A description of the second class in the cotilting cotorsion pair in terms of cofiltrations

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Algebra seminar, Charles Univ., Prague (via Zoom)

October 12, 2020

Basic Definition: Cotilting Modules

Let R be an associative ring. For any R-module M, we denote by $\mathrm{Prod}(M) \subset R\text{-}\mathrm{Mod}$ the class of all direct summands of products of copies of M in $R\text{-}\mathrm{Mod}$.

Let $n \geqslant 0$ be an integer. An R-module U is said to be *n-cotilting* if the following conditions hold:

1. (C1) the injective dimension of the R-module U does not exceed n;
2. (C2) $\mathrm{Ext}^i_R(U^X, U) = 0$ for any set X and all $i > 0$;
3. (C3) for some (equivalently, for any) injective generator J of $R\text{-}\mathrm{Mod}$ and some finite integer r (equivalently, for $r = n$) there exists an exact sequence of R-modules

$$0 \longrightarrow U_r \longrightarrow U_{r-1} \longrightarrow \cdots \longrightarrow U_0 \longrightarrow J \longrightarrow 0$$

with $U_i \in \mathrm{Prod}(U)$.

In particular, an R-module is 0-cotilting if and only if it is an injective cogenerator of $R\text{-}\mathrm{Mod}$.

Background material: Cotorsion Pairs

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be two classes of R-modules. Denote by $\mathcal{F}^{\perp_1} \subset R\text{-}\mathrm{Mod}$ the class of all modules $X \in R\text{-}\mathrm{Mod}$ such that $\mathrm{Ext}^1_R(F, X) = 0$ for all $F \in \mathcal{F}$, and by ${}^{\perp_1}\mathcal{C} \subset R\text{-}\mathrm{Mod}$ the class of all modules $Y \in R\text{-}\mathrm{Mod}$ such that $\mathrm{Ext}^1_R(Y, C) = 0$ for all $C \in \mathcal{C}$.

A pair of classes $(\mathcal{F}, \mathcal{C})$ is said to be a cotorsion pair if $\mathcal{C} = \mathcal{F}^{\perp_1}$ and $\mathcal{F} = {}^{\perp_1}\mathcal{C}$.

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be complete if for every module $M \in R\text{-}\mathrm{Mod}$ there exist short exact sequences

$$0 \longrightarrow C' \longrightarrow F \longrightarrow M \longrightarrow 0 \qquad (1)$$

$$0 \longrightarrow M \longrightarrow C \longrightarrow F' \longrightarrow 0 \qquad (2)$$

with F, $F' \in \mathcal{F}$ and C, $C' \in \mathcal{C}$. The short exact sequence (1) is called a special precover sequence and the short exact sequence (2) is called a special preenvelope sequence. The sequences (1–2) are also called the approximation sequences.

Cotilting Cotorsion Pair

Let $\mathcal{S} \subset R\text{-}\mathrm{Mod}$ be a class of R-modules. The cotorsion pair $(\mathcal{F}', \mathcal{C}')$ with $\mathcal{C}' = \mathcal{S}^{\perp_1}$ and $\mathcal{F}' = {}^{\perp_1}\mathcal{C}'$ (so $\mathcal{S} \subset \mathcal{F}'$) is said to be generated by $\mathcal{S}$. The cotorsion pair $(\mathcal{F}'', \mathcal{C}'')$ with $\mathcal{F}'' = {}^{\perp_1}\mathcal{S}$ and $\mathcal{C}'' = \mathcal{F}''^{\perp_1}$ (so $\mathcal{S} \subset \mathcal{C}''$) is said to be cogenerated by $\mathcal{S}$.

Given an R-module M, choose its projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ and injective coresolution $0 \longrightarrow M \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$. Then the image of the morphism $P_i \longrightarrow P_{i-1}$ is called the i-th syzygy module of M and denoted by $\Omega^i M$. The image of the morphism $J^{i-1} \longrightarrow J^i$ is called the i-th cosyzygy module of M and denoted by $\mho^i M$.

Let U be an n-cotilting R-module. The cotilting cotorsion pair induced by U is defined as the cotorsion pair $(\mathcal{F}, \mathcal{C})$ with

$$\mathcal{F} = {}^{\perp_{>0}}U = \{F \in R\text{-}\mathrm{Mod} \mid \mathrm{Ext}^i_R(F, U) = 0 \ \forall i > 0\}.$$

Equivalently, one can say that $(\mathcal{F}, \mathcal{C})$ is the cotorsion pair cogenerated by all the cosyzygy modules U, $\mho^1 U$, ..., $\mho^{n-1} U$ of the R-module U. All the cotilting cotorsion pairs are complete.

Description of the Left and Right Cotilting Classes in terms of Coresolutions and Resolutions

The left class $\mathcal{F}$ in the cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ induced by an n-cotilting module U is called the n-cotilting class. Both the classes $\mathcal{F}$ and $\mathcal{C}$ can be described as follows.

Proposition

The n-cotilting class $\mathcal{F}$ consists of all the R-modules F admitting a coresolution by modules from $\mathrm{Prod}(U)$, $0 \longrightarrow F \longrightarrow U^1 \longrightarrow U^2 \longrightarrow U^3 \longrightarrow \cdots$. Equivalently, $F \in \mathcal{F}$ if and only if an exact sequence of R-modules $0 \longrightarrow F \longrightarrow U^1 \longrightarrow U^2 \longrightarrow \cdots \longrightarrow U^n$ with $U^i \in \mathrm{Prod}(U)$ exists.

Proposition

The second (right) class $\mathcal{C}$ in the n-cotilting cotorsion pair consists of all the R-modules C admitting a finite resolution of some length r (equivalently, of length $r = n$) by modules from $\mathrm{Prod}(U)$, $0 \longrightarrow U_r \longrightarrow U_{r-1} \longrightarrow \cdots \longrightarrow U_0 \longrightarrow C \longrightarrow 0$ $(U_i \in \mathrm{Prod}(U))$.

Background material: Filtrations and Cofiltrations

Let M be an R-module and α be an ordinal. An α-indexed filtration of M is a collection of submodules $F_iM \subset M$ indexed by the ordinals $0 \leqslant i \leqslant \alpha$ such that

- $F_0M = 0$, $F_\alpha M = M$;
- $F_jM \subset F_iM$ for all $0 \leqslant j \leqslant i \leqslant \alpha$;
- $F_iM = \bigcup_{j<i} F_jM$ for all limit ordinals $i \leqslant \alpha$.

One says that the module M is filtered by the modules $F_{i+1}M/F_iM$, $0 \leqslant i < \alpha$.

An α-indexed cofiltration of M is a collection of R-modules G_iM indexed by $0 \leqslant i \leqslant \alpha$ and surjective R-module morphisms $G_iM \longrightarrow G_jM$ given for all $0 \leqslant j < i \leqslant \alpha$ such that

- the triangle diagram $G_iM \longrightarrow G_jM \longrightarrow G_kM$ is commutative for all $0 \leqslant k < j < i \leqslant \alpha$;
- $G_0M = 0$, $G_\alpha M = M$;
- $G_iM = \varprojlim_{j<i} G_jM$ for all limit ordinals $i \leqslant \alpha$.

One says that the module M is cofiltered by the modules $\ker(G_{i+1}M \to G_iM)$, $0 \leqslant i < \alpha$.

The Šaroch–Trlifaj Description for $n = 1$

According to the Eklof–Trlifaj theorem (2001), for any set of R-modules $\mathcal{S}$, the left class $\mathcal{F}$ of the cotorsion pair generated by $\mathcal{S}$ can be described as the class of all direct summands of R-modules filtered by modules from $\mathcal{S} \cup \{{}_R R\}$.

No comparable description of the right class $\mathcal{C}$ of the cotorsion pair cogenerated by $\mathcal{S}$ in terms of cofiltrations is available in general. Only one implication is known: all the direct summands of R-modules cofiltered by modules from $\mathcal{S} \cup \{J\}$ (where J is any injective cogenerator of $R\text{-}\mathrm{Mod}$) belong to $\mathcal{C}$.

The following recent theorem describes the second (right) class of any 1-cotilting cotorsion pair in terms of cofiltrations.

Theorem (Šaroch and Trlifaj, 2019)

Let U be a 1-cotilting R-module. Then the right class $\mathcal{C}$ of the induced 1-cotilting cotorsion pair consists of all direct summands C of the R-modules D admitting a short exact sequence $0 \longrightarrow U' \longrightarrow D \longrightarrow J' \longrightarrow 0$ with $U' \in Prod(U)$, $J' \in \mathrm{Prod}(J)$.

Dual Bongartz Lemma

The proof of the Šaroch–Trlifaj theorem is a simple application of the dual Bongartz lemma.

Lemma ("dual Bongartz lemma")

Let S be an R-module such that $\mathrm{Ext}^1_R(S^X, S) = 0$ for all sets X. Then the cotorsion pair $(\mathcal{F}, \mathcal{C})$ cogenerated by S is complete, and the class $\mathcal{C}$ consists of all the direct summands C of R-modules D admitting a short exact sequence $0 \longrightarrow S' \longrightarrow D \longrightarrow J' \longrightarrow 0$ with $S' \in \mathrm{Prod}(S)$ and an injective R-module J'.

Sketch of proof.

For any R-module M, let X be the underlying set of $\mathrm{Ext}^1_R(M, S)$. Then the middle term of $0 \longrightarrow S^X \longrightarrow F \longrightarrow M \longrightarrow 0$ belongs to $\mathcal{F}$, providing a special precover sequence. A special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ for any R-module N is then obtained from the Salce lemma (choose $0 \to N \to J' \to M \to 0$). When $N \in \mathcal{C}$, it follows that N is a direct summand of D. □

Historical Remarks on Bongartz' Lemma

The dual Bongartz lemma is the dual assertion to the "Bongartz lemma", which comes from a classical 1981 paper of Bongartz. The relevant lemma in Bongartz' paper claimed (in the present-day terminology) that a finite-dimensional partial tilting module over a finite-dimensional algebra over a field is a direct summand of a finite-dimensional tilting module. The related result in infinitely generated tilting theory requires more complicated assumptions than in Bongartz' 1981 paper.

What is now called the "Bongartz lemma", that is the dual assertion to the previous slide, was abstracted from the key technical step in Bongartz' proof of his lemma and generalized to infinitely generated modules.

The Dual n-Bongartz Lemma

Lemma

Let $\mathcal{S} = \{S_0, S_1, \dots, S_n\}$ be a collection of $n+1$ R-modules such that $S_0 = J$ is an injective cogenerator, and $\mathrm{Ext}^1_R(S_j^X, S_i) = 0$ for all $0 \leqslant i \leqslant j \leqslant n$ and all sets X. Then the cotorsion pair $(\mathcal{F}, \mathcal{C})$ cogenerated by $\mathcal{S}$ is complete, and $\mathcal{C}$ is the class of all direct summands of R-modules D admitting a finite cofiltration $D = G_{n+1}D \twoheadrightarrow G_nD \twoheadrightarrow \cdots \twoheadrightarrow G_1D \twoheadrightarrow G_0D = 0$ such that $\ker(G_{i+1}D \to G_iD) \in \mathrm{Prod}(S_i)$ for every $0 \leqslant i \leqslant n$.

Proof.

Let $\mathcal{D}$ be the class of all R-modules D admitting a cofiltration as above. First we will construct a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$ for every R-module M.

Proof of dual n-Bongartz lemma cont'd.

Put $G_1F = M$. Denote by X_1 the underlying set of $\operatorname{Ext}^1_R(M, S_1)$, and let G_2F be the middle term of the related short exact sequence $0 \longrightarrow S_1^{X_1} \longrightarrow G_2F \longrightarrow G_1F \longrightarrow 0$. From the long exact sequence $\operatorname{Hom}_R(S_1^{X_1}, S_1) \twoheadrightarrow \operatorname{Ext}^1_R(G_1F, S_1) \longrightarrow \operatorname{Ext}^1_R(G_2F, S_1) \longrightarrow \operatorname{Ext}^1_R(S_1^{X_1}, S_1) = 0$ we see that $\operatorname{Ext}^1_R(G_2F, S_1) = 0$.

Denote by X_2 the underlying set of $\operatorname{Ext}^1_R(G_2F, S_2)$, and let G_3F be the middle term of the related short exact sequence $0 \longrightarrow S_2^{X_2} \longrightarrow G_3F \longrightarrow G_2F \longrightarrow 0$. Then $\operatorname{Ext}^1_R(G_3F, S_1) = 0$ since $\operatorname{Ext}^1_R(G_2F, S_1) = 0 = \operatorname{Ext}^1_R(S_2^{X_2}, S_1)$, and $\operatorname{Ext}^1_R(G_3F, S_2) = 0$ by construction (since $\operatorname{Ext}^1_R(S_2^{X_2}, S_2) = 0$).

Basically, at the passage from G_jF to $G_{j+1}F$ we kill all elements of $\operatorname{Ext}^1_R(-, S_j)$. This creates no new elements in $\operatorname{Ext}^1_R(-, S_i)$ for $i \leqslant j$ due to the Ext^1_R-self-orthogonality conditions imposed on the modules S_i.

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \dots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration $F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\mathrm{Ext}^1_R(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Denoting by D' the kernel of the surjective map $F = G_{n+1}F \to G_1F = M$, we obtain a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$. Here the R-module D' is endowed with a cofiltration G as desired, with the additional property that $G_1D' = 0$.

Proof of dual n-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$, so there is a short exact sequence $0 \longrightarrow N \longrightarrow S_0^{X_0} \longrightarrow M \longrightarrow 0$.

Argue as usually in the Salce lemma, using the pullback diagram for the pair of surjective morphisms $S_0^{X_0} \longrightarrow M$ and $F \longrightarrow M$. Then it is clear from the short exact sequence $0 \longrightarrow D' \longrightarrow D \longrightarrow S_0^{X_0} \longrightarrow 0$ that the R-module D has a cofiltration of the desired form.

Finally, if $N \in \mathcal{C}$, then $\mathrm{Ext}_R^1(F, N) = 0$, so N is a direct summand of D. □

Recall that the cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by the n-cotilting module U and its cosyzygy modules $\mho^1 U, \ldots, \mho^{n-1} U$. One has $\mathrm{Ext}^1_R((\mho^i U)^X, U) \simeq \mathrm{Ext}^{i+1}_R(U^X, U) = 0$ for all $i \geqslant 0$ and all sets X. However, it may well happen that $\mathrm{Ext}^1_R(\mho^1 U, \mho^1 U) \neq 0$.

One observes that if $0 \longrightarrow U \longrightarrow J^0 \longrightarrow \mho^1 U \longrightarrow 0$ is a short exact sequence of R-modules with an injective R-module J^0, then $\mathrm{Ext}^1_R(\mho^1 U, \mho^1 U) \simeq \mathrm{Ext}^2_R(\mho^1 U, U) \simeq \mathrm{Ext}^2_R(J^0, U)$, and there is no apparent reason for this Ext group to vanish.

In fact, there is a counterexample due to D'Este (2005). Over any field k, she constructs a quiver algebra A of global dimension 2, with 4 vertices, 4 edges, and 2 relations. Then ${}_AA$ is a 2-cotilting A-module (since A is a finite-dimensional algebra of global dimension 2), but $\mathrm{Ext}^1_A(\mho^1 A, \mho^1 A) \neq 0$ (for the minimal cosyzygy module $\mho^1 A$ of the free A-module A).

Therefore, the dual n-Bongartz lemma is not applicable to the sequence of cosyzygy modules $\mho^i U$ of an n-cotilting R-module U, generally speaking.

Associated Cotilting Modules and Classes

For every R-module M and integer $i \geqslant 0$, denote by ${}^{\perp_{>i}}M \subset R\text{-}\mathrm{Mod}$ the class of all R-modules F such that $\mathrm{Ext}_R^n(F, M) = 0$ for all $n > i$.

Proposition (Bazzoni)

Let U be an n-cotilting R-module. Then for every $0 \leqslant i \leqslant n$ there exists an $(n-i)$-cotilting R-module U_i such that ${}^{\perp_{>0}}U_i = {}^{\perp_{>i}}U$. In other words, ${}^{\perp_{>i}}U$ is an $(n-i)$-cotilting class.

In particular, one can take $U_0 = U$, while U_n is a 0-cotilting module, i.e., an injective cogenerator of $R\text{-}\mathrm{Mod}$.

The proof of the proposition is based on a theorem and a lemma, which are formulated below.

Associated Cotilting Modules and Classes

Theorem (Angeleri Hügel–Coelho, 2001)

Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$. Then $(\mathcal{F},\mathcal{C})$ is an m-cotilting cotorsion pair if and only if it is hereditary (i.e., $\mathcal{F} \subset {}^{\perp_{>0}}\mathcal{C}$), $\mathcal{F}$ is closed under infinite products, and $\mathcal{C}$ consists of modules of injective dimension $\leqslant m$.

Lemma (Bazzoni, 2004)

Let U be an n-cotilting R-module. Then for every $0 \leqslant i \leqslant n$ the class of R-modules ${}^{\perp_{>i}}U$ is closed under infinite products.

The proposition follows immediately from the theorem and the lemma.

Right Cotilting Class Described in terms of Cofiltrations

The applicability of the dual n-Bongartz lemma to the sequence of associated cotilting modules $U_n = J,\ U_{n-1},\ \dots,\ U_1,\ U_0 = U$ is based on two lemmas.

Lemma 1

For all $1 \leqslant i \leqslant j \leqslant n$ *and every set* X*, one has* $\operatorname{Ext}^1_R(U_i^X, U_j) = 0$.

Proof.

In fact, $\operatorname{Ext}^n_R(U_i^X, U_j) = 0$ for all $n \geqslant 1$, since

$$U_i^X \in {}^{\perp_{>0}}U_i = {}^{\perp_{>i}}U \subset {}^{\perp_{>j}}U = {}^{\perp_{>0}}U_j.$$

□

Right Cotilting Class Described in terms of Cofiltrations

Lemma 2

The n-cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ *induced by an n-cotilting module U is cogenerated by the modules* U, U_1, …, U_{n-1}, *that is* ${}^{\perp_{>0}}U = {}^{\perp_1}\{U, U_1, \dots, U_{n-1}\}$.

Proof.

For any $i, j \geqslant 0$ we have ${}^{\perp_{>i}}U_j = {}^{\perp_{>i+j}}U$, because an R-module F belongs to ${}^{\perp_{>i}}U_j$ if and only if the i-th syzygy R-module $\Omega^i F$ belongs to ${}^{\perp_{>0}}U_j$, which by the definition of U_j means that $\Omega^i F$ belongs to ${}^{\perp_{>j}}U$, which holds if and only if F belongs to ${}^{\perp_{>i+j}}U$.

In particular, it follows that ${}^{\perp_{>1}}U_j = {}^{\perp_{>j+1}}U = {}^{\perp_{>0}}U_{j+1}$.

Now one proceeds by decreasing induction in $0 \leqslant j \leqslant n$ proving that ${}^{\perp_1}\{U_j, \dots, U_n\} = {}^{\perp_{>0}}U_j$, since ${}^{\perp_1}U_j \cap {}^{\perp_1}\{U_{j+1}, \dots, U_n\} = {}^{\perp_1}U_j \cap {}^{\perp_{>0}}U_{j+1} = {}^{\perp_1}U_j \cap {}^{\perp_{>1}}U_j = {}^{\perp_{>0}}U_j$. □

Right Cotilting Class Described in terms of Cofiltrations

The following theorem is our main result.

Theorem

Let U be an n-cotilting module over an associative ring R, and let $(\mathcal{F},\mathcal{C})$ be the induced n-cotilting cotorsion pair. Then the class $\mathcal{C}$ consists precisely of all the direct summands C of R-modules D admitting an $(n+1)$-step cofiltration
$D = G_{n+1}D \twoheadrightarrow G_nD \twoheadrightarrow \cdots \twoheadrightarrow G_1D \twoheadrightarrow G_0D = 0$ *such that*
$G_1D \in \mathrm{Prod}(J) = \mathrm{Prod}(U_n)$, $\ker(G_{i+1}D \to G_iD) \in \mathrm{Prod}(U_{n-i})$ *for every* $0 \leqslant i \leqslant n$, *and* $\ker(G_{n+1}D \to G_nD) \in \mathrm{Prod}(U)$.

Proof.

By Lemma 2, the cotorsion pair $(\mathcal{F},\mathcal{C})$ is cogenerated by the modules $S_i = U_{n-i}$. By Lemma 1, the assumptions of the dual n-Bongartz lemma are satisfied. Hence the theorem follows as a particular case of the dual n-Bongartz lemma. □

Dual Setting: Tilting Modules

For any R-module M, denote by $\mathrm{Add}(M) \subset R\text{-}\mathrm{Mod}$ the class of all direct summands of (infinite) direct sums of copies of M in $R\text{-}\mathrm{Mod}$.

An R-Module T is said to be *n-tilting* if the following conditions hold:

1. (T1) the projective dimension of the R-module T does not exceed n;
2. (T2) $\mathrm{Ext}_R^i(T, T^{(X)}) = 0$ for any set X and all $i > 0$ (where $T^{(X)}$ denotes the direct sum of X copies of T);
3. (T3) for some finite integer r (equivalently, for $r = n$) there exists an exact sequence of R-modules

$$0 \longrightarrow R \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0$$

with $T^i \in \mathrm{Add}(T)$.

In particular, an R-module is 0-tilting if and only if it is a projective generator of $R\text{-}\mathrm{Mod}$.

Tilting Cotorsion Pair

Let T be an n-tilting R-module. The tilting cotorsion pair induced by T is the cotorsion pair $(\mathcal{F}, \mathcal{C})$ with

$$\mathcal{C} = T^{\perp_{>0}} = \{C \in R\text{-}\mathrm{Mod} \mid \mathrm{Ext}^i_R(T, C) = 0 \ \forall i > 0\}.$$

Equivalently, one can say that $(\mathcal{F}, \mathcal{C})$ is the cotorsion pair generated by all the syzygy modules T, $\Omega^1 T$, …, $\Omega^{n-1} T$ of the R-module T.

By the Eklof–Trlifaj theorem, any cotorsion pair generated by a set of modules is complete. In particular, all the tilting cotorsion pairs are complete.

Description of the Left and Right Tilting Classes in terms of Resolutions and Coresolutions

The right class $\mathcal{C}$ in the tilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ induced by an n-tilting module T is called the n-tilting class. Both the classes $\mathcal{F}$ and $\mathcal{C}$ can be described as follows.

Proposition

The n-tilting class $\mathcal{C}$ consists of all the R-modules C admitting a resolution by modules from $\mathrm{Add}(T)$, $\cdots \longrightarrow T_3 \longrightarrow T_2 \longrightarrow T_1 \longrightarrow C \longrightarrow 0$. Equivalently, $C \in \mathcal{C}$ if and only if there exists an exact sequence of R-modules $T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow C \longrightarrow 0$ with $T_i \in \mathrm{Add}(T)$.

Proposition

The second (left) class $\mathcal{F}$ in the n-tilting cotorsion pair consists of all the R-modules F admitting a finite coresolution of some length r (equivalently, of length $r = n$) by modules from $\mathrm{Add}(T)$, $0 \longrightarrow C \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0$ $(T^i \in \mathrm{Add}(T))$.

The n-Bongartz Lemma

The Eklof–Trlifaj theorem provides the description of the left class $\mathcal{F}$ in an n-tilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ as the class of all direct summands of modules filtered by T, $\Omega^1 T$, …, $\Omega^{n-1} T$, and R.

An approach based on the n-Bongartz lemma leads to a more concrete description of $\mathcal{F}$ in terms of filtrations, in that the shape of the filtrations involved is more precisely specified. However, one needs to use the associated tilting modules T_i in place of the syzygy modules $\Omega^i T$.

Lemma (n-Bongartz lemma)

Let $\mathcal{S} = \{S_0, S_1, \dots, S_n\}$ be a collection of $n+1$ R-modules such that S_0 is a projective generator and $\operatorname{Ext}^1_R(S_i, S_j^{(X)}) = 0$ for all $0 \leqslant i \leqslant j \leqslant n$ and all sets X. Then the left class $\mathcal{F}$ in the cotorsion pair generated by $\mathcal{S}$ is the class of all direct summands of R-modules G admitting a finite $(n+1)$-step filtration $0 = F_0 G \subset F_1 G \subset \cdots \subset F_n G \subset F_{n+1} G = G$ such that $F_{i+1}G/F_iG \in \mathrm{Add}(S_i)$ for every $0 \leqslant i \leqslant n$.

Associated Tilting Modules and Classes

For every R-module M and integer $i \geqslant 0$, denote by $M^{\perp_{>i}} \subset R\text{-}\mathrm{Mod}$ the class of all R-modules C such that $\mathrm{Ext}^n_R(M, C) = 0$ for all $n > i$.

Proposition (Bazzoni–Šťovíček, 2007)

Let T be an n-tilting R-module. Then for every $0 \leqslant i \leqslant n$ there exists an $(n-i)$-tilting R-module T_i such that $T_i^{\perp_{>0}} = T^{\perp_{>i}}$. In other words, $T^{\perp_{>i}}$ is an $(n-i)$-tilting class.

In particular, one can take $T_0 = T$, while T_n is a 0-tilting module, i.e., an projective generator of $R\text{-}\mathrm{Mod}$.

The proof of the proposition is based on a theorem and a lemma.

Associated Tilting Modules and Classes

Theorem (Angeleri Hügel–Coelho, 2001; Šťovíček–Trlifaj, 2007)

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$. Then $(\mathcal{F}, \mathcal{C})$ is an m-tilting cotorsion pair if and only if it is hereditary, $\mathcal{C}$ is closed under infinite direct sums, and $\mathcal{F}$ consists of modules of projective dimension $\leqslant m$.

Lemma (Bazzoni, 2004)

Let T be an n-tilting R-module. Then for every $0 \leqslant i \leqslant n$ the class of R-modules $T^{\perp_{>i}}$ is closed under infinite direct sums.

Left Tilting Class Described in terms of Filtrations of Specific Shape

The applicability of the n-Bongartz lemma to the sequence of associated tilting modules $S_0 = T_n$, $S_1 = T_{n-1}$, $\dots$, $S_{n-1} = T_1$, $S_n = T_0 = T$ is based on two lemmas.

Lemma 1

For all $1 \leqslant i \leqslant j \leqslant n$ and every set X, one has $\mathrm{Ext}^1_R(T_j, T_i^{(X)}) = 0$.

Lemma 2

The n-tilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ induced by an n-tilting module T is generated by the modules T, T_1, $\dots$, T_{n-1}, that is $T^{\perp_{>0}} = \{T, T_1, \dots, T_{n-1}\}^{\perp_1}$.

Left Tilting Class Described in terms of Filtrations of Specific Shape

The following theorem is our main result for tilting cotorsion pairs.

Theorem

Let T be an n-tilting module over an associative ring R, and let $(\mathcal{F}, \mathcal{C})$ be the induced n-tilting cotorsion pair. Then the class $\mathcal{F}$ consists precisely of all the direct summands of R-modules G admitting a finite $(n+1)$-step filtration $0 = F_0G \subset F_1G \subset \cdots \subset F_nG \subset F_{n+1}G = G$ such that $F_{i+1}G/F_iG \in \mathrm{Add}(T_{n-i})$ for every $0 \leqslant i \leqslant n$. □

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