Contramodules: their History, and Applications in Commutative and Noncommutative Algebra

Leonid Positselski – IM AV ČR

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Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers". Some of the missing pieces were defined or hinted at in the 1960's and 1970's, then left undeveloped or completely forgotten. Some of the pieces were only invented in the 1990's or 00's, even in the 2010's.
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Examples include modules, comodules, and contramodules; quasi-coherent sheaves and contraherent cosheaves; curved DG-modules, DG-comodules, and DG-contramodules; derived, coderived, and contraderived categories; relative, mixed or intermediate forms: mixtures of modules with comodules, mixtures of modules with contramodules, semiderived and pseudo-derived categories, etc.
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Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables. I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts). It turned out that there is a nonhomogeneous quadratic duality construction connecting algebras with quadratic-linear relations with quadratic DG-algebras. A similar, but more general construction connects algebras with quadratic-linear-scalar relations with what I called quadratic curved DG-algebras.
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A similar, but more general construction connects algebras with quadratic-linear-scalar relations with what I called quadratic curved DG-algebras.
Curved DG-algebras

A DG-algebra \((A, d)\) is a graded associative algebra

\[ A = \bigoplus_{i = -\infty}^{\infty} A_i \]

endowed with an operator \(d: A_i \to A_{i+1}\) for all \(i \in \mathbb{Z}\)

satisfying Leibniz rule for the derivative of a product with signs:

\[ d(ab) = d(a)b + (-1)^i ad(b) \]

if \(a \in A_i, b \in A_j\)

and such that \(d^2: A_i \to A_{i+2}\) is the zero map for all \(i\).

For example, for any finite-dimensional Lie algebra \(g\) the map \(g^* \to \bigwedge^2 g^*\) dual to the bracket map \([-,-]: \bigwedge^2 g \to g\) extends to a differential \(d\) on the exterior algebra \(\bigwedge g^*\) of the vector space dual to \(g\), endowing \(\bigwedge g^*\) with a DG-algebra structure.

This is called the cohomological Chevalley–Eilenberg complex or the Chevalley–Eilenberg DG-algebra of \(g\).
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where \(k\) is the ground field and \(d = \dim \mathfrak{g}\). This is called the cohomological Chevalley–Eilenberg complex.
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Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

\[ B = \bigoplus_{i = -\infty}^{\infty} B_i \]

endowed with an operator \( d : B_i \to B_{i+1} \), \( i \in \mathbb{Z} \), satisfying the Leibniz rule with signs and with a curvature element \( h \in B_2 \) such that

\[ d^2 (b) = [h, b] \]

for all \( b \in B \) and \( d (h) = 0 \).

Given an element \( a \in B_1 \), one can transform the differential and the curvature element of a CDG-algebra \( B \) by the rules

\[ d' (b) = d (b) + [a, b], \]

where \([a, b] = ab - (-1)^j ba\) for \( b \in B_j \) and

\[ h' = h + d (a) + a^2. \]

The CDG-algebras \((B, d, h)\) and \((B, d', h')\) are considered to be isomorphic (the category of CDG-algebras is defined so that they are).

The element \( a \) is called a change-of-connection element.
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Curved DG-algebras: Example

Let $M$ be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential $d$ is a DG-algebra.

Let $E$ be a vector bundle on $M$ and $\nabla_E$ be a connection in $E$. Then there is an induced connection $\nabla_{\text{End}(E)}$ on the vector bundle of endomorphisms $\text{End}(E) = E^* \otimes E$.

This is also a bundle of associative algebras, so the differential forms on $M$ with coefficients in $\text{End}(E)$ form a graded algebra $\Omega(M, \text{End}(E))$.

The graded algebra $\Omega(M, \text{End}(E))$ with the de Rham differential $d = \nabla_{\text{End}(E)}$ corresponding to the connection on $\text{End}(E)$ and the curvature element $h = h_{\nabla_E} \in \Omega^2(M, \text{End}(E))$ (equal to the curvature of the connection $\nabla_E$) is a curved DG-algebra.

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Changing the connection in $\mathcal{E}$ leads to an isomorphic CDG-algebra.
Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example:

\[
\begin{align*}
xy - 2yx &= 3z \\
zy - 2zy &= 3x \\
xz - 2xz &= 3y
\end{align*}
\]

More invariantly, to define a quadratic algebra one needs to specify a vector space of generators \( V \) and a subspace of quadratic relations \( R \subset V \otimes V \). Then \( A \) is the graded algebra with the components:

\[A_0 = k\] (the ground field),
\[A_1 = V\]
\[A_2 = V^\otimes 2 / R\]
\[A_n = V^\otimes n / \sum_{i=1}^{n-1} (V^\otimes i - 1 \otimes R \otimes V^\otimes n-i)\]

The quadratic dual algebra \( A^! \) has the space of generators \( V^* \) and the space of relations \( R^\perp \subset V^* \otimes V^* \) (the orthogonal complement to \( R \subset V \otimes V \) in \( V^* \otimes V^* = (V \otimes V)^* \)).
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A_0 &= k \\
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A_n &= V \otimes V / \sum_{i=1}^{n-1} (V \otimes V / R) \otimes V \otimes V / (V \otimes V / R) 
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Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example:

\[
\begin{align*}
xy - yx &= z \\
yz - zy &= x \\
zx - xz &= y
\end{align*}
\]
or

\[
\begin{align*}
x^2 &= -1 \\
xy + yx &= 0 \\
y^2 &= -1
\end{align*}
\]

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense." For example, \{\(xy = y - 1\), \(yx = y\)\} looks fine until one realizes that it implies \((xy)x = (y - 1)x = yx - x = y - x\) and \(x(yx) = xy = y - 1\), hence \(x = 1\).

Substituting \(x = 1\) into \(xy = y - 1\), one comes to \(1 = 0\).
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\begin{cases}
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A delicate point is that not every system of nonhomogeneous quadratic relations “makes sense”. For example

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Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

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\[
x, y, z + y, z, x + z, x, y = 0, x, y, z \in g\]

is the self-consistency equation for the system of nonhomogeneous quadratic relations \(xy - yx = [x, y], x, y \in g\) defining the universal enveloping algebra \(U(g)\).
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Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\tilde{A}$, one needs to specify a vector space of generators $V$ and a subspace of nonhomogeneous quadratic relations $\tilde{R} \subset V \otimes 2 \oplus V \oplus k$.

Taking the projection of the subspace $\tilde{R}$ onto the direct summand $V \otimes 2$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V \otimes 2$ defining a quadratic graded algebra $A$.

The subspace $\tilde{R} \subset V \otimes 2 \oplus V \oplus k$ can be then described in terms of two linear maps $R \rightarrow V$ and $R \rightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to $A$. Then one has $B_1 = V^*$ and $B_2 \sim = R^*$.

Dualizing the maps $R \rightarrow V$ and $R \rightarrow k$ defining the linear and the scalar parts of the relations in $\tilde{A}$, one obtains a linear map $d_1: B_1 \rightarrow B_2$ and an element $h \in B_2$.

The self-consistency equations on the coefficients of the relations guarantee that the map $d_1$ extends to a well-defined differential $d: B_i \rightarrow B_{i+1}$ for all $i \geq 0$ satisfying the Leibniz rule with signs, and that $(B, d, h)$ is a curved DG-algebra.
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Curved DG-algebras and $A_\infty$-algebras

An $A_\infty$-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i = -\infty}^\infty A_i$ endowed with a sequence of higher multiplications $m_n: A^\otimes n \to A$, $n \geq 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making $A$ a complex, the map $m_2$ is a (nonassociative) multiplication, and the higher multiplications $m_n$, $n \geq 3$, are a sequence of corrections to the nonassociativity of $m_2$.

A curved DG-algebra can be thought of as an algebra with $m_0$, $m_1$, and $m_2$, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and $m_2$ is the multiplication.

The curvature element $m_0$ is a correction to the failure of the differential $m_1$ to have zero square.

This point of view is due to Getzler and Jones (Illinois J. Math. 1990), who defined what is now usually called a curved $A_\infty$-algebra (with the operations $m_n$, $n \geq 0$).
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Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(g)$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(g^*)$. The question was: How can one recover the derived category of $g$-modules $D(g\text{-mod})$ from the DG-algebra $\bigwedge(g^*)$?

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Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $C$ over a field $k$ is a vector space endowed with linear maps $\mu : C \rightarrow C \otimes C$ and $\epsilon : C \rightarrow k$, called the comultiplication and counit maps. The two maps have to satisfy the coassociativity and counitality equations: the two compositions $C \xrightarrow{\mu} C \otimes C \xrightarrow{\mu^*} C \otimes C \otimes C$ should be equal to each other, and the two compositions $C \xrightarrow{\mu} C \otimes C \xrightarrow{\epsilon^*} C$ should be equal to the identity map $\text{id}_C$. 
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A left \( C \)-comodule \( \mathcal{M} \) is a vector space endowed with a linear map \( \nu : \mathcal{M} \rightarrow C \otimes \mathcal{M} \).

A left \( C \)-contramodule \( \mathcal{P} \) is a vector space endowed with a linear map \( \pi : \text{Hom}(C, \mathcal{P}) \rightarrow \mathcal{P} \), called the left contraaction map. The contraaction map has to satisfy the contraassociativity and contraunitality equations:

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\text{Hom}(C, \text{Hom}(C, \mathcal{P})) = \text{Hom}(C \otimes C, \mathcal{P}) \Rightarrow \text{Hom}(C, \mathcal{P}) \rightarrow \mathcal{P} \\
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For any coalgebra $C$ over a field $k$, left $C$-comodules form an abelian category

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Left $C$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

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The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it “BRST.”

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov’s construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures are properly associated with an algebra object in the tensor category of bicomodules over a coalgebra. I called such structures semialgebras (meaning “half algebra and half coalgebra”).
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The semi-infinite homology (or the semi-infinite Tor spaces)
Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules.
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The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one.
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The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra.

Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too, should include the derived categories of DG-modules, the coderived categories of CDG-comodules, and the contraderived categories of CDG-contramodules.
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Coalgebras over a field $k$ can be described in terms of their dual pro-finite-dimensional topological algebras $C^*$. In particular, let $C$ be the coalgebra whose dual topological algebra $C^*$ is the algebra of formal power series $k[[t]]$ in one variable. Explicitly, $C$ is a $k$-vector space with the basis $1^*, t^*, t^2^*, t^3^*, \ldots$ endowed with the comultiplication $\mu(t^n^*) = \sum_{i+j=n} t^i^* \otimes t^j^*$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^n^*) = 0$ for $n \geq 1$. 
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The coaction map $\nu: \mathcal{M} \rightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t_i^* \otimes t_i m$$

where $t: \mathcal{M} \rightarrow \mathcal{M}$ is a certain linear operator.

Since the sum in the right-hand side must be finite, it follows that $t$ must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

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In other words, this means that a $C$-contramodule $\mathbb{P}$ is a $k$-vector space endowed with an infinite summation operation assigning to any sequence of elements $p_0, p_1, p_2, \ldots \in \mathbb{P}$ an element denoted formally by

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which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t_n (ap_n + bq_n) = a \sum_{n=0}^{\infty} t_n p_n + b \sum_{n=0}^{\infty} t_n q_n,$$

unitality:

$$\sum_{n=0}^{\infty} t_n p_n = p_0$$

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Contramodules over Coalgebras: a Counterexample

For the same coalgebra $C$, for any $C$-contramodule $P$, an element $p \in P$, and an integer $n \geq 0$, one can define

$$t_n p = 1 \cdot 0 + \cdots + t_{n-1} \cdot 0 + t_n + t_{n+1} \cdot 0 + \cdots \in P.$$

Then there exists a $C$-contramodule $P$ and a sequence of elements $p_0, p_1, p_2, \ldots \in P$ such that $t_n p_n = 0$ for every $n \geq 0$, but $\sum_{n=0}^{\infty} t_n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t_n p_n$ belongs to $t_m P$ for every $m \geq 0$, so the $t$-adic topology on $P$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit of finite partial sums. This is a new concept of infinite sum in mathematics, quite different from the ones usually studied in analysis.

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Then there exists a $C$-contramodule $\mathcal{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathcal{P}$ such that $t^n p_n = 0$ for every $n \geq 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

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Let $R$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $R$-contramodule $C$ can be defined as a set endowed with the following infinite summation operations. For every set $X$, every $X$-indexed family of elements $r_x \in R$ converging to zero in the topology of $R$, and every $X$-indexed family of elements $c_x \in C$, an element denoted formally by $\sum_{x \in X} r_x c_x \in C$ must be specified. Here a family of elements $(r_x \in R)_{x \in X}$ is said to converge to zero in $R$ if for every neighborhood of zero $U \subset R$ one has $r_x \in U$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\sum_{x \in X} (\sum_{y \in Y} r_x s_y) c_x = \sum_{x \in X} \sum_{y \in Y} (r_x s_y) c_x$$

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Contramodules: History and Applications 35 / 53
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For every set $X$, every $X$-indexed family of elements $r_x \in \mathcal{R}$ converging to zero in the topology of $\mathcal{R}$, and every $X$-indexed family of elements $c_x \in \mathcal{C}$, an element denoted formally by $\sum_{x \in X} r_x c_x \in \mathcal{C}$ must be specified. Here a family of elements $(r_x \in \mathcal{R})_{x \in X}$ is said to converge to zero in $\mathcal{R}$ if for every neighborhood of zero $U \subset \mathcal{R}$ one has $r_x \in U$ for all but a finite subset of $x \in X$.

These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\sum_{x \in X, y} r_x s_y c_x y = \sum_{y \in Y, x} (\sum_{x \in X} r_x s_y) c_x y,$$

$$\sum_{x \in X} r_x \sum_{y \in Y} c_y = \sum_{y \in Y} (\sum_{x \in X} r_x) c_y,$$

for $r_x \in \mathcal{R}$, $c_y \in \mathcal{C}$.
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Some Bits of History


From the beginning of 1980s, after the works of L. Salce and E. Enochs, the word “cotorsion” started to mean something else (not unrelated, but quite different).
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For the purposes of this talk, let us say that an $R$-module $C$ is Matlis $S$-cotorsion if $\text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the $R$-module $S^{-1}R$ has projective dimension at most 1.

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After 1980s, people started to call “Matlis cotorsion” (or “weakly cotorsion”, which is a better term)
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Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring $R$ with the elements from $S$ inverted. For the purposes of this talk, let us say that an $R$-module $C$ is **Matlis $S$-cotorsion** if $\text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the $R$-module $S^{-1}R$ has projective dimension at most 1.

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Example: the Adic Topology

Let $R$ be a commutative ring and $I \subset R$ be a finitely-generated ideal.

Denote by $\hat{R}$ the $I$-adic completion of the ring $R$, that is $\hat{R} = \lim_{\leftarrow} n \geq 1 R/I^n$, and endow $\hat{R}$ with the projective limit (or $I$-adic) topology.

Consider the abelian category $\text{R-\text{-contra}}$ of contramodules over the topological ring $\hat{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\text{R-\text{-contra}} \rightarrow \text{R-\text{-mod}}$ is fully faithful, so $\text{R-\text{-contra}}$ is a full subcategory in $\text{R-\text{-mod}}$.

How to describe this full subcategory?

Following the above terminology, given an element $s \in R$ we will say that an $R$-module $C$ is Matlis $s$-cotorsion if it is Matlis $S$-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \ldots\}$.

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Then one can prove (L.P., 2008–16) that the forgetful functor $\mathcal{K}$-contra $\rightarrow R$-mod is fully faithful, so $\mathcal{K}$-contra is a full subcategory in $R$-mod.
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Then one can prove (L.P., 2008–16) that the forgetful functor $\mathcal{R}$-contra $\rightarrow R$-mod is fully faithful, so $\mathcal{R}$-contra is a full subcategory in $R$-mod.

How to describe this full subcategory?
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Let \( R \) be a commutative ring and \( I \subset R \) be a finitely-generated ideal. Denote by \( \mathcal{A} \) the \( I \)-adic completion of the ring \( R \), that is \( \mathcal{A} = \lim_{\leftarrow n \geq 1} R/I^n \), and endow \( \mathcal{A} \) with the projective limit (\( = I \)-adic) topology. Consider the abelian category \( \mathcal{A}\)-contra of contramodules over the topological ring \( \mathcal{A} \).

Then one can prove (L.P., 2008–16) that the forgetful functor \( \mathcal{A}\text{-contra} \rightarrow R\text{-mod} \) is fully faithful, so \( \mathcal{A}\text{-contra} \) is a full subcategory in \( R\text{-mod} \).

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Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis $I$-cotorsion $R$-modules in $R\text{-mod}$.
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Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis $I$-cotorsion $R$-modules in $R\text{-mod}$.

**Theorem (L.P., 2008–2012)**

Assume that $R$ is Noetherian.

For any finitely generated ideal $I$ in a commutative ring $R$, one has $R\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

For so-called weakly proregular ideals $I$, one has $R\text{-contra} = R\text{-mod}_{I\text{-mcot}}$.

Generally speaking, any Matlis $I$-cotorsion $R$-module is an extension of two $R$-contramodules.
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Some bits of personal history

I first heard the word "cotorsion" from Jan Řestock in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs. These were cotorsion modules in Enochs' sense, of course.

I first looked into Harrison's 1959 paper on cotorsion abelian groups only in 2012, and learned about Matlis' work on cotorsion modules as late as in 2015–16.
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“Consider the exact category of flat modules over a ring. Does it have enough injective objects?”

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called “flat cover conjecture”, and the name of the key author was Enochs. These were cotorsion modules in Enochs’ sense, of course.

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I first looked into Harrison’s 1959 paper on cotorsion abelian groups only in 2012, and learned about Matlis’ work on cotorsion modules as late as in 2015–16.
Terminological Conclusion
Terminological Conclusion

What Harrison and Matlis called cotorsion modules

What I call here “Matlis $S$-cotorsion” and “Matlis $I$-$R$-modules”, are actually called “$S$-contramodule $R$-modules” and “$I$-contramodule $R$-modules” in my papers.
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Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved. The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums). In Spring 2012 I solved this problem by inventing the definition of the exact category of contraherent cosheaves, which is assigned to any algebraic variety and has exact functors of infinite product.
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Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let $U$ be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \to S$.

In the simplest case of a principal affine open subscheme, one has $S = R[\frac{1}{f}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $M$, the modules of sections over $U$ and $V$ are connected by an isomorphism $M(V) = S \otimes_R M(U)$.

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The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme $V$ in an affine scheme $U$ the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$. For the definition of a contraherent cosheaf to work similarly, one would need $S$ to be a projective $R$-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian. Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the $R$-module $S$ never exceeds 1, and there are other properties. A narrow class of $R$-modules to which modules like $S$ belong is called the class of very flat $R$-modules.
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The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme \( V \) in an affine scheme \( U \) the ring \( S = \mathcal{O}(V) \) is a \textit{flat} module over the ring \( R = \mathcal{O}(U) \).

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Quasi-Coherent Sheaves and Contraherent Cosheaves

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A narrow class of $R$-modules to which modules like $S$ belong is called the class of very flat $R$-modules.
Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of \( R \)-modules \((F, C)\) such that an \( R \)-module \( F \) belongs to \( F \) if and only if \( \text{Ext}^1_R(F, C) = 0 \) for all \( C \in C \) and vice versa. This concept was introduced in the paper L. Salce, "Cotorsion theories for abelian groups", Symposia Mathematica XXIII, 1979. After the paper P.C. Eklof, J. Trlifaj, "How to make Ext vanish", Bull. London Math. Soc. 33, 2001, it gradually became one of the most powerful technical tools in homological algebra of rings and modules.
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It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of \(R\)-modules \(S\) containing the \(R\)-module \(R\), the class \(\mathcal{F}\) in the cotorsion pair \((\mathcal{F}, \mathcal{C})\) generated by \(S\) can be described as follows. An \(R\)-module \(F\) belongs to \(\mathcal{F}\) if and only if it is a direct summand of an \(S\)-filtered \(R\)-module \(G\). The latter condition means that there exists an ordinal \(\alpha\) and an increasing filtration \(G_i\) of \(G\) indexed by the ordinals \(0 \leq i \leq \alpha\) such that every successive quotient module \(G_{i+1}/G_i\) is isomorphic to a module from \(S\).
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Let $R$ be a commutative ring, and let $S$ denote the set of all $R$-modules of the form $S = R[s^{-1}]$, where $s \in R$.

Let $(F, C)$ be the cotorsion pair generated by $S$. So $C$ is the class of all $R$-modules such that $\text{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $F$ is the class of all direct summands of $R$-modules filtered by the $R$-modules $R[s^{-1}]$.

$R$-modules from $F$ are called very flat, while $R$-modules from $C$ are called contraadjusted (which means “adjusted to contraherent cosheaves”).

Let $T$ be a finitely presented commutative $R$-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$ by a finitely generated ideal.

Assume that $T$ is a flat $R$-module. The Very Flat Conjecture (now theorem) claims that $T$ is then a very flat $R$-module.
Let $R$ be a commutative ring.
Very Flat Modules and Very Flat Conjecture

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The Very Flat Conjecture (now theorem) claims that $T$ is then a very flat $R$-module.
Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012). In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and ˇSt ˇov ˇ íˇccek in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited. Soon it was agreed that I would come to Brno and Prague again as an ECI visitor. There was a masters student in Prague, Alexander Slavík his name, who was going to start studying very flat and contraadjusted modules.
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In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)
For any Noetherian commutative ring $R$ with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat $R$-module is a direct summand of an $R$-module filtered by $S_j^R$, $j \geq 1$.

When $R$ has finite Krull dimension $d$, a finite collection of at most $m = 2^{(d+1)/2}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem, while surprisingly completely different, are both based on a heavy use of Matlis $S$-cotorsion $R$-modules (called $S$-contramodule $R$-modules in the paper).
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Theorem (L.P., 2018)
Let $R \to U$ be a homomorphism of associative rings such that $U$ is a flat left $R$-module and the multiplication map $U \otimes_R U \to U$ is an isomorphism.

Consider the filter $G$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $G$ has a countable base.

Then the left $R$-module $U$ has projective dimension at most 1.

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Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999. Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis. The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years, from 1999 to 2012 or even 2015–17, to discover and understand the connection. This is what made the modern applications possible.

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