Contramodules: their History, and Applications in Commutative and Noncommutative Algebra

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Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers".

Some of the missing pieces were defined or hinted at in the 1960's and 1970's, then left undeveloped or completely forgotten. Some of the pieces were only invented in the 1990's or 00's, even in the 2010's.

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;
- relative, mixed or intermediate forms: mixtures of modules with comodules, mixtures of modules with contramodules, semiderived and pseudo-derived categories, etc.

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

It turned out that there is a nonhomogeneous quadratic duality construction connecting algebras with quadratic-linear relations with quadratic DG-algebras.

A similar, but more general construction connects algebras with quadratic-linear-scalar relations with what I called quadratic curved DG-algebras.

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

$$0 \longrightarrow k \xrightarrow{0} \mathfrak{g}^* \longrightarrow \bigwedge\nolimits^2 \mathfrak{g}^* \longrightarrow \bigwedge\nolimits^3 \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \bigwedge\nolimits^d \mathfrak{g}^* \longrightarrow 0,$$

where k is the ground field and $d = \dim \mathfrak{g}$. This is called the cohomological Chevalley–Eilenberg complex or the Chevalley–Eilenberg DG-algebra of $\mathfrak{g}$.

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$
- and $h' = h + d(a) + a^2$.

The CDG-algebras (B, d, h) and (B, d', h') are considered to be isomorphic (the category of CDG-algebras is defined so that they are). The element a is called a change-of-connection element.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

The graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ corresponding to the connection on $\mathcal{E}nd(\mathcal{E})$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$ (equal to the curvature of the connection $\nabla_{\mathcal{E}}$) is a curved DG-algebra.

Changing the connection in $\mathcal{E}$ leads to an isomorphic CDG-algebra.

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \textstyle\sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$ has the space of generators V^* and the space of relations $R^\perp \subset V^* \otimes V^*$ (the orthogonal complement to $R \subset V \otimes V$ in $V^* \otimes V^* = (V \otimes V)^*$).

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy = y - 1$, hence $x = 1$. Substituting $x = 1$ into $xy = y - 1$, one comes to $1 = 0$.

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \qquad x, y, z \in \mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$ is the self-consistency equation for the system of nonhomogeneous quadratic relations

$$xy - yx = [x, y], \qquad x, y \in \mathfrak{g}$$

defining the universal enveloping algebra $U(\mathfrak{g})$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$.
The self-consistency equations on the coefficients of the relations guarantee that the map d^1 extends to a well-defined differential $d : B^i \longrightarrow B^{i+1}$ for all $i \geqslant 0$ satisfying the Leibniz rule with signs, and that (B, d, h) is a curved DG-algebra.

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction to the failure of the differential m_1 to have zero square.

This point of view is due to Getzler and Jones (Illinois J. Math. 1990), who defined what is now usually called a curved A_∞-algebra (with the operations m_n, $n \geqslant 0$).

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

In a somewhat more complicated form, this generalizes to an equivalence between the derived categories of graded modules over a quadratic algebra A and its quadratic dual algebra $A^!$, provided that A has the so-called Koszul property.

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*),d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra. The cohomology algebra $H^*(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains too little information about $\mathfrak{g}$, and there is no hope of recovering the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the algebra $H^*(\mathfrak{g})$.

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

I called them the derived categories of the second kind. Starting from about 2006, I am now calling them the coderived and the contraderived category.

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.
- "Foundations of relative homological algebra", *Memoirs of the American Math. Society* **55**, 1965.
- "Homology and fibrations. I. Coalgebras, cotensor product and its derived functors", *Commentarii Mathematici Helvetici* **40**, 1966.

What I would consider as the last paper in this series was written somewhat later by a different group of authors:

- D. Husemoller, J.C. Moore, and J. Stasheff, "Differential homological algebra and homogeneous spaces", *Journ. of Pure and Appl. Algebra* **5**, 1974.

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor, one would resolve one or both of the DG-(co)modules by a complex of DG-(co)modules, take the Hom or (co)tensor product, and totalize the resulting bicomplex by taking direct sums or direct products along the diagonals.

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind took a DG-(co)module to be a deformation of its cohomology. The differential derived functors of the second kind took a DG-(co)module to be a deformation of itself with the differential forgotten. This was reflected in the (weak) convergence vs. divergence of the related spectral sequences.

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors of the first and second kind. Hence the name derived categories of the second kind for the exotic derived categories important for the derived nonhomogeneous Koszul duality purposes that I had constructed.

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore, which I found in the library. It contained the definitions of two kinds of module objects over a coalgebra: the comodules and the contramodules.

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations: the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\mu_*}{\rightrightarrows} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

should be equal to each other, and the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\epsilon_*}{\rightrightarrows} \mathcal{C}$$

should be equal to the identity map $\mathrm{id}_{\mathcal{C}}$.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a vector space endowed with a linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, called the left contraaction map. The contraaction map has to satisfy the contraassociativity and contraunitality equations:

$$\operatorname{Hom}(\mathcal{C}, \operatorname{Hom}(\mathcal{C}, \mathfrak{P})) = \operatorname{Hom}(\mathcal{C} \otimes \mathcal{C},\ \mathfrak{P}) \rightrightarrows \operatorname{Hom}(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}$$

$$\mathfrak{P} \xrightarrow{\epsilon^*} \operatorname{Hom}(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}$$

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

The abelian category of left $\mathcal{C}$-comodules has enough injective objects, but may have no projectives. The abelian category of left $\mathcal{C}$-contramodules has enough projective objects, but may have no injectives.

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = ...") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding, this leads to 11 actual papers mentioning contramodules (of which 4 are mine). The first one after 1970 is dated 2009.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures are properly associated with an algebra object in the tensor category of bicomodules over a coalgebra. I called such structures semialgebras (meaning "half algebra and half coalgebra").

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too, should include the derived categories of DG-modules, the coderived categories of CDG-comodules, and the contraderived categories of CDG-contramodules.

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication

$$\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$$

and the counit

$$\varepsilon(1^*) = 1, \qquad \varepsilon(t^{n*}) = 0 \quad \text{for } n \geqslant 1.$$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Thus a $\mathcal{C}$-comodule, for the coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[t]]$, is the same thing as a t-torsion $k[t]$-module.

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n \sum_{i+j=n} p_{ij}.$$

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit of finite partial sums. This is a new concept of infinite sum in mathematics, quite different from the ones usually studied in analysis.

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$ as assigned to the topological algebra $\mathcal{C}^*$ dual to $\mathcal{C}$, and then try to extend this assignment to topological rings of more general nature.

In particular, here is a definition of contramodules over the topological ring of p-adic integers $\mathbb{Z}_p$ for a prime number p, based on the analogy between $\mathbb{Z}_p$ and $k[[t]]$.

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, $\ldots \in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

- unitality + compatibility: $\sum_{n=0}^{\infty} p^n c_n = c_0 + pc_1$ when $c_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} p^j c_{ij} = \sum_{n=0}^{\infty} p^n \sum_{i+j=n} c_{ij}.$$

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology did not exist in my seminar talks and writings of the first half of 00's. I picked it up sometime in 2005–06 from B. Keller's unpublished 2003 note "Koszul duality and coderived categories (after K. Lefèvre)", which is still available from his homepage.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring, which we will discuss below, was invented between 2007–08 and first appeared in a remark in the June 2008 update of this preprint. This work was published as a book in September 2010.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}, \quad r_x, s_{x,y} \in \mathfrak{R},\ c_{x,y} \in \mathfrak{C},$$

$$\textstyle\sum_{x,y} r_{x,y} c_x = \sum_x (\sum_y r_{x,y}) c_x, \quad r_{x,y} \in \mathfrak{R},\ c_x \in \mathfrak{C}.$$

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, "1-dimensional Cohen–Macaulay rings", *Lecture Notes in Math.* **327**, Springer, 1973.

From the beginning of 1980s, after the works of L. Salce and E. Enochs, the word "cotorsion" started to mean something else (not unrelated, but quite different).

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

After 1980s, people started to call "Matlis cotorsion" (or "weakly cotorsion", which is a better term) the R-modules C such that $\mathrm{Ext}^1_R(S^{-1}R, C) = 0$ (dropping the condition of Hom vanishing).

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings, they appear in subsequent topological literature, such as

- M. Hovey, N.P. Strickland, "Morava K-theories and localization", *Memoirs Amer. Math. Soc.* **139**, #666, 1999.

In the greater generality of weakly proregular finitely generated ideals I in commutative rings R, they were studied by A. Yekutieli and collaborators (since 2010) under the name of cohomologicaly I-adically complete modules.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion if it is Matlis S-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \dots\}$. An R-module C is Matlis I-cotorsion if it is Matlis s-cotorsion for all $s \in I$. (One can prove that it suffices to check this condition for any given set of generators s_1, …, s_m of the ideal I.)

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has $\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$. For so-called weakly proregular ideals I, one has $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}}$. Generally speaking, any Matlis I-cotorsion R-module is an extension of two $\mathfrak{R}$-contramodules.

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs. These were cotorsion modules in Enochs' sense, of course.

I first looked into Harrison's 1959 paper on cotorsion abelian groups only in 2012, and learned about Matlis' work on cotorsion modules as late as in 2015–16.

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called "weakly" or "cohomologically" complete modules, from the contemporary point of view are properly considered as species of contramodules.

What I call here "Matlis S-cotorsion" and "Matlis I-cotorsion R-modules", are actually called "S-contramodule R-modules" and "I-contramodule R-modules" in my papers.

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums).

In Spring 2012 I solved this problem by inventing the definition of the exact category of contraherent cosheaves, which is assigned to any algebraic variety and has exact functors of infinite product.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism $\mathcal{M}(V) = S \otimes_R \mathcal{M}(U)$. In a contraherent cosheaf $\mathfrak{P}$, it is an isomorphism $\mathfrak{P}[V] = \mathrm{Hom}_R(S, \mathfrak{P}[U])$.

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1, and there are other properties.

A narrow class of R-modules to which modules like S belong is called the class of very flat R-modules.

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\mathrm{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa. This concept was introduced in the paper

- L. Salce, "Cotorsion theories for abelian groups", *Symposia Mathematica* **XXIII**, 1979.

After the paper

- P.C. Eklof, J. Trlifaj, "How to make Ext vanish", *Bull. London Math. Soc.* **33**, 2001

it gradually became one of the most powerful technical tools in homological algebra of rings and modules.

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if it is a direct summand of an $\mathcal{S}$-filtered R-module G. The latter condition means that there exists an ordinal α and an increasing filtration G_i of G indexed by the ordinals $0 \leqslant i \leqslant \alpha$ such that every successive quotient module G_{i+1}/G_i is isomorphic to a module from $\mathcal{S}$.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$ by a finitely generated ideal. Assume that T is a flat R-module.

The Very Flat Conjecture (now theorem) claims that T is then a very flat R-module.

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Soon it was agreed that I would come to Brno and Prague again as an ECI visitor. There was a masters student in Prague, Alexander Slávik his name, who was going to start studying very flat and contraadjusted modules.

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017. The preprint appeared on the arXiv in August.

The argument was based on a heavy use of what are above called Matlis s-cotorsion and Matlis I-cotorsion R-modules (called the s-contramodule and I-contramodule R-modules in the paper).

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem, while surpisingly completely different, are both based on a heavy use of Matlis S-cotorsion R-modules (called S-contramodule R-modules in the paper).

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base. Then the left R-module U has projective dimension at most 1.

The proof is based on a heavy use of contramodules over a certain topological ring $\mathfrak{R}$ (namely, the completion of R with respect to the topology where $\mathbb{G}$ is a base of neighborhoods of zero).

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years, from 1999 to 2012 or even 2015–17, to discover and understand the connection. This is what made the modern applications possible.

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