A description of the second class in the cotilting cotorsion pair in terms of cofiltrations

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Algebra seminar, Charles Univ., Prague (via Zoom)

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In particular, an R-module is 0-cotilting if and only if it is an injective cogenerator of $R\text{-}\mathrm{Mod}$.

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No comparable description of the right class $\mathcal{C}$ of the cotorsion pair cogenerated by $\mathcal{S}$ in terms of cofiltrations is available in general. Only one implication is known: all the direct summands of R-modules cofiltered by modules from $\mathcal{S} \cup \{J\}$ (where J is any injective cogenerator of $R\text{-}\mathrm{Mod}$) belong to $\mathcal{C}$.

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What is now called the "Bongartz lemma", that is the dual assertion to the previous slide, was abstracted from the key technical step in Bongartz' proof of his lemma and generalized to infinitely generated modules.

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Let $\mathcal{D}$ be the class of all R-modules D admitting a cofiltration as above.

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Proof.

Let $\mathcal{D}$ be the class of all R-modules D admitting a cofiltration as above. First we will construct a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$ for every R-module M.

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Proof of dual n-Bongartz lemma cont'd.

Put $G_1F = M$. Denote by X_1 the underlying set of $\mathrm{Ext}^1_R(M, S_1)$, and let G_2F be the middle term of the related short exact sequence $0 \longrightarrow S_1^{X_1} \longrightarrow G_2F \longrightarrow G_1F \longrightarrow 0$. From the long exact sequence $\mathrm{Hom}_R(S_1^{X_1}, S_1) \twoheadrightarrow \mathrm{Ext}^1_R(G_1F, S_1) \longrightarrow \mathrm{Ext}^1_R(G_2F, S_1) \longrightarrow \mathrm{Ext}^1_R(S_1^{X_1}, S_1) = 0$ we see that $\mathrm{Ext}^1_R(G_2F, S_1) = 0$.

Denote by X_2 the underlying set of $\mathrm{Ext}^1_R(G_2F, S_2)$, and let G_3F be the middle term of the related short exact sequence $0 \longrightarrow S_2^{X_2} \longrightarrow G_3F \longrightarrow G_2F \longrightarrow 0$. Then $\mathrm{Ext}^1_R(G_3F, S_1) = 0$ since $\mathrm{Ext}^1_R(G_2F, S_1) = 0 = \mathrm{Ext}^1_R(S_2^{X_2}, S_1)$, and $\mathrm{Ext}^1_R(G_3F, S_2) = 0$ by construction (since $\mathrm{Ext}^1_R(S_2^{X_2}, S_2) = 0$).

Basically, at the passage from G_jF to $G_{j+1}F$ we kill all elements of $\mathrm{Ext}^1_R(-, S_j)$.

Proof of dual n-Bongartz lemma cont'd.

Put $G_1F = M$. Denote by X_1 the underlying set of $\mathrm{Ext}^1_R(M, S_1)$, and let G_2F be the middle term of the related short exact sequence $0 \longrightarrow S_1^{X_1} \longrightarrow G_2F \longrightarrow G_1F \longrightarrow 0$. From the long exact sequence $\mathrm{Hom}_R(S_1^{X_1}, S_1) \twoheadrightarrow \mathrm{Ext}^1_R(G_1F, S_1) \longrightarrow \mathrm{Ext}^1_R(G_2F, S_1) \longrightarrow \mathrm{Ext}^1_R(S_1^{X_1}, S_1) = 0$ we see that $\mathrm{Ext}^1_R(G_2F, S_1) = 0$.

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Basically, at the passage from G_jF to $G_{j+1}F$ we kill all elements of $\mathrm{Ext}^1_R(-, S_j)$. This creates no new elements in $\mathrm{Ext}^1_R(-, S_i)$ for $i \leqslant j$

Proof of dual n-Bongartz lemma cont'd.

Put $G_1F = M$. Denote by X_1 the underlying set of $\mathrm{Ext}^1_R(M, S_1)$, and let G_2F be the middle term of the related short exact sequence $0 \longrightarrow S_1^{X_1} \longrightarrow G_2F \longrightarrow G_1F \longrightarrow 0$. From the long exact sequence $\mathrm{Hom}_R(S_1^{X_1}, S_1) \twoheadrightarrow \mathrm{Ext}^1_R(G_1F, S_1) \longrightarrow \mathrm{Ext}^1_R(G_2F, S_1) \longrightarrow \mathrm{Ext}^1_R(S_1^{X_1}, S_1) = 0$ we see that $\mathrm{Ext}^1_R(G_2F, S_1) = 0$.

Denote by X_2 the underlying set of $\mathrm{Ext}^1_R(G_2F, S_2)$, and let G_3F be the middle term of the related short exact sequence $0 \longrightarrow S_2^{X_2} \longrightarrow G_3F \longrightarrow G_2F \longrightarrow 0$. Then $\mathrm{Ext}^1_R(G_3F, S_1) = 0$ since $\mathrm{Ext}^1_R(G_2F, S_1) = 0 = \mathrm{Ext}^1_R(S_2^{X_2}, S_1)$, and $\mathrm{Ext}^1_R(G_3F, S_2) = 0$ by construction (since $\mathrm{Ext}^1_R(S_2^{X_2}, S_2) = 0$).

Basically, at the passage from G_jF to $G_{j+1}F$ we kill all elements of $\mathrm{Ext}^1_R(-, S_j)$. This creates no new elements in $\mathrm{Ext}^1_R(-, S_i)$ for $i \leqslant j$ due to the Ext^1_R-self-orthogonality conditions imposed on the modules S_i.

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account,

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration

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$F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \dots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration
$F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that
$\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration
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Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration $F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\mathrm{Ext}^1_R(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \dots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration $F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\mathrm{Ext}_R^1(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Denoting by D' the kernel of the surjective map $F = G_{n+1}F \to G_1F = M$,

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration $F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\operatorname{Ext}^1_R(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Denoting by D' the kernel of the surjective map $F = G_{n+1}F \to G_1F = M$, we obtain a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$.

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \dots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration
$F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\operatorname{Ext}^1_R(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Denoting by D' the kernel of the surjective map $F = G_{n+1}F \to G_1F = M$, we obtain a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$. Here the R-module D' is endowed with a cofiltration G as desired,

Proof of dual n-Bongartz lemma cont'd.

Proceeding in this way until all the modules $S_1, \ldots, S_n$ have been taken into account, we construct an R-module F with an $(n+1)$-step cofiltration $F = G_{n+1}F \twoheadrightarrow G_nF \twoheadrightarrow \cdots \twoheadrightarrow G_1F \twoheadrightarrow G_0F = 0$ such that $\ker(G_{i+1}F \to G_iF) = S_i^{X_i}$ for $1 \leqslant i \leqslant n$ and $G_1F = M$. We also have $\mathrm{Ext}^1_R(F, S_i) = 0$ for all $0 \leqslant i \leqslant n$, so $F \in \mathcal{F}$.

Denoting by D' the kernel of the surjective map $F = G_{n+1}F \to G_1F = M$, we obtain a special precover sequence $0 \longrightarrow D' \longrightarrow F \longrightarrow M \longrightarrow 0$ with $D' \in \mathcal{D}$ and $F \in \mathcal{F}$. Here the R-module D' is endowed with a cofiltration G as desired, with the additional property that $G_1D' = 0$.

Proof of dual n-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N,

Proof of dual n-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$,

Proof of dual *n*-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$, so there is a short exact sequence $0 \longrightarrow N \longrightarrow S_0^{X_0} \longrightarrow M \longrightarrow 0$.

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To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$, so there is a short exact sequence $0 \longrightarrow N \longrightarrow S_0^{X_0} \longrightarrow M \longrightarrow 0$.

Argue as usually in the Salce lemma, using the pullback diagram for the pair of surjective morphisms $S_0^{X_0} \longrightarrow M$ and $F \longrightarrow M$.

Proof of dual n-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$, so there is a short exact sequence $0 \longrightarrow N \longrightarrow S_0^{X_0} \longrightarrow M \longrightarrow 0$.

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Finally, if $N \in \mathcal{C}$, then $\mathrm{Ext}^1_R(F, N) = 0$,

Proof of dual n-Bongartz lemma fin'd.

To construct a special preenvelope sequence $0 \longrightarrow N \longrightarrow D \longrightarrow F \longrightarrow 0$ (with $D \in \mathcal{D}$ and $F \in \mathcal{F}$) for an R-module N, choose a set X_0 such that N is a submodule in $S_0^{X_0} = J^{X_0}$, so there is a short exact sequence $0 \longrightarrow N \longrightarrow S_0^{X_0} \longrightarrow M \longrightarrow 0$.

Argue as usually in the Salce lemma, using the pullback diagram for the pair of surjective morphisms $S_0^{X_0} \longrightarrow M$ and $F \longrightarrow M$. Then it is clear from the short exact sequence $0 \longrightarrow D' \longrightarrow D \longrightarrow S_0^{X_0} \longrightarrow 0$ that the R-module D has a cofiltration of the desired form.

Finally, if $N \in \mathcal{C}$, then $\mathrm{Ext}^1_R(F, N) = 0$, so N is a direct summand of D. □

Cosyzygy Modules of a Cotilting Module

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Recall that the cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by the n-cotilting module U and its cosyzygy modules $\mho^1 U, \ldots, \mho^{n-1} U$.

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One observes that if $0 \longrightarrow U \longrightarrow J^0 \longrightarrow \mho^1 U \longrightarrow 0$ is a short exact sequence of R-modules with an injective R-module J^0, then $\mathrm{Ext}_R^1(\mho^1 U, \mho^1 U) \simeq \mathrm{Ext}_R^2(\mho^1 U, U) \simeq \mathrm{Ext}_R^2(J^0, U)$, and there is no apparent reason for this Ext group to vanish.

In fact, there is a counterexample due to D'Este (2005). Over any field k, she constructs a quiver algebra A of global dimension 2, with 4 vertices, 4 edges, and 2 relations.

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In fact, there is a counterexample due to D'Este (2005). Over any field k, she constructs a quiver algebra A of global dimension 2, with 4 vertices, 4 edges, and 2 relations. Then ${}_A A$ is a 2-cotilting A-module (since A is a finite-dimensional algebra of global dimension 2),

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In fact, there is a counterexample due to D'Este (2005). Over any field k, she constructs a quiver algebra A of global dimension 2, with 4 vertices, 4 edges, and 2 relations. Then ${}_A A$ is a 2-cotilting A-module (since A is a finite-dimensional algebra of global dimension 2), but $\mathrm{Ext}^1_A(\mho^1 A, \mho^1 A) \neq 0$ (for the minimal cosyzygy module $\mho^1 A$ of the free A-module A).

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One observes that if $0 \longrightarrow U \longrightarrow J^0 \longrightarrow \mho^1 U \longrightarrow 0$ is a short exact sequence of R-modules with an injective R-module J^0, then $\mathrm{Ext}^1_R(\mho^1 U, \mho^1 U) \simeq \mathrm{Ext}^2_R(\mho^1 U, U) \simeq \mathrm{Ext}^2_R(J^0, U)$, and there is no apparent reason for this Ext group to vanish.

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Therefore, the dual n-Bongartz lemma is not applicable to the sequence of cosyzygy modules $\mho^i U$ of an n-cotilting R-module U,

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Recall that the cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by the n-cotilting module U and its cosyzygy modules $\mho^1 U, \dots, \mho^{n-1} U$. One has $\operatorname{Ext}^1_R((\mho^i U)^X, U) \simeq \operatorname{Ext}^{i+1}_R(U^X, U) = 0$ for all $i \geqslant 0$ and all sets X. However, it may well happen that $\operatorname{Ext}^1_R(\mho^1 U, \mho^1 U) \neq 0$.

One observes that if $0 \longrightarrow U \longrightarrow J^0 \longrightarrow \mho^1 U \longrightarrow 0$ is a short exact sequence of R-modules with an injective R-module J^0, then $\operatorname{Ext}^1_R(\mho^1 U, \mho^1 U) \simeq \operatorname{Ext}^2_R(\mho^1 U, U) \simeq \operatorname{Ext}^2_R(J^0, U)$, and there is no apparent reason for this Ext group to vanish.

In fact, there is a counterexample due to D'Este (2005). Over any field k, she constructs a quiver algebra A of global dimension 2, with 4 vertices, 4 edges, and 2 relations. Then ${}_A A$ is a 2-cotilting A-module (since A is a finite-dimensional algebra of global dimension 2), but $\operatorname{Ext}^1_A(\mho^1 A, \mho^1 A) \neq 0$ (for the minimal cosyzygy module $\mho^1 A$ of the free A-module A).

Therefore, the dual n-Bongartz lemma is not applicable to the sequence of cosyzygy modules $\mho^i U$ of an n-cotilting R-module U, generally speaking.

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The proof of the proposition is based on a theorem and a lemma, which are formulated below.

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The proposition follows immediately from the theorem and the lemma.

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The n-cotilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ induced by an n-cotilting module U is cogenerated by the modules U, U_1, …, U_{n-1}, that is ${}^{\perp_{>0}}U = {}^{\perp_1}\{U, U_1, \dots, U_{n-1}\}$.

Proof.

For any $i, j \geqslant 0$ we have ${}^{\perp_{>i}}U_j = {}^{\perp_{>i+j}}U$, because an R-module F belongs to ${}^{\perp_{>i}}U_j$ if and only if the i-th syzygy R-module $\Omega^i F$ belongs to ${}^{\perp_{>0}}U_j$, which by the definition of U_j means that $\Omega^i F$ belongs to ${}^{\perp_{>j}}U$, which holds if and only if F belongs to ${}^{\perp_{>i+j}}U$.

In particular, it follows that ${}^{\perp_{>1}}U_j = {}^{\perp_{>j+1}}U = {}^{\perp_{>0}}U_{j+1}$.

Now one proceeds by decreasing induction in $0 \leqslant j \leqslant n$ proving that ${}^{\perp_1}\{U_j, \dots, U_n\} = {}^{\perp_{>0}}U_j$, since ${}^{\perp_1}U_j \cap {}^{\perp_1}\{U_{j+1}, \dots, U_n\} = {}^{\perp_1}U_j \cap {}^{\perp_{>0}}U_{j+1} = {}^{\perp_1}U_j \cap {}^{\perp_{>1}}U_j = {}^{\perp_{>0}}U_j$.

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Proof.

By Lemma 2, the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by the modules $S_i = U_{n-i}$.

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By Lemma 2, the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by the modules $S_i = U_{n-i}$. By Lemma 1, the assumptions of the dual n-Bongartz lemma are satisfied. Hence the theorem follows as a particular case of the dual n-Bongartz lemma. □

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In particular, an R-module is 0-tilting if and only if it is a projective generator of $R\text{-}\mathrm{Mod}$.

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Lemma (n-Bongartz lemma)

Let $\mathcal{S} = \{S_0, S_1, \dots, S_n\}$ be a collection of $n+1$ R-modules

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The proof of the proposition is based on a theorem and a lemma.

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The n-tilting cotorsion pair $(\mathcal{F}, \mathcal{C})$ induced by an n-tilting module T is generated by the modules T, T_1, $\dots$, T_{n-1}, that is $T^{\perp_{>0}} = \{T, T_1, \dots, T_{n-1}\}^{\perp_1}$.

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