ-generated object is γ-presentable, cf. [13] 9.19. or 2) every $F \in \mathbb{F}_c$ preserves finite limits, e.g. in the algebraic case 3.2' II).

3.22 Theorem Let $\underline{A}$ be a locally presentable category with a data M , R and $\mathbb{F}$ for bialgebras (3.1). Assume there is a regular cardinal β such that

1) every $F \in \mathbb{F}$ preserves β-filtered colimits
2) every β-wellordered colimit of monomorphisms in $\underline{A}$ is again a monomorphism.

Let $\gamma > \beta$ be any regular cardinal such that

3) card(M) < γ and card(R) < γ
4) $\underline{A}$ is locally γ-noetherian and if $U \in \underline{A}$ is γ-presentable, then FU is γ-presentable for every $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Instead of 4) one can assume

4)' $\underline{A}$ is locally γ-generated (cf. [13] 9.1,) and every $F \in \mathbb{F}_c$ preserves finite limits; moreover if $U \in \underline{A}$ is γ-generated, then FU is γ-generated for every $F \in \mathbb{F}_d$ (in this case the assumption $\text{card}(R) < \gamma$ is redundant).

Then the following hold.

a) If (A,M,R) is a bialgebra and $U \in A$ is a γ-generated subobject of A, then there is a sub-bialgebra $(U',M,R) \hookrightarrow (A,M,R)$ such that U' contains U and U' is also γ-generated.

b) A bialgebra (X,M,R) is γ-generated in $\text{Bialg}(\underline{A})$ iff X is γ-generated in $\underline{A}$.

c) A bialgebra (A,M,R) is the γ-filtered colimit in $\text{Bialg}(\underline{A})$ of its γ-generated sub-bialgebras.

d) If $\underline{A}$ is locally γ-noetherian, then every γ-generated bialgebra is γ-presentable in $\text{Bialg}(\underline{A})$; in particular if $\text{Bialg}(\underline{A})$ is co-complete (cf. 3.24 a), b) and 3.27 below), then $\text{Bialg}(\underline{A})$ is locally γ-noetherian.

3.23 Remarks

a) Note that γ has to be strictly bigger than β, hence $\gamma \geq \aleph_1$. If the codomain of every $F \in \mathbb{F}_d$ is locally presentable, then by 3.7 resp. 5.1 there is always a cardinal $\gamma > \beta$ such that the above conditions 3) and 4) hold (resp. the second half of 4)'). The point is of course to choose γ as small as possible. The most useful situation seems $\gamma = \aleph_1$ and $\beta = \aleph_0$. This happens in any of the following cases.

I Every $F \in \mathbb{F}$ preserves filtered colimits, $\text{card}(M) \leq \aleph_0 \geq \text{card}(R)$, $\underline{A}$ is locally finitely noetherian (resp. $\underline{A}$ is locally finitely generated), every $F \in \mathbb{F}_c$ takes finitely generated objects into countably presentable objects (resp. into countably generated objects and every $F \in \mathbb{F}_c$ preserves finite limits), cf. Corollary to 3.7.

II Every $F \in \mathbb{F}$ preserves filtered colimits and every countable colimit of monomorphisms in $\underline{A}$ is again a monomorphism, $card(M) \leq \aleph_0 \geq card(R)$, $\underline{A}$ is locally $\aleph_1$-noetherian (resp. $\underline{A}$ is locally $\aleph_1$-generated), every $F \in \mathbb{F}_d$ takes countably generated objects into countably presentable objects (resp. into countably generated objects, and every $F \in \mathbb{F}_c$ preserves finite limits), cf. Corollary to 3.7.

III Every $F \in \mathbb{F}$ preserves filtered colimits and every countable colimit of monomorphisms in $\underline{A}$ is again a monomorphism, $card(M) \leq \aleph_0 \geq card(R)$, $\underline{A}$ is locally $\aleph_1$-noetherian (resp. $\underline{A}$ is locally $\aleph_1$-generated).
Every $F \in \mathbb{F}_d$ has a right adjoint which preserves countably filtered colimits (resp. every $F \in \mathbb{F}_d$ has a right adjoint which preserves monomorphic countably filtered colimits, and every $F \in \mathbb{F}_c$ preserves finite limits), cf. 3.6.

b) As before in 3.8 the existence of arbitrary colimits in $\underline{A}$ is not needed for 3.22 (cf. 3.11 a), b)).

<u>Proof of 3.22</u> The proof is the same as for 3.8 with the following obvious modifications. First for 3.12 - 3.16:

In 3.12 the morphisms $f : U \longrightarrow A$ and $h_t : U_t \longrightarrow U$, $t \in T$, are monomorphisms and $\varepsilon(U) \leq \gamma \geq \varepsilon(U_t)$. In 3.13 the category $\underline{D}_A$ consists of all γ-generated subobjects of A and likewise $\underline{D}_{A,f}$ consists of all γ-generated subobjects of A containing $f : U \longrightarrow A$ (clearly both categories are γ-filtered, [13] 9.1 - 9.3).
With this the proof (3.14 - 3.16) of 3.12 goes through without change because either by assumption 4) in 3.22) one has $\pi(F_{d\mu}U) \leq \gamma$, for every $\mu \in M$ and every $U \in \underline{A}$ with $\varepsilon(U) \leq \gamma$ or by 4)' in 3.22 one has $\varepsilon(F_{d\mu}U) \leq \gamma$, for every $U \in \underline{A}$ with $\varepsilon(U) \leq \gamma$ and the transition morphisms in $F_{c\mu}A = \varinjlim F_{c\mu}U_i$ are monomorphic for every $\mu \in M$. Note

that by assumption 2) in 3.22 the induced morphism $\varinjlim_{\lambda<\beta} f_\lambda : \varinjlim_{\lambda<\beta} U_\lambda \longrightarrow A$ is again a monomorphism.

Second for 3.17 - 3.2o:

In 3.17 and 3.18 the categories $\underline{D}_{(A,M)}$ and $D_{(A,M,R)}$ consist of all sub-bialgebras of (A,M) (resp. sub-bialgebras of (A,M,R)) whose underlying object in $\underline{A}$ is γ-generated. In 3.2o the underlying morphism of $f : (U,M) \longrightarrow (A,M,R)$ in A is a monomorphism, and the category $\underline{D}_{(A,M),f}$ consists of all sub-prebialgebras of (A,M,R) which contain $f : (U,M) \longrightarrow (A,M,R)$ and whose underlying object in $\underline{A}$ is γ-generated.

With this the arguments in 3.17 - 3.2o go through without change. Note that as above by assumption 2) in 3.22 the induced morphism $\varinjlim_{\lambda<\beta} f_\lambda : \varinjlim_{\lambda<\beta}(U_\lambda,M) \longrightarrow (A,M)$ in 3.2o is again a monomorphism. Also note that in the presence of the assumption 1), 2), 3) and 4)' the cofinality argument in 3.2o is redundant because by 4)' a sub-prebialgebra of a bialgebra satisfies the relations automatically (whence the assumption card(R) < γ is not needed). Moreover in 3.19 a bialgebra (X,M,R) is γ-generated because of the remark following 3.5.

With these modifications it follows from 3.18 that the assertions a), b) and c) in 3.22 hold. As for d) it suffices to show that a γ-generated bialgebra (X,M,R) is γ-presentable in Bialg($\underline{A}$) . By 3.22 b) X is γ-generated in $\underline{A}$ and hence also γ-presentable because $\underline{A}$ is locally γ-noetherian. By 3.5 and assumption 4) in 3.22 (X,M,R) is γ-presentable in Bialg($\underline{A}$) .

We now investigate the completeness and cocompleteness of Bialg($\underline{A}$). Basically this occurs when the given data M , R and $\mathbb{F}$ for bialgebras (3.1) has one of the following properties: 1) every $F \in \mathbb{F}_c$ preserves limits (algebraic case, cf. 3.2 II), 2) every $F \in \mathbb{F}_d$ preserves colimits (coalgebraic case, cf. 3.2 II), and 3) the data M , R and $\mathbb{F}$ can be decomposed into one of type 1) and one of type 2) (roughly speaking every operation is algebraic or coalgebraic and every relation is algebraic or coalgebraic or a distributive law between an algebraic and coalgebraic operation).

3.24 Theorem Let $\underline{A}$ be a locally presentable category with a data M, R and $\mathbb{F}$ for bialgebras (cf. 3.1). Assume there is a regular cardinal β such that every $F \in \mathbb{F}$ preserves β-filtered colimits.
Let $\gamma > \beta$ be any regular cardinal such that

1) $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ and $\underline{A}$ is locally γ-presentable,
2) if $U \in \underline{A}$ is γ-presentable, then FU is γ-presentable for every $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Then the following hold.

a) If every $F \in \mathbb{F}_d$ preserves colimits, then $\mathrm{Bialg}(\underline{A})$ is locally γ-presentable and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is cotripleable. The right adjoint $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ of V preserves γ-filtered colimits (cF = cofree functor).

b) If every $F \in \mathbb{F}_c$ preserves limits, then $\mathrm{Bialg}(\underline{A})$ is locally $\sup(\beta, \pi(\underline{A}))$-presentable and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and preserves β-filtered colimits. (The left adjoint $F : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ of V is the free functor).

Remark Note the asymmetry between $\sup(\beta, \pi(\underline{A}))$ and γ in a) and b). For the locally γ-noetherian case see 3.22 d). For conditions guaranteeing $\beta = \aleph_0$ and $\gamma = \aleph_1$ see the remark following 3.3.

3.25 Corollary Let $\underline{A}$ be a Grothendieck category (resp. a topos) with a data M, R and $\mathbb{F}$ for bialgebras. If every $F \in \mathbb{F}_d$ preserves colimits and every $F \in \mathbb{F}_c$ finite limits, then $\mathrm{Bialg}(\underline{A})$ is again a Grothendieck category (resp. a topos). This follows from 3.24 a), 4.11 and 3.3.

Proof a) It follows from 3.8 and 3.4 b) that $\mathrm{Bialg}(\underline{A})$ is locally γ-presentable. By the special adjoint functor theorem the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ has a right adjoint $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$, whence by 3.4 b) V is cotripleable. For every γ-presentable object $(U,M,R) \in \mathrm{Bialg}(\underline{A})$ the functors $[(U,M,R), cF-]$ and $[U,-]$ are equivalent by adjointness, and by 3.8 U is γ-presentable. For a γ-filtered colimit $X = \varinjlim_{\nu} X_{\nu}$ the canonical morphism $\varphi : \varinjlim_{\nu} cFX_{\nu} \longrightarrow cF(\varinjlim_{\nu} X_{\nu})$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
[(U,M,R), \varinjlim_{\nu} cFX_{\nu}] & \xrightarrow{[(U,M,R),\varphi]} & [(U,M,R), cF(\varinjlim_{\nu} X_{\nu})] & \xrightarrow{\cong} & [U, \varinjlim_{\nu} X_{\nu}] \\
\downarrow \cong & & & & \downarrow \cong \\
\varinjlim_{\nu} [(U,M,R), cFX_{\nu}] & & \xrightarrow{\cong} & & \varinjlim_{\nu} [U, X_{\nu}]
\end{array}$$

Hence $[(U,M,R), \varphi]$ is a bijection for every γ-presentable object $(U,M,R) \in \mathrm{Bialg}(\underline{A})$. Since these objects forma set of (dense) generators in $\mathrm{Bialg}(\underline{A})$ (cf. 3.9), it follows that φ is an isomorphism. Thus $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ preserves γ-filtered colimits.

b) By 3.4 a), c) $\mathrm{Bialg}(\underline{A})$ has limits and β-filtered colimits and $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ preserves and reflects them. In order to show that V has a left adjoint, we verify the solution set condition. For every object $U \in \underline{A}$ there is a regular cardinal δ such that U is δ-presentable. By 3.7 there is a regular cardinal γ such that $\delta \leq \gamma > \beta$ and the conditions a) and b) of 3.8 hold for γ . Since the category $\underline{A}(\gamma)$ of γ-presentable objects in $\underline{A}$ is small, it follows from 3.8 and 3.1 that the same holds for $\underline{Y}(\gamma)$ (see 3.9), where $\underline{Y} = \mathrm{Bialg}(\underline{A})$. It then follows from 3.8 that a set of representatives of $\underline{Y}(\gamma)$ - i.e. a skeleton - is a solution set for U . Hence $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ has a left adjoint F and is tripleable by 3.4 a). The composite $V \circ F : \underline{A} \longrightarrow \underline{A}$ preserves β-filtered colimits and it therefore follows from Gabriel-Ulmer [13] 1o.3 that $\mathrm{Bialg}(\underline{A})$ is locally $\sup(\beta, \pi(\underline{A}))$-presentable.

3.26 It is well known that the category of commutative (resp. cocommutative) Hopfalgebras over a commutative ring Λ can be viewed as the category of cogroup (resp. group) objects in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras). Similarly the category of commutative (resp. cocommutative) Λ-bialgebras can be viewed as the category of comonoid (resp. monoid) objects in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras). In both cases this rests on the fact that in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras) the categorical coproduct (resp. product) is the tensor product lifted from $\underline{\mathrm{Mod}}_{\Lambda}$. Thus theorem 3.24 can be applied twice - first a) and then b) or vice versa- and it follows that any of the above categories is locally $\aleph_1$-presentable. However the category of arbitrary Λ-bialgebras (resp. Λ-Hopfalgebras) cannot be expressed this way because the tensor product lifted to the category of Λ-algebras or Λ-coalgebras is not the categorical coproduct or product. The following is motivated to rectify this, at least in part.

3.27 <u>Definition</u> Let M, R and $\mathbb{F}$ be a data for bialgebras in $\underline{A}$ (3.1). A decomposition of M, R and $\mathbb{F}$ into an algebraic and coalgebraic part consists of a data $\bar{M}$, $\bar{R}$ and $\bar{\mathbb{F}}$ in $\underline{A}$ and a data $\bar{\bar{M}}$, $\bar{\bar{R}}$ and $\bar{\bar{\mathbb{F}}}$ in $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ with the following properties:

1) $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{\bar{\bar{M}},\bar{\bar{R}}}(\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}))$,

2) every $\bar{F} \in \bar{\mathbb{F}}_c$ preserves limits,

3) every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_d$ preserves colimits.

Likewise a decomposition of M, R and $\mathbb{F}$ into a coalgebraic and algebraic part consists of a data $\bar{M}$, $\bar{R}$ and $\bar{\mathbb{F}}$ in $\underline{A}$ and a data $\bar{\bar{M}}$, $\bar{\bar{R}}$ and $\bar{\bar{F}}$ in $\mathrm{Bialg}_{\bar{M},\bar{R}}(A)$ with the properties

1) $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{\bar{\bar{M}},\bar{\bar{R}}}(\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}))$

2) every $\bar{F} \in \bar{\mathbb{F}}_d$ preserves colimits,

3) every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_c$ preserves limits.

For example to express the category Λ-Bialg of arbitrary Λ-bialgebras in this way let $\underline{A} = \underline{Mod}_\Lambda$ and choose $\bar{M}$ to consist of a multiplication $\mu : id_{\underline{A}} \otimes id_{\underline{A}} \longrightarrow id_{\underline{A}}$ and a unit $\mu : const_\Lambda \longrightarrow id_{\underline{A}}$ and $\bar{R}$ of the associative and unitary laws. Then $\underline{B} = Bialg_{\bar{M},\bar{R}}(\underline{A})$ is obviously the category of Λ-algebras and the tensor product lifts from $\underline{Mod}_\Lambda$ to $\underline{B}$. Let $\bar{\bar{M}}$ in $\underline{B}$ consist of a comultiplication $\Delta : id_{\underline{B}} \longrightarrow id_{\underline{B}} \otimes_\Lambda id_{\underline{B}}$ and a counit $\varepsilon : id_{\underline{B}} \longrightarrow const_\Lambda$ and let $\bar{\bar{R}}$ consist likewise of the coassociative and counitary laws. With this one readily checks that $Bialg_{\bar{\bar{M}},\bar{\bar{R}}}(\underline{B})$ is canonically isomorphic with $\Lambda\text{-}Bialg(\underline{A})$, cf. 4.4 for details. Unfortunately it doesn't seem possible to express the category of arbitrary Λ-Hopfalgebras in a similar way. While the antipode can be viewed as a morphism $s : id_{\underline{B}} \longrightarrow id_{\underline{B}}^{opp}$ I don't know how to express the relations involving s in $\underline{B}$. One would have to show that for a Λ-bialgebra $(M,\mu,u,\Delta,\varepsilon)$ the composites

$M \xrightarrow{\Delta} M \otimes_\Lambda M \xrightarrow{s \otimes id} M \otimes_\Lambda M \xrightarrow{\mu} M$ and

$M \xrightarrow{\Delta} M \otimes_\Lambda M \xrightarrow{id \otimes s} M \otimes_\Lambda M \xrightarrow{\mu} M$ which are defined in $\underline{Mod}_\Lambda$ are multiplicative or antimultiplicative whithout using that they coincide with $M \xrightarrow{\varepsilon} \Lambda \xrightarrow{u} M$.

3.28 Theorem Let $\underline{A}$ be a locally presentable category. Let M , R and $\mathbb{F}$ be a data for bialgebras in $\underline{A}$ which admits a decomposition into an algebraic part $\bar{M}$, $\bar{R}$, $\bar{\mathbb{F}}$ and a coalgebraic part $\bar{\bar{M}}$, $\bar{\bar{R}}$, $\bar{\bar{\mathbb{F}}}$ (cf. 3.27). Assume there is a regular cardinal β such that every $\bar{F} \in \bar{\mathbb{F}}$ and every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}$ preserve β-filtered colimits. Let $\gamma > \beta$ be any regular cardinal such that

1) $card(\bar{M}) < \gamma$, $card(\bar{\bar{M}}) < \gamma$, $card(\bar{R}) < \gamma$, $card(\bar{\bar{R}}) < \gamma$ and $\underline{A}$ is locally γ-presentable,
2) if $U \in \underline{A}$ and $(X,\bar{M},\bar{R}) \in Bialg_{\bar{M},\bar{R}}(\underline{A})$ are γ-presentable, then $\bar{F}U$ and $\bar{\bar{F}}(X,\bar{M},\bar{R})$ are γ-presentable for every $\bar{F} \in \bar{\mathbb{F}}_d$ and $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Then $Bialg_{M,R}(\underline{A}) \cong Bialg_{\bar{\bar{M}},\bar{\bar{R}}}(Bialg_{\bar{M},\bar{R}}(\underline{A}))$ is locally γ-presentable

<u>and the underlying functors</u>

$$\mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \quad \text{and} \quad \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \longrightarrow \underline{A}$$

<u>are cotripleable and tripleable respectively.</u>

<u>Proof</u> By 3.24 b) the underlying functor $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ is locally γ-presentable. Likewise by 3.24 a) the underlying functor $\mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ is cotripleable and $\mathrm{Bialg}_{M,R}(\underline{A})$ is locally γ-presentable.

3.29 <u>Remark</u> In the same way one considers morphism between algebraic theories and the corresponding algebraic functors (cf. Lawvere [21]), one can study morphisms between data for bialgebras. For a given data M, R in $\underline{A}$ and a subset $M' \subset M$ there is an obvious relative forgetful functor

$$V_{rel} : \text{P-}\mathrm{Bialg}_M(\underline{A}) \longrightarrow \text{P-}\mathrm{Bialg}_{M'}(\underline{A}) \ , \ (A,\mu(A))_{\mu\in M} \rightsquigarrow (A,\mu(A))_{\mu \in M'} \ .$$

Let R' be a set of relations on $\text{P-}\mathrm{Bialg}_{M'}(\underline{A})$ which hold in $\mathrm{Bialg}_{M,R}(\underline{A})$, i.e. $V_{rel}(A,M,R) \in \mathrm{Bialg}_{M',R'}(\underline{A})$ for every $(A,M,R) \in \mathrm{Bialg}_{M,R}(\underline{A})$. Then there is also an induced forgetful functor

$$V_{rel} : \mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{M',R'}(\underline{A})$$

One can easily generalize the results of this chapter – in particular 3.8, 3.22, 3.24, 3.28 – to this situation. But in general it is difficult to find a data M'', R'' in $\mathrm{Bialg}_{M'R'}(\underline{A})$ – hopefully simpler than M, R – such that $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{M'',R''}(\mathrm{Bialg}_{M',R'}(\underline{A}))$; (see 3.26, 3.27 for cases like $\mathrm{Bialg}(\underline{A}) \cong \mathrm{Coalg}(\mathrm{Alg}(\underline{A}))$.