A construction of complete cotorsion pairs in the relative context

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Basic Definitions

Let A be an associative ring and $A\text{-}\mathrm{Mod}$ the category of left A-modules. A pair of classes of left A-modules $\mathcal{F}$ and $\mathcal{C} \subset A\text{-}\mathrm{Mod}$ is said to be an Ext^1-orthogonal pair if $\mathrm{Ext}^1_A(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

We denote by $\mathcal{F}^{\perp_1} \subset A\text{-}\mathrm{Mod}$ the class of all modules $X \in A\text{-}\mathrm{Mod}$ such that $\mathrm{Ext}^1_A(F, X) = 0$ for all $F \in \mathcal{F}$, and by ${}^{\perp_1}\mathcal{C} \subset A\text{-}\mathrm{Mod}$ the class of all modules $Y \in A\text{-}\mathrm{Mod}$ such that $\mathrm{Ext}^1_A(Y, C) = 0$ for all $C \in \mathcal{C}$. A pair of classes $(\mathcal{F}, \mathcal{C})$ is said to be a cotorsion pair if $\mathcal{C} = \mathcal{F}^{\perp_1}$ and $\mathcal{F} = {}^{\perp_1}\mathcal{C}$.

Idiosyncratic terminology: a class of left A-modules $\mathcal{F}$ is called resolving if it is closed under extensions and the kernels of epimorphisms, and every left A-module is a quotient of a module from $\mathcal{F}$. A class $\mathcal{C}$ is called coresolving if it is closed under extensions and the cokernels of monomorphisms, and every left A-module is a submodule of a module from $\mathcal{C}$.

Basic Definitions

Idiosyncratic terminology: an Ext^1-orthogonal pair of classes $(\mathcal{F}, \mathcal{C})$ is said to admit approximation sequences if for every module $M \in A\text{-}\mathrm{Mod}$ there exist short exact sequences

$$0 \longrightarrow C' \longrightarrow F \longrightarrow M \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow M \longrightarrow C \longrightarrow F' \longrightarrow 0 \tag{2}$$

with F, $F' \in \mathcal{F}$ and C, $C' \in \mathcal{C}$.

The short exact sequence (1) is called a special precover sequence and the short exact sequence (2) is called a special preenvelope sequence.

Lemma (Salce, 1979)

Assume that every left A-module is a quotient module of a module from $\mathcal{F}$ and a submodule of a module from $\mathcal{C}$. Assume further that both the classes $\mathcal{F}$ and $\mathcal{C}$ are closed under extensions. Then the pair $(\mathcal{F}, \mathcal{C})$ admits special precover sequences if and only if it admits special preenvelope sequences. □

Basic Definitions

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called complete if it admits approximation sequences.

For any class of left A-modules $\mathcal{B} \subset A\text{-}\mathrm{Mod}$, we denote by $\mathcal{B}^{\oplus} \subset A\text{-}\mathrm{Mod}$ the class of all direct summands of A-modules from $\mathcal{B}$.

Lemma ("direct summand lemma")

Let $(\mathcal{F}, \mathcal{C})$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Then the pair of classes $(\mathcal{F}^{\oplus}, \mathcal{C}^{\oplus})$ is a complete cotorsion pair. □

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called hereditary if any one of the following equivalent conditions holds:

- the class $\mathcal{F}$ is resolving (i.e., closed under kers of epis);
- the class $\mathcal{C}$ is coresolving (i.e., closed under cokers of monos);
- $\mathrm{Ext}^2_A(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$;
- $\mathrm{Ext}^n_A(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$, and $n \geqslant 1$.

Basic Definitions

Let $\mathcal{F} \subset A\text{-}\mathrm{Mod}$ be a resolving class. A left A-module M is said to have $\mathcal{F}$-resolution dimension $\leqslant k$ if there exists an exact sequence of left A-modules

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with $F_i \in \mathcal{F}$ for all $i = 0, \ldots, k$.

Lemma

Let M be a left A-module of $\mathcal{F}$-resolution dimension $\leqslant k$, and let $0 \longrightarrow G_k \longrightarrow G_{k-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ be an exact sequence with $G_i \in \mathcal{F}$ for all $0 \leqslant i \leqslant k-1$. Then $G_k \in \mathcal{F}$. □

Lemma

Choose $k \geqslant 0$, and consider the class $\mathcal{F}(k)$ of all left A-modules of $\mathcal{F}$-resolution dimension $\leqslant k$. Then the class $\mathcal{F}(k)$ is resolving. □

The definition of the $\mathcal{C}$-coresolution dimension for a coresolving class $\mathcal{C} \subset A\text{-}\mathrm{Mod}$ is dual, and it has similar/dual properties.

Basic Definitions

Let M be an A-module and α be an ordinal. An α-filtration of M is a collection of submodules $F_iM \subset M$ indexed by the ordinals $0 \leqslant i \leqslant \alpha$ such that

- $F_0M = 0$, $F_\alpha M = M$;
- $F_jM \subset F_iM$ for all $0 \leqslant j \leqslant i \leqslant \alpha$;
- $F_iM = \bigcup_{j<i} F_jM$ for all limit ordinals $i \leqslant \alpha$.

We say that the module M is α-filtered by the modules $F_{i+1}M/F_iM$, $0 \leqslant i < \alpha$.

Given a class of left A-modules $\mathcal{S}$, we denote by $\mathrm{Fil}_\alpha(\mathcal{S})$ the class of all left A-modules M admitting an α-filtration F such that $F_{i+1}M/F_iM \in \mathcal{S}$ for all $0 \leqslant i < \alpha$. We denote by $\mathrm{Fil}(\mathcal{S})$ the union of the classes $\mathrm{Fil}_\alpha(\mathcal{S})$ taken over all the ordinals α.

Lemma (Eklof)

For any class of left A-modules $\mathcal{S}$, one has $\mathrm{Fil}(\mathcal{S})^{\perp_1} = \mathcal{S}^{\perp_1}$. □

Let M be a left A-module and α be an ordinal. An α-cofiltration of M is a collection of left A-modules G_iM indexed by $0 \leqslant i \leqslant \alpha$ and surjective A-module morphisms $G_iM \longrightarrow G_jM$ given for all $0 \leqslant j < i \leqslant \alpha$ such that

- the triangle diagram $G_iM \longrightarrow G_jM \longrightarrow G_kM$ is commutative for all $0 \leqslant k < j < i \leqslant \alpha$;
- $G_0M = 0, \ \ G_\alpha M = M$;
- $G_iM = \varprojlim_{j<i} G_jM$ for all limit ordinals $i \leqslant \alpha$.

We say that the module M is α-cofiltered by the modules $\ker(G_{i+1}M \to G_iM), \ \ 0 \leqslant i < \alpha$.

Let $\mathcal{T}$ be a class of left A-modules. The notation $\mathrm{Cof}(\mathcal{T})$ and $\mathrm{Cof}_\alpha(\mathcal{T})$ stands for the classes of all modules cofiltered or α-cofiltered by $\mathcal{T}$, similarly to the filtered modules above.

Lemma (Lukas or "dual Eklof")

For any class of left A-modules $\mathcal{T}$, one has ${}^{\perp_1}\mathrm{Cof}(\mathcal{T}) = {}^{\perp_1}\mathcal{T}$. □

Fundamental Result

Let $\mathcal{S}$ and $\mathcal{T}$ be two classes of left A-modules. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $A\text{-}\mathrm{Mod}$ is said to be generated by $\mathcal{S}$ if $\mathcal{C} = \mathcal{S}^{\perp_1}$. $(\mathcal{F}, \mathcal{C})$ is said to be cogenerated by $\mathcal{T}$ if $\mathcal{F} = {}^{\perp_1}\mathcal{T}$.

Theorem (Eklof–Trlifaj, 2001)

Let $\mathcal{S}$ be a set (rather than a class) of left A-modules. Then
(a) *the cotorsion pair $(\mathcal{F}, \mathcal{C})$ generated by $\mathcal{S}$ is complete;*
(b) *the class $\mathcal{F}$ can be described as $\mathcal{F} = \mathrm{Fil}(\mathcal{S} \cup \{{}_AA\})^{\oplus}$.* □

The proof is based on a version of the small object argument. Since modules are usually not cosmall, the dual version of this argument does not work for modules, and in fact it is known that the dual assertion to the Eklof–Trlifaj theorem is not true.

More precisely, it is consistent with ZFC+GCH that the cotorsion pair cogenerated by $\mathcal{T} = \{\mathbb{Z}\}$ in $\mathbb{Z}\text{-}\mathrm{Mod}$ is not complete.

Posing the Problem

Let $R \longrightarrow A$ be a homomorphism of associative rings, and let $(\mathcal{F}, \mathcal{C})$ be a complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Let $\mathcal{F}_A$ be the class of all left A-modules whose underlying R-modules belong to $\mathcal{F}$.

Questions:

1. Is $\mathcal{F}_A$ the left part of a cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ in $A\text{-}\mathrm{Mod}$?
2. Assuming a cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ exists, is it complete?
3. Assuming a cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ exists, can the class $\mathcal{C}_A$ be explicitly described?

Answering Question 1

Proposition

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$, and let $R \longrightarrow A$ and $\mathcal{F}_A$ be as above. Then a cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ in $A\text{-}\mathrm{Mod}$ exists if and only if the left R-module A belongs to $\mathcal{F}$.

Proof of "Only if".

For any cotorsion pair in $A\text{-}\mathrm{Mod}$, all projective left A-modules must belong to the left class. Now ${}_AA \in \mathcal{F}_A$ means ${}_RA \in \mathcal{F}$. □

Proof of "If".

Suppose that the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is cogenerated by a class $\mathcal{T} \subset R\text{-}\mathrm{Mod}$, that is $\mathcal{F} = {}^{\perp_1}\mathcal{T}$. One can always take $\mathcal{T} = \mathcal{C}$. Let $\mathrm{Hom}_R(A, \mathcal{T})$ denote the class of all left A-modules $\mathrm{Hom}_R(A, T)$ with $T \in \mathcal{T}$. Then we claim that $(\mathcal{F}_A, \mathcal{C}_A)$ is the cotorsion pair cogenerated by $\mathrm{Hom}_R(A, \mathcal{T})$, that is $\mathcal{F}_A = {}^{\perp_1}\mathrm{Hom}_R(A, \mathcal{T})$.

Homological Formulas

Lemma

Let $R \longrightarrow A$ be a ring homomorphism and $n \geqslant 1$ be an integer. Then (a) *for any left R-module S such that $\mathrm{Tor}_i^R(A, S) = 0$ for $1 \leqslant i \leqslant n$, and for any left A-module C, one has $\mathrm{Ext}_A^n(A \otimes_R S,\ C) \simeq \mathrm{Ext}_R^n(S, C)$;*
(b) *for any left R-module T such that $\mathrm{Ext}_R^i(A, T) = 0$ for $1 \leqslant i \leqslant n$, and for any left A-module F, one has $\mathrm{Ext}_A^n(F, \mathrm{Hom}_R(A, T)) \simeq \mathrm{Ext}_R^n(F, T)$.*

Proof of part (b).

If $0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ is an injective coresolution of the R-module T, then $0 \longrightarrow \mathrm{Hom}_R(A, I^0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(A, I^{n+1})$ is an initial fragment of an injective coresolution of the A-module $\mathrm{Hom}_R(A, T)$. This fragment can be extended to a full injective resolution and used to compute $\mathrm{Ext}_R^n(F, \mathrm{Hom}_R(A, T))$. □

Answering Question 1

End of proof of the "If" implication.

We have $\mathrm{Ext}^1_R(A, T) = 0$ for all $T \in \mathcal{T}$, since ${}_RA \in \mathcal{F}$. By Lemma (b) for $n = 1$, it follows that for any left A-module F we have $\mathrm{Ext}^1_A(F, \mathrm{Hom}_R(A, T)) \simeq \mathrm{Ext}^1_R(F, T)$.

So any one of these two Ext group vanishes if and only if the other one does. $\square$

Answering Question 2

Proposition

Let $(\mathcal{F}, \mathcal{C})$ be a (complete) cotorsion pair in $R\text{-}\mathrm{Mod}$ generated by a set $\mathcal{S}$. Assume that ${}_RA \in \mathcal{F}$. Then the cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ is also generated by some set $\mathcal{S}_A$, hence also complete.

Proof.

By the Eklof–Trlifaj theorem, the class $\mathcal{F}$ can be described as the class of all direct summands of R-modules filtered by $\mathcal{S} \cup \{{}_RR\}$, that is $\mathcal{F} = \mathrm{Fil}(\mathcal{S} \cup \{{}_RR\})^{\oplus}$. It follows that $\mathcal{F}$ is deconstructible, i.e., there exists a set $\mathcal{S}'$ such that $\mathcal{F} = \mathrm{Fil}(\mathcal{S}')$. Using the Hill lemma, one shows that the class $\mathcal{F}_A$ is deconstructible, too; so $\mathcal{F}_A = \mathrm{Fil}(\mathcal{S}_A)$ for some set of left A-modules $\mathcal{S}_A$. By assumption, we have ${}_AA \in \mathcal{F}_A$. Applying the Eklof–Trlifaj theorem again, one can conclude that $(\mathcal{F}_A, \mathcal{C}_A)$ is generated by $\mathcal{S}_A$ and complete. $\square$

Answering Question 3

We have seen that the cotorsion pair $(\mathcal{F}_A, \mathcal{C}_A)$ is cogenerated by $\mathrm{Hom}_R(A, \mathcal{C})$. It follows that $\mathrm{Hom}_R(A, \mathcal{C}) \subset \mathcal{C}_A$, and consequently also $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus} \subset \mathcal{C}_A$.

The main part of this talk starts at this point. Under various assumptions, admittedly rather restrictive, we will prove that $\mathcal{C}_A = \mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. In fact, depending on the specific assumptions, we will show that $\mathcal{C}_A = \mathrm{Cof}_\beta(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ for some rather small ordinal β, such as $\beta < \omega$, or $\beta = \omega$, or $\beta < \omega + \omega$. Here ω denotes the ordinal of natural numbers (the smallest infinite ordinal).

The following technical assumption will be used throughout:

(††) The class $\mathcal{F}$ is preserved by the functor $\mathrm{Hom}_R(A, -)$. In other words, for any left R-module $F \in \mathcal{F}$, the underlying R-module of the left A-module $\mathrm{Hom}_R(A, F)$ belongs to $\mathcal{F}$.

Finite $\mathcal{F}$-resolution dimension case

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any left R-module is $\leqslant k$, where k is a finite integer. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences as well.

In fact, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then, following the proof of the Salce lemma, we will produce special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))$.

Lemma

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Let M be a left R-module of $\mathcal{F}$-resolution dimension $\leqslant d$. Then the $\mathcal{F}$-resolution dimension of the left R-module $\mathrm{Hom}_R(A, M)$ also does not exceed d.

Proof.

Choose a special precover sequence for M, then a special precover sequence for the kernel, etc. Proceeding in this way, we obtain an exact sequence $0 \longrightarrow C_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, where $F_i \in \mathcal{F}$ for all $0 \leqslant i \leqslant d-1$, $C_d \in \mathcal{C}$, and the image C_i of the morphism $F_i \longrightarrow F_{i-1}$ belongs to $\mathcal{C}$ for all $1 \leqslant i \leqslant d-1$. Since the $\mathcal{F}$-resolution dimension of M is $\leqslant d$, it follows that $C_d \in \mathcal{F}$. As $A \in {}^{\perp_1}\mathcal{C}$ and (††) is assumed, applying $\mathrm{Hom}_R(A, -)$ to our exact sequence produces a resolution of $\mathrm{Hom}_R(A, M)$ by modules from $\mathcal{F}$, of length d. □

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition.

Let M be a left A-module. Then there is a natural (adjunction) left A-module morphism $\nu_M\colon M \longrightarrow \mathrm{Hom}_R(A, M)$ given by the formula $\nu_M(m)(a) = am$ for all $m \in M$ and $a \in A$. The map ν_M is always injective, and as a morphism of left R-modules it is split injective. In fact, the map $\phi_M\colon \mathrm{Hom}_R(A, M) \longrightarrow M$ given by $\phi_M(f) = f(1)$ for all $f \in \mathrm{Hom}_R(A, M)$ is R-linear and satisfies $\phi_M \circ \nu_M = \mathrm{id}_M$.

Let $0 \longrightarrow C'(M) \longrightarrow F(M) \longrightarrow M \longrightarrow 0$ be a special precover sequence for the underlying left R-module of M; so $C'(M) \in \mathcal{C}$ and $F(M) \in \mathcal{F} \subset R\text{-}\mathrm{Mod}$. Since $A \in {}^{\perp_1}\mathcal{C}$, the coinduced A-module map $\mathrm{Hom}_R(A, F(M)) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective. Denote by $Q(M)$ the pullback of the pair of morphisms $M \longrightarrow \mathrm{Hom}_R(A, M)$ and $\mathrm{Hom}_R(A, F(M)) \longrightarrow \mathrm{Hom}_R(A, M)$ in $A\text{-}\mathrm{Mod}$.

Finite $\mathcal{F}$-resolution dimension case

We get a commutative diagram of A-module morphisms, where all the three-term sequences are short exact:

$$\begin{array}{ccccc}
M & \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \| \\
Q(M) & \longrightarrow & \mathrm{Hom}_R(A, F(M)) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}_R(A, C'(M)) & = & \mathrm{Hom}_R(A, C'(M)) & &
\end{array}$$

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

Let $\operatorname{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N. The claim is that $\operatorname{rd}_{\mathcal{F}} Q(M) < \operatorname{rd}_{\mathcal{F}} M$ whenever $0 < \operatorname{rd}_{\mathcal{F}} M < \infty$.

Indeed, $\operatorname{rd}_{\mathcal{F}} \operatorname{Hom}_R(A, M) \leqslant \operatorname{rd}_{\mathcal{F}} M$ by the lemma. Since the short exact sequence

$$0 \longrightarrow M \longrightarrow \operatorname{Hom}_R(A, M) \longrightarrow \operatorname{Hom}_R(A, M)/M \longrightarrow 0$$

splits over R, it follows that $\operatorname{rd}_{\mathcal{F}} \operatorname{Hom}_R(A, M)/M \leqslant \operatorname{rd}_{\mathcal{F}} M$. From the short exact sequence

$$0 \longrightarrow Q(M) \longrightarrow \operatorname{Hom}_R(A, F(M)) \longrightarrow \operatorname{Hom}_R(A, M)/M \longrightarrow 0$$

we conclude that $\operatorname{rd}_{\mathcal{F}} Q(M) < \operatorname{rd}_{\mathcal{F}}(M)$, since $\operatorname{Hom}_R(A, F(M)) \in \mathcal{F}$ by (††).

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\mathrm{rd}_{\mathcal{F}}\, Q(M) \leqslant k$ by assumption, we have $Q^k(M) \in \mathcal{F}$.

From the commutative diagram above, we see that $\ker(Q(M) \to M) = \mathrm{Hom}_R(A, C'(M))$. Thus the kernel of the map $Q^k(M) \longrightarrow M$ is cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^{k-1}(M)))$. So $\ker(Q^k(M) \to M) \in \mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C}))$, as desired.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition fin'd.

The construction above provides the special precover sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$.

The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))$ are produced from these using (the construction from the proof of) the Salce lemma and the fact that any A-module can be embedded into an A-module from $\mathrm{Hom}_R(A, \mathcal{C})$ (since any R-module can be embedded into an R-module from $\mathcal{C}$). □

Finite $\mathcal{F}$-resolution dimension case

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k, where $k \geqslant 0$ is a finite integer. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

Proof.

Follows from Proposition and the direct summand lemma. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. □

The case when $\mathcal{F}$ is closed under products

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.

As in the previous proof, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then the construction of the Salce lemma will provide the special preenvelope sequences for the same pair of classes.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition.

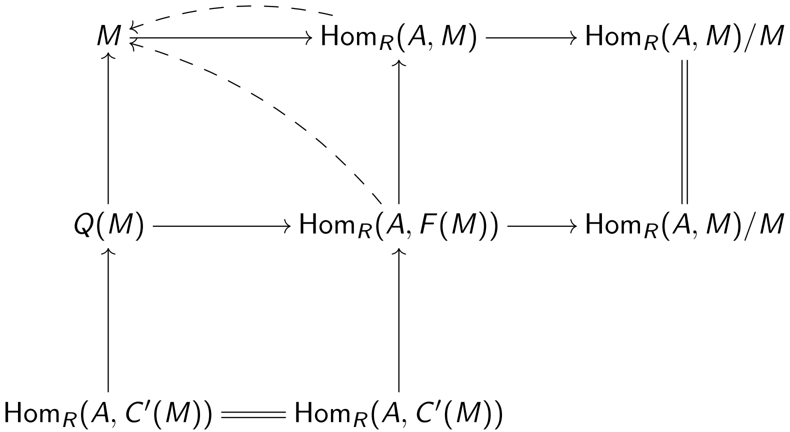
Let M be a left A-module.We use the construction of the surjective left A-module morphism $Q(M) \longrightarrow M$ from the previous proof, but now we iterate it over the ordinal ω

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

and consider the projective limit $\varprojlim_n Q^n(M)$.

The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$ is ω-cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^n(M)))$, … So this kernel belongs to $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$.

Let us show that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$. Look on the diagram from the previous proof again.



The injective morphism of A-modules $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M \colon \mathrm{Hom}_R(A, M) \longrightarrow M$. Therefore, the morphism of left A-modules $Q(M) \longrightarrow M$, viewed as a morphism of left R-modules, factors through $\mathrm{Hom}_R(A, F(M))$.

The dashed arrows show R-module maps between A-modules.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms $\operatorname{Hom}_R(A, F(M)) \longleftarrow \operatorname{Hom}_R(A, F(Q(M))) \longleftarrow \cdots \longleftarrow \operatorname{Hom}_R(A, F(Q^n(M))) \longleftarrow \cdots$.

Hence the derived projective limit $\varprojlim_n^1 \operatorname{Hom}_R(A, F(Q^n(M)))$ vanishes, $\varprojlim_n^1 \operatorname{Hom}_R(A, F(Q^n(M))) = \varprojlim_n^1 Q^n(M) = 0$, since the maps $Q^n(M) \longrightarrow Q^{n-1}(M)$ are surjective.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

As the underlying R-module of the A-module $\varprojlim_n Q^n(M)$ is isomorphic to $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M)))$, we can conclude that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$, as desired. This finishes the construction of the special precover sequences. The rest is the Salce lemma. □

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. □

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences.

Proof.

Let M be a left A-module. In order to construct a special precover sequence for M, we start from the projective system of left A-modules

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

indexed by the ordinal ω.

Combined result

Proof of Proposition cont'd.

Put $N = \varprojlim_n Q^n(M)$. As we have seen, the underlying R-module of N is isomorphic to $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M)))$. Furthermore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

The left R-modules $\mathrm{Hom}_R(A, F(Q^n(M)))$ belong to $\mathcal{F}$ by (††), hence the left R-module $\prod_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M))))$ has $\mathcal{F}$-resolution dimension $\leqslant k$ by assumption.

As the class of all left R-modules of $\mathcal{F}$-resolution dimension $\leqslant k$ is resolving by lemma, it follows that the $\mathcal{F}$-resolution dimension of $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \simeq N$ does not exceed k.

Combined result

Proof of Proposition fin'd.

Now we make a further sequence of k iterations, producing surjective A-module morphisms

$$N \longleftarrow Q(N) \longleftarrow Q(Q(N)) \longleftarrow \cdots \longleftarrow Q^k(N).$$

Following the above argument, applying Q lowers the $\mathcal{F}$-resolution dimension, hence $Q^k(N) \in \mathcal{F}_A$.

Finally, we have surjective A-module maps $Q^k(N) \longrightarrow N = \varprojlim_n Q^n(M) \longrightarrow M$. The kernel of the map $Q^k(N) \longrightarrow N$ belongs to $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C}))$ and the kernel of the map $\varprojlim_n Q^n(M) \longrightarrow M$ belongs to $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$. Thus the kernel of composition $Q^k(N) \longrightarrow M$ belongs to $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))$. □

Combined result

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus} = \mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. □

Dual results

The dual results to the above three theorems are also provable using the dual constructions, but they are less surprising given the Eklof–Trlivaj theorem.

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$. Denote by $\mathcal{C}^A$ the class of all left A-modules whose underlying left R-modules belong to $\mathcal{C}$. Then there exists a cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ in $A\text{-}\mathrm{Mod}$ if and only if the left R-module $A^+ = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ belongs to $\mathcal{C}$. If this is the case (which we assume in the sequel) then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $A \otimes_R \mathcal{F}$ of all left A-modules of the form $A \otimes_R F$ with $F \in \mathcal{F}$.

If the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A \otimes_R \mathcal{S}$. In particular, if $(\mathcal{F}, C)$ is generated by a set $\mathcal{S}$, then $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the set $\mathcal{S}^A$. Hence $(\mathcal{F}^A, \mathcal{C}^A)$ is complete and $\mathcal{F}^A = \mathrm{Fil}(\mathcal{S}^A \cup \{{}_AA\})^\oplus$.

Dual results

Our approach does not need the assumption that $(\mathcal{F}, \mathcal{C})$ is generated by a set, but it uses other rather restrictive assumptions, most importantly

(†) The class $\mathcal{C}$ is preserved by the functor $A \otimes_R -$. In other words, for any left R-module $C \in \mathcal{C}$, the underlying R-module of the left A-module $A \otimes_R C$ belongs to $\mathcal{C}$.

Using various specific assumptions on top of (†), our approach allows to describe $\mathcal{F}^A$ as the class of all direct summands of β-filtered modules for small ordinals β. In particular:

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA^+ \in \mathcal{C}$ and the condition (†) holds. Assume further that the $\mathcal{C}$-coresolution dimension of any countable direct sum of modules from $\mathcal{C}$ does not exceed a fixed finite integer k. Then the pair of classes $\mathrm{Fil}_{\omega+k}(A \otimes_R \mathcal{F})^{\oplus}$ and $\mathcal{C}^A$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Dual results

Under more restrictive assumptions one can say more:

Theorem

Let $(\mathcal{F}, \mathcal{C})$ *be a hereditary complete cotorsion pair in* $R\text{-}\mathrm{Mod}$*. Assume that* ${}_RA^+ \in \mathcal{C}$ *and the condition* $(\dagger)$ *holds. Assume further that the class* $\mathcal{C}$ *is closed under countable direct sums in* $R\text{-}\mathrm{Mod}$*. Then the pair of classes* $\mathrm{Fil}_\omega(A \otimes_R \mathcal{F})^\oplus$ *and* $\mathcal{C}^A$ *is a hereditary complete cotorsion pair in* $A\text{-}\mathrm{Mod}$*.* □

Theorem

Let $(\mathcal{F}, \mathcal{C})$ *be a hereditary complete cotorsion pair in* $R\text{-}\mathrm{Mod}$*. Assume that* ${}_RA^+ \in \mathcal{C}$ *and the condition* $(\dagger)$ *holds. Assume further that the* $\mathcal{C}$*-coresolution dimension of any left* R*-module does not exceed* k*. Then the pair of classes* $\mathrm{Fil}_{k+1}(A \otimes_R \mathcal{F})^\oplus$ *and* $\mathcal{C}^A$ *is a hereditary complete cotorsion pair in* $A\text{-}\mathrm{Mod}$*.* □

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