Leonid Positselski - Moscow

Expanded version of presentation given in Třešť, Czech Republic, on April 12, 2014

April 15, 2014

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The coalgebra plays the role of a dualizing complex over itself.



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[Getzler-Jones '90, L.P. '93]



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- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$ -graded CDG-ring ($B=B^0,\ d=0,\ h=w$), where B^0 is an associative ring and $w\in B^0$ is a central element.

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Except in the so-called "weakly curved" case, it is generally only the derived categories of the second kind that are well-defined for CDG-modules.

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Then there are two spectral sequences converging to the same limit

$${}^{\prime}E_{2}^{pq} = R^{p}F(H^{q}C^{\bullet}) \Longrightarrow \mathbb{H}^{p+q}(C^{\bullet});$$
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Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

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[Grothendieck, Verdier, Deligne, . . . '60s –]

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Sometimes one wants to use their mixtures—the semiderived categories.

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[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, . . . '98 –]



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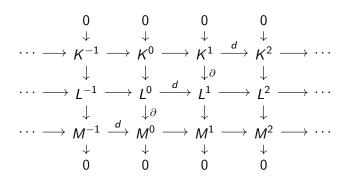
In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Warning: derived category of the second kind comes in several versions. The largest one is called the absolute derived category. The two most important definitions, dual to each other, are called the coderived and the contraderived category.

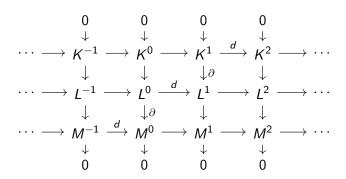
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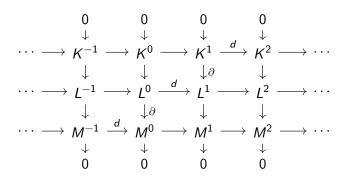


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Form the total CDG-module $\mathsf{Tot}(K \to L \to M)$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.

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For unbounded complexes of modules already, these categories can differ from the conventional derived category and from each other.

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Every category of comodules is typically accompanied by a closely related (but much less familiar) category of contramodules.



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The derived comodule-contramodule correspondences are covariant equivalences of triangulated categories.

Let A be an associative ring (with unit). A coring C over A is

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A coalgebra over a commutative ring A (most typically over a field) is a coring whose left and right A-module structures coincide.

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A right \mathcal{C} -comodule \mathcal{N} is a right A-module endowed with a coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_{\mathcal{A}} \mathcal{C}$ satisfying the similar conditions.



Let $\mathcal C$ be a coring over a ring A. A left $\mathcal C$ -contramodule $\mathfrak P$ is

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[Eilenberg-Moore '65]



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[Eilenberg-Moore '65] (almost forgotten between 1970-2000)

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In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \ge 0$, so the *t*-adic topology on \mathfrak{P} is not separated.



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The graded dual vector space $C^* = \bigoplus_{i=-\infty}^{\infty} C^{-i*}$ to a CDG-coalgebra over k is a CDG-algebra.



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Corollary

For any CDG-coalgebra C over a field k, there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}}\colon \mathrm{D^{co}}(\mathcal{C}\text{-}\mathrm{comod}^{\mathrm{cdg}})\simeq \mathrm{D^{ctr}}(\mathcal{C}\text{-}\mathrm{contra}^{\mathrm{cdg}}):\mathbb{L}\Phi_{\mathcal{C}}.$$

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}} \colon \mathcal{C}\text{-}\mathrm{comod}^\mathrm{cdg} \longrightarrow \mathcal{C}\text{-}\mathrm{contra}^\mathrm{cdg}$ and $\Phi_{\mathcal{C}} \colon \mathcal{C}\text{-}\mathrm{contra}^\mathrm{cdg} \longrightarrow \mathcal{C}\text{-}\mathrm{comod}^\mathrm{cdg}$

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One restricts these functors to graded-injective CDG-comodules and graded-projective CDG-contramodules in order to construct the derived functors.



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- to any set X one assigns the set R[X] of all formal linear combinations of elements of X with coefficients in R;
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- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m: R[M] \longrightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

• and the unity equation $m \circ \varepsilon_X = id_M$

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performing infinite summations in the conventional sense of the topology of \mathfrak{R} to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X \colon X \longrightarrow \mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X \longmapsto \mathfrak{R}[[X]] \colon \operatorname{Sets} \longrightarrow \operatorname{Sets}$.

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The composition of the contraaction map $\pi \colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left \mathfrak{R} -module structure on every left \mathfrak{R} -contramodule.

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For any discrete right \mathfrak{R} -module \mathcal{N} and any abelian group U, the left \mathfrak{R} -module $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{N},U)$ has a natural left \mathfrak{R} -contramodule structure.

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[Jørgensen, Krause, Iyengar-Krause '05-'06]



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Denote by $\Re\text{-discr}$ the abelian category of discrete $\Re\text{-modules}.$



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