Contramodules: their History, and Applications in Commutative and Noncommutative Algebra

Leonid Positselski – IM AV ČR

IM external meeting, Zvánovice, ČR

October 17–19, 2018

Overview

Overview

Contramodules are an importance piece in a larger puzzle

Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra"

Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers".

Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers".

Some of the missing pieces were defined or hinted at in the 1960's and 1970's,

Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers".

Some of the missing pieces were defined or hinted at in the 1960's and 1970's, then left undeveloped or completely forgotten.

Overview

Contramodules are an importance piece in a larger puzzle which can be called "a missing half of algebra" or "a half of homological algebra that was either overlooked by the classical authors or forgotten by their followers".

Some of the missing pieces were defined or hinted at in the 1960's and 1970's, then left undeveloped or completely forgotten. Some of the pieces were only invented in the 1990's or 00's, even in the 2010's.

Overview

The familiar basic elements of (homological) algebra include:

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;
- relative, mixed or intermediate forms:

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;
- relative, mixed or intermediate forms: mixtures of modules with comodules,

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;
- relative, mixed or intermediate forms: mixtures of modules with comodules, mixtures of modules with contramodules,

Overview

The familiar basic elements of (homological) algebra include:

- modules and sheaves (sometimes also comodules);
- complexes of modules/sheaves and DG-modules;
- and derived categories.

The full picture should include:

- modules, comodules, and contramodules;
- quasi-coherent sheaves and contraherent cosheaves;
- curved DG-modules, DG-comodules, and DG-contramodules;
- derived, coderived, and contraderived categories;
- relative, mixed or intermediate forms: mixtures of modules with comodules, mixtures of modules with contramodules, semiderived and pseudo-derived categories, etc.

Some bits of personal history

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality.

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality.
This is the construction that connects the algebra of polynomials in several variables

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts)

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

It turned out that there is a nonhomogeneous quadratic duality construction

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

It turned out that there is a nonhomogeneous quadratic duality construction connecting algebras with quadratic-linear relations with quadratic DG-algebras.

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

It turned out that there is a nonhomogeneous quadratic duality construction connecting algebras with quadratic-linear relations with quadratic DG-algebras.

A similar, but more general construction connects algebras with quadratic-linear-scalar relations

Some bits of personal history

Sometime around 1990 I learned about the quadratic duality. This is the construction that connects the algebra of polynomials in several variables with the exterior algebra in the dual variables.

I wanted to extend this construction to algebras with nonhomogeneous quadratic relations, like the universal enveloping algebra (whose relations have quadratic and linear parts) or the Clifford algebra (whose relations have quadratic and scalar parts).

It turned out that there is a nonhomogeneous quadratic duality construction connecting algebras with quadratic-linear relations with quadratic DG-algebras.

A similar, but more general construction connects algebras with quadratic-linear-scalar relations with what I called quadratic curved DG-algebras.

Curved DG-algebras

A *DG-algebra* (A, d) is

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d \colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i a d(b)$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i a d(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i a d(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

$$0 \longrightarrow k \xrightarrow{0} \mathfrak{g}^* \longrightarrow \bigwedge^2 \mathfrak{g}^* \longrightarrow \bigwedge^3 \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \bigwedge^d \mathfrak{g}^* \longrightarrow 0,$$

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

$$0 \longrightarrow k \xrightarrow{0} \mathfrak{g}^* \longrightarrow \bigwedge\nolimits^2 \mathfrak{g}^* \longrightarrow \bigwedge\nolimits^3 \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \bigwedge\nolimits^d \mathfrak{g}^* \longrightarrow 0,$$

where k is the ground field and $d = \dim \mathfrak{g}$.

Curved DG-algebras

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i ad(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

$$0 \longrightarrow k \xrightarrow{0} \mathfrak{g}^* \longrightarrow \textstyle\bigwedge^2 \mathfrak{g}^* \longrightarrow \bigwedge^3 \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \bigwedge^d \mathfrak{g}^* \longrightarrow 0,$$

where k is the ground field and $d = \dim \mathfrak{g}$. This is called the cohomological Chevalley–Eilenberg complex

A *DG-algebra* (A, d) is

- a graded associative algebra $A = \bigoplus_{i=-\infty}^{\infty} A^i$
- endowed with an operator $d\colon A^i \longrightarrow A^{i+1}$ for all $i \in \mathbb{Z}$
- satisfying Leibniz rule for the derivative of a product with signs: $d(ab) = d(a)b + (-1)^i a d(b)$ if $a \in A^i$, $b \in A^j$
- and such that $d^2\colon A^i \longrightarrow A^{i+2}$ is the zero map for all i.

For example, for any finite-dimensional Lie algebra $\mathfrak{g}$ the map $g^* \longrightarrow \bigwedge^2 \mathfrak{g}^*$ dual to the bracket map $[-,-]\colon \bigwedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$ extends to a differential d on the exterior algebra $\bigwedge \mathfrak{g}^*$ of the vector space dual to $\mathfrak{g}$, endowing $\bigwedge \mathfrak{g}^*$ with a DG-algebra structure

$$0 \longrightarrow k \xrightarrow{0} \mathfrak{g}^* \longrightarrow \textstyle\bigwedge^2 \mathfrak{g}^* \longrightarrow \bigwedge^3 \mathfrak{g}^* \longrightarrow \cdots \longrightarrow \bigwedge^d \mathfrak{g}^* \longrightarrow 0,$$

where k is the ground field and $d = \dim \mathfrak{g}$. This is called the cohomological Chevalley–Eilenberg complex or the Chevalley–Eilenberg DG-algebra of $\mathfrak{g}$.

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$,

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$,

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$
- and $h' = h + d(a) + a^2$.

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$
- and $h' = h + d(a) + a^2$.

The CDG-algebras (B, d, h) and (B, d', h') are considered to be isomorphic

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$
- and $h' = h + d(a) + a^2$.

The CDG-algebras (B, d, h) and (B, d', h') are considered to be isomorphic (the category of CDG-algebras is defined so that they are).

Curved DG-algebras

A curved DG-algebra (CDG-algebra) is

- a graded associative algebra $B = \bigoplus_{i=-\infty}^{\infty} B^i$
- endowed with an operator $d\colon B^i \longrightarrow B^{i+1}$, $i \in \mathbb{Z}$, satisfying the Leibniz rule with signs
- and with a curvature element $h \in B^2$
- such that $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Given an element $a \in B^1$, one can transform the differential and the curvature element of a CDG-algebra B by the rules

- $d'(b) = d(b) + [a, b]$, where $[a, b] = ab - (-1)^j ba$ for $b \in B^j$
- and $h' = h + d(a) + a^2$.

The CDG-algebras (B, d, h) and (B, d', h') are considered to be isomorphic (the category of CDG-algebras is defined so that they are). The element a is called a change-of-connection element.

Curved DG-algebras: Example

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety).

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras,

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

The graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ corresponding to the connection on $\mathcal{E}nd(\mathcal{E})$

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

The graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ corresponding to the connection on $\mathcal{E}nd(\mathcal{E})$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$ (equal to the curvature of the connection $\nabla_{\mathcal{E}}$)

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

The graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ corresponding to the connection on $\mathcal{E}nd(\mathcal{E})$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$ (equal to the curvature of the connection $\nabla_{\mathcal{E}}$) is a curved DG-algebra.

Curved DG-algebras: Example

Let M be a smooth manifold (or a nonsingular affine algebraic variety). Then the algebra of differential forms $\Omega(M)$ with the de Rham differential d is a DG-algebra.

Let $\mathcal{E}$ be a vector bundle on M and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Then there is an induced connection $\nabla_{\mathcal{E}nd(\mathcal{E})}$ on the vector bundle of endomorphisms $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$. This is also a bundle of associative algebras, so the differential forms on M with coefficients in $\mathcal{E}nd(\mathcal{E})$ form a graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$.

The graded algebra $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ corresponding to the connection on $\mathcal{E}nd(\mathcal{E})$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$ (equal to the curvature of the connection $\nabla_{\mathcal{E}}$) is a curved DG-algebra.

Changing the connection in $\mathcal{E}$ leads to an isomorphic CDG-algebra.

Homogeneous Quadratic Duality

Homogeneous Quadratic Duality

A quadratic algebra is

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$.

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field)

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$,

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \textstyle\sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$ has the space of generators V^*

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$ has the space of generators V^* and the space of relations $R^\perp \subset V^* \otimes V^*$

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \textstyle\sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$ has the space of generators V^* and the space of relations $R^\perp \subset V^* \otimes V^*$ (the orthogonal complement to $R \subset V \otimes V$

Homogeneous Quadratic Duality

A quadratic algebra is an associative algebra defined by quadratic relations between noncommutative variables, like for example

$$\begin{cases} xy - 2yx = 3z^2 \\ yz - 2zy = 3x^2 \\ zx - 2xz = 3y^2 \end{cases}$$

More invariantly, to define a quadratic algebra A one needs to specify a vector space of generators V and a subspace of quadratic relations $R \subset V \otimes V$. Then A is the graded algebra with the components $A_0 = k$ (the ground field), $A_1 = V$, $A_2 = V^{\otimes 2}/R$, and

$$A_n = V^{\otimes n} / \sum_{i=1}^{n-1} (V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}).$$

The quadratic dual algebra $A^!$ has the space of generators V^* and the space of relations $R^\perp \subset V^* \otimes V^*$ (the orthogonal complement to $R \subset V \otimes V$ in $V^* \otimes V^* = (V \otimes V)^*$).

Nonhomogeneous Quadratic Duality

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases}$$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense".

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \quad \text{or} \quad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations “makes sense”. For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \quad \text{or} \quad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \quad \text{or} \quad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$
$= yx - x = y - x$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy = y - 1$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy = y - 1$, hence $x = 1$.

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy = y - 1$, hence $x = 1$. Substituting $x = 1$ into $xy = y - 1$

Nonhomogeneous Quadratic Duality

Roughly speaking, a nonhomogeneous quadratic algebra is an associative algebra defined by nonhomogeneous quadratic relations, like for example

$$\begin{cases} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{cases} \qquad \text{or} \qquad \begin{cases} x^2 = -1 \\ xy + yx = 0 \\ y^2 = -1 \end{cases}$$

A delicate point is that not every system of nonhomogeneous quadratic relations "makes sense". For example

$$\begin{cases} xy = y - 1 \\ yx = y \end{cases}$$

looks fine until one realizes that it implies $(xy)x = (y-1)x$ $= yx - x = y - x$ and $x(yx) = xy = y - 1$, hence $x = 1$. Substituting $x = 1$ into $xy = y - 1$, one comes to $1 = 0$.

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations.

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \qquad x,y,z\in\mathfrak{g}$$

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \qquad x,y,z \in \mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \qquad x, y, z \in \mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$ is the self-consistency equation

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \qquad x,y,z\in\mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$ is the self-consistency equation for the system of nonhomogeneous quadratic relations

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \qquad x, y, z \in \mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$ is the self-consistency equation for the system of nonhomogeneous quadratic relations

$$xy - yx = [x, y], \qquad x, y \in \mathfrak{g}$$

Nonhomogeneous Quadratic Duality

So, for a system of nonhomogeneous quadratic relations to make sense, its coefficients must, in turn, themselves satisfy a certain system of equations, called the self-consistency equations. For example, the Jacobi identity

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \qquad x,y,z\in\mathfrak{g}$$

for the bracket of a Lie algebra $\mathfrak{g}$ is the self-consistency equation for the system of nonhomogeneous quadratic relations

$$xy-yx=[x,y], \qquad x,y\in\mathfrak{g}$$

defining the universal enveloping algebra $U(\mathfrak{g})$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.

The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.

The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$.

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1\colon B^1 \longrightarrow B^2$ and an element $h \in B^2$.
The self-consistency equations on the coefficients of the relations

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A. The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$. The self-consistency equations on the coefficients of the relations guarantee that the map d^1 extends to a well-defined differential

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A.
The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$.
The self-consistency equations on the coefficients of the relations guarantee that the map d^1 extends to a well-defined differential $d : B^i \longrightarrow B^{i+1}$ for all $i \geqslant 0$

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A. The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$. The self-consistency equations on the coefficients of the relations guarantee that the map d^1 extends to a well-defined differential $d : B^i \longrightarrow B^{i+1}$ for all $i \geqslant 0$ satisfying the Leibniz rule with signs

Nonhomogeneous Quadratic Duality: the Construction

To define a nonhomogeneous quadratic algebra $\widetilde{A}$, one needs to specify a vector space of generators V and a subspace of nonhomogeneous quadratic relations $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$. Taking the projection of the subspace $\widetilde{R}$ onto the direct summand $V^{\otimes 2}$ one obtains the associated subspace of homogeneous quadratic relations $R \subset V^{\otimes 2}$ defining a quadratic graded algebra A. The subspace $\widetilde{R} \subset V^{\otimes 2} \oplus V \oplus k$ can be then described in terms of two linear maps $R \longrightarrow V$ and $R \longrightarrow k$.

Let $B = A^!$ be the quadratic dual algebra to A. Then one has $B^1 = V^*$ and $B^2 \cong R^*$. Dualizing the maps $R \longrightarrow V$ and $R \longrightarrow k$ defining the linear and the scalar parts of the relations in $\widetilde{A}$, one obtains a linear map $d^1 \colon B^1 \longrightarrow B^2$ and an element $h \in B^2$. The self-consistency equations on the coefficients of the relations guarantee that the map d^1 extends to a well-defined differential $d : B^i \longrightarrow B^{i+1}$ for all $i \geqslant 0$ satisfying the Leibniz rule with signs, and that (B, d, h) is a curved DG-algebra.

Curved DG-algebras and A_∞-algebras

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra)

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n],\ \ n \geqslant 1$

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n \colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex,
the map m_2 is a (nonassociative) multiplication

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n \colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n \colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication.

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction to the failure of the differential m_1 to have zero square.

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n \colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction to the failure of the differential m_1 to have zero square.

This point of view is due to Getzler and Jones (Illinois J. Math. 1990)

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n \colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction to the failure of the differential m_1 to have zero square.

This point of view is due to Getzler and Jones (Illinois J. Math. 1990), who defined what is now usually called a curved A_∞-algebra

Curved DG-algebras and A_∞-algebras

An A_∞-algebra (or homotopy associative algebra) is a graded vector space $A = \bigoplus_{i=-\infty}^{\infty} A^i$ endowed with a sequence of higher multiplications $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n \geqslant 1$, satisfying a certain sequence of higher associativity equations.

The map $d = m_1$ is the differential, making A a complex, the map m_2 is a (nonassociative) multiplication, and the higher multiplications m_n, $n \geqslant 3$, are a sequence of corrections to the nonassociativity of m_2.

A curved DG-algebra can be thought of as an algebra with m_0, m_1, and m_2, where $m_0 = h$ is the curvature, $m_1 = d$ is the differential, and m_2 is the multiplication. The curvature element m_0 is a correction to the failure of the differential m_1 to have zero square.

This point of view is due to Getzler and Jones (Illinois J. Math. 1990), who defined what is now usually called a curved A_∞-algebra (with the operations m_n, $n \geqslant 0$).

Derived Homogeneous Koszul duality

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.)

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism.

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects.

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978)

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

In a somewhat more complicated form

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

In a somewhat more complicated form, this generalizes to an equivalence between the derived categories of graded modules

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

In a somewhat more complicated form, this generalizes to an equivalence between the derived categories of graded modules over a quadratic algebra A and its quadratic dual algebra $A^!$

Derived Homogeneous Koszul duality

The derived category $\mathrm{D}(\mathrm{A})$ of an abelian category A (like, e.g., the category of modules over an associative ring, etc.) is the category of complexes in A up to quasi-isomorphism. Here a morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of the cohomology modules/objects. These definitions go back to Grothendieck and Verdier (1960s).

The classical Bernstein–Gelfand–Gelfand duality (1978) provides an equivalence between the derived category of finitely-generated graded modules over the algebra of polynomials $\mathrm{Sym}(V)$ and the derived category of finite-dimensional graded modules over the exterior algebra in the dual variables $\bigwedge(V^*)$.

In a somewhat more complicated form, this generalizes to an equivalence between the derived categories of graded modules over a quadratic algebra A and its quadratic dual algebra $A^!$, provided that A has the so-called Koszul property.

Some bits of personal history

Some bits of personal history

Since about 1992 I was thinking about the problem of developing

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory.

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$.

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was:

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d)

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A, d)\text{-mod})$ for any DG-algebra (A, d), but this produces a wrong category for the above problem.

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is,

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-}\mathrm{mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-}\mathrm{mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-}\mathrm{mod})$ only depends

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*),d)$ is formal

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-}\mathrm{mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-}\mathrm{mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-}\mathrm{mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*),d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra.

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*),d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra. The cohomology algebra $H^*(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*),d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra. The cohomology algebra $H^*(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains too little information about $\mathfrak{g}$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*), d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra. The cohomology algebra $H^*(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains too little information about $\mathfrak{g}$, and there is no hope of recovering the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$

Some bits of personal history

Since about 1992 I was thinking about the problem of developing the derived nonhomogeneous Koszul duality theory. A thematic example would be the duality between the enveloping algebra $U(\mathfrak{g})$ and the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$. The question was: How can one recover the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the DG-algebra $\bigwedge(\mathfrak{g}^*)$?

It is easy to define the derived category of DG-modules $\mathrm{D}((A,d)\text{-mod})$ for any DG-algebra (A,d), but this produces a wrong category for the above problem. The situation is, the derived category $\mathrm{D}((A,d)\text{-mod})$ only depends on the quasi-isomorphism class of the DG-algebra (A,d).

In particular, for a semisimple Lie algebra $\mathfrak{g}$, the Chevalley–Eilenberg DG-algebra $(\bigwedge(\mathfrak{g}^*), d)$ is formal , i.e., it is quasi-isomorphic to its cohomology algebra. The cohomology algebra $H^*(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains too little information about $\mathfrak{g}$, and there is no hope of recovering the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ from the algebra $H^*(\mathfrak{g})$.

Derived Nonhomogeneous Koszul Duality

Derived Nonhomogeneous Koszul Duality

Thus the problem was:

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d)

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-}\mathrm{mod})$?

I solved this problem in 1999.

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

I called them the derived categories of the second kind.

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

I called them the derived categories of the second kind. Starting from about 2006

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

I called them the derived categories of the second kind. Starting from about 2006, I am now calling them the coderived

Derived Nonhomogeneous Koszul Duality

Thus the problem was: How does one define, in a natural way, an exotic derived category of a DG-algebra (A, d), so that for the Chevalley–Eilenberg DG-algebra $\bigwedge(\mathfrak{g}^*)$ this category is equivalent to the derived category of $\mathfrak{g}$-modules $\mathrm{D}(\mathfrak{g}\text{-mod})$?

I solved this problem in 1999. It turned out that there is not one, but two such natural constructions of an exotic derived category, dual to each other, both producing the derived category $\mathrm{D}(\mathfrak{g}\text{-mod})$ out of the DG-algebra $\bigwedge(\mathfrak{g}^*)$.

I called them the derived categories of the second kind. Starting from about 2006, I am now calling them the coderived and the contraderived category.

Some Bits of History

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.
- "Foundations of relative homological algebra", *Memoirs of the American Math. Society* **55**, 1965.

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.
- "Foundations of relative homological algebra", *Memoirs of the American Math. Society* **55**, 1965.
- "Homology and fibrations. I. Coalgebras, cotensor product and its derived functors", *Commentarii Mathematici Helvetici* **40**, 1966.

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.
- "Foundations of relative homological algebra", *Memoirs of the American Math. Society* **55**, 1965.
- "Homology and fibrations. I. Coalgebras, cotensor product and its derived functors", *Commentarii Mathematici Helvetici* **40**, 1966.

What I would consider as the last paper in this series was written somewhat later by a different group of authors:

Some Bits of History

Between 1962–1966, there was an series of papers published, on several closely related topics, by S. Eilenberg and J.C. Moore, including, in particular:

- "Limits and spectral sequences", *Topology* **1**, 1962.
- "Foundations of relative homological algebra", *Memoirs of the American Math. Society* **55**, 1965.
- "Homology and fibrations. I. Coalgebras, cotensor product and its derived functors", *Commentarii Mathematici Helvetici* **40**, 1966.

What I would consider as the last paper in this series was written somewhat later by a different group of authors:

- D. Husemoller, J.C. Moore, and J. Stasheff, "Differential homological algebra and homogeneous spaces", *Journ. of Pure and Appl. Algebra* **5**, 1974.

Some Bits of History

In the first paper in this series

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes.

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense)

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors.

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor, one would resolve one or both of the DG-(co)modules by a complex of DG-(co)modules

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor, one would resolve one or both of the DG-(co)modules by a complex of DG-(co)modules, take the Hom or (co)tensor product

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor, one would resolve one or both of the DG-(co)modules by a complex of DG-(co)modules, take the Hom or (co)tensor product, and totalize the resulting bicomplex

Some Bits of History

In the first paper in this series, the authors studied the problem of convergence of spectral sequences, including the spectral sequences of unbounded bicomplexes. It was realized that the two spectral sequences of an unbounded bicomplex both converge (at least in some weak sense) but, generally speaking, to two different limits.

In the subsequent papers in the series, the authors applied this understanding to the particular case of the so-called differential derived functors. This meant Ext or Tor between two DG-modules (or, as another alternative, Cotor between two DG-comodules — this was relevant in the context of what is now known as the Eilenberg–Moore spectral sequence in topology).

In order to construct the differential derived functor, one would resolve one or both of the DG-(co)modules by a complex of DG-(co)modules, take the Hom or (co)tensor product, and totalize the resulting bicomplex by taking direct sums or direct products along the diagonals.

Some Bits of History

The key issue was

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff)

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind.

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways:

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten).

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind took a DG-(co)module to be a deformation of its cohomology.

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind took a DG-(co)module to be a deformation of its cohomology. The differential derived functors of the second kind

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind took a DG-(co)module to be a deformation of its cohomology. The differential derived functors of the second kind took a DG-(co)module to be a deformation of itself with the differential forgotten.

Some Bits of History

The key issue was whether to use infinite direct sums or infinite products in order to totalize a particular unbounded bicomplex.

In the last paper of the series (by Husemoller, Moore, and Stasheff), the author discussed what they called differential derived functors of the first and second kind. The difference between the two consisted in choosing the direct sums or the direct products for the totalization.

The basic philosophy was that a DG-module can be thought of in two ways: as a deformation of its graded module of cohomology, or as a deformation of its underlying graded module (with the differentials forgotten). The differential derived functors of the first kind took a DG-(co)module to be a deformation of its cohomology. The differential derived functors of the second kind took a DG-(co)module to be a deformation of itself with the differential forgotten. This was reflected in the (weak) convergence vs. divergence of the related spectral sequences.

Some bits of personal history

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors of the first and second kind.

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors of the first and second kind. Hence the name derived categories of the second kind

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors of the first and second kind. Hence the name derived categories of the second kind for the exotic derived categories important for the derived nonhomogeneous Koszul duality purposes

Some bits of personal history

Having discovered the solution of the problem of derived nonhomogeneous Koszul duality in the Spring 1999, I went to the library in order to search for relevant literature and found this series of papers by Eilenberg–Moore and Husemoller–Moore–Stasheff.

In particular, I found the classical definitions of the differential derived functors of the first and second kind. Hence the name derived categories of the second kind for the exotic derived categories important for the derived nonhomogeneous Koszul duality purposes that I had constructed.

Some bits of personal history

I had already known since mid-90's

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality.

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra.

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra).

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore, which I found in the library.

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore, which I found in the library. It contained the definitions

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore, which I found in the library. It contained the definitions of two kinds of module objects over a coalgebra

Some bits of personal history

I had already known since mid-90's that coalgebras are important in Koszul duality. When a quadratic algebra A is infinitely generated, i.e., its space of generators V is infinite-dimensional, its quadratic dual algebra $A^!$ is properly viewed as a coalgebra.

For example, when a Lie algebra $\mathfrak{g}$ is infinite-dimensional, it makes little sense to view its cohomological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g}^*)$ as an abstract or discrete DG-algebra. One can consider $\bigwedge(\mathfrak{g}^*)$ as a pro-finite-dimensional topological algebra ("linearly compact" or "pseudo-compact" algebra). Or, better yet, one can work with the homological Chevalley–Eilenberg complex $\bigwedge(\mathfrak{g})$ instead, viewing it as a DG-coalgebra.

Back in the Spring of 1999, I also looked through the 1965 AMS Memoir "Foundations of relative homological algebra" of Eilenberg and Moore, which I found in the library. It contained the definitions of two kinds of module objects over a coalgebra: the comodules and the contramodules.

Coalgebras, Comodules, and Contramodules

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations: the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\mu_*}{\rightrightarrows} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C}\otimes\mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations: the two compositions

$$\mathcal{C} \xrightarrow{\ \mu\ } \mathcal{C}\otimes\mathcal{C} \overset{\mu_*}{\rightrightarrows} \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C}$$

should be equal to each other

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations: the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\mu_*}{\rightrightarrows} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

should be equal to each other, and the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\epsilon_*}{\rightrightarrows} \mathcal{C}$$

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a vector space endowed with linear maps $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$, called the comultiplication and counit maps.

The two maps have to satisfy the coassociativity and counitality equations: the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\mu_*}{\rightrightarrows} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$$

should be equal to each other, and the two compositions

$$\mathcal{C} \xrightarrow{\mu} \mathcal{C} \otimes \mathcal{C} \overset{\epsilon_*}{\rightrightarrows} \mathcal{C}$$

should be equal to the identity map $\mathrm{id}_{\mathcal{C}}$.

Coalgebras, Comodules, and Contramodules

A left $\mathcal{C}$-comodule $\mathcal{M}$

Coalgebras, Comodules, and Contramodules

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$

Coalgebras, Comodules, and Contramodules

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map.

Coalgebras, Comodules, and Contramodules

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a vector space endowed with a linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a vector space endowed with a linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, called the left contraaction map.

Coalgebras, Comodules, and Contramodules

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a vector space endowed with a linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, called the left contraaction map. The contraaction map has to satisfy the contraassociativity and contraunitality equations:

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a vector space endowed with a linear map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$, called the left coaction map. The coaction map also has to satisfy the coassociativity and counitality equations:

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \rightrightarrows \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{M}$$

$$\mathcal{M} \xrightarrow{\nu} \mathcal{C} \otimes \mathcal{M} \xrightarrow{\epsilon_*} \mathcal{M}$$

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a vector space endowed with a linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, called the left contraaction map. The contraaction map has to satisfy the contraassociativity and contraunitality equations:

$$\operatorname{Hom}(\mathcal{C}, \operatorname{Hom}(\mathcal{C}, \mathfrak{P})) = \operatorname{Hom}(\mathcal{C} \otimes \mathcal{C},\ \mathfrak{P}) \rightrightarrows \operatorname{Hom}(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}$$

$$\mathfrak{P} \xrightarrow{\epsilon^*} \operatorname{Hom}(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}$$

Thus the definition of a contramodule is very similar to that of a comodule, up to duality.

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

Coalgebras, Comodules, and Contramodules

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products.

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products.

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

Coalgebras, Comodules, and Contramodules

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

The abelian category of left $\mathcal{C}$-comodules has enough injective objects

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

The abelian category of left $\mathcal{C}$-comodules has enough injective objects, but may have no projectives.

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

The abelian category of left $\mathcal{C}$-comodules has enough injective objects, but may have no projectives. The abelian category of left $\mathcal{C}$-contramodules has enough projective objects

Thus the definition of a contramodule is very similar to that of a comodule, up to duality. So are their properties.

For any coalgebra $\mathcal{C}$ over a field k, left $\mathcal{C}$-comodules form an abelian category with infinite direct sums and products. The functors of infinite direct sum are exact and agree with the direct sums of vector spaces (but the products aren't and don't).

Left $\mathcal{C}$-contramodules also form an abelian category with infinite direct sums and products. The functors of infinite product are exact and agree with the products of vector spaces (but the direct sums aren't and don't).

The abelian category of left $\mathcal{C}$-comodules has enough injective objects, but may have no projectives. The abelian category of left $\mathcal{C}$-contramodules has enough projective objects, but may have no injectives.

Some Bits of History

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules.

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history"

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules $= 797$,

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules $= 797$,
- contramodules $= 3$.

The three publications on contramodules that this statistics referred to

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules $=$ 797,
- contramodules $=$ 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir)

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish)

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled “A bit of history”, which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits (“Anywhere = ...”) is

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = ...") is:

- comodules = 1323,
- contramodules = 14.

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = ...") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = . . .") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding, this leads to 11 actual papers mentioning contramodules

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = ...") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding, this leads to 11 actual papers mentioning contramodules (of which 4 are mine).

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = . . . ") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding, this leads to 11 actual papers mentioning contramodules (of which 4 are mine). The first one after 1970

Some Bits of History

At a conference in Split, Croatia, in September 2007, Tomasz Brzeziński gave a talk on contramodules. At this talk, he had a slide titled "A bit of history", which contained the following statistics of MathSciNet search hits:

- comodules = 797,
- contramodules = 3.

The three publications on contramodules that this statistics referred to were dated 1965 (the Eilenberg–Moore memoir), 1965 (an obscure Mexican paper by Vázquez García, in Spanish), and 1970 (a rather remarkable paper by Barr in Math. Zeitschrift).

Eleven years later, the current statistic of MathSciNet search hits ("Anywhere = ...") is:

- comodules = 1323,
- contramodules = 14.

Double count and accidental hits excluding, this leads to 11 actual papers mentioning contramodules (of which 4 are mine). The first one after 1970 is dated 2009.

Some bits of personal history

Some bits of personal history

The semi-infinite homology

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures are properly associated with an algebra object in the tensor category of bicomodules over a coalgebra.

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures are properly associated with an algebra object in the tensor category of bicomodules over a coalgebra. I called such structures semialgebras

Some bits of personal history

The semi-infinite homology of some infinite-dimensional Lie algebras was defined by Feigin in 1984. The same concept was discovered by physicists, who call it "BRST".

The problem of defining the semi-infinite homology of associative algebras was posed by Feigin at his seminar in Moscow sometime in 1994–95. The first solution was suggested by Arkhipov, who wrote a series of papers about it between 1996–2002.

Over the years, I tried to understand Arkhipov's construction, in order to generalize it and reformulate in more aesthetically appealing terms. A breakthrough in my understanding came in the Summer 2000, when I realized that semi-infinite (co)homology of associative algebraic structures are properly associated with an algebra object in the tensor category of bicomodules over a coalgebra. I called such structures semialgebras (meaning "half algebra and half coalgebra").

The semi-infinite homology

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces)

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra.

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces)

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a “semimodule” meant “half module and half comodule”.

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule.

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a “semimodule” meant “half module and half comodule”.

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a “semicontramodule” meant

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a “semimodule” meant “half module and half comodule”.

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a “semicontramodule” meant “half module and half contramodule”.

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules.

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too, should include the derived categories of DG-modules

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too, should include the derived categories of DG-modules, the coderived categories of CDG-comodules

Some bits of personal history

The semi-infinite homology (or the semi-infinite Tor spaces) were assigned to a pair of semimodules, a right one and a left one, over the semialgebra. Here a "semimodule" meant "half module and half comodule".

The semi-infinite cohomology (or the semi-infinite Ext spaces) were assigned to a left semimodule and a left semicontramodule. Here a "semicontramodule" meant "half module and half contramodule".

Thus it turned out that the proper definition of semi-infinite cohomology of associative algebraic structures required contramodules. Forgotten for 30 years, and accidentally found in my library searches in 1999, contramodules found their first uses in the semi-infinite homological algebra.

Soon I realized that the derived nonhomogeneous Koszul duality theory, too, should include the derived categories of DG-modules, the coderived categories of CDG-comodules, and the contraderived categories of CDG-contramodules.

Coalgebras, Comodules and Contramodules: an Example

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, …

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, ... endowed with the comultiplication

$$\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$$

Coalgebras, Comodules and Contramodules: an Example

Coalgebras $\mathcal{C}$ over a field k can be described in terms of their dual pro-finite-dimensional topological algebras $\mathcal{C}^*$.

In particular, let $\mathcal{C}$ be the coalgebra whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication

$$\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$$

and the counit

$$\varepsilon(1^*) = 1, \qquad \varepsilon(t^{n*}) = 0 \quad \text{for } n \geqslant 1.$$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows.

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator.

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Thus a $\mathcal{C}$-comodule

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Thus a $\mathcal{C}$-comodule, for the coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[t]]$

Coalgebras, Comodules and Contramodules: an Example

Comodules $\mathcal{M}$ over this coalgebra $\mathcal{C}$ can be described as follows. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes \mathcal{M}$ has the form

$$\nu(m) = \sum_{i=0}^{\infty} t^{i*} \otimes t^i m$$

where $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ is a certain linear operator. Since the sum in the right-hand side must be finite, it follows that t must be locally nilpotent, that is for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Thus a $\mathcal{C}$-comodule, for the coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[t]]$, is the same thing as a t-torsion $k[t]$-module.

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]]$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i$$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} =$$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n$$

Coalgebras, Comodules and Contramodules: an Example

For the same coalgebra $\mathcal{C}$, a $\mathcal{C}$-contramodule is vector space $\mathfrak{P}$ endowed with a linear map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) = \mathfrak{P}[[t]] \longrightarrow \mathfrak{P}$.

In other words, this means that a $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, $\ldots \in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- which must satisfy the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n \sum_{i+j=n} p_{ij}.$$

Contramodules over Coalgebras: a Counterexample

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p =$

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$,

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0,\ p_1,\ p_2\ \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0,\ p_1,\ p_2\ \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 ... $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 ... $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit of finite partial sums.

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit of finite partial sums. This is a new concept of infinite sum in mathematics

Contramodules over Coalgebras: a Counterexample

For the same coalgebra $\mathcal{C}$, for any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define $t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Thus the contramodule infinite summation operation cannot be understood as any kind of limit of finite partial sums. This is a new concept of infinite sum in mathematics, quite different from the ones usually studied in analysis.

Example: Contramodules over the p-Adic Integers

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$ as assigned to the topological algebra $\mathcal{C}^*$ dual to $\mathcal{C}$

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$ as assigned to the topological algebra $\mathcal{C}^*$ dual to $\mathcal{C}$, and then try to extend this assignment to topological rings of more general nature.

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$ as assigned to the topological algebra $\mathcal{C}^*$ dual to $\mathcal{C}$, and then try to extend this assignment to topological rings of more general nature.

In particular, here is a definition of contramodules over the topological ring of p-adic integers $\mathbb{Z}_p$ for a prime number p

Example: Contramodules over the p-Adic Integers

One can view the abelian category of left $\mathcal{C}$-contramodules for a coalgebra $\mathcal{C}$ as assigned to the topological algebra $\mathcal{C}^*$ dual to $\mathcal{C}$, and then try to extend this assignment to topological rings of more general nature.

In particular, here is a definition of contramodules over the topological ring of p-adic integers $\mathbb{Z}_p$ for a prime number p, based on the analogy between $\mathbb{Z}_p$ and $k[[t]]$.

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, … $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

- unitality + compatibility

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

- unitality + compatibility: $\sum_{n=0}^{\infty} p^n c_n = c_0 + pc_1$ when $c_i = 0$ for all $i \geqslant 2$,

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

- unitality + compatibility: $\sum_{n=0}^{\infty} p^n c_n = c_0 + pc_1$ when $c_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} p^j c_{ij} =$$

Example: Contramodules over p-Adic Integers

A $\mathbb{Z}_p$-contramodule $\mathfrak{C}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements c_0, c_1, c_2, ... $\in \mathfrak{C}$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n \in \mathfrak{C}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} p^n(ac_n + bd_n) = a\sum_{n=0}^{\infty} p^n c_n + b\sum_{n=0}^{\infty} p^n d_n,$$

- unitality + compatibility: $\sum_{n=0}^{\infty} p^n c_n = c_0 + pc_1$ when $c_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} p^j c_{ij} = \sum_{n=0}^{\infty} p^n \sum_{i+j=n} c_{ij}.$$

Some bits of personal history

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras.

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com).

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras.

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology did not exist in my seminar talks and writings of the first half of 00's.

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology did not exist in my seminar talks and writings of the first half of 00's. I picked it up sometime in 2005–06

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology did not exist in my seminar talks and writings of the first half of 00's. I picked it up sometime in 2005–06 from B. Keller's unpublished 2003 note "Koszul duality and coderived categories (after K. Lefèvre)"

Some bits of personal history

In Summer 2000 I wrote a series of letters to S. Arkhipov and R. Bezrukavnikov about the semi-infinite (co)homology of associative algebras. In Summer 2002 I wrote a second series of letters, and put them all on the internet (in my blog on livejournal.com). The last letter in the first series contained an extensive discussion of contramodules over coalgebras and (what are now called) semicontramodules over semialgebras. The second series discussed what are now called the semiderived categories.

The "coderived/contraderived categories" terminology did not exist in my seminar talks and writings of the first half of 00's. I picked it up sometime in 2005–06 from B. Keller's unpublished 2003 note "Koszul duality and coderived categories (after K. Lefèvre)", which is still available from his homepage.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters).

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring, which we will discuss below

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring, which we will discuss below, was invented between 2007–08

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring, which we will discuss below, was invented between 2007–08 and first appeared in a remark in the June 2008 update of this preprint.

Some bits of personal history

The above definition of a contramodule over the p-adic integers I invented sometime in the Summer 2003.

My letters about the semi-infinite (co)homology of associative algebras were written in transliterated Russian (with Latin letters). Brzeziński was able to read them, and he refers to them, with a link to my blog, from a January 2007 version of one of his arXiv preprints. A 2006 paper of Gaitsgory and Kazhdan referred to the same material as "private communications".

The first version of my monograph on semi-infinite homological algebra appeared on the arXiv in August 2007. The modern general definition of a contramodule over a topological ring, which we will discuss below, was invented between 2007–08 and first appeared in a remark in the June 2008 update of this preprint. This work was published as a book in September 2010.

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x \in X} r_x c_x$$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum\nolimits_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x \in X}$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x \in X}$ is said to converge to zero in $\mathfrak{R}$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x \in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x \in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum_{x \in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x \in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y}$$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}$$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}, \quad r_x, s_{x,y} \in \mathfrak{R}, \ c_{x,y} \in \mathfrak{C},$$

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}, \quad r_x, s_{x,y} \in \mathfrak{R},\ c_{x,y} \in \mathfrak{C},$$

$$\textstyle\sum_{x,y} r_{x,y} c_x$$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}, \quad r_x, s_{x,y} \in \mathfrak{R},\ c_{x,y} \in \mathfrak{C},$$

$$\textstyle\sum_{x,y} r_{x,y} c_x = \sum_x (\sum_y r_{x,y}) c_x$$

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Then a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations. For every set X, every X-indexed family of elements $r_x \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, and every X-indexed family of elements $c_x \in \mathfrak{C}$, an element denoted formally by

$$\textstyle\sum_{x\in X} r_x c_x \in \mathfrak{C}$$

must be specified. Here a family of elements $(r_x \in \mathfrak{R})_{x\in X}$ is said to converge to zero in $\mathfrak{R}$ if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ one has $r_x \in \mathfrak{U}$ for all but a finite subset of $x \in X$. These infinite summation operations must satisfy certain axioms, such as the contraassociativity and the distributivity

$$\textstyle\sum_x r_x \sum_y s_{x,y} c_{x,y} = \sum_{x,y} (r_x s_{x,y}) c_{x,y}, \quad r_x, s_{x,y} \in \mathfrak{R},\ c_{x,y} \in \mathfrak{C},$$

$$\textstyle\sum_{x,y} r_{x,y} c_x = \sum_x (\sum_y r_{x,y}) c_x, \quad r_{x,y} \in \mathfrak{R},\ c_x \in \mathfrak{C}.$$

Some Bits of History

Some Bits of History

What was then called

Some Bits of History

What was then called cotorsion modules

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s.

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, “Infinite abelian groups and homological methods”, *Annals of Math.* **69**, 1959.
- R.J. Nunke, “Modules of extensions over Dedekind rings”, *Illinois Journ. of Math.* **3**, 1959.

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, "1-dimensional Cohen–Macaulay rings", *Lecture Notes in Math.* **327**, Springer, 1973.

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, "1-dimensional Cohen–Macaulay rings", *Lecture Notes in Math.* **327**, Springer, 1973.

From the beginning of 1980s

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, "1-dimensional Cohen–Macaulay rings", *Lecture Notes in Math.* **327**, Springer, 1973.

From the beginning of 1980s, after the works of L. Salce and E. Enochs

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, “Infinite abelian groups and homological methods”, *Annals of Math.* **69**, 1959.
- R.J. Nunke, “Modules of extensions over Dedekind rings”, *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, “Cotorsion modules”, *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, “1-dimensional Cohen–Macaulay rings”, *Lecture Notes in Math.* **327**, Springer, 1973.

From the beginning of 1980s, after the works of L. Salce and E. Enochs, the word “cotorsion” started to mean something else

Some Bits of History

What was then called cotorsion modules were introduced and studied from the end of 1950s to the first half of 1970s. The key authors and publications were

- D.K. Harrison, "Infinite abelian groups and homological methods", *Annals of Math.* **69**, 1959.
- R.J. Nunke, "Modules of extensions over Dedekind rings", *Illinois Journ. of Math.* **3**, 1959.
- E. Matlis, "Cotorsion modules", *Memoirs of the American Math. Society* **49**, 1964.
- E. Matlis, "1-dimensional Cohen–Macaulay rings", *Lecture Notes in Math.* **327**, Springer, 1973.

From the beginning of 1980s, after the works of L. Salce and E. Enochs, the word "cotorsion" started to mean something else (not unrelated, but quite different).

Some Bits of History

Some Bits of History

Let R be a commutative ring

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset.

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted.

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0$

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules.

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

After 1980s, people started to call "Matlis cotorsion"

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

After 1980s, people started to call "Matlis cotorsion" (or "weakly cotorsion", which is a better term)

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\operatorname{Hom}_R(S^{-1}R, C) = 0 = \operatorname{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\operatorname{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

After 1980s, people started to call "Matlis cotorsion" (or "weakly cotorsion", which is a better term) the R-modules C such that $\operatorname{Ext}^1_R(S^{-1}R, C) = 0$

Some Bits of History

Let R be a commutative ring and $S \subset R$ be a multiplicative subset. Let $S^{-1}R$ be ring R with the elements from S inverted. For the purposes of this talk, let us say that an R-module C is Matlis S-cotorsion if $\mathrm{Hom}_R(S^{-1}R, C) = 0 = \mathrm{Ext}^1_R(S^{-1}R, C)$.

This definition captures what Harrison and Matlis meant by cotorsion modules. It is well-behaved when the R-module $S^{-1}R$ has projective dimension at most 1.

After the paper

- E. Enochs, "Flat covers and flat cotorsion modules", *Proceedings of the Amer. Math. Soc.* **92**, 1984

by a cotorsion module people generally mean a left module C over an associative ring R such that $\mathrm{Ext}^1_R(F, C) = 0$ for all flat left R-modules F.

After 1980s, people started to call "Matlis cotorsion" (or "weakly cotorsion", which is a better term) the R-modules C such that $\mathrm{Ext}^1_R(S^{-1}R, C) = 0$ (dropping the condition of Hom vanishing).

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings, they appear in subsequent topological literature, such as

- M. Hovey, N.P. Strickland, "Morava K-theories and localization", *Memoirs Amer. Math. Soc.* **139**, #666, 1999.

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings, they appear in subsequent topological literature, such as

- M. Hovey, N.P. Strickland, "Morava K-theories and localization", *Memoirs Amer. Math. Soc.* **139**, #666, 1999.

In the greater generality of weakly proregular finitely generated ideals I in commutative rings R

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings, they appear in subsequent topological literature, such as

- M. Hovey, N.P. Strickland, "Morava K-theories and localization", *Memoirs Amer. Math. Soc.* **139**, #666, 1999.

In the greater generality of weakly proregular finitely generated ideals I in commutative rings R, they were studied by A. Yekutieli and collaborators (since 2010)

Some Bits of History

In the case of the ring of integers $R = \mathbb{Z}$ and a prime number $s = p$, what I call below Matlis s-cotorsion modules were discussed under the names of Ext-p-complete or weakly p-complete abelian groups in the topology book and the algebraic geometry paper

- A.K. Bousfield, D.M. Kan, "Homotopy limits, completions and localizations", *Lecture Notes in Math.* **304**, 1972.
- U. Jannsen, "Continuous étale cohomology", *Math. Annalen* **280**, 1988.

In the generality of regular maximal ideals in commutative Noetherian rings, they appear in subsequent topological literature, such as

- M. Hovey, N.P. Strickland, "Morava K-theories and localization", *Memoirs Amer. Math. Soc.* **139**, #666, 1999.

In the greater generality of weakly proregular finitely generated ideals I in commutative rings R, they were studied by A. Yekutieli and collaborators (since 2010) under the name of cohomologicaly I-adically complete modules.

Example: the Adic Topology

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($=$ I-adic) topology.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-}\mathrm{contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit $(= I\text{-adic})$ topology. Consider the abelian category $\mathfrak{R}\text{-}\mathrm{contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16)

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit $(= I$-adic$)$ topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory?

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n\geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($=$ I-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n\geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion if it is Matlis S-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \dots\}$.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($=$ I-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion if it is Matlis S-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \dots\}$. An R-module C is Matlis I-cotorsion

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($= I$-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion if it is Matlis S-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \dots\}$. An R-module C is Matlis I-cotorsion if it is Matlis s-cotorsion for all $s \in I$.

Example: the Adic Topology

Let R be a commutative ring and $I \subset R$ be a finitely-generated ideal. Denote by $\mathfrak{R}$ the I-adic completion of the ring R, that is $\mathfrak{R} = \varprojlim_{n \geqslant 1} R/I^n$, and endow $\mathfrak{R}$ with the projective limit ($=$ I-adic) topology. Consider the abelian category $\mathfrak{R}\text{-contra}$ of contramodules over the topological ring $\mathfrak{R}$.

Then one can prove (L.P., 2008–16) that the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful, so $\mathfrak{R}\text{-contra}$ is a full subcategory in $R\text{-mod}$.

How to describe this full subcategory? Following the above terminology, given an element $s \in R$ we will say that an R-module C is Matlis s-cotorsion if it is Matlis S-cotorsion for the multiplicative subset $S = \{1, s, s^2, s^2, \dots\}$. An R-module C is Matlis I-cotorsion if it is Matlis s-cotorsion for all $s \in I$. (One can prove that it suffices to check this condition for any given set of generators $s_1, \dots, s_m$ of the ideal I.)

Example: the Adic Topology

Denote by $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-}\mathrm{mod}$.

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian.

Example: the Adic Topology

Denote by $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-}\mathrm{mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-}\mathrm{contra} = R\text{-}\mathrm{mod}_{I\text{-}\mathrm{mcot}} \subset R\text{-}\mathrm{mod}.$

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}.$

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has
$\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}.$

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has $\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$. *For so-called weakly proregular ideals I*

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}.$

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has $\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$. *For so-called weakly proregular ideals I, one has* $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}}$.

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has $\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$. *For so-called weakly proregular ideals I, one has* $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}}$. *Generally speaking, any Matlis I-cotorsion R-module*

Example: the Adic Topology

Denote by $R\text{-mod}_{I\text{-mcot}}$ the full subcategory of Matlis I-cotorsion R-modules in $R\text{-mod}$.

Theorem (L.P., 2008–2012)

Assume that R is Noetherian. Then one has
$\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$.

Theorem (L.P., 2017)

For any finitely generated ideal I in a commutative ring R, one has $\mathfrak{R}\text{-contra} \subset R\text{-mod}_{I\text{-mcot}} \subset R\text{-mod}$. For so-called weakly proregular ideals I, one has $\mathfrak{R}\text{-contra} = R\text{-mod}_{I\text{-mcot}}$. Generally speaking, any Matlis I-cotorsion R-module is an extension of two $\mathfrak{R}$-contramodules.

Some bits of personal history

Some bits of personal history

I first heard the word "cotorsion"

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn.

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring.

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules

Some bits of personal history

I first heard the word “cotorsion” from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

“Consider the exact category of flat modules over a ring. Does it have enough injective objects?”

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called “flat cover conjecture”

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs.

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs. These were cotorsion modules in Enochs' sense, of course.

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs. These were cotorsion modules in Enochs' sense, of course.

I first looked into Harrison's 1959 paper on cotorsion abelian groups only in 2012

Some bits of personal history

I first heard the word "cotorsion" from Jan Šťovíček in May 2009, during a workshop in Paderborn. Several of us participants of the workshop were sitting together in a classroom and discussing various mathematics, and I asked the following question, motivated by my work on semi-infinite homological algebra:

"Consider the exact category of flat modules over a ring. Does it have enough injective objects?"

Jan answered that it did, that these were called flat cotorsion modules, that the key result in this connection was called "flat cover conjecture", and the name of the key author was Enochs. These were cotorsion modules in Enochs' sense, of course.

I first looked into Harrison's 1959 paper on cotorsion abelian groups only in 2012, and learned about Matlis' work on cotorsion modules as late as in 2015–16.

Terminological Conclusion

Terminological Conclusion

What Harrison and Matlis called cotorsion modules

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called “weakly” or “cohomologically” complete modules

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called “weakly” or “cohomologically” complete modules, from the contemporary point of view are properly considered

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called "weakly" or "cohomologically" complete modules, from the contemporary point of view are properly considered as species of contramodules.

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called "weakly" or "cohomologically" complete modules, from the contemporary point of view are properly considered as species of contramodules.

What I call here "Matlis S-cotorsion" and "Matlis I-cotorsion R-modules"

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called "weakly" or "cohomologically" complete modules, from the contemporary point of view are properly considered as species of contramodules.

What I call here "Matlis S-cotorsion" and "Matlis I-cotorsion R-modules", are actually called "S-contramodule R-modules" and "I-contramodule R-modules"

Terminological Conclusion

What Harrison and Matlis called cotorsion modules, and what some later authors called "weakly" or "cohomologically" complete modules, from the contemporary point of view are properly considered as species of contramodules.

What I call here "Matlis S-cotorsion" and "Matlis I-cotorsion R-modules", are actually called "S-contramodule R-modules" and "I-contramodule R-modules" in my papers.

Some bits of personal history

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry.

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact)

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum.

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums).

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums).

In Spring 2012 I solved this problem

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums).

In Spring 2012 I solved this problem by inventing the definition of the exact category of contraherent cosheaves

Some bits of personal history

The notion of a quasi-coherent sheaf over an algebraic variety is the main technical tool of algebraic geometry. Quasi-coherent sheaves form an abelian category with exact direct limits, and in particular exact direct sums, but infinite products of quasi-coherent sheaves are not well-behaved.

The construction of coderived category makes sense for any abelian (or exact) category with exact functors of infinite direct sum. Dually, the contraderived category is well-defined for any abelian or exact category with exact functors of infinite product.

Since Spring 2009, I wanted to assign to every algebraic variety a geometric module category similar to but different from the quasi-coherent sheaves, in that it would have exact functors of infinite product (but possibly nonexact direct sums).

In Spring 2012 I solved this problem by inventing the definition of the exact category of contraherent cosheaves, which is assigned to any algebraic variety and has exact functors of infinite product.

Quasi-Coherent Sheaves and Contraherent Cosheaves

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme")

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or “scheme”) is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or “scheme”) is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of “restriction of functions”) $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism $\mathcal{M}(V) = S \otimes_R \mathcal{M}(U)$.

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism $\mathcal{M}(V) = S \otimes_R \mathcal{M}(U)$. In a contraherent cosheaf $\mathfrak{P}$

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism $\mathcal{M}(V) = S \otimes_R \mathcal{M}(U)$. In a contraherent cosheaf $\mathfrak{P}$, it is an isomorphism

Quasi-Coherent Sheaves and Contraherent Cosheaves

A nonaffine algebraic variety (or "scheme") is obtained by gluing together affine pieces, and both the quasi-coherent sheaves and the contraherent cosheaves are the result of gluing modules over the rings of functions over an affine cover of the scheme. How does the gluing construction work?

Let U be an affine variety/scheme and $V \subset U$ be an affine open subscheme. Let $R = \mathcal{O}(U)$ and $S = \mathcal{O}(V)$ be their rings of functions. Then there is a ring homomorphism (of "restriction of functions") $R \longrightarrow S$. In the simplest case of a principal affine open subscheme, one has $S = R[f^{-1}]$, where $f \in R$ is an element.

In a quasi-coherent sheaf $\mathcal{M}$, the modules of sections over U and V are connected by an isomorphism $\mathcal{M}(V) = S \otimes_R \mathcal{M}(U)$. In a contraherent cosheaf $\mathfrak{P}$, it is an isomorphism $\mathfrak{P}[V] = \mathrm{Hom}_R(S, \mathfrak{P}[U])$.

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module.

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not.

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory.

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general.

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1, and there are other properties.

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1, and there are other properties.

A narrow class of R-modules

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1, and there are other properties.

A narrow class of R-modules to which modules like S belong

Quasi-Coherent Sheaves and Contraherent Cosheaves

The definition of a quasi-coherent sheaf works nicely, and provides an abelian category, because for every affine open subscheme V in an affine scheme U the ring $S = \mathcal{O}(V)$ is a flat module over the ring $R = \mathcal{O}(U)$.

For the definition of a contraherent cosheaf to work similarly, one would need S to be a projective R-module. But it is not. This is the reason why the category of contraherent cosheaves is only exact, and not abelian.

Thus homological properties of the $\mathcal{O}(U)$-modules $\mathcal{O}(V)$ are very important in the contraherent cosheaf theory. While not projective, these modules have much better properties than flat modules in general. In particular, the projective dimension of the R-module S never exceeds 1, and there are other properties.

A narrow class of R-modules to which modules like S belong is called the class of very flat R-modules.

Some Bits of History

Some Bits of History

In the contemporary language

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\operatorname{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\operatorname{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa.

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\mathrm{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa. This concept was introduced in the paper

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\mathrm{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa. This concept was introduced in the paper

- L. Salce, "Cotorsion theories for abelian groups", *Symposia Mathematica* **XXIII**, 1979.

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\operatorname{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa. This concept was introduced in the paper

- L. Salce, "Cotorsion theories for abelian groups", *Symposia Mathematica* **XXIII**, 1979.

After the paper

- P.C. Eklof, J. Trlifaj, "How to make Ext vanish", *Bull. London Math. Soc.* **33**, 2001

Some Bits of History

In the contemporary language, a cotorsion pair means a pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ such that an R-module F belongs to $\mathcal{F}$ if and only if $\mathrm{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and vice versa. This concept was introduced in the paper

- L. Salce, "Cotorsion theories for abelian groups", *Symposia Mathematica* **XXIII**, 1979.

After the paper

- P.C. Eklof, J. Trlifaj, "How to make Ext vanish", *Bull. London Math. Soc.* **33**, 2001

it gradually became one of the most powerful technical tools in homological algebra of rings and modules.

Cotorsion Pairs

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\operatorname{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class)

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if it is a direct summand of an $\mathcal{S}$-filtered R-module G.

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if it is a direct summand of an $\mathcal{S}$-filtered R-module G. The latter condition means that there exists an ordinal α

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\operatorname{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if it is a direct summand of an $\mathcal{S}$-filtered R-module G. The latter condition means that there exists an ordinal α and an increasing filtration G_i of G indexed by the ordinals $0 \leqslant i \leqslant \alpha$

Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is said to be generated by a class of modules $\mathcal{S}$ if $\mathcal{C}$ is the class of all R-modules C such that $\mathrm{Ext}^1_R(S, C) = 0$ for all $S \in \mathcal{S}$.

It is (essentially) proved in the paper of Eklof and Trlifaj that, for any set (rather than a proper class) of R-modules $\mathcal{S}$ containing the R-module R, the class $\mathcal{F}$ in the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ can be described as follows.

An R-module F belongs to $\mathcal{F}$ if and only if it is a direct summand of an $\mathcal{S}$-filtered R-module G. The latter condition means that there exists an ordinal α and an increasing filtration G_i of G indexed by the ordinals $0 \leqslant i \leqslant \alpha$ such that every successive quotient module G_{i+1}/G_i is isomorphic to a module from $\mathcal{S}$.

Very Flat Modules and Very Flat Conjecture

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\operatorname{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\operatorname{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\operatorname{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$ by a finitely generated ideal.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \dots, x_m]$ by a finitely generated ideal. Assume that T is a flat R-module.

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\operatorname{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$ by a finitely generated ideal. Assume that T is a flat R-module.

The Very Flat Conjecture (now theorem)

Very Flat Modules and Very Flat Conjecture

Let R be a commutative ring, and let $\mathcal{S}$ denote the set of all R-modules of the form $S = R[s^{-1}]$, where $s \in R$. Let $(\mathcal{F}, \mathcal{C})$ be the cotorsion pair generated by $\mathcal{S}$.

So $\mathcal{C}$ is the class of all R-modules such that $\operatorname{Ext}^1_R(R[s^{-1}], C) = 0$ for all $s \in R$, and $\mathcal{F}$ is the class of all direct summands of R-modules filtered by the R-modules $R[s^{-1}]$.

R-modules from $\mathcal{F}$ are called very flat, while R-modules from $\mathcal{C}$ are called contraadjusted (which means "adjusted to contraherent cosheaves").

Let T be a finitely presented commutative R-algebra, that is, a quotient ring of a ring of polynomials $R[x_1, \ldots, x_m]$ by a finitely generated ideal. Assume that T is a flat R-module.

The Very Flat Conjecture (now theorem) claims that T is then a very flat R-module.

Some bits of personal history

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings)

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs.

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University.

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Soon it was agreed that I would come to Brno and Prague again as an ECI visitor.

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Soon it was agreed that I would come to Brno and Prague again as an ECI visitor. There was a masters student in Prague

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Soon it was agreed that I would come to Brno and Prague again as an ECI visitor. There was a masters student in Prague, Alexander Slávik his name

Some bits of personal history

The Very Flat Conjecture was formulated (for Noetherian rings), and some particular cases of it were proved, in the 4th, February 2014 version of my long preprint on contraherent cosheaves (the first version of which was dated September 2012).

In mid-March 2014 I suddenly landed in Czech Republic for the first time in my life, just as a tourist, taking unpaid vacations from my Moscow jobs. I contacted Rosický in Brno and Šťovíček in Prague, and came to the Department of Algebra of Charles University. It turned out that Jan Trlifaj was working there, and that local people had noticed my preprint on contraherent cosheaves, where the paper of Eklof and Trlifaj was cited.

Soon it was agreed that I would come to Brno and Prague again as an ECI visitor. There was a masters student in Prague, Alexander Slávik his name, who was going to start studying very flat and contraadjusted modules.

Some Bits of History

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017.

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017. The preprint appeared on the arXiv in August.

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017. The preprint appeared on the arXiv in August.

The argument was based on a heavy use of

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017. The preprint appeared on the arXiv in August.

The argument was based on a heavy use of what are above called Matlis s-cotorsion and Matlis I-cotorsion R-modules

Some Bits of History

The paper of Slávik and Trlifaj on very flat, locally very flat, and contraadjusted modules was published in Journ. of Pure and Applied Algebra in 2016.

A proof of the Very Flat Conjecture was found, jointly by Slávik and me, during my visit to Prague in June 2017. The preprint appeared on the arXiv in August.

The argument was based on a heavy use of what are above called Matlis s-cotorsion and Matlis I-cotorsion R-modules (called the s-contramodule and I-contramodule R-modules in the paper).

Description of Flat Modules

Description of Flat Modules

In a companion paper

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$.

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem, while surpisingly completely different

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem, while surpisingly completely different, are both based on a heavy use of Matlis S-cotorsion R-modules

Description of Flat Modules

In a companion paper, we obtained the following relatively explicit description of flat modules over commutative Noetherian rings with countable spectrum (e.g., countable Noetherian rings).

Theorem (A. Slávik and L.P., 2017)

For any Noetherian commutative ring R with countable spectrum, there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of an R-module filtered by $S_j^{-1}R$, $j \geqslant 1$. When R has finite Krull dimension d, a finite collection of at most $m = 2^{(d+1)^2/4}$ multiplicative subsets is sufficient.

The proofs of the two assertions of the theorem, while surpisingly completely different, are both based on a heavy use of Matlis S-cotorsion R-modules (called S-contramodule R-modules in the paper).

Flat Ring Epimorphisms of Countable Type

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism.

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base.

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base. Then the left R-module U has projective dimension

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base. Then the left R-module U has projective dimension at most 1.

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base. Then the left R-module U has projective dimension at most 1.

The proof is based on a heavy use of contramodules over a certain topological ring $\mathfrak{R}$

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ be a homomorphism of associative rings such that U is a flat left R-module and the multiplication map $U \otimes_R U \longrightarrow U$ is an isomorphism. Consider the filter $\mathbb{G}$ of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$, and assume that the filter $\mathbb{G}$ has a countable base. Then the left R-module U has projective dimension at most 1.

The proof is based on a heavy use of contramodules over a certain topological ring $\mathfrak{R}$ (namely, the completion of R with respect to the topology

Flat Ring Epimorphisms of Countable Type

Here is a recent application of contramodules to a presently somewhat popular area of noncommutative algebra.

Theorem (L.P., 2018)

Let $R \longrightarrow U$ *be a homomorphism of associative rings such that* U *is a flat left* R*-module and the multiplication map* $U \otimes_R U \longrightarrow U$ *is an isomorphism. Consider the filter* $\mathbb{G}$ *of all right ideals* $I \subset R$ *such that* $R/I \otimes_R U = 0$*, and assume that the filter* $\mathbb{G}$ *has a countable base. Then the left* R*-module* U *has projective dimension at most* 1*.*

The proof is based on a heavy use of contramodules over a certain topological ring $\mathfrak{R}$ (namely, the completion of R with respect to the topology where $\mathbb{G}$ is a base of neighborhoods of zero).

Historical Conclusion

Historical Conclusion

Various species of contramodules

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally).

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years, from 1999 to 2012 or even 2015–17

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years, from 1999 to 2012 or even 2015–17, to discover and understand the connection.

Historical Conclusion

Various species of contramodules fill a big gap in the big picture of the present-day homological algebra (or even algebra generally). They interplay with such concepts as curved DG-algebras, contraderived categories, and contraherent cosheaves. Introduced originally in a 1965 AMS Memoir of Eilenberg and Moore, contramodules over coalgebras were completely forgotten for three decades, until I found them in this memoir in 1999.

Cotorsion abelian groups were introduced by Harrison in 1959, and cotorsion modules were studied by Matlis in his 1964 AMS Memoir. This work of Matlis was not forgotten, but his ideas were not fully developed. It appears that people did not quite know what to do with cotorsion modules in the sense of Matlis.

The contramodules of Eilenberg–Moore and the cotorsion modules of Harrison and Matlis are two closely related, sometimes equivalent concepts. It took me about 15 years, from 1999 to 2012 or even 2015–17, to discover and understand the connection. This is what made the modern applications possible.

L. Positselski. Nonhomogeneous quadratic duality and curvature. *Funct. Anal. Appl.* **27**, #3, p. 197–204, 1993. arXiv:1411.1982 [math.RA]

L. Positselski. Seriya pisem pro polubeskonechnye (ko)gomologii associativnyh algebr (“A series of letters about the semi-infinite (co)homology of associative algebras”, transliterated Russian). 2000, 2002. Available from http://positselski.livejournal.com/314.html or http://posic.livejournal.com/413.html .

L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne IMPAN, vol. 70, Birkhäuser/Springer Basel, 2010,bxxiv+349 pp. arXiv:0708.3398 [math.CT]

- L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs Amer. Math. Soc.* **212**, #996, 2011, vi+133 pp. arXiv:0905.2621 [math.CT]
- L. Positselski. Weakly curved A_∞-algebras over a topological local ring. Electronic preprint arXiv:1202.2697 [math.CT], 2012–18.
- L. Positselski. Contraherent cosheaves. Electronic preprint arXiv:1209.2995 [math.CT], 2012–17.
- L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. *Moscow Math. J.* **17**, #3, p. 385–455, 2017. arXiv:1605.03934 [math.CT]
- L. Positselski. Triangulated Matlis equivalence. *J. Algebra Appl.* **17**, #4, article ID 1850067, 2018. arXiv:1605.08018 [math.CT]

L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint arXiv:1705.04960 [math.CT], 2017–18.

L. Positselski, A. Slávik. Flat morphisms of finite presentation are very flat. Electronic preprint arXiv:1708.00846 [math.AC], 2017–18.

L. Positselski, A. Slávik. On strongly flat and weakly cotorsion modules. Electronic preprint arXiv:1708.06833 [math.AC], 2017–18. To appear in *Math. Zeitschrift*.

L. Positselski. Flat ring epimorphisms of countable type. Electronic preprint arXiv:1808.00937 [math.RA].