§ 4 Examples of bialgebras in locally presentable categories

In this section we give a first series of examples of bialgebras and apply the main results of § 3. A second series can be found in § 6. In the following we discuss universal algebras (4.1), universal co-algebras (4.2), coalgebras over a commutative ring Λ (4.3), Λ-bialgebras and Λ-Hopfalgebras and generalizations (4.4 - 4.7), comodules over a Λ-coalgebra (4.8), bimodules over a Λ-bialgebra (4.9), coalgebras over a cotriple (4.10 - 4.12), algebras over a triple (4.13), données de recollement and descent data (4.14 - 4.18) and more generally sections and cartesian closed sections with respect to a fibration or cofibration (4.19 - 4.26). Although some of these cases are dual to each other as far as the data for bialgebras is concerned, the assertions resulting from 3.7, 3.8, 3.9, 3.22, 3.24 and 3.28 are not and can be quite different. We always assume the base category $\underline{A}$ to be locally presentable although, as for 3.8, 3.9 and 3.22 the existence of arbitrary colimits in $\underline{A}$ is not needed. We mostly leave the generalization by means of 3.11 to the reader.

We use the following notation for a data (3.1) of bialgebras $M, R, \mathbb{F}$: For an operation $\mu \in M$ and a relation $r \in R$ we write $\mu : F_{d\mu} \dashrightarrow F_{c\mu}$ and $r : F_{dr} \rightrightarrows F_{cr}$ respectively. A data will often be given by first specifying the set $\mathbb{F}$ of support functors and then indicating the operations and relations in this form.

4.1 universal algebra.

Let $\underline{A}$ be a category with finite products. Let Θ be a finitary algebraic theory in the sense of Lawvere [21] (or Birkhoff), e.g. groups, rings, algebras... . Let M be a set of defining operations and R a set of defining relations for Θ in the usual sense. For $\mu \in M$ let $F_{c\mu} = id_{\underline{A}}$ and let $F_{d\mu} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \prod_{n_\mu} A$ which assigns to an object its n_μ-fold product, where n_μ is the arity of μ . A pre-bialgebra (A,M) is an object $A \in \underline{A}$ together with a morphism $\prod_{n_\mu} A \longrightarrow A$ for every $\mu \in M$. For a relation $r \in R$ let $F_{cr} = id_{\underline{A}}$ and let $F_{dr} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \prod_{n_r} A$, where n_r denotes the arity of r . The functor F_{dr} is also denoted with $id_{\underline{A}}^{n_r}$. Since relations are built up of operations and projections, for every pre-bialgebra (A,M) and every relation $r \in R$ there is a morphism pair $r(A,M) : \prod_{n_r} A \rightrightarrows A$ which is natural in (A,M) - i.e. with respect to pre-bialgebra morphisms. Note that $\mathbb{F}_c = \{id_{\underline{A}}\}$ and $\mathbb{F}_d = \{id_{\underline{A}}^{0}, id_{\underline{A}}, id_{\underline{A}}^{2} \ldots\}$. It is straight forward that Bialg($\underline{A}$) is isomorphic with the category Θ-$\underline{\text{Alg}}(\underline{A})$ of Θ-algebras in $\underline{A}$ (i.e. the category of product preserving functors $\underline{\Theta \rightarrow A}$) .

Assume that $\underline{A}$ is locally α-presentable. Let β be the least regular cardinal such that β-filtered colimits commute with finite products, whence $\beta \leq \alpha$ by [13] 7.12. By an obvious cofinality argument for every $n \geq 0$ the functor $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \prod_n A$ preserves β-filtered colimits, moreover it is right adjoint to $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_n A$. Thus by 3.24 b) and 3.7 (remark) Θ-Alg($\underline{A}$) is locally α-presentable and the forgetful functor $V : \Theta\text{-Alg}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and preserves β-filtered colimits (cf. also [13] 11.4).

Let γ be a regular cardinal such that

1) $\beta < \gamma \geq \alpha$, 2) card(M) $< \gamma >$ card(R) and 3) if $A \in \underline{A}$ is γ-presentable, then so is $\prod_n A$ for every finite $n \geq 0$ (cf. 3.7 remarks). Then by 3.8 a Θ-algebra (X,M,R) is γ-presentable in Θ-Alg($\underline{A}$) iff X is γ-presentable in $\underline{A}$.

Likewise, if γ is a regular cardinal such that

1) $\beta < \gamma \geq \alpha$, 2) card(M) $< \gamma$ and 3) if $A \in \underline{A}$ is γ-generated, then so is $\prod_n A$ for every finite $n \geq 0$, then a Θ-algebra (A,M,R) is γ-generated in Θ-Alg($\underline{A}$) iff A is γ-generated in $\underline{A}$ (cf. 3.22).

If in addition $\underline{A}$ is locally γ-noetherian, then so is Θ-Alg($\underline{A}$) .

The generalizations to non-finitary theories in the sense of Linton [23] with rank or to partial operations are obvious generalizations and left to the reader (see also 6.14). The above can be generalized to categories $\underline{A}$ which are dual to a locally presentable category. By means of 4.2 below and $\Theta\text{-Alg}(\underline{A}) \cong (\Theta\text{-Coalg}(\underline{A}^o))^o$ it follows that the category of Θ-algebras in the dual of a locally presentable category is itself the dual of a locally presentable category. In particular if $\underline{A} = \underline{\text{Comp}}$ (compact spaces) or $\underline{A}$ is any Grothendieck AB 5)* category with cogenerators, then $\Theta\text{-Alg}(\underline{A})$ is the dual of a locally presentable category.

4.2 <u>universal coalgebra</u>

Let $\underline{A}$ be a category with finite coproducts. Let Θ be a finitary algebraic theory and let M and R be sets of defining operations and relations as above. For $\mu \in M$ let $F_{d\mu} = id_{\underline{A}}$ and let $F_{c\mu} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \coprod_n A$. A pre-bialgebra (A,M) is an object $A \in \underline{A}$ together with a morphism $A \longrightarrow \coprod_{n_\mu} A$ for every $\mu \in M$. Likewise for a relation $r \in R$ let $F_{dr} = id_{\underline{A}}$ and let $F_{cr} : \underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_{n_r} A$. As above there is for every pre-bialgebra (A,M) a morphism pair $r(A,M) : A \rightrightarrows \coprod_{n_r} A$ and the category $\text{Bialg}(\underline{A})$ is isomorphic with the category $\Theta\text{-Coalg}(\underline{A})$ of Θ-coalgebras in $\underline{A}$. Note that $\mathbb{F}_d = \{id_{\underline{A}}\}$ and $\mathbb{F}_c = \{id_{\underline{A}}^{(0)}, id_{\underline{A}}^{(1)}, id_{\underline{A}}^{(2)}, \}$, where $id_{\underline{A}}^{(n)}$ denotes the functor $A \rightsquigarrow \coprod_n A$. If $\underline{A}$ has finite products, then $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_n A$ is left adjoint to $A \rightsquigarrow \prod_n A$.

<u>Assume that $\underline{A}$ is locally presentable and let</u>

$$\gamma \geq \sup \{\aleph_1, \pi(\underline{A}), \text{card}(M)^+, \text{card}(R)^+\} .$$

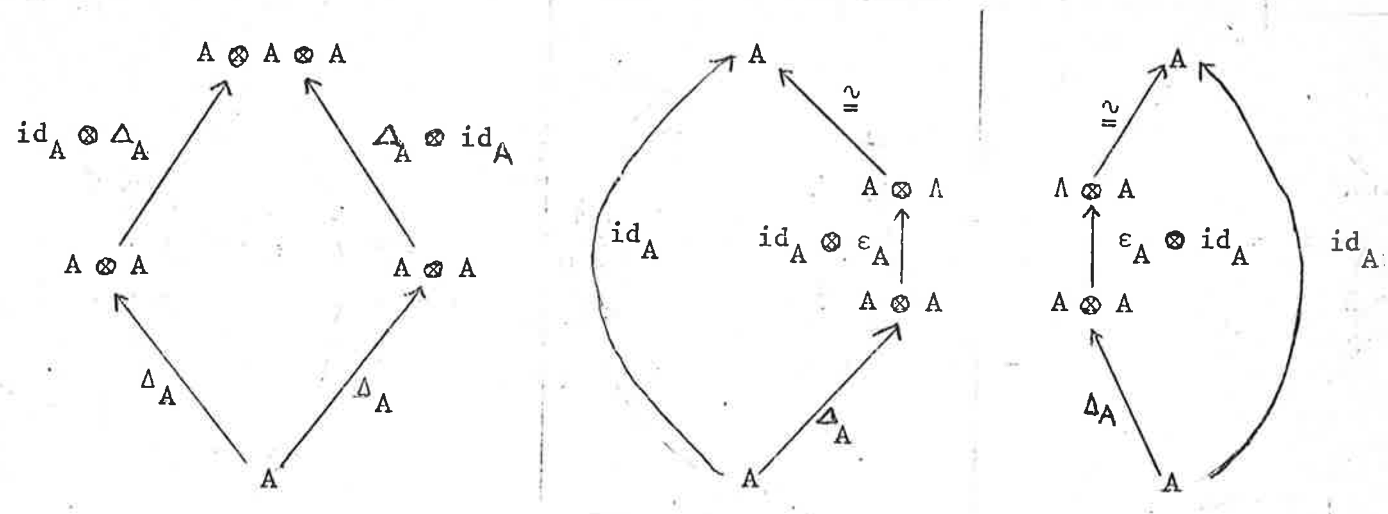
(Recall that δ^+ denotes the least regular cardinal $> \delta$.) <u>Since $\underline{A}(\gamma)$ is closed in $\underline{A}$ under finite coproducts</u> (cf. 2.8), <u>it follows from 3.24 a) that the category $\Theta\text{-Coalg}(\underline{A})$ is locally γ-presentable and the underlying functor $V : \Theta\text{-Coalg}(\underline{A}) \longrightarrow \underline{A}$ is cotripleable and its right adjoint $cF : \underline{A} \longrightarrow \Theta\text{-Coalg}(\underline{A})$ preserves γ-filtered colimits. Moreover by 3.8 a Θ-coalgebra</u> (X,M,R) <u>is γ-presentable in</u> $\Theta\text{-Coalg}(\underline{A})$

iff X is γ-presentable in $\underline{A}$, in particular a morphism $U \dashrightarrow (A,M,R)$ with $\pi(U) \leq \gamma$ admits a decomposition into a morphism $U \longrightarrow U'$ and a Θ-coalgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that $\pi(U') \leq \gamma$. Likewise if $\underline{A}$ is locally γ-noetherian and if in $\underline{A}$ β-filtered colimits of monomorphisms are monomorphic for some $\beta < \gamma$, then by 3.22 d) $\Theta\text{-Coalg}(\underline{A})$ is locally γ-noetherian. In addition every γ-generated subobject of a Θ-coalgebra is contained in a Θ-subcoalgebra whose underlying object is also γ-generated. (Note if $\underline{A}$ is not locally γ-noetherian, then the latter need not hold, in particular a Θ-coalgebra (X,M,R) need not be γ-generated in $\Theta\text{-Coalg}(\underline{A})$ if X is γ-generated in $\underline{A}$, and conversely.)

The generalizations to non-finitary theories in the sense of Linton [23] with rank or to partial co-operations are obvious and left to the reader (see also 6.14 - 6.16). The above can be generalized to categories $\underline{A}$ which are dual to a locally presentable category. This is done in some way as in 4.1 by means of $\Theta\text{-Coalg}(\underline{A}) \cong (\Theta\text{-Alg}(\underline{A}^o))^o$.

4.3 Coalgebras over a commutative ring.

Let $\underline{A} = \underline{Mod}_\Lambda$ be the category of Λ-modules over a commutative ring Λ. Let $\mathbb{F} = \{const_\Lambda, id, id \otimes id, id \otimes id \otimes id\}$, where id is the identity functor of $\underline{Mod}_\Lambda$ and $const_\Lambda : \underline{Mod}_\Lambda \longrightarrow \underline{Mod}_\Lambda$ is the constant functor $A \rightsquigarrow \Lambda$. The tensor product is taken over Λ. Let $M = \{\Delta, \varepsilon\}$, where $\Delta : id \dashrightarrow id \otimes id$ and $\varepsilon : id \dashrightarrow const_\Lambda$ are operations called comultiplication and counit. A pre-bialgebra is a Λ-module A together with homomorphisms $\Delta_A : A \longrightarrow A \otimes A$ and $\varepsilon_A : A \longrightarrow \Lambda$. Let $R = \{r_1, r_2, r_3\}$, where $r_1 : id \Rrightarrow id \otimes id \otimes id$ and $r_2 : id \Rrightarrow id$, $r_3 : id \Rrightarrow id$ are relations, called coassociative and counitary laws, which for a pre-bialgebra $(A, \Delta_A, \varepsilon_A)$ are given by the diagrams



It is straight forward that r_1, r_2 and r_3 are relations and that Bialg($\underline{Mod}_\Lambda$) is the category Λ-Coalg of Λ-coalgebras. Note that $\mathbb{F}_d = \{id\}$. Recall that $\underline{Mod}_\Lambda$ is locally γ-noetherian (i.e. every γ-generated module is γ-presentable) for some $\gamma \geq \aleph_0$ iff every ideal $I \subset \Lambda$ is γ-generated. If Λ has this property, it is called γ-noetherian; in particular for $\gamma = \aleph_0$ the notion $\aleph_0$-noetherian coincides with noetherian in the usual sense. Clearly if Λ is noetherian, then it is γ-noetherian for any $\gamma \geq \aleph_0$.

By 3.24 a) the category Λ-Coalg is locally $\aleph_1$-presentable and by 3.8 for $\gamma \geq \aleph_1$ a coalgebra $(X, \Delta_X, \varepsilon_X)$ is γ-presentable in Λ-Coalg iff its underlying module X is γ-presentable in $\underline{Mod}_\Lambda$. In particular a Λ-homomorphism $U \longrightarrow (A, \Delta_A, \varepsilon_A)$ with $\pi(U) \leq \gamma$ admits a decomposition into a Λ-homomorphism $U \longrightarrow U'$ and a coalgebra morphism $(U', \Delta_{U'}, \varepsilon_{U'}) \longrightarrow (A, \Delta_A, \varepsilon_A)$ such that $\pi(U') \leq \gamma$.

Likewise, if Λ is γ-noetherian for some $\gamma \geq \aleph_1$, then by 3.22 Λ-Coalg is locally γ-noetherian and a γ-generated Λ-submodule of a coalgebra is contained in a subcoalgebra whose underlying Λ-module is γ-generated. Moreover a coalgebra is γ-generated in Λ-Coalg iff its underlying module is γ-generated in $\underline{Mod}_\Lambda$. (Note that these assertions need not hold if Λ is not γ-noetherian.)

The same results hold for the category of cocommutative Λ-coalgebras.

For by adding to the above data of bialgebras a relation expressing the cocommutativity of $\Delta: id \dashrightarrow id \otimes id$ one obtains instead the category of cocommutative Λ-coalgebras.

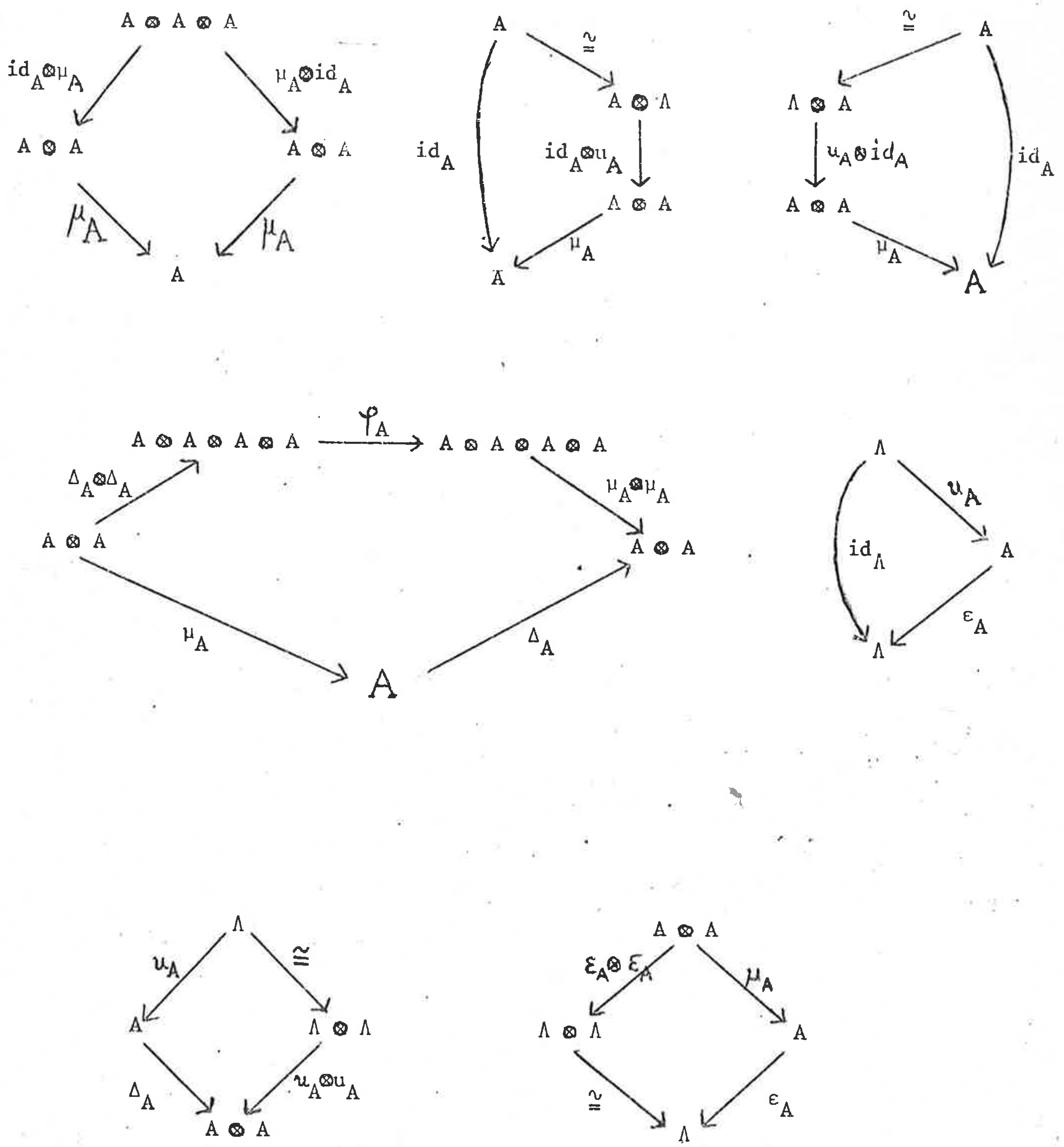
The above improves the results of M. Barr [1] considerably. He showed that for $\delta \geq \sup(card(\Lambda)^+, \aleph_1)$ every δ-generated submodule of a Λ-coalgebra is contained in a subcoalgebra whose underlying module is also δ-generated; in particular the coalgebras whose underlying module is $\sup(card(\Lambda)^+), \aleph_1)$-generated, form a set of generators in Λ-Coalg . As shown above these problems have something to do with the (minimal) number of generators for ideals $I \subset \Lambda$ and not with the cardinality of Λ. The latter enters his argument for a different reason. A submodule of a coalgebra which is closed under the comultiplication need not be a subcoalgebra because it need not be coassociative. If however the submodule is pure, then the coassociativity carries over. Therefore he considered only pure submodules and embedded the given submodule of the coalgebra into a pure submodule. In this way the cardinality of Λ comes in and the "generated" subcoalgebra can become much bigger than necessary.

As for Fox's [8] generalization of Barr's results see 4.7 below.

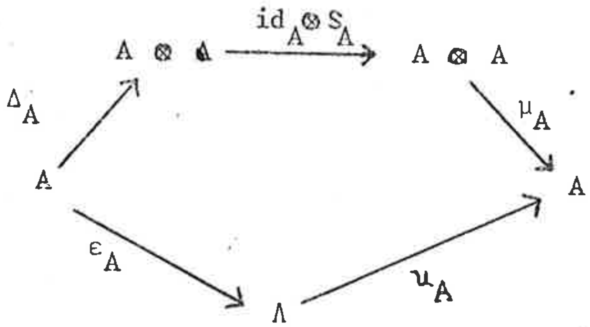
4.4 Bialgebras, Hopfalgebras over a commutative ring, generalizations to Props and locally presentable categories.

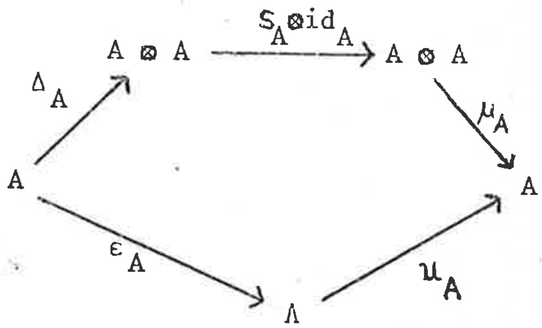
Let $\underline{A} = \underline{Mod}_\Lambda$ be the category of modules over a commutative ring Λ. The data $M, \mathbb{R}, \mathbb{F}$ for Λ-bialgebras is as follows. Let $\mathbb{F} = \{const_\Lambda , id , id \otimes id , id \otimes id \otimes id\}$ be as above for coalgebras (4.3). Let $M = \{\Delta , \varepsilon , \mu , u\}$ be operations, where $\Delta : id \dashrightarrow id \otimes id$, $\varepsilon : id \dashrightarrow const_\Lambda$, $\mu : id \otimes id \dashrightarrow id$ and $u : const_\Lambda \dashrightarrow id$ are operations called comultiplication, counit, multiplication and unit respectively. Thus a pre-bialgebra is a Λ-module A together with homomorphisms $\Delta_A : A \longrightarrow A \otimes A$, $\varepsilon_A : A \longrightarrow \Lambda$, $\mu_A : A \otimes A \longrightarrow A$ and $u_A : \Lambda \longrightarrow A$. Let $R = \{r_1, r_2, .., r_{10}\}$, where

$r_1 : \mathrm{id} \dashrightarrow \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}$, $r_2 : \mathrm{id} \dashrightarrow \mathrm{id}$, $r_3 : \mathrm{id} \dashrightarrow \mathrm{id}$ are as above in 4.3 and the relations $r_4 : \mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id} \dashrightarrow \mathrm{id}$, $r_5 : \mathrm{id} \dashrightarrow \mathrm{id}$, $r_6 : \mathrm{id} \dashrightarrow \mathrm{id}$, $r_7 : \mathrm{id} \otimes \mathrm{id} \dashrightarrow \mathrm{id} \otimes \mathrm{id}$, $r_8 : \mathrm{const}_\Lambda \dashrightarrow \mathrm{const}_\Lambda$, $r_9 : \mathrm{const}_\Lambda \dashrightarrow \mathrm{id}\otimes\mathrm{id}$, $r_{10} : \mathrm{id} \otimes \mathrm{id} \dashrightarrow \mathrm{const}_\Lambda$ are given for a pre-bialgebra $(A, \Delta_A, \varepsilon_A, \mu_A, u_A)$ by the diagrams



where $\rho_A : A \otimes A \otimes A \otimes A \longrightarrow A \otimes A \otimes A \otimes A$ is the homomorphism interchanging the two inner factors. It is straight forward that $r_1, r_2, \ldots, r_{10}$ are relations on P-Bialg($\underline{Mod}_\Lambda$) and that Bialg($\underline{Mod}_\Lambda$) is the category Λ-$\underline{Bialg}$ of Λ-bialgebras. In order to express the category Λ-$\underline{Hopf}$ of Λ-Hopfalgebras as bialgebras in $\underline{Mod}_\Lambda$ one adds to M an operation $S : id \dashrightarrow id$ (= the antipode) and to R two relations $r_{11} : id \rightrightarrows id$, $r_{12} : id \rightrightarrows id$ which for a pre-bialgebra $(A, \Delta_A, \varepsilon_A, \mu_A, u_A, S_A)$ are given by the diagrams





Likewise by adding relations expressing the commutativity of μ or the cocommutativity of Δ or both one can obtain the categories of commutative Λ-bialgebras and Λ-Hopfalgebras, cocommutative Λ-bialgebras and Λ-Hopfalgebras and bicommutative Λ-bialgebras and Λ-Hopfalgebras. Note that $\mathbb{F}_d = \mathbb{F}$ and that $\pi(A \otimes A) \leq \pi(A)$ and likewise $\varepsilon(A \otimes A) \leq \varepsilon(A)$ for every $A \in \underline{Mod}_\Lambda$.

4.5 Thus for $\gamma \geq \aleph_1$ it follows from 3.8 that a Λ-bialgebra (X,M,R) is γ-presentable in Λ-$\underline{Bialg}$ iff its underlying module X is γ-presentable in $\underline{Mod}_\Lambda$. Moreover a Λ-homomorphism $U \longrightarrow (A,M,R)$ with $\pi(U) \leq \gamma$ admits a decomposition into a Λ-homomorphism $U \longrightarrow U'$ and a Λ-bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that $\pi(U') \leq \gamma$; in particular the Λ-bialgebras whose underlying module is $\aleph_1$-presentable form a set of dense generators in Λ-$\underline{Bialg}$ (cf. 3.8 and [13] 3.1).

If in addition Λ is γ-noetherian for some $\gamma \geq \aleph_1$ (cf. 4.3), then by 3.22 a γ-generated submodule of a bialgebra is contained in a subbialgebra whose underlying module is also γ-generated. Moreover a Λ-bialgebra is γ-generated in Λ-Bialg iff its underlying module is γ-generated in Mod_Λ .

The same assertions hold for the categories of commutative Λ-bialgebras and Λ-Hopfalgebras, cocommutative Λ-bialgebras and Λ-Hopfalgebras and bicommutative Λ-bialgebras and Λ-Hopfalgebras.

4.6 With the exception of arbitrary Λ-Hopfalgebras all of the above categories are locally $\aleph_1$-presentable. In addition the various relative forgetful functors have left adjoints resp. right adjoints. If Λ is γ-noetherian for some $\gamma \geq \aleph_1$, then the above categories are also locally $\aleph_1$-noetherian.

The data of bialgebras for these categories admit a decomposition into algebraic and coalgebraic parts, cf. 3.27. Thus the first assertion follows from 3.28 and the last from 3.22 d) while the one concerning adjoints is a consequence of either 2.9 or the special adjoint functor theorem. For more details see 3.26 and the discussions following 3.27.

4.7 Generalizations Let P be a prop in the sense of Mac Lane [24] Section 24, and assume that it can be defined by a countable number of operations and relations (see M. Barr [ ] p. 605/606 for a discussion). It is clear that the tensor product preserving functors $P \to Mod_\Lambda$ can be expressed as bialgebras and therefore the assertions in 4.5 carry over to this situation. Likewise if the prop P is algebraic or coalgebraic ([1] 6.1) or admits a decomposition as in 3.2 , then the category of tensor product preserving functors $P \to Mod_\Lambda$ is locally $\aleph_1$-presentable, and if Λ is γ-noetherian for $\gamma \geq \aleph_1$, it is locally γ-noetherian, etc.

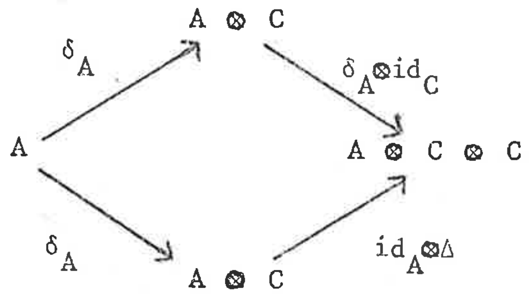
More generally let A be a category equipped with a bifunctor $\otimes : A \times A \to A$ which is coherently associative, symmetric and unitary. Then for an arbitrary prop P one can express tensor product preserving functors as bialgebras as above. If $\underline{A}$ is locally presentable and $\otimes$ preserves α-filtered colimits in both variables for some $\alpha \geq \aleph_0$, then 3.8 (resp. 3.7) and 3.22 (resp. 5.1) apply. Moreover if the prop $\underline{P}$ is of algebraic or coalgebraic type (cf. [1] 6.1) or admits a decomposition like in 3.27, then the category of tensor product preserving functors $\underline{P} \to \underline{A}$ is again locally presentable etc. (see 3.28 and 3.22 d)). In particular this applies to the coalgebraic situation considered by Fox [8]. We leave it to the reader to specify the minimal cardinals in 3.7 - 3.28 for tensor product preserving functors $\underline{P} \to \underline{A}$ (note that the case $\alpha = \aleph_0$ is particularly simple and useful).

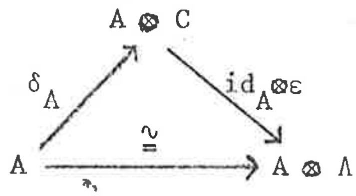
While props give rise to data of bialgebras, the converse is not true, not even for $\underline{A} = \underline{Mod}_\Lambda$ and $\mathbb{F} = \{const_\Lambda, id, id \otimes id, id \otimes id \otimes id\}$. For instance, as M. Barr pointed out to me, Lie algebras over Λ cannot be expressed as tensor product preserving functors $\underline{P} \to \underline{Mod}_\Lambda$ for some prop $\underline{P}$ because the Jacoby identity involves addition of structure morphisms. However they can easily be described as bialgebras, the Jacoby identity is given by the relation $A \otimes A \otimes A \overset{f_A}{\underset{g_A}{\rightrightarrows}} A$, where $f(x,y,z) = 0$ and $g(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$. (Note that f_A and g_A are obviously natural with respect to Λ-homomorphisms preserving the bracket). The notion of bialgebras allows more flexibility as far as relations are concerned. It is also more natural and its simplicity should be compared with the technical problems involved with a prop $\underline{P}$ and the coherence aparatus for $\otimes$ and $\underline{P}$.
I should add that these prop problems prompted me to look for something simpler. When I met M.Barr and T. Fox in the fall of 1975 I had ten "different" proofs for the same theorem (namely 3.8); one for Σ-cocontinuous functors, one for coalgebras over a cotriple, one for descent data, one for Λ-coalgebras, one for comodules over a coalgebra, one for Λ-bialgebras,.... On the other hand Fox [8] had a proof for

a tensored locally presentable category and a coalgebraic prop, but no reasonable size estimates for the generators he constructed. In order to obtain that and also to cover the case of non-coalgebraic props I had to look for an "eleventh proof" of 3.8 considering interlocking operations and relations which turned to be very technical and extremely laborious. Fox [6] had got around this problem in the same way as Barr [1] by using purity (see 4.3 above). The use of purity however makes good size estimates impossible and thus something else had to be found. In this way I was led to the notion of pre-bialgebras and bialgebras as defined in 3.1, the above mentioned example of Lie algebras served as a guide. The unification of the eleven proofs of 3.8 was a somewhat "unexpected fringe benefit".

4.8 Comodules over a Λ-coalgebra.

Let $\underline{A} = \underline{Mod}_\Lambda$ be the category of modules over a commutative ring Λ and let C be a Λ-coalgebra with comultiplication $\Delta : C \longrightarrow C\otimes C$ and counit $\varepsilon : C \longrightarrow \Lambda$ (cf. 4.3). Recall that a right C-comodule is a Λ-module A together with a Λ-homomorphism $\delta_A : A \longrightarrow A\otimes C$ such that the diagrams





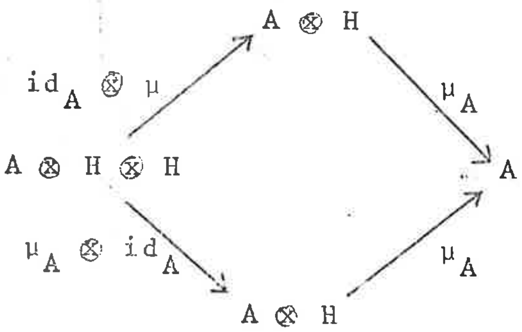
commute. The tensor product is over Λ. A right C-comodule morphism $(A,\delta_A) \longrightarrow (A',\delta_{A'})$ is a Λ-homomorphism $f : A \longrightarrow A'$ with the property $\delta_{A'} \circ f = (f \otimes id_C) \circ \delta_A$. The category of right C-comodules is denoted with $\underline{Comod}_C$ (cf. Demazure-Gabriel [4] p. 174, Sweedler [28] 30/31). To express $\underline{Comod}_C$ as bialgebras in $\underline{Mod}_\Lambda$ let $\mathbb{F} = \{id, \otimes C, \otimes C \otimes C\}$ and $M = \{\delta\}$, where id is the identity of $\underline{Mod}_\Lambda$ and δ an operation $id \dashrightarrow \otimes C$. Let $R = \{r_1, r_2\}$, where $r_1 : id \rightrightarrows \otimes C \otimes C$ and $r_2 : id \rightrightarrows id$ are given by the above diagrams. Clearly r_1 and r_2 are relations on P-Bialg($\underline{Mod}_\Lambda$) and $\underline{Comod}_C$ = Bialg($\underline{Mod}_\Lambda$) holds. Note that $\mathbb{F}_d = \{id\}$.

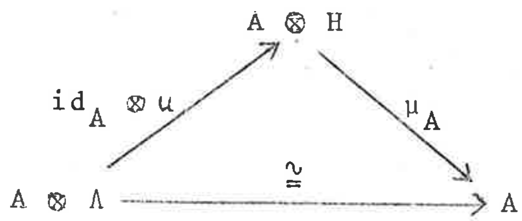
Thus by 3.22 and 3.8 the category $\underline{Comod}_C$ is locally $\aleph_1$-presentable and for $\gamma \geq \aleph_1$ a comodule (X, δ_X) is γ-presentable in $\underline{Comod}_C$ if X is γ-presentable in $\underline{Mod}_\Lambda$, in particular a Λ-homomorphism $U \longrightarrow (A, \delta_A)$ with $\pi(U) \leq \gamma$ factors into a Λ-homomorphism $U \longrightarrow U'$ and a comodule morphism $(U', \delta_{U'}) \longrightarrow (A, \delta_A)$ such that $\pi(U') \leq \gamma$. Likewise if Λ is γ-noetherian for some $\gamma \geq \aleph_1$ (cf. 4.3), then by 3.22 $\underline{Comod}_C$ is locally γ-noetherian and a comodule is γ-generated in $\underline{Comod}_C$ iff its underlying module is γ-generated in $\underline{Mod}_\Lambda$. In addition a γ-generated Λ-submodule of a comodule is contained in a subcomodule whose underlying module is γ-generated. The last assertion was first proved by Wischnewsky [36] under the additional assumption that $\gamma > card(\Lambda)$. Following Barr [1] he used purity arguments which in general make the "generated" subcomodule bigger than necessary.
If C is Λ-flat one can easily show that $\underline{Comod}_C$ is a locally $\aleph_1$-presentable Grothendieck category and that for $\alpha \geq \aleph_1$ a comodule is α-generated iff its underlying module is, etc. (cf. 3.25, 3.22, see also [36])

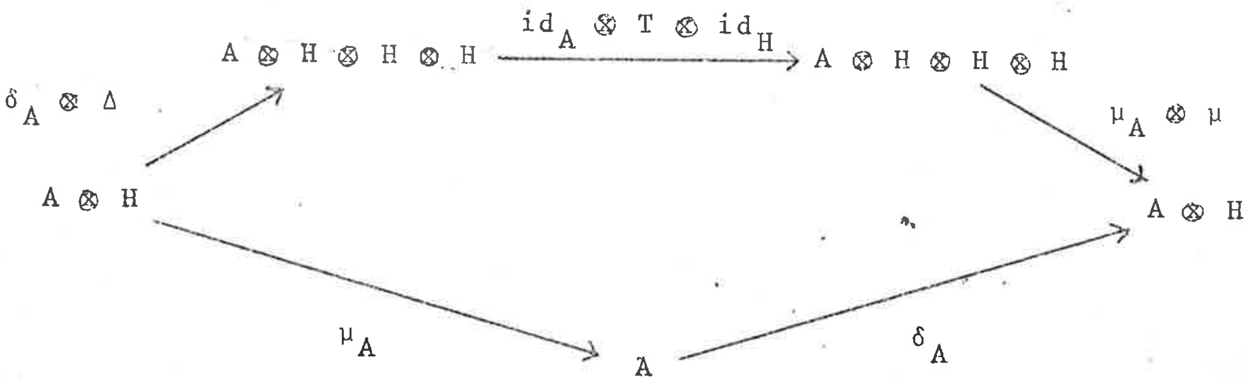
4.9 Bimodules over a Λ-bialgebra

Let $\underline{Mod}_\Lambda$ be as above and let H be a Λ-bialgebra with multiplication $\mu : H \otimes H \longrightarrow H$, unit $u : \Lambda \longrightarrow H$, comultiplication $\Delta : H \longrightarrow H \otimes H$ and counit $\varepsilon : H \longrightarrow \Lambda$ (cf. 4.4). Recall that a bimodule over H is a Λ-module A together with Λ-homomorphisms $\mu_A : A \otimes H \longrightarrow A$ and $\delta_A : A \longrightarrow A \otimes H$ such that 1) μ_A defines a right H-module structure on A with H being viewed as a Λ-algebra 2) δ_A defines a right H-comodule structure on A with H being viewed as a Λ-coalgebra 3) δ_A is H-linear, where the right H-structure on $A \otimes H$ is given by $\Delta : H \longrightarrow H \otimes H$, i.e. if $\Delta(g) = \Sigma\, g_i' \otimes g_i''$, then

$(m \otimes h)g = \Sigma\, mg_i' \otimes hg_i''$ (cf. Sweedler [28] 4.1). For instance, if $X \in \underline{Mod}_\Lambda$ then $(X \otimes H\,,\ id \otimes \mu\,,\ id \otimes \Delta)$ is a H-bimodule. A morphism between H-bimodules is a Λ-homomorphism which is compatible with both structures. Let $\underline{Bimod}_H$ denote the category of bimodules over H. To express $\underline{Bimod}_H$ as bialgebras in $\underline{Mod}_\Lambda$ let $\mathbb{F} = \{id, \otimes H, \otimes H \otimes H\}$ and let $M = \{\delta, \mu\}$ consist of operations $\delta : id \dashrightarrow \otimes H$ and $\mu : \otimes H \dashrightarrow id$, where id is the identity functor of $\underline{Mod}_\Lambda$. Let $R = \{r_1, r_2, r_3, r_4, r_5\}$ consist of relations $r_1 : id \Rrightarrow \otimes H \otimes H$, $r_2 : id \Rrightarrow id$, $r_3 : \otimes H \otimes H \Rrightarrow id$, $r_4 : id \Rrightarrow id$, $r_5 : \otimes H \Rrightarrow \otimes H$, where r_1 and r_2 are as above in 4.3 and r_3, r_4, r_5 are given for a pre-bialgebra (A, δ_A, μ_A) by the diagrams







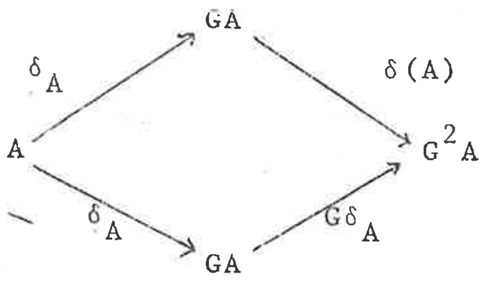
with $T : H \otimes H \longrightarrow H \otimes H$ being the twist homomorphism $h \otimes h' \rightsquigarrow h' \otimes h$. One easily checks that $r_1, .., r_5$ are relations on P-Bialg($\underline{Mod}_\Lambda$) and that Bialg($\underline{Mod}_\Lambda$) = $\underline{Bimod}_H$. Note that $\mathbb{F}_d = \mathbb{F}$ and that every functor in $\mathbb{F}_d$ is colimit preserving. Moreover for every $A \in \underline{Mod}_\Lambda$ it follows from $[A \otimes H, -] \cong [A, [H, -]]$ and $[A \otimes H \otimes H, -] \cong [A \otimes H, [H, -]]$ that $\pi(A \otimes H) \le \sup(\pi(A), \pi(H)) \ge \pi(A \otimes H \otimes H)$ and likewise

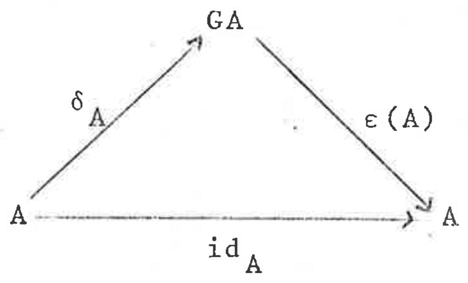
$\varepsilon(A \otimes H) \leq \sup(\varepsilon(A), \varepsilon(H)) \geq \varepsilon(A \otimes H \otimes H)$.

Thus by 3.24 a) $\underline{\text{Bimod}}_H$ is locally $\sup(\aleph_1, \pi(H))$-presentable and for $\gamma \geq \sup(\aleph_1, \pi(H))$ it follows from 3.8 that a bimodule (X, δ_X, μ_X) is γ-presentable in $\underline{\text{Bimod}}_H$ iff X is γ-presentable in $\underline{\text{Mod}}_\Lambda$; in particular a Λ-homomorphism $U \longrightarrow (A, \delta_A, \mu_A)$ with $\pi(U) \leq \gamma$ factors into a Λ-homomorphism $U \longrightarrow U'$ and a bimodule morphism $(U', \delta_{U'}, \mu_{U'}) \longrightarrow (A, \delta_A, \mu_A)$ such that $\pi(U') \leq \gamma$.
Likewise if Λ is γ-noetherian (cf. 4.3) for some $\gamma \geq \sup(\aleph_1, \pi(H))$, then $\underline{\text{Bimod}}_H$ is locally γ-noetherian and a bimodule is γ-generated in $\underline{\text{Bimod}}_H$ iff its underlying module is γ-generated in $\underline{\text{Mod}}_\Lambda$. In addition a γ-generated Λ-submodule of a bimodule is contained in a sub-bimodule whose underlying Λ-module is γ-generated.
Actually $\underline{\text{Bimod}}_H$ is locally $\aleph_1$-presentable and locally δ-noetherian where δ is the least regular cardinal $\geq \aleph_1$ such that every right ideal of H is δ-generated (in the category of right H-modules).
This follows from 3.28 resp. 3.28 and 3.24 a) because there is a decomposition $\underline{\text{Bimod}}_H \cong \underline{\text{Comod}}_H(\underline{\text{Mod}}_H)$ in the sense of 3.27. In more detail the algebraic part of $M = \{\delta, \mu\}$ and $R = \{r_1, r_2, r_3, r_4, r_5\}$ is $M' = \{\mu\}$ and $R' = \{r_3, r_4\}$ whence $\text{Bialg}_{M', R'}(\underline{\text{Mod}}_\Lambda) \cong \underline{\text{Mod}}_H$. There is a functor $\otimes H : \underline{\text{Mod}}_H \longrightarrow \underline{\text{Mod}}_H$, $A \rightsquigarrow A \otimes_\Lambda H$ (see 3) above) together with natural transformations $\otimes \varepsilon : \otimes H \longrightarrow \text{id}_{\underline{\text{Mod}}_H}$ and $\otimes \Delta : \otimes H \longrightarrow \otimes H \otimes H$, where ε is the counit of H and Δ the comultiplication. (The verification that $\otimes \Delta$ is well defined is somewhat laborious but straight forward.) With this one can define the coalgebraic part of M and R as $M'' = \{\delta : \text{id} \dashrightarrow \otimes H\}$ and $R'' = \{r_1 : \text{id} \rightrightarrows \otimes H \otimes H, \ r_2 : \text{id} \rightrightarrows \text{id}\}$, where id is the identity of $\underline{\text{Mod}}_H$ and r_1 and r_2 are defined exactly as in 4.8. It is now routine to show that $\text{Bialg}_{M'', R''}(\underline{\text{Mod}}_H) \cong \underline{\text{Bimod}}_H$. Note that if H is flat over Λ, then $\underline{\text{Bimod}}_H$ is a locally $\aleph_1$-presentable Grothendieck category and for $\alpha \geq \sup(\aleph_1, \varepsilon(H))$ a bimodule is α-generated iff its underlying module is, etc. (cf. 3.25, 3.22).

4.1o Coalgebras over a cotriple.

Recall that a cotriple $\mathbb{G} = (G,\delta,\varepsilon)$ in a category $\underline{A}$ consists of a functor $G : \underline{A} \longrightarrow \underline{A}$ and natural transformations $\delta : G \longrightarrow G^2$ (= comultiplication), $\varepsilon : G \longrightarrow id_{\underline{A}}$ (=counit) satisfying $G\delta \cdot \delta = \delta G \cdot \delta$ (coassociative law) and $G\varepsilon \cdot \delta = id_G = \varepsilon G \cdot \delta$ (counitary law). A $\mathbb{G}$-coalgebra in $\underline{A}$ is a pair (A,ξ), where $\xi : A \longrightarrow GA$ is a morphism satisfying $\varepsilon(A) \circ \xi = id_A$ and $G\xi \circ \xi = \delta(A) \circ \xi$. A morphism $(A,\xi) \longrightarrow (A',\xi')$ of $\mathbb{G}$-coalgebras is a morphism $f : A \longrightarrow A'$ satisfying $\xi' \circ f = Gf \circ \xi$. The category of all $\mathbb{G}$-coalgebras is denoted with $\underline{A}_{\mathbb{G}}$. The underlying functor $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$, $(A,\xi) \rightsquigarrow A$ is left adjoint to the cofree functor $\underline{A} \longrightarrow \underline{A}_{\mathbb{G}}$, $A \rightsquigarrow (GA,\delta(A))$. Given a cotriple $\mathbb{G} = (G,\delta,\varepsilon)$ in $\underline{A}$ it is easy to describe $\underline{A}_{\mathbb{G}}$ in terms of bialgebras. Let $\mathbb{F} = \{id_{\underline{A}}, G, G^2\}$ and let $M = \{\delta\}$ be an operation $\delta : id_{\underline{A}} \dashrightarrow G$. Thus a pre-bialgebra is an object $A \in \underline{A}$ together with a morphism $\delta_A : A \longrightarrow GA$. Let $R = \{r_1, r_2\}$ be the relations $r_1 : id_{\underline{A}} \rightrightarrows G^2$ and $r_2 : id_A \rightrightarrows id_A$ which for a pre-bialgebra (A,δ_A) are given by the diagrams





Clearly r_1 and r_2 are relations on P-Bialg($\underline{A}$) and $\underline{A}_{\mathbb{G}} \cong$ Bialg($\underline{A}$) holds. Note that $\mathbb{F}_d = \{id_{\underline{A}}\}$.

Assume $\underline{A}$ is locally presentable and that G has rank (cf. 2.1) and let $\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$. Then by 3.24 a) $\underline{A}_{\mathbb{G}}$ is locally $\sup(\aleph_1, \pi(\underline{A}), \pi(G))$-presentable and by 3.8 a coalgebra (X,δ_X) is γ-presentable in $\underline{A}_{\mathbb{G}}$ iff X is γ-presentable in $\underline{A}$; in particular a morphism $U \longrightarrow (A,\delta_A)$ with $\pi(U) \leq \gamma$ admits a decomposition into a morphism $U \longrightarrow U'$ and a coalgebra morphism $(U',\delta_{U'}) \longrightarrow (A,\delta_A)$ such that $\pi(U') \leq \gamma$. Likewise if $\underline{A}$ is locally γ-noetherian for some

$\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$ and if in $\underline{A}$ β-filtered colimits of monomorphisms are monomorphic for some $\beta < \gamma$, then $\underline{A}_{\mathbb{G}}$ is locally γ-noetherian and a coalgebra (X, δ_X) is γ-generated in $\underline{A}_{\mathbb{G}}$ iff X is γ-generated in $\underline{A}$. Also a γ-generated subobject U of a coalgebra (A, δ_A) is contained in a subcoalgebra $(U', \delta_{U'})$ such that U' is γ-generated.

4.11 Corollary Let $\mathbb{G} = (G, \delta, \varepsilon)$ be a cotriple in a topos $\underline{A}$ (resp. Grothendieck category). Equivalent are

(i) $\underline{A}_{\mathbb{G}}$ is a topos (resp. Grothendieck category) and the left adjoint $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$, $(A, \delta_A) \rightsquigarrow A$, preserves finite limits.

(ii) $G : \underline{A} \longrightarrow \underline{A}$ preserves finite limits and has rank.

Moreover iff i) holds, then $\underline{A}_{\mathbb{G}}$ is a locally $\sup(\aleph_1, \pi(\underline{A}), \pi(G))$-presentable topos (resp. Grothendieck category) and for $\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$ a coalgebra (X, δ_X) is γ-generated in $\underline{A}_{\mathbb{G}}$ iff X is γ-generated in $\underline{A}$, etc. (see 3.25 and 3.22 for $\beta = \aleph_0$).

Proof (i) $\Longrightarrow$ (ii) The first assertion is trivial and the second follows from 2.9.

(ii) $\Longrightarrow$ (i) By 4.1o $\underline{A}_{\mathbb{G}}$ is locally presentable. The underlying functor $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$ preserves and creates colimits. The same holds with respect to finite limits because G is finite limit preserving. This implies that $\underline{A}_{\mathbb{G}}$ is a Grothendieck category provided $\underline{A}$ is. Likewise if $\underline{A}$ is a topos, one readily checks with this that $\underline{A}_{\mathbb{G}}$ satisfies the conditions [13] 12.13 a) - d) (=Giraud's axioms) and hence is a topos.
The last assertion follows from 3.25 and 3.22 for $\beta = \aleph_0$.

4.12 Remarks a) If G does not preserve finite limits, then $\underline{A}_{\mathbb{G}}$ need not be a Grothendieck category (resp. topos), if $\underline{A}$ is. For instance, let $\underline{U} \hookrightarrow \underline{Ab.Gr.}$ be the inclusion of the full subcategory consisting of all finite p-groups for some prime p. Let $\underline{A} = [\underline{U}, \underline{Ab.Gr.}]_+$ be the category of additive functors and let $\underline{X} \subset \underline{A}$ be the full subcategory of all cocontinuous functors. By 6.16 below the inclusion $\underline{X} \subset \underline{A}$ has a right adjoint and the resulting cotriple on $\underline{A}$ has obviously the property $\underline{A}_{\mathbb{G}} \cong \underline{X}$. By 6.25 c) below $\underline{A}_{\mathbb{G}}$ is isomorphic with the category of p-adic complete abelian groups which is not a Grothendieck category (e.g. the colimit of the system $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p\cdot} \dots$ is zero because in the category of all abelian groups it is the Prüfer group $\mathbb{Z}(p^\infty)$ whose completion is zero.)

b) Corollary 4.11 may sound like the well known theorem "If $\mathbb{C}$ is a left exact cotriple in an elementary topos $\mathcal{E}$, then $\mathcal{E}_{\mathbb{G}}$ is again an elementary topos" but in fact in has little to do with it. The main ingredient in 4.11 is the existence of generators in $\underline{A}_{\mathbb{G}}$ which is not contained in the assertion concerning elementary topoi. Also in the latter there is no rank assumption on the cotriple which is necessary for the existence of generators.

4.13 Algebras over a triple. Let $\mathbb{T} = (T,\mu,u)$ be a triple in a category $\underline{A}$ and let $\underline{A}^{\mathbb{T}}$ denote the category of $\mathbb{T}$-algebras in $\underline{A}$, cf. [13] § 1o. The description of $\underline{A}^{\mathbb{T}}$ as bialgebras in $\underline{A}$ is dual to 4.1o, i.e. if $\mathbb{F} = \{id_{\underline{A}}, T, T^2\}$, $M = \{\mu : T \dashrightarrow id_{\underline{A}}\}$, and $R = \{r_1 : T^2 \rightrightarrows id_{\underline{A}},\ r_2 : id_{\underline{A}} \rightrightarrows id_{\underline{A}}\}$ are dual to the data for bialgebras in 4.1o, then $\underline{A}^{\mathbb{T}} = \mathrm{Bialg}(\underline{A})$. Note that $\mathbb{F}_c = \{id_{\underline{A}}\}$ and $\mathbb{F}_d = \{id_{\underline{A}}, T, T^2\}$. Assume $\underline{A}$ is locally presentable and T has rank (2.1). Then by [ ] § 1o $\underline{A}^{\mathbb{T}}$ is locally $\sup(\pi(\underline{A}),\pi(T))$-presentable. Let $\gamma \geq \aleph_1$ be a regular cardinal such that $\gamma \geq \pi(\underline{A})$, $\gamma > \pi(T)$ and that $\pi(U) \leq \gamma$ implies $\pi(TU) \leq \gamma$ for $U \in \underline{A}$. (Note that by 3.7 such cardinals exist.) Thus by 3.8 a $\mathbb{T}$-algebra (X,μ_X) is γ-presentable in $\underline{A}^{\mathbb{T}}$ iff X is γ-presentable in $\underline{A}$. Likewise if $\gamma \geq \aleph_1$ is a regular cardinal such that $\gamma \geq \varepsilon(\underline{A})$, $\gamma > \varepsilon(T)$ and that $\varepsilon(U) \leq \gamma$ implies $\varepsilon(TU) \leq \gamma$ for $U \in \underline{A}$ (cf. 5.1), then by 3.22 a $\mathbb{T}$-algebra (X,μ_X) is γ-generated in $\underline{A}^{\mathbb{T}}$ iff X is γ-generated in $\underline{A}$. If in addition $\underline{A}$ is locally γ-noetherian, then so is $\underline{A}^{\mathbb{T}}$. (Note that for $\beta \geq \sup(\pi(\underline{A}),\pi(T))$ a morphism $U \longrightarrow (A,\mu_A)$ with $\pi(U) \leq \beta$ obviously factors into a morphism $U \longrightarrow V$ and a $\mathbb{T}$-algebra morphism $(V,\mu_V) \longrightarrow (A,\mu_A)$ such that (V,μ_V) is β-presentable in $\underline{A}^{\mathbb{T}}$, namely the one given by the free $\mathbb{T}$-algebra on U, but V need not be β-presentable in $\underline{A}$).

4.14 Descent data and données de recollements.

We follow Grothendieck [16] but limit ourselve to descent data. The case of données de recollement is almost identical (but simpler) and the obvious modifications are left to the reader. It should be noted that the following is a special case of cartesian closed sections below in 4.19. Let $\mathcal{F}$ be a fibration with base $\underline{C}$, i.e. for each $X \in \underline{C}$ there is a category $\underline{\mathcal{F}}_X$ (= the fibre over X) and for each morphism $f : X \to Y$ a functor $f^* : \underline{\mathcal{F}}_Y \longrightarrow \underline{\mathcal{F}}_X$ (= the inverse image of f) and for each composite $X \xrightarrow{f} Y \xrightarrow{g} C$ a natural equivalence $c_{f,g} : (gf)^* \longrightarrow f^* g^*$ subject to the usual compatibility conditions (see [16] Def. 1.1 or [14]). Let $\alpha : S_o \to S$ be a morphism in $\underline{C}$ and assume that the fibre products $S_o \underset{S}{\times} S_o$ and $S_o \underset{S}{\times} S_o \underset{S}{\times} S_o$ exist. Let $S_1 = S_o \underset{S}{\times} S_o$ and let $p_i : S_1 \to S_o$ denote the projection on the i-th factor, i = 1,2. Likewise let $S_2 = S_o \underset{S}{\times} S_o \underset{S}{\times} S_o$ and let $p_{ij} : S_2 \to S_1$ denote the partial projection on the i-th and j-th factor, where $(i,j) = (3,1), (3,2), (2,1)$. Clearly $p_1 p_{31} = p_1 p_{21}$, $p_2 p_{31} = p_2 p_{32}$ and $p_2 p_{21} = p_1 p_{32}$ hold and these morphisms together with the diagonal $\Delta : S_o \longrightarrow S_1$ give rise to a diagram

$$\underline{\mathcal{F}}_S \xrightarrow{\alpha^*} \underline{\mathcal{F}}_{S_o} \;\overset{\Delta^*}{\underset{p_2^*}{\overset{\longleftarrow}{\underset{\longrightarrow}{\overset{p_1^*}{\longrightarrow}}}}}\; \underline{\mathcal{F}}_{S_1} \;\overset{p_{31}^*}{\underset{p_{21}^*}{\overset{\longrightarrow}{\underset{\longrightarrow}{\overset{p_{32}^*}{\longrightarrow}}}}}\; \underline{\mathcal{F}}_{S_2}$$

and natural equivalences $c_{\Delta,p_1} : id \xrightarrow{\cong} \Delta^* p_1^*$, $c_{\Delta,p_2} : id \xrightarrow{\cong} \Delta^* p_2^*$,

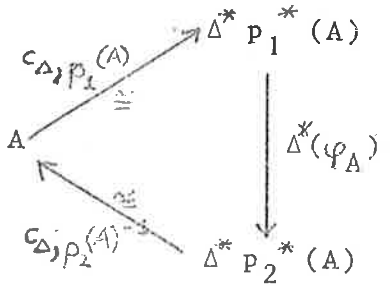
$c_{p_{31},p_1} : (p_1 p_{31})^* \xrightarrow{\cong} p_{31}^* p_1^*$, $c_{p_{21},p_1} : (p_1 p_{21})^* \xrightarrow{\cong} p_{21}^* p_1^*$,

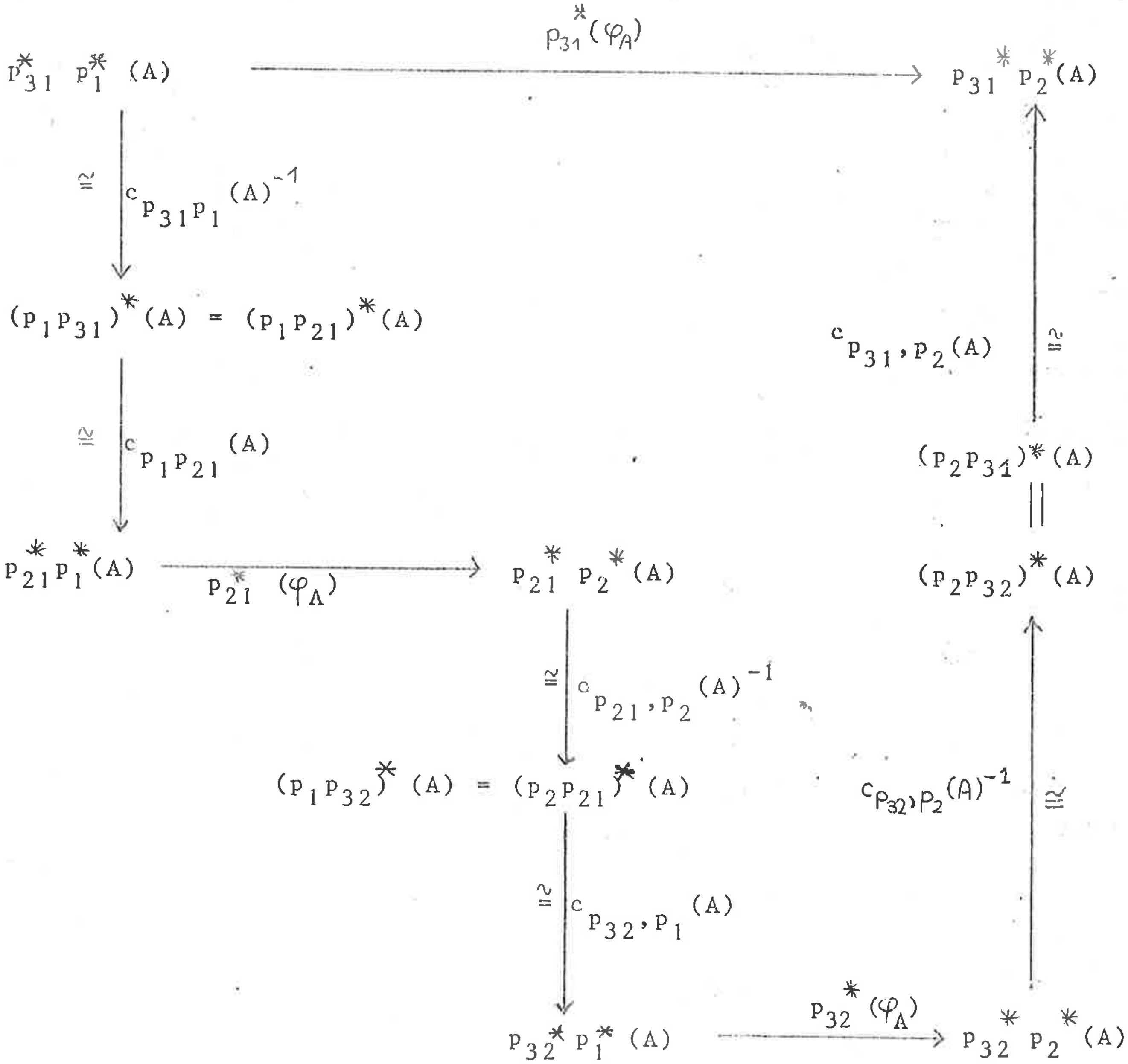
$c_{p_{31},p_2} : (p_2 p_3)^* \xrightarrow{\cong} p_{31}^* p_2^*$, $c_{p_{32},p_2} : (p_2 p_{32})^* \xrightarrow{\cong} p_{32}^* p_2^*$,

$c_{p_{21},p_2} : (p_2 p_{21})^* \xrightarrow{\cong} p_{21}^* p_2^*$ and $c_{p_{31},p_1} : (p_1 p_{31})^* \xrightarrow{\cong} p_{31}^* p_1^*$.

Recall that a descent datum on an object $A \in \underline{\mathcal{F}}_S$ is an isomorphism

$\varphi_A : p_1^{*} A \xrightarrow{\cong} p_2^{*} A$ with the properties $\Delta^{*}(\varphi_A) = id_A$ and $p_{31}^{*}(\varphi_A) = p_{32}^{*}(\varphi_A) \circ p_{21}^{*}(\varphi_A)$ modulo equivalence, i.e. the diagrams





commute. In the following we mean by a descent datum also a pair (A,φ_A) satisfying the above conditions. A morphism $(A,\varphi_A) \longrightarrow (A', \varphi_{A'})$ between descent data is a morphism $\xi : A \longrightarrow A'$ in $\underline{\mathfrak{F}}_{S_o}$ with the property $p_2^*(\xi) \circ \varphi_A = \varphi_{A'} \circ p_1^*(\xi)$. The resulting category of descent data is denoted with $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o})$. To express descent data as bialgebras in $\underline{\mathfrak{F}}_{S_o}$ let $\mathbb{F} = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathfrak{F}}_{S_o}}, \Delta^* p_2^*, p_{31}^* p_1^*, p_{31}^* p_2^*\}$ and let $M = \{\varphi, \bar{\varphi}\}$ consist of operations $\varphi : p_1^* \dashrightarrow p_2^*$ and $\bar{\varphi} : p_2 \dashrightarrow p_1$. Likewise let $R = \{r_1, r_2, r_3, r_4\}$ consist of relations $r_1 : p_1^* \Rrightarrow p_1^*$, $r_2 : p_2^* \Rrightarrow p_2^*$, $r_3 : \mathrm{id}_{\underline{\mathfrak{F}}_{S_o}} \Rrightarrow \mathrm{id}_{\underline{\mathfrak{F}}_{S_o}}$ and $r_4 : p_{31}^* p_1^* \Rrightarrow p_{31}^* p_2^*$ which for a pre-bialgebra $(A, \varphi_A, \bar{\varphi}_A)$ are given by

$$p_1^*A \underset{\bar{\varphi}_A \circ \varphi_A}{\overset{\mathrm{id}_{p_1^*A}}{\rightrightarrows}} p_1^*A \qquad\qquad p_2^*A \underset{\varphi_A \circ \bar{\varphi}_A}{\overset{\mathrm{id}_{p_2^*A}}{\rightrightarrows}} p_2^*A$$

and the two diagrams above. With this it is immediate that $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o}) = \mathrm{Bialg}(\underline{\mathfrak{F}}_{S_o})$. Note that $\mathbb{F}_d = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathfrak{F}}_{S_o}}, p_{31}^* p_1^*\}$ and $\mathbb{F}_c = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathfrak{F}}_{S_o}}, p_{31}^* p_2^*\}$. Thus by 3.3 $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o})$ has colimits (resp. limits) and the forgetful functor $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o}) \longrightarrow \underline{\mathfrak{F}}_{S_o}$, $(A, \varphi_A) \leadsto A$ preserves them provided $\underline{\mathfrak{F}}_{S_o}$ has colimits (resp. limits) and the inverse image functors p_1^*, p_2^* and p_{31}^* preserve them. Likewise if $\underline{\mathfrak{F}}_{S_o}$ has γ-filtered colimits for some $\gamma \geq \aleph_o$ and the above functors preserve them, then $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_o})$ has γ-filtered colimits and the forgetful functor preserves them.

4.15 Assume that $\underline{\mathfrak{F}}_{S_o}$, $\underline{\mathfrak{F}}_{S_1}$ and $\underline{\mathfrak{F}}_{S_2}$ are locally presentable and that the inverse image functors p_1^*, p_2^* and p_{31}^* have rank (2.1). Let $\gamma > \beta$ be cardinals such that

1) $\underline{\mathfrak{F}}_{S_o}$ is locally γ-presentable and

2) the functors p_1^*, p_2^* and p_{31}^* preserve β-filtered colimits and take γ-presentable objects into γ-presentable objects (the existence of such γ's follows from 3.7, see also 3.6).

Then by 3.8 for every descent datum (A,φ_A) and every morphism $f : U \to A$ in $\underline{\mathcal{F}}_{S_o}$ with $\pi(U) \leq \gamma$ there is a decomposition of f into a morphism $U \to U'$ and a morphism $(U', \varphi_{U'}) \to (A, \varphi_A)$ of descent data such that $\pi(U') \leq \gamma$. Moreover a descent datum (X, φ_X) is γ-presentable in $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o})$ iff X is γ-presentable in $\underline{\mathcal{F}}_{S_o}$. If in addition the inverse image functors p_1^*, p_2^* and p_{31}^* preserve colimits (resp. limits), then by 3.24 $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o})$ is locally γ-presentable (resp. β-presentable) and the forgetful functor $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o}) \to \underline{\mathcal{F}}_{S_o}$, $(A, \varphi_A) \rightsquigarrow A$ is cotripleable (resp. tripleable). In particular the canonical functor (cf. [16] 1.4)

$$\Phi : \underline{\mathcal{F}}_S \to \mathrm{Desc}(\underline{\mathcal{F}}_{S_o}),\ Y \rightsquigarrow (\alpha^*(Y), c_{p_2,\alpha}(Y) \circ c_{p_1,\alpha}(Y)^{-1})$$

is an equivalence iff $\alpha^* : \underline{\mathcal{F}}_S \to \underline{\mathcal{F}}_{S_o}$ is cotripleable (resp. tripleable). The relationship between descent and tripleability (resp. cotripleability) was first noticed by J. Beck and J. Benabou.

4.16 If $\underline{\mathcal{F}}_{S_o}$, $\underline{\mathcal{F}}_{S_1}$ and $\underline{\mathcal{F}}_{S_2}$ are Grothendieck categories (resp. topoi) and the inverse image functors p_1^*, p_2^* and p_{31}^* preserve colimits and finite limits, then $\mathrm{Desc}(\underline{\mathcal{F}}_{S_0})$ is again a Grothendieck category (resp. topos). This follows from 3.25.

4.17 The version of 4.15 for generated objects is as follows. Assume that $\underline{\mathcal{F}}_{S_0}$, $\underline{\mathcal{F}}_{S_1}$ and $\underline{\mathcal{F}}_{S_2}$ are locally presentable and that p_1^*, p_2^* and p_{31}^* have rank (2.1). Let $\gamma > \beta$ be cardinals such that

1) $\underline{\mathcal{F}}_{S_0}$ is locally γ-noetherian (resp. $\underline{\mathcal{F}}_{S_0}$ is locally γ-generated)

2) every β-well ordered colimit of monomorphisms in $\underline{\mathcal{F}}_{S_0}$ is again monomorphic

3) the functors p_1^*, p_2^* and p_{31}^* preserve β-filtered colimits and take γ-presentable objects into γ-presentable objects (resp. they preserve β-filtered colimits and finite limits and take γ-generated objects into γ-generated objects, cf. 5.1)

Then by 3.22 for every descent datum (A, φ_A) and every γ-generated subobject U of A there is a γ-generated subobject $U' \subset A$ containing U and a descent datum $(U', \varphi_{U'})$ such that the inclusion $U' \hookrightarrow A$ is a morphism of descent data. Moreover a descent datum (X, φ_X) is γ-generated in $\mathrm{Des}(\underline{\mathcal{F}}_{S_0})$ iff X is γ-generated in $\underline{\mathcal{F}}_{S_0}$, etc.

4.18 A possible application of the above is the following. If descent data are effective on small objects in $\underline{\mathcal{F}}_{S_0}$, then they are effective on all objects. In more detail let $\Phi : \underline{\mathcal{F}}_S \to \mathrm{Desc}(\underline{\mathcal{F}}_{S_0})$ be the canonical functor defined in 4.15. Recall that $\alpha : S_0 \to S$ is called of $\mathcal{F}$-descent type (resp. of strict $\mathcal{F}$-descent type) if Φ is full and faithful (resp. an equivalence), cf. [16] Def. 1.7. In addition to the assumptions made for the first half of 4.15 we assume that $\underline{\mathcal{F}}_S$ has γ-filtered colimits and that $\alpha^* : \underline{\mathcal{F}}_S \to \underline{\mathcal{F}}_{S_0}$ preserves γ-filtered colimits. Then $\alpha : S_0 \to S$ is of strict $\mathcal{F}$-descent type provided it is of $\mathcal{F}$-descent type and every descent data (U, φ_U) with U γ-presentable in $\underline{\mathcal{F}}_{S_0}$ is effective (i.e. in the image of Φ). This follows from 4.15 and 3.9 which imply that every descent datum (A, φ_A) in $\mathrm{Desc}(\underline{\mathcal{F}}_{S_0})$ is the γ-filtered colimit of descent data (U_i, φ_{U_i}) with $\pi(U_i) \le \gamma$; whence if $\Phi(Y_i) = (U_i, \varphi_{U_i})$, then $\Phi(\varinjlim_i Y_i) \cong \varinjlim_i \Phi(Y_i) = \varinjlim_i (U_i, \varphi_{U_i}) = (A, \varphi_A)$.

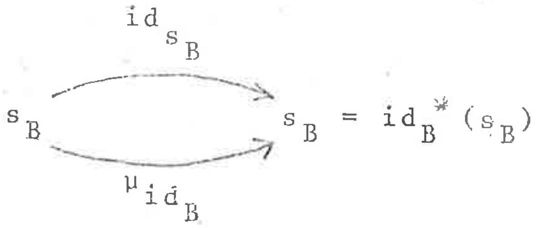
The smallest cardinals γ and β which are possible for this (and 4.15) are $\aleph_1$ and $\aleph_0$. Thus, if $\underline{\mathcal{F}}_{S_0}$ is locally countably presentable (or finitely presentable) and $\underline{\mathcal{F}}_S$ has countably filtered colimits and the inverse image functors α^*, p_1^*, p_2^* and p_{31}^* preserve filtered colimits and take countably presentable objects into countably presentable objects, then every descent datum is effective provided descent data are effective on countably presentable objects and $\alpha^* : \underline{\mathcal{F}}_S \longrightarrow \underline{\mathcal{F}}_{S_0}$ is of $\mathcal{F}$-descent type.

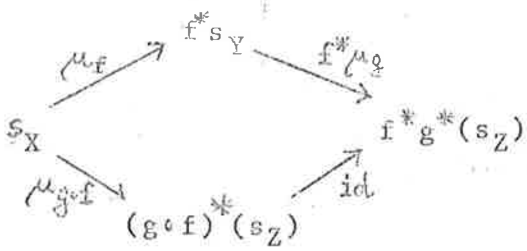
4.19 Sections and cartesian closed sections with respect to a fibration. The following is based on, or rather prompted by exposé I in SGA 4 by

A. Grothendieck and J.L. Verdier [17] (mainly p. 138-179). At my talk [34] M. Tierney suggested to compare the notion of a bialgebra (3.1) with the notion of a section (resp. cartesian closed section) for a fibration, and theorem 3.8 with theorem I 9.25 in [17]. (Both apply to descent data and récollements in the context of Grothendieck categories or topoi and yield the existence of generators.) In order to facilitate the comparison we essentially use the notion and notation for a fibration $p : \underline{E} \longrightarrow \underline{B}$ as defined in [17] p. 160, although it differs from the one used in 4.14 above (for an exposé on the different ways to look at fibrations see Giraud [14] or SGA 1 exposé VI). Let $p : \underline{E} \to \underline{B}$ be a fibration with small base $\underline{B}$. For an object $B \in \underline{B}$ the fibre $p^{-1}(B)$ is denoted with $\underline{E}_B$ and the inverse image functor for a morphism $f : A \to B$ in $\underline{B}$ with $f^* : \underline{E}_B \to \underline{E}_A$. Let $Hom_{\underline{B}}(\underline{B},\underline{E})$ denote the category of sections with respect to $p : \underline{E} \to \underline{B}$, i.e. the full subcategory of $[\underline{B},\underline{E}]$ consisting of all functors $s : \underline{B} \to \underline{E}$ with the property $ps = id_{\underline{B}}$, cf. [17] p. 161. Likewise let $Homcart_{\underline{B}}(\underline{B},\underline{E})$ be the full subcategory of $Hom_{\underline{B}}(\underline{B},\underline{E})$ of all cartesian closed sections, i.e. all sections $s : \underline{B} \to \underline{E}$ such that for every morphism $f : B \to A$ the canonical morphism $s(B) \to f^*(A)$ is an isomorphism. The main theorems of section I.9 in [17] concern the existence of generators in $Hom_{\underline{B}}(\underline{B},\underline{E})$ resp. $Homcart_{\underline{B}}(\underline{B},\underline{E})$ and implicite size estimates in terms of "filtrations cardinales". Without loss of generality one can assume that the objects of $\underline{B}$ form a set whose cardinality is the same as that of a skeleton of $\underline{B}$; this can always be achieved by pulling back the fibration $p : \underline{E} \longrightarrow \underline{B}$ along a full inclusion $\overset{\circ}{\underline{B}} \xrightarrow{\simeq} \underline{B}$ (for skeleton $\overset{\circ}{\underline{B}}$ of $\underline{B}$ see [26]...). In order to express sections and cartesian closed sections as bialgebras let $\underline{A} = \prod_{B \in \underline{B}} \underline{E}_B$ and let $\mathbb{F} = \{f^* \circ p_Y \mid (X \xrightarrow{f} Y) \in Mor\ \underline{B}\}$ consist of all composites $\prod_{B \in \underline{B}} \underline{E}_B \xrightarrow[p_Y]{} \underline{E}_Y \xrightarrow{f^*} \underline{E}_X$, where f runs through all morphisms of $\underline{B}$ and p_Y denotes the canonical projection onto $\underline{E}_Y$. Let $M = \{\mu_f \mid f \in Mor\ \underline{B}\}$, where $f : X \to Y$ runs through all morphisms of $\underline{B}$ and μ_f is an operation $p_X \dashrightarrow f^* p_Y$. Thus a pre-bialgebra consists of a family $(s_B)_{B\in \underline{B}}$ of objects in $\underline{E}$ together with a family $\{\mu_f : s_X \longrightarrow f^* s_Y\}_{(X \xrightarrow{f} Y)\in \mathrm{Mor}\,\underline{B}}$ of morphisms. Let

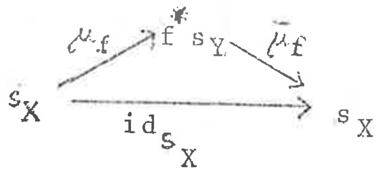
$$R = \{r_{id_B} \mid B\in \underline{B}\} \cup \{r_{f,g} \mid f,g \in \mathrm{Mor}\,\underline{B} \text{ and } g\circ f \text{ defined}\}$$

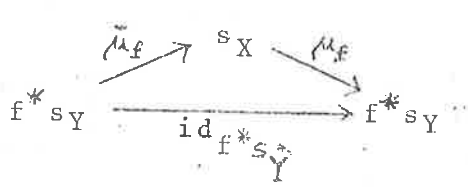
consist of relations $r_{id_B} : p_B \dashrightarrow p_B$ for every $B \in \underline{B}$ and $r_{f,g} : p_X \dashrightarrow g^* f^* p_Z$ for every composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ in B, which for a pre-bialgebra $(s_B, \mu_f)_{B\in \underline{B}, f\in \mathrm{Mor}\,\underline{B}}$ are given by the diagrams





With this it is straight forward that $\mathrm{Bialg}(\underline{A}) = \mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$. Clearly $\mathbb{F}_c = \mathbb{F}$, $\mathbb{F}_d = \{p_B \mid B \in \underline{B}\}$ and every projection p_B preserves all (existing) colimits and limits. In order to obtain cartesian closed sections one adds to M for every morphism $f : X \longrightarrow Y$ an operation $\bar{\mu}_f : f^* p_Y \dashrightarrow p_X$ and to R two relations which for a pre-bialgebra $(s_B, \mu_f, \bar{\mu}_f)_{B,f}$ are given by the diagrams





With this $\mathrm{Bialg}(\underline{A}) = \mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$. Note however that in this case $\mathbb{F} = \mathbb{F}_c = \mathbb{F}_d$. Thus the functors in $\mathbb{F}_d$ preserve only those colimits (resp. limits) which are preserved by all inverse image functors f^*, $f \in \mathrm{Mor}\,\underline{B}$. The above shows that sections and cartesian closed sections are special cases of bialgebras. The converse, i.e. that for a given data M , R and $\mathbb{F}$ of bialgebras in a category $\underline{X}$ there is a fibration $\underline{E} \dashrightarrow \underline{B}$ such that either $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E}) = \mathrm{Bialg}(\underline{X})$ or $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E}) = \mathrm{Bialg}(\underline{X})$, is very unlikely.

Such a hypothetical fibration would have to have weird properties (cf. discussion below in ...)

4.2o Assume that the fibre $\underline{E}_B$ is locally presentable for every $B \in \underline{B}$ and that the inverse image functor $f^* : \underline{E}_Y \to \underline{E}_X$ has rank (2.1) for every morphism $f : X \to Y$ in $\underline{B}$. Let $\gamma \geq \aleph_1$ be any regular cardinal such that 1) $\gamma \geq \pi(\underline{E}_B)$ for every $B \in \underline{B}$ 2) $\gamma > \pi(f^*)$ for every $f : X \to Y$ and 3) $\gamma > \text{Mor}\,\underline{B}$. Let δ be the least regular cardinal $\delta \geq \aleph_1$ satisfying 1) - 3). Then in $\underline{A} = \prod_{B \in B} \underline{E}_B$ an object $(s_B)_{B \in \underline{B}}$ is γ-presentable iff s_B is γ-presentable in $\underline{E}_B$ for every $B \in \underline{B}$, in particular $\underline{A}$ is locally δ-presentable and the projections p_B take γ-presentable objects into γ-presentable objects and likewise for γ-generated objects. Moreover for every $f : X \to Y$ the functor $f^* p_Y : \prod_{B \in B} \underline{E}_B \to \underline{E}_X$ preserves δ-filtered colimits.

4.21 Assume 4.2o. Then by 3.8 for every section $s \in \text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$, for every family $(t_B)_{B \in \underline{B}}$ of objects and every family $(f_B : t_B \to sB)_{B \in \underline{B}}$ of morphisms such that $\pi(t_B) \leq \gamma$ in $\underline{E}_B$ for every $B \in \underline{B}$ there is a section $t' \in \text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ together with a natural transformation $\varphi : t' \to s$ such that $t'B$ is γ-presentable in $\underline{E}_B$ and $f_B : t_B \to sB$ admits a decomposition $t_B \to t'B \xrightarrow{\varphi(B)} sB$ for every $B \in \underline{B}$. Moreover a section $s : \underline{B} \to \underline{E}$ is γ-presentable in $\text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ iff sB is γ-presentable in $\underline{E}_B$ for every $B \in \underline{B}$.

Assume in addition to 4.2o that there is a regular cardinal $\beta < \gamma$ such that for every $B \in \underline{B}$ β-well ordered colimits of monomorphisms in $\underline{E}_B$ are monomorphic and that either $\underline{E}_B$ is locally γ-noetherian or all inverse image functors f^*, where $f \in \text{Mor}\,\underline{B}$, preserve finite limits. Then for every section $s : \underline{B} \to \underline{E}$ and every family $(f_B : t_B \hookrightarrow sB)_{B \in \underline{B}}$ of γ-generated subobjects there is a subsection $\varphi : t' \hookrightarrow s$ such that $t'B$ contains t_B and $t'B$ is γ-generated in $\underline{E}_B$ for every $B \in \underline{B}$. Moreover a section $s : \underline{B} \to \underline{E}$ is γ-generated in $\text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ iff sB is γ-generated in $\underline{E}_B$ for every $B \in \underline{B}$.

4.22 Assume 4.2o. Then by 3.24 a) $Hom_{\underline{B}}(\underline{B},\underline{E})$ is locally δ-presentable and the functor

$$Hom_{\underline{B}}(\underline{B},\underline{E}) \dashrightarrow \prod_{B \in \underline{B}} \underline{E}_B \ , \quad s \rightsquigarrow (sB)_{B \in \underline{B}}$$

is cotripleable. Its right adjoint preserves δ-filtered colimits. Assume in addition to 4.2o that every inverse image functor f^*, where $f \in Mor\ \underline{B}$, preserves finite limits and that every fibre $\underline{E}_B$, where $B \in \underline{B}$, is a Grothendieck category (resp. a topos). Then by 3.25 $Hom_{\underline{B}}(\underline{B},\underline{E})$ is also a Grothendieck category (resp. a topos).

4.23 Assume 4.2o and that for every $f \in Mor\ \underline{B}$ the inverse image functor f^* preserves limits. Let δ' be the least regular cardinal such that every f^* preserves δ'-filtered colimits. Then by 3.24 b) $Hom_{\underline{B}}(\underline{B},\underline{E})$ is locally δ'-presentable and the functor

$$Hom_{\underline{B}}(\underline{B},\underline{E}) \longrightarrow \prod_{B \in \underline{B}} \underline{E}_B, \quad s \rightsquigarrow (sB)_{B \in \underline{B}}$$

is tripleable and preserves δ'-filtered colimits. (Note that in contrast to 4.22 the case $\delta' = \aleph_0$ is possible, e.g. if $Mor\ \underline{B}$ is finite, every fibre $\underline{E}_B$, $B \in \underline{B}$, is locally finitely presentable and f^* preserves filtered colimits for every $f \in Mor\ \underline{B}$.)

4.24 The situation for $Homcart_{\underline{B}}(\underline{B},\underline{E})$ is different because $\mathbb{F}_d$ consists of all composites $\prod_{B \in \underline{B}} \underline{E}_B \xrightarrow{p_Y} \underline{E}_Y \xrightarrow{f^*} \underline{E}_X$. In general these functors neither take γ-presentable objects into γ-presentable objects nor do they preserve colimits. Therefore additional conditions are needed to guarantee the validity of 4.21 - 4.23 for $Homcart_{\underline{B}}(\underline{B},\underline{E})$. They are as follows.

For the first half of 4.21 one has to assume in addition to 4.20 that for every $f \in Mor\ \underline{B}$ the inverse image functor f^* takes γ-presentable objects into γ-presentable objects, and for the second half of 4.21 that every f^* takes γ-generated objects into γ-generated objects.

For 4.22 (both the first and second half) one has to assume in addition to 4.2o that for every $f \in \mathrm{Mor}\ \underline{B}$ the inverse image functor f^* preserves colimits and takes γ-presentable objects into γ-presentable objects. (Note that by the special adjoint functor theorem every f^* has a right adjoint f_*. Thus it follows from $[X, f_* -] \cong [f^* X, -]$ that the functor f^* takes γ-presentable objects into γ-presentable objects iff f_* preserves γ-filtered colimits for every $f \in \mathrm{Mor}\ \underline{B}$.)

For 4.23 no additional assumptions to 4.2o are needed. The functor f^* may not take δ'-presentable objects into δ'-presentable objects with δ' as in 4.23, but by 3.7 there is a regular cardinal $\tilde{\delta} \geq \delta'$ such that every f^* takes $\tilde{\delta}$-presentable objects into $\tilde{\delta}$-presentable objects. Thus by 3.24 b) $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ is locally δ'-presentable with δ' as in 4.23 and $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E}) \longrightarrow \prod_{B \in \underline{B}} \underline{E}_B$, $s \rightsquigarrow (sB)_{B \in \underline{B}}$ is tripleable and preserves δ'-filtered colimits.

4.25 Remark For the first half of 4.21 (in particular the existence of γ-presentable generators in $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$) the assumptions 4.2o are not fully used, in particular the existence of arbitrary colimits in the fibres $\underline{E}_B$, $B \in \underline{B}$, is not needed. Instead of 4.2o it suffices to assume that there are regular cardinals $\gamma > \beta \geq \aleph_0$ such that the following conditions hold

1) card(Mor $\underline{B}$) $< \gamma$

2) for every $B \in \underline{B}$ the fibre $\underline{E}_B$ has β-filtered colimits and for every $f \in \mathrm{Mor}\ \underline{B}$ the inverse image functor f^* preserves β-filtered colimits

3) for every $B \in \underline{B}$ and every $A \in \underline{E}_B$ the category $\underline{E}_B(\gamma)/A$ of γ-presentable objects over A is γ-filtered and A is the colimit of $\underline{E}_B(\gamma)/A \longrightarrow \underline{A}$, $(U \longrightarrow A) \rightsquigarrow U$.

This follows from 3.11 a), b) and 4.19.

Likewise the first half of 4.21 holds also for $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ provided in addition to the above conditions 1) - 3) the following is satisfied.

4) for every $f \in \text{Mor}\,\underline{B}$ the inverse image functor f^* takes γ-presentable objects into γ-presentable objects.

4.26 Comparison with SGA 4 I.9. The main theorem I. 9.25 asserts the following. Let $p : \underline{E} \longrightarrow \underline{B}$ be a fibration with a small base $\underline{B}$ and assume that for every morphism f in $\underline{B}$ the inverse image functor f^* has rank (2.1). Then both $\text{Hom}_{\underline{B}}(\underline{B},\underline{E})$ and $\text{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ admit a set of strict generators provided the following four conditions are satisfied a) every fibre $\underline{E}_B$, $B \in \underline{B}$, has a set of strict*) generators b) every fibre $\underline{E}_B$, $B \in \underline{B}$, has colimits and pullbacks c) for every $B \in \underline{B}$ the kernel functor $\text{Mor}^2\underline{E}_B \longrightarrow \underline{E}_B$, $(u,v) \rightsquigarrow \ker(u,v)$, has rank (2.1), where $\text{Mor}^2\underline{E}_B$ denotes the category of morphism pairs in $\underline{E}_B$ with common domain and codomain d) in every fibre $\underline{E}_B$, $B \in \underline{B}$, the pullback of a strict epimorphism*) is again a strict epimorphism.

The conditions a) - d) imply that for every $B \in \underline{B}$ the fibre $\underline{E}_B$ is locally presentable. (This is because by [17] I. 9.11 every object in $\underline{E}_B$ is presentable.) The converse is not true. A locally presentable category satisfies a), b) and c) but not d) in general; e.g. the category Cat of small categories does not satisfy d). Grothendieck and Verdier do not give explicite size estimates for the generators and the ones which result from their proof are not very effective. For instance if the fibres are locally finitely or locally $\aleph_1$-presentable and the inverse image functors preserve filtered colimits and if the set of morphisms of $\underline{B}$ is countable, then their proof yields that all sections $s \in \text{Hom}_{\underline{B}}(\underline{B},\underline{E})$ with $\pi(sB) < (2^{\aleph_0})^+$ for every $B \in \underline{B}$ form a small generating (even dense) subcategory. (Recall that $(2^{\aleph_0})^+$ denotes the least regular cardinal $> 2^{\aleph_0}$). In contrast it follows from 4.21 above

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*) An epimorphism $f : A \longrightarrow B$ is called strict if it is the cokernel of its kernel pair $A \times_B A \rightrightarrows A$. In Gabriel-Ulmer [13] § 1 strict epimorphisms are called regular.

that already all sections $s \in \mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ such that sB is countably presentable for every $B \in \underline{B}$ form a small generating (even dense) subcategory.

The proof of Grothendieck and Verdier (cf. I. 9.22 - I. 9.26) for the existence of generators in $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ is not correct. The error is on p. 173 in [17], where they claim that the indicated composed morphism $f^*(X(\beta)_i) \longrightarrow f^*(X(\beta)) \xrightarrow{X(f)^{-1}} X(\alpha)$ factors through a canonical morphism $X(\alpha)_j \longrightarrow \varinjlim_{j\in I}(X_j) = X(\alpha)$ for some j, assuming that I is c^+-filtered (= I est grand devant c) and $X(\beta)_i$ is c^+-presentable (= c-accessible). This need not be so because in general f^* does not take c^+-presentable objects into c^+-presentable objects! As a matter of fact with c as in [17] p. 173 the cardinal $\pi(f^*(\beta)_i)$ can be arbitrary large although f^* has rank. (As a guidance for this phenomenon we mention the filtered colimit preserving forgetful functor $\underline{\mathrm{Mod}}_\Lambda \rightarrow \underline{\mathrm{Sets}}$ for some ring Λ, which takes finitely presentable objects into $\mathrm{card}(\Lambda)^+$-presentable objects, cf. also 3.5 - 3.7.) As a consequence of this error the lemmata 9.21.16 (ii) and 9.21.19 are incorrect and the "filtrations" of $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ given in I. 9.22-I. 9.26 are not "filtrations cardinales" as claimed.