Comodule-Contramodule Correspondence

Leonid Positselski – Moscow

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The coalgebra plays the role of a dualizing complex over itself.

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[Getzler–Jones '90, L.P. '93]

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A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N\colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring (B, d, h) is naturally neither a left, nor a right CDG-module over itself. But it has a natural structure of CDG-bimodule over itself.

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- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$-graded CDG-ring $(B = B^0,\ d = 0,\ h = w)$, where B^0 is an associative ring and $w \in B^0$ is a central element.

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Except in the so-called "weakly curved" case, it is generally only the derived categories of the second kind that are well-defined for CDG-modules.

Derived categories of the first and second kind

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two hypercohomology spectral sequences

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Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^\bullet) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet);$$
$$''E_2^{pq} = H^p(R^qF(C^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet).$$

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For unbounded complexes $C^\bullet$, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*.

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two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^\bullet) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet);$$
$$''E_2^{pq} = H^p(R^qF(C^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet).$$

For unbounded complexes $C^\bullet$, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*. The same problem occurs for (even totally finite-dimensional) DG-modules.

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Sometimes one wants to use their mixtures—the semiderived categories.

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[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, . . . '98 –]

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Coderived and contraderived categories of CDG-modules

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\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \xrightarrow{d} & K^2 & \longrightarrow & \cdots \\
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Comodules and contramodules

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A *coalgebra* over a commutative ring A (most typically over a field) is a coring whose left and right A-module structures coincide.

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A right $\mathcal{C}$-comodule $\mathcal{N}$ is a right A-module endowed with a coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C}$ satisfying the similar conditions.

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For a coalgebra $\mathcal{C}$ over k, the datum of a map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ is quite different from that of a map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$.

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The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

The category $\mathcal{C}\text{-contra}$ of left $\mathcal{C}$-contramodules is abelian provided that $\mathcal{C}$ is a projective left A-module. In this case, the category $\mathcal{C}\text{-contra}$ has exact functors of infinite product and enough projectives.

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In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

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Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

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The graded dual vector space $\mathcal{C}^* = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^{-i*}$ to a CDG-coalgebra over k is a CDG-algebra.

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

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The homotopy (H^0) categories of these DG-subcategories are denoted by $\mathrm{Hot}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}})$ and $\mathrm{Hot}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}})$, as usually.

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Corollary

For any CDG-coalgebra $\mathcal{C}$ over a field k, there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}} \colon \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}) : \mathbb{L}\Phi_{\mathcal{C}}.$$

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The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$

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The right adjoint functor $\Psi_{\mathcal{C}}$ is simply

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One restricts these functors to graded-injective CDG-comodules and graded-projective CDG-contramodules in order to construct the derived functors.

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- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
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The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

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For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any abelian group U, the left $\mathfrak{R}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, U)$ has a natural left $\mathfrak{R}$-contramodule structure.

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In particular, $\mathbb{Z}_\ell$-contramodules = weakly ℓ-complete (Ext-ℓ-complete) abelian groups

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neighborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

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In particular, $\mathbb{Z}_\ell$-contramodules = weakly ℓ-complete (Ext-ℓ-complete) abelian groups [Bousfield–Kan '72, Jannsen '88].

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The choice of a dualizing complex $D_R^\bullet$ for a Noetherian commutative ring R induces an equivalence of triangulated categories $\mathbb{R}\Psi_{D_R^\bullet}\colon \mathrm{D}^{\mathrm{co}}(R\text{-mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod}) :\!\mathbb{L}\Phi_{D_R^\bullet}$.

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[Jørgensen, Krause, Iyengar–Krause '05–'06]

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Denote by $\mathfrak{R}\text{-}\mathrm{discr}$ the abelian category of discrete $\mathfrak{R}$-modules.

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Corollary

Any compatible system $\mathcal{D}^\bullet_{\mathfrak{R}}$ of choices of dualizing complexes $D^\bullet_{R_n}$ for the Noetherian rings R_n, $n \geqslant 0$, induces an equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{D}^\bullet_{\mathfrak{R}}} : \mathrm{D^{co}}(\mathfrak{R}\text{-discr}) \simeq \mathrm{D^{ctr}}(\mathfrak{R}\text{-contra}) : \mathbb{L}\Phi_{\mathcal{D}^\bullet_{\mathfrak{R}}}.$$

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