

Comodule-Contramodule Correspondence

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The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality between representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)
[Feigin–Fuchs '83, Rocha-Caridi—Wallach '84, Arkhipov '96–'99, L.P. '02–'10]
- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory
[Iyengar–Krause '06, Neeman–Murfet '07–'08, L.P. '11–'14]

Maximal natural generality not found yet (maybe does not exist because the phenomenon is too general)

The **co-contraction correspondence** is known to happen for or over

- (coassociative) coalgebras over fields, curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes, separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension (= noncommutative smooth semi-separated stacks);
- corings over rings with dualizing complexes (= noncommutative semi-separated stacks with dualizing complexes);
- curved DG-rings with Gorenstein underlying graded rings (including the de Rham complexes of smooth affine varieties);
- complete Noetherian rings in the adic topology (= affine Noetherian formal schemes) with dualizing complexes;
- pro-Noetherian rings (= ind-affine ind-Noetherian ind-schemes) with dualizing complexes;

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The latter two versions are a bit different from the previous ones (in that a different kind of derived category construction is used; this is called the *semimodule-semicontramodule correspondence*).

The version for quasi-compact semi-separated schemes also differs a bit (the conventional derived category is used here).

Main ingredients:

- curved DG-structures;
- comodules and contra-modules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;
- of the second kind (exotic) better behaved for coalgebras.

Taking derived category of the second kind along algebra variables, a dualizing complex is needed for the co-contra correspondence.

The coalgebra plays the role of a dualizing complex over itself.

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d: B^i \rightarrow B^{i+1}$,
 $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all $a, b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

h is called the *curvature element*.

An A_{∞} -algebra is a graded vector space with the operations
 $m_n: A^{\otimes n} \rightarrow A[2-n]$, $n = 1, 2, \dots$

A CDG-algebra has $m_0 = h$, $m_1 = d$, and m_2 .

[Getzler–Jones '90, L.P. '93]

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B -module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B -derivation $d_M: M^i \rightarrow M^{i+1}$,
 $d_M(bm) = d_B(b)m + (-1)^{|b|}bd_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B -module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B -derivation $d_N: N^i \rightarrow N^{i+1}$,
 $d_N(nb) = d_N(n)b + (-1)^{|n|}nd_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring (B, d, h) is naturally neither a left, nor a right CDG-module over itself. But it has a natural structure of CDG-bimodule over itself.

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, \mathcal{E} is a vector bundle on M , and $\nabla_{\mathcal{E}}$ is a connection in \mathcal{E} , then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of \mathcal{E} is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$ -graded CDG-ring $(B = B^0, d = 0, h = w)$, where B^0 is an associative ring and $w \in B^0$ is a central element.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a) , where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ (the supercommutator) for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor DG-rings \longrightarrow CDG-rings is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings: CDG-modules over a CDG-ring form a DG-category. (In particular, the DG-categories of DG-modules over CDG-isomorphic DG-rings are isomorphic.)

Curved DG-structures

To be precise, the complex of morphisms $\text{Hom}_B^\bullet(K, L)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B\text{-mod}^{\text{cdg}}$ of left CDG-modules over a CDG-ring B is defined by the rules:

- $\text{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f: K \rightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|}bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\text{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{n|k|}f(d_K(k)) \in \text{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$, so $\text{Hom}_B^\bullet(K, L)$ is indeed a complex.

Replacing a CDG-ring (B, d, h) with an isomorphic CDG-ring (B, d', h') via a connection change $d'(b) = d(b) + [a, b]$ and $h' = h + d(a) + a^2$ with $a \in B^1$, one transforms the differentials in left CDG-modules M by the rule $d'(m) = d(m) + am$ to establish an isomorphism between the two DG-categories of CDG-modules.

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\text{Hot}(B\text{-mod}^{\text{cdg}}) = H^0(B\text{-mod}^{\text{cdg}})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

In particular, the conventional derived categories of DG-modules over two CDG-isomorphic DG-rings can be very different and entirely unrelated to each other.

Except in the so-called “weakly curved” case, it is generally only the derived categories of the second kind that are well-defined for CDG-modules.

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that \mathcal{A} has enough injectives).

Let $0 \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots$ be a complex in \mathcal{A} .

Then there are two spectral sequences converging to the same limit

$$\begin{aligned} 'E_2^{pq} &= R^p F(H^q C^\bullet) \implies \mathbb{H}^{p+q}(C^\bullet); \\ ''E_2^{pq} &= H^p(R^q F(C^\bullet)) \implies \mathbb{H}^{p+q}(C^\bullet). \end{aligned}$$

For unbounded complexes C^\bullet , the two spectral sequences converge (perhaps in some weak sense) to *two different limits*. The same problem occurs for (even totally finite-dimensional) DG-modules.

Hence [differential derived functors of the first](#) and the [second kind](#) [Eilenberg–Moore '62 — Husemoller–Moore–Stasheff '74].

Derived categories of the first and second kind

Classical homological algebra

Let \mathcal{A} be an abelian category with enough projectives and injectives. Then the derived category of complexes over \mathcal{A} bounded above or below can be alternatively described as

- $D^+(\mathcal{A}) = \text{Hot}^+(\mathcal{A})/\text{Acycl}^+(\mathcal{A}) \simeq \text{Hot}^+(\mathcal{A}_{\text{inj}})$;
- $D^-(\mathcal{A}) = \text{Hot}^-(\mathcal{A})/\text{Acycl}^-(\mathcal{A}) \simeq \text{Hot}^-(\mathcal{A}_{\text{proj}})$.

Not true for unbounded complexes.

Example: let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (the ring of dual numbers) over a field k . Then

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

is an unbounded complex of projective, injective Λ -modules. It is acyclic, but not contractible.

Derived Categories of the First and Second Kind

The complex

$$\dots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \dots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not “projective” or “injective” (not suitable for computing the derived functors)
derived category of the first kind (conventional)
- “projective” and/or “injective” (adjusted for computing derived functors), representing a nontrivial object in the derived category
derived category of the second kind (exotic)

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, ... '60s–]

Classical homological algebra can be defined as encompassing all the settings in which there is no difference between the theories of the first and of the second kind.

This includes

- bounded or unbounded complexes over abelian or exact categories of finite homological dimension;
- appropriately bounded above or below complexes over arbitrary abelian or exact categories;

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^0 A^i$, $d: A^i \rightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d: A^i \rightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

In most other situations (including, e.g., DG-modules over the de Rham DG-algebra or the standard cohomological complex of a Lie algebra, etc.) one has to choose between derived categories of the first and of the second kind.

Sometimes one wants to use their mixtures—the semiderived categories.

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described
(being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions
(homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described
(depending only on the underlying graded module structure, irrespective of the differentials on complexes)
- complicated descriptions of equivalence relations on complexes
(more delicate than the conventional quasi-isomorphism)

[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, . . . '98 –]

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Warning: derived category of the second kind comes in several versions. The largest one is called the **absolute derived** category. The two most important definitions, dual to each other, are called the **coderived** and the **contraderived** category.

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring. Suppose

$0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ is a short exact sequence (in the abelian category) of left CDG-modules over B :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \xrightarrow{d} & K^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \partial & & \downarrow & & \\
 \cdots & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \xrightarrow{d} & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \partial & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

Form the total CDG-module $\text{Tot}(K \rightarrow L \rightarrow M)$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.

Coderived and Contraderived Categories of CDG-modules

The totalization $\text{Tot}(K \rightarrow L \rightarrow M)$ of a short exact sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 \pm (\partial d - d\partial) + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be **absolutely acyclic** if it belongs to the minimal thick subcategory of the homotopy category $\text{Hot}(B\text{-mod}^{\text{cdg}})$ containing the CDG-modules $\text{Tot}(K \rightarrow L \rightarrow M)$ for all the short exact sequences $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$:

$$\text{Acycl}^{\text{abs}}(B\text{-mod}^{\text{cdg}}) = \langle \text{Tot}(K \rightarrow L \rightarrow M) \rangle \subset \text{Hot}(B\text{-mod}^{\text{cdg}}).$$

The triangulated quotient category

$$D^{\text{abs}}(B\text{-mod}^{\text{cdg}}) = \text{Hot}(B\text{-mod}^{\text{cdg}}) / \text{Acycl}^{\text{abs}}(B\text{-mod}^{\text{cdg}})$$

is called the **absolute derived category** of left CDG-modules over B .

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called **coacyclic** if it belongs to the minimal triangulated subcategory of the homotopy category $\text{Hot}(B\text{-mod}^{\text{cdg}})$ containing the CDG-modules $\text{Tot}(K \rightarrow L \rightarrow M)$ and closed under infinite direct sums:

$$\text{Acycl}^{\text{co}}(B\text{-mod}^{\text{cdg}}) = \langle \text{Tot}(K \rightarrow L \rightarrow M) \rangle_{\oplus} \subset \text{Hot}(B\text{-mod}^{\text{cdg}}).$$

A left CDG-module over B is called **contraacyclic** if it belongs to the minimal triangulated subcategory of $\text{Hot}(B\text{-mod}^{\text{cdg}})$ containing the CDG-modules $\text{Tot}(K \rightarrow L \rightarrow M)$ and closed under infinite products:

$$\text{Acycl}^{\text{ctr}}(B\text{-mod}^{\text{cdg}}) = \langle \text{Tot}(K \rightarrow L \rightarrow M) \rangle_{\Pi} \subset \text{Hot}(B\text{-mod}^{\text{cdg}}).$$

Coderived and contraderived categories

The triangulated quotient category

$$D^{\text{co}}(B\text{-mod}^{\text{cdg}}) = \text{Hot}(B\text{-mod}^{\text{cdg}})/\text{Acycl}^{\text{co}}(B\text{-mod}^{\text{cdg}})$$

is called the coderived category of left CDG-modules over B .

The quotient category

$$D^{\text{ctr}}(B\text{-mod}^{\text{cdg}}) = \text{Hot}(B\text{-mod}^{\text{cdg}})/\text{Acycl}^{\text{ctr}}(B\text{-mod}^{\text{cdg}})$$

is called the contraderived category of left CDG-modules over B .

The absolute derived category is perfectly well-defined for any (Quillen) exact category, and in fact even for an “exact DG-category” (like that of CDG-modules).

The coderived (respectively, contraderived) category is defined for any exact (DG-) category with exact functors of infinite direct sum (resp., exact functors of infinite product).

For unbounded complexes of modules already, these categories can differ from the conventional derived category *and from each other*.

Coderived and contraderived categories

Example 1: the acyclic complex $\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$ of modules over the algebra of dual numbers $\Lambda = k[\varepsilon]/(\varepsilon^2)$ is neither coacyclic, nor contraacyclic.

Let us decompose this complex in two halves. The acyclic complex of Λ -modules

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \twoheadrightarrow k \rightarrow 0$$

is contraacyclic, but not coacyclic.

The acyclic complex of Λ -modules

$$0 \rightarrow k \rightarrow \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

is coacyclic, but not contraacyclic.

Coderived and contraderived categories

Example 2: let \mathfrak{g} be a finite-dimensional reductive Lie algebra (e.g., one-dimensional abelian, or \mathfrak{sl}_n , etc.) over a field k of characteristic zero. Let M be a finite-dimensional irreducible \mathfrak{g} -module.

Then the standard cohomological (Chevalley–Eilenberg cochain) complex $C^\bullet(\mathfrak{g}) = C^\bullet(\mathfrak{g}, k)$ is a finite-dimensional DG-algebra over k , and the complex $C^\bullet(\mathfrak{g}, M)$ is a finite-dimensional DG-module over $C^\bullet(\mathfrak{g})$.

When an irreducible \mathfrak{g} -module M is nontrivial, the DG-module $C^\bullet(\mathfrak{g}, M)$ over the DG-algebra $C^\bullet(\mathfrak{g})$ is acyclic. But it is neither coacyclic, nor contraacyclic.

In fact, $C^\bullet(\mathfrak{g}, M)$ does not belong to the minimal triangulated subcategory of the homotopy category of DG-modules $\text{Hot}(C^\bullet(\mathfrak{g})\text{-mod}^{\text{dg}})$ containing the total DG-modules $\text{Tot}(K \rightarrow L \rightarrow M)$ of all the short exact sequences of DG-modules over $C^\bullet(\mathfrak{g})$ and closed under *both* the infinite direct sums and infinite products.

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more “comodule-like” abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or “smooth” modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs, modules of the Bernstein–Gelfand–Gelfand category “ \mathcal{O} ”;
- sheaves generally, or at least certainly quasi-coherent sheaves on schemes or algebraic stacks.

Every category of comodules is typically accompanied by a closely related (but much less familiar) category of contramodules.

Comodules and Contramodules

While comodules are “torsion” modules, contramodules are defined as modules with **infinite summation operations** and feel like being in some sense “complete”.

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in “naturally appearing” abelian categories.

The derived comodule-contramodule correspondences are covariant equivalences of (exact or triangulated) triangulated categories.

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* \mathcal{C} over A is

- an A - A -bimodule endowed with
- a comultiplication map $\mu: \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon: \mathcal{C} \longrightarrow A$,
- which must be morphisms of A - A -bimodules
- satisfying the coassociativity equation

$$(\mu \otimes \text{id}) \circ \mu = (\text{id} \otimes \mu) \circ \mu$$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

- and the counity equations $(\varepsilon \otimes \text{id}) \circ \mu = \text{id}_{\mathcal{C}} = (\text{id} \otimes \varepsilon) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C}.$$

A *coalgebra* over a commutative ring A (most typically over a field) is a coring whose left and right A -module structures coincide.

Comodules and contramodules over corings

Let \mathcal{C} be a coring over a ring A . A left \mathcal{C} -comodule \mathcal{M} is

- a left A -module endowed with
- a coaction map $\nu: \mathcal{M} \rightarrow \mathcal{C} \otimes_A \mathcal{M}$,
- which must be a morphism of left A -modules
- satisfying the coassociativity equation
$$(\mu \otimes \text{id}) \circ \nu = (\text{id} \otimes \nu) \circ \nu$$

$$\mathcal{M} \rightarrow \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$$

- and the counity equation $(\varepsilon \otimes \text{id}) \circ \nu = \text{id}_{\mathcal{M}}$

$$\mathcal{M} \rightarrow \mathcal{C} \otimes_A \mathcal{M} \rightarrow \mathcal{M}.$$

A right \mathcal{C} -comodule \mathcal{N} is a right A -module endowed with a coaction map $\mathcal{N} \rightarrow \mathcal{N} \otimes_A \mathcal{C}$ satisfying the similar conditions.

Comodules and contramodules over corings

Let \mathcal{C} be a coring over a ring A . A *left \mathcal{C} -contramodule* \mathfrak{P} is

- a left A -module endowed with
- a contraaction map $\pi: \text{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A -modules
- satisfying the contraassociativity equation
$$\pi \circ \text{Hom}(\mu, \text{id}) = \pi \circ \text{Hom}(\text{id}, \pi)$$

$$\text{Hom}_A(\mathcal{C} \otimes_A \mathcal{C}, \mathfrak{P}) \simeq$$

$$\text{Hom}_A(\mathcal{C}, \text{Hom}_A(\mathcal{C}, \mathfrak{P})) \rightrightarrows \text{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

- and the counity equation $\pi \circ \text{Hom}(\varepsilon, \text{id}) = \text{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \text{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}.$$

[Eilenberg–Moore '65] (almost forgotten between 1970–2000)

Comodules and contramodules over corings

Example: let \mathcal{N} be a right \mathcal{C} -comodule endowed with a left action of a ring B by right \mathcal{C} -comodule endomorphisms. Let U be a left B -module. Then the left A -module $\text{Hom}_B(\mathcal{N}, U)$ has a natural left \mathcal{C} -contramodule structure:

$$\text{Hom}_A(\mathcal{C}, \text{Hom}_B(\mathcal{N}, U)) \simeq \text{Hom}_B(\mathcal{N} \otimes_A \mathcal{C}, U) \xrightarrow{\nu^*} \text{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k . Then the structure of a left B -module on a k -vector space L can be defined alternatively as a map $B \otimes_k L \rightarrow L$ or a map $L \rightarrow \text{Hom}_k(B, L)$.

For a coalgebra \mathcal{C} over k , the datum of a map $\mathcal{M} \rightarrow \mathcal{C} \otimes_k \mathcal{M}$ is quite different from that of a map $\text{Hom}_k(\mathcal{C}, \mathfrak{F}) \rightarrow \mathfrak{F}$.

Comodules and contra-modules over corings

Let \mathcal{C} be a coring over a ring A . Then the category \mathcal{C} -comod of left \mathcal{C} -comodules is abelian provided that \mathcal{C} is a flat right A -module. In this case, \mathcal{C} -comod is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and (consequently) enough injectives.

The injective left \mathcal{C} -comodules are the direct summands of the \mathcal{C} -comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A -modules J .

The category \mathcal{C} -contra of left \mathcal{C} -contra-modules is abelian provided that \mathcal{C} is a projective left A -module. In this case, the category \mathcal{C} -contra has exact functors of infinite product and enough projectives.

The projective left \mathcal{C} -contra-modules are the direct summands of the \mathcal{C} -contra-modules $\text{Hom}_A(\mathcal{C}, F)$ induced from free (or projective) left A -modules F .

Comodules and contramodules over coalgebras (over fields)

Example: let \mathcal{C} be the coalgebra over a field k whose dual topological algebra \mathcal{C}^* is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, \mathcal{C} is a k -vector space with the basis $1^*, t^*, t^{2*}, t^{3*}, \dots$ endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1, \varepsilon(t^{n*}) = 0$ for $n \geq 1$.

Then a \mathcal{C} -comodule \mathcal{M} is

- a k -vector space with a linear operator $t: \mathcal{M} \rightarrow \mathcal{M}$
- which must be locally nilpotent, i.e., for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Comodules and contramodules over coalgebras (over fields)

Example: \mathcal{C} is a k -coalgebra with a basis $1^*, t^*, t^{2*}, t^{3*}, \dots$ and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A \mathcal{C} -contramodule \mathfrak{P} is

- a k -vector space endowed with an infinite summation operation assigning to any sequence of elements $p_0, p_1, p_2, \dots \in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n (ap_n + bq_n) = a \sum_{n=0}^{\infty} t^n p_n + b \sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geq 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n \sum_{i+j=n} p_{ij}.$$

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let \mathcal{C} be the k -coalgebra with a basis $1^*, t^*, t^{2*}, t^{3*}, \dots$ and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any \mathcal{C} -contramodule \mathfrak{P} , an element $p \in \mathfrak{P}$, and an integer $n \geq 0$, one can define

$$t^n p = 1 \cdot 0 + \dots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \dots \in \mathfrak{P}.$$

Then there exists a \mathcal{C} -contramodule \mathfrak{P} and a sequence of elements $p_0, p_1, p_2 \dots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geq 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geq 0$, so the t -adic topology on \mathfrak{P} is not separated.

Co-contra correspondence for coalgebras over fields

Let \mathcal{C} be a coalgebra over a field k . Then the injective objects of the category of \mathcal{C} -comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced \mathcal{C} -comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of \mathcal{C} -contra modules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced \mathcal{C} -contra modules* $\text{Hom}_k(\mathcal{C}, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left \mathcal{C} -comodules and induced left \mathcal{C} -contra modules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\text{Hom}_k(\mathcal{C}, U)$ and back:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k U, \mathcal{C} \otimes_k V) &\simeq \text{Hom}_k(\mathcal{C} \otimes_k U, V) \simeq \\ &\text{Hom}_k(U, \text{Hom}_k(\mathcal{C}, V)) \simeq \text{Hom}^{\mathcal{C}}(\text{Hom}_k(\mathcal{C}, U), \text{Hom}_k(\mathcal{C}, V)). \end{aligned}$$

This generalizes to comodules and contra modules over any coring, and is a particular case of the abstract-categorical *equivalence of Kleisli categories* for an adjoint monad and comonad [connection noticed by Böhm–Brzeziński–Wisbauer '09].

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\text{inj}} \simeq \mathcal{C}\text{-contra}_{\text{proj}}$$

$$\Psi_{\mathcal{C}}: \mathcal{C}\text{-comod}_{\text{inj}} \simeq \mathcal{C}\text{-contra}_{\text{proj}} : \Phi_{\mathcal{C}}$$

for any coassociative coalgebra \mathcal{C} over a field k .

Theorem

For any coassociative coalgebra \mathcal{C} over k , the natural functors induce equivalences of triangulated categories

- $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \simeq D^{\text{co}}(\mathcal{C}\text{-comod})$;
- $\text{Hot}(\mathcal{C}\text{-contra}_{\text{proj}}) \simeq D^{\text{ctr}}(\mathcal{C}\text{-contra})$.

Corollary

For any coassociative coalgebra \mathcal{C} over a field k , there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}}: D^{\text{co}}(\mathcal{C}\text{-comod}) \simeq D^{\text{ctr}}(\mathcal{C}\text{-contra}) : \mathbb{L}\Phi_{\mathcal{C}}. \quad \square$$

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings \mathcal{C} over associative rings A of finite homological dimension;
- all corings \mathcal{C} over Gorenstein associative rings A (i.e., such that the classes of left A -modules of finite projective dimension and of finite injective dimension coincide)

assuming only that \mathcal{C} is a projective left and a flat right A -module (to make the categories \mathcal{C} -comod and \mathcal{C} -contra abelian).

There are further generalizations to

- corings over rings with dualizing complexes, endowed a lifting of the ring's dualizing complex to (a complex of bicomodules over) the coring;
- corings \mathcal{C} that are only flat left and right A -modules, when one has to restrict the class of contra modules under consideration.

Co-contra correspondence for coalgebras over fields

Example: let \mathcal{C} be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \dots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \text{Sym}(W)$ is the symmetric coalgebra of a vector space W such that x_1, \dots, x_m is a basis in W^* .

Consider the one-dimensional trivial \mathcal{C} -comodule k ; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$. Have a right injective (Koszul) resolution

$$0 \longrightarrow k \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \bigwedge_k^2 W \longrightarrow \dots$$

in the category of \mathcal{C} -comodules. Applying $\Psi_{\mathcal{C}}$, obtain the complex of projective \mathcal{C} -contramodules

$$\begin{aligned} \text{Hom}_k(\mathcal{C}, k) &\longrightarrow \text{Hom}_k(\mathcal{C}, W) \longrightarrow \text{Hom}_k(\mathcal{C}, \bigwedge_k^2 W) \\ &\longrightarrow \dots \longrightarrow \text{Hom}_k(\mathcal{C}, \bigwedge_k^{m-1} W) \longrightarrow \text{Hom}_k(\mathcal{C}, \bigwedge_k^m W). \end{aligned}$$

This is a left projective \mathcal{C} -contramodule resolution of the one-dimensional trivial \mathcal{C} -contramodule $\bigwedge_k^m W$ placed at the cohomological degree m . So we get $\mathbb{R}\Psi_{\mathcal{C}}(k) \simeq k[-m]$.

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k -vector space. Then our Koszul resolution of the \mathcal{C} -comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$. Thus the infinite complex of \mathcal{C} -contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \bigwedge_k^2 W) \longrightarrow \dots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $D^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

This complex is known as “a left projective resolution of the one-dimensional trivial \mathcal{C} -contramodule $\bigwedge_k^{\infty} W$ placed at the cohomological degree $+\infty$ ”.

This phenomenon appears in the representation theory of infinite-dimensional Lie algebras such as the Virasoro [B. Feigin, mid-'80s].

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k -coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon: \mathcal{C}^0 \longrightarrow k$
- endowed with an odd coderivation $d: \mathcal{C} \longrightarrow \mathcal{C}$ with the components $d^i: \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a curvature linear function $h: \mathcal{C}^{-2} \longrightarrow k$
- satisfying the equations dual to those of a CDG-algebra over k .

The graded dual vector space $\mathcal{C}^* = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^{-i*}$ to a CDG-coalgebra over k is a CDG-algebra.

Co-contr correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left \mathcal{C} -comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components
$$\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$$
- endowed with a $d_{\mathcal{C}}$ -coderivation $d_{\mathcal{M}}: \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d_{\mathcal{M}}^i: \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$
- satisfying the equations dual to those of a CDG-module over a CDG-algebra over k ,
- i.e., in particular, the operator $d_{\mathcal{M}}^2: \mathcal{M} \longrightarrow \mathcal{M}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in \mathcal{M} .

Any graded left \mathcal{C} -comodule has a natural (induced) structure of a graded left \mathcal{C}^* -module. The mentioned equations can be rewritten as saying that $(\mathcal{M}, d_{\mathcal{M}})$ is a CDG-module over \mathcal{C}^* .

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left \mathcal{C} -contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraction map π with the components $\prod_{j-i=n} \text{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \rightarrow \mathfrak{P}^n$
- endowed with a $d_{\mathcal{C}}$ -contraderivation $d_{\mathfrak{P}}: \mathfrak{P} \rightarrow \mathfrak{P}$ with the components $d_{\mathfrak{P}}^i: \mathfrak{P}^i \rightarrow \mathfrak{P}^{i+1}$
- satisfying the equations similar to those of a CDG-module over a CDG-algebra over k ,
- i.e., in particular, the operator $d_{\mathfrak{P}}^2: \mathfrak{P} \rightarrow \mathfrak{P}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in \mathfrak{P} .

Any graded left \mathcal{C} -contramodule has a natural (underlying) structure of a graded left \mathcal{C}^* -module. The mentioned equations can be rewritten as saying that $(\mathfrak{P}, d_{\mathfrak{P}})$ is a CDG-module over \mathcal{C}^* .

Co-contr correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\text{cdg}}$ and $\mathcal{C}\text{-contra}^{\text{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $D^{\text{co}}(\mathcal{C}\text{-comod}^{\text{cdg}})$ and $D^{\text{ctr}}(\mathcal{C}\text{-contra}^{\text{cdg}})$.

Denote by $\mathcal{C}\text{-comod}_{\text{inj}}^{\text{cdg}} \subset \mathcal{C}\text{-comod}^{\text{cdg}}$ and $\mathcal{C}\text{-contra}_{\text{proj}}^{\text{cdg}} \subset \mathcal{C}\text{-contra}^{\text{cdg}}$ the DG-subcategories of CDG-comodules with injective underlying graded \mathcal{C} -comodules and CDG-contramodules with projective underlying graded \mathcal{C} -contramodules.

The homotopy (H^0) categories of these DG-subcategories are denoted by $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}^{\text{cdg}})$ and $\text{Hot}(\mathcal{C}\text{-contra}_{\text{proj}}^{\text{cdg}})$, as usually.

Co-contr correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k , the natural functors induce equivalences of triangulated categories

- $\text{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}^{\text{cdg}}) \simeq \text{D}^{\text{co}}(\mathcal{C}\text{-comod}^{\text{cdg}});$
- $\text{Hot}(\mathcal{C}\text{-contra}_{\text{proj}}^{\text{cdg}}) \simeq \text{D}^{\text{ctr}}(\mathcal{C}\text{-contra}^{\text{cdg}}).$

Corollary

For any CDG-coalgebra \mathcal{C} over a field k , there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}} : \text{D}^{\text{co}}(\mathcal{C}\text{-comod}^{\text{cdg}}) \simeq \text{D}^{\text{ctr}}(\mathcal{C}\text{-contra}^{\text{cdg}}) : \mathbb{L}\Phi_{\mathcal{C}}.$$

Co-contr correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}: \mathcal{C}\text{-comod}^{\text{cdg}} \rightarrow \mathcal{C}\text{-contra}^{\text{cdg}}$ and $\Phi_{\mathcal{C}}: \mathcal{C}\text{-contra}^{\text{cdg}} \rightarrow \mathcal{C}\text{-comod}^{\text{cdg}}$ between the whole abelian DG-categories of left CDG-comodules and CDG-contramodules over \mathcal{C} .

The right adjoint functor $\Psi_{\mathcal{C}}$ is simply

$$\Psi_{\mathcal{C}}(\mathcal{M}) = \text{Hom}_{\mathcal{C}}(\mathcal{C}, -),$$

while the left adjoint functor $\Phi_{\mathcal{C}}$ is the *contratensor product*

$$\Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}.$$

One restricts these functors to graded-injective CDG-comodules and graded-projective CDG-contramodules in order to construct the derived functors.

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R :

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R ;
- the functor $X \mapsto R[X]$ is a monad on the category of sets
- with the “parentheses opening” map $\phi_X: R[R[X]] \rightarrow R[X]$
- and the “point measure” map $\varepsilon_X: X \rightarrow R[X]$;
- define left R -modules as algebras/modules over this monad on Sets, that is
- a left R -module M is a set
- endowed with a map of sets $m: R[M] \rightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

- and the unity equation $m \circ \varepsilon_X = \text{id}_M$

$$M \longrightarrow R[M] \longrightarrow M.$$

Contramodules over topological rings

Let \mathfrak{R} be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X , denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of \mathfrak{R} , i.e., for any neighborhood of zero $\mathcal{U} \subset \mathfrak{R}$ the set $\{x \mid r_x \notin \mathcal{U}\}$ must be finite.

It follows from the conditions on the topology of \mathfrak{R} that there is a well-defined “parentheses opening” map

$$\phi_X: \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of \mathfrak{R} to compute the coefficients. There is also the obvious “point measure” map $\varepsilon_X: X \longrightarrow \mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X \longmapsto \mathfrak{R}[[X]]: \text{Sets} \longrightarrow \text{Sets}$.

Contramodules over Topological Rings

Let \mathfrak{K} be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A **left contramodule over the topological ring** \mathfrak{K} is an algebra/module over the monad $X \mapsto \mathfrak{K}[[X]]$ on Sets, that is

- a set \mathfrak{P}
- endowed with a contraaction map $\pi: \mathfrak{K}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{K}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{K}[[\mathfrak{K}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{K}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \text{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{K}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}.$$

The composition of the contraaction map $\pi: \mathfrak{K}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{K}[\mathfrak{P}] \longrightarrow \mathfrak{K}[[\mathfrak{P}]]$ defines the underlying left \mathfrak{K} -module structure on every left \mathfrak{K} -contramodule.

Contramodules over topological rings

Let \mathfrak{R} be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left \mathfrak{R} -contramodules is abelian with exact functors of infinite products and enough projectives (which are the direct summands of the free \mathfrak{R} -contramodules $\mathfrak{R}[[X]]$).

The forgetful functor $\mathfrak{R}\text{-contra} \rightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

A right \mathfrak{R} -module \mathcal{N} is called *discrete* if the action map $\mathcal{N} \times \mathfrak{R} \rightarrow \mathcal{N}$ is continuous in the given topology of \mathfrak{R} and the discrete topology of \mathcal{N} , i.e., if the annihilator of any element of \mathcal{N} is open in \mathfrak{R} .

For any discrete right \mathfrak{R} -module \mathcal{N} and any abelian group U , the left \mathfrak{R} -module $\text{Hom}_{\mathbb{Z}}(\mathcal{N}, U)$ has a natural left \mathfrak{R} -contramodule structure.

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ -adic integers. A discrete \mathbb{Z}_ℓ -module is just an ℓ^∞ -torsion abelian group.

A \mathbb{Z}_ℓ -contramodule \mathfrak{P} is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements $p_0, p_1, p_2, \dots \in \mathfrak{P}$ an element denoted by $\sum_{n=0}^{\infty} \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} \ell^n (ap_n + bq_n) = a \sum_{n=0}^{\infty} \ell^n p_n + b \sum_{n=0}^{\infty} \ell^n q_n,$$

- unitality + compatibility: $\sum_{n=0}^{\infty} \ell^n p_n = p_0 + \ell p_1$ when $p_i = 0$ for all $i \geq 2$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} \ell^i \sum_{j=0}^{\infty} \ell^j p_{ij} = \sum_{n=0}^{\infty} \ell^n \sum_{i+j=n} p_{ij}.$$

Contramodules over topological rings

Nakayama's lemma: let \mathfrak{R} be a topological ring (complete and separated, with open right ideals forming a base of neighborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that is topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geq 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let \mathfrak{P} be a nonzero left \mathfrak{R} -contramodule. Then the contraction map $\mathfrak{m}[[\mathfrak{P}]] \rightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \dots, s_m \in R$, and let $\mathfrak{R} = \widehat{R}_I$ be the I -adic completion of R (endowed with the I -adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \rightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that $\text{Ext}_R^*(R[s_i^{-1}], P) = 0$ for all $i = 1, \dots, m$.

In particular, \mathbb{Z}_ℓ -contramodules = weakly ℓ -complete (Ext- ℓ -complete) abelian groups [Bousfield–Kan '72, Jannsen '88].

Classical (contravariant) Serre–Grothendieck duality (affine case)

Let R be a commutative Noetherian ring. Let $D^b(R\text{-mod}^{\text{fg}})$ denote the bounded derived category of finitely generated R -modules.

A bounded complex of injective R -modules D_R^\bullet with finitely generated cohomology R -modules is called a *dualizing complex* for R if the natural map $R \rightarrow \text{Hom}_R(D_R^\bullet, D_R^\bullet)$ is a quasi-isomorphism.

Let $R \rightarrow S$ be a surjective homomorphism of commutative Noetherian rings and D_R^\bullet be a dualizing complex for the ring R . Then the maximal subcomplex of S -modules $\text{Hom}_R(S, D_R^\bullet)$ in D_R^\bullet is a dualizing complex for the ring S .

The following is a classical result [Hartshorne '66]: for any commutative Noetherian ring R with a dualizing complex D_R^\bullet , the functor $\text{Hom}_R(-, D_R^\bullet)$ is an involutive auto-anti-equivalence of the derived category $D^b(R\text{-mod}^{\text{fg}})$:

$$D^b(R\text{-mod}^{\text{fg}})^{\text{op}} \simeq D^b(R\text{-mod}^{\text{fg}}).$$

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\text{Hot}(R\text{-mod}_{\text{inj}}) \simeq D^{\text{co}}(R\text{-mod})$;
- $\text{Hot}(R\text{-mod}_{\text{proj}}) \simeq D^{\text{abs}}(R\text{-mod}_{\text{flat}}) \simeq D^{\text{ctr}}(R\text{-mod})$.

Corollary

The choice of a dualizing complex D_R^\bullet for a Noetherian commutative ring R induces an equivalence of triangulated categories $\mathbb{R}\Psi_{D_R^\bullet} : D^{\text{co}}(R\text{-mod}) \simeq D^{\text{ctr}}(R\text{-mod}) : \mathbb{L}\Phi_{D_R^\bullet}$.

Here the functors to be derived are $\Psi_{D_R^\bullet}(M^\bullet) = \text{Hom}_R(D_R^\bullet, M^\bullet)$ and $\Phi_{D_R^\bullet}(P^\bullet) = D_R^\bullet \otimes_R P^\bullet$.

[Jørgensen, Krause, Iyengar–Krause '05–'06]

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any \mathfrak{R} -contramodule \mathfrak{P} , denote by $\overline{\mathfrak{P}}_n$ the maximal quotient \mathfrak{R} -contramodule of \mathfrak{P} whose \mathfrak{R} -contramodule structure comes from an R_n -module structure. An \mathfrak{R} -contramodule \mathfrak{P} is called *flat* if

- the R_n -module $\overline{\mathfrak{P}}_n$ is flat for every $n \geq 0$,
- and the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

The class $\mathfrak{R}\text{-contra}_{\text{flat}}$ of flat \mathfrak{R} -contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-contra}$, so in particular $\mathfrak{R}\text{-contra}_{\text{flat}}$ inherits an exact category structure from $\mathfrak{R}\text{-contra}$.

Denote by $\mathfrak{R}\text{-discr}$ the abelian category of discrete \mathfrak{R} -modules.

Co-contr correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\text{Hot}(\mathfrak{R}\text{-discr}_{\text{inj}}) \simeq D^{\text{co}}(\mathfrak{R}\text{-discr});$
- $D^{\text{ctr}}(\mathfrak{R}\text{-contra}_{\text{flat}}) \simeq D^{\text{ctr}}(\mathfrak{R}\text{-contra}).$

When the Krull dimensions of the rings R_n are uniformly bounded, one has $\text{Hot}(\mathfrak{R}\text{-contra}_{\text{proj}}) \simeq D^{\text{abs}}(\mathfrak{R}\text{-contra}_{\text{flat}}) \simeq D^{\text{ctr}}(\mathfrak{R}\text{-contra})$. This is not necessary for the following corollary.

Corollary

Any compatible system $\mathcal{D}_{\mathfrak{R}}^{\bullet}$ of choices of dualizing complexes $D_{R_n}^{\bullet}$ for the Noetherian rings R_n , $n \geq 0$, induces an equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{D}_{\mathfrak{R}}^{\bullet}} : D^{\text{co}}(\mathfrak{R}\text{-discr}) \simeq D^{\text{ctr}}(\mathfrak{R}\text{-contra}) : \mathbb{L}\Phi_{\mathcal{D}_{\mathfrak{R}}^{\bullet}}.$$

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