A construction of complete cotorsion pairs in the relative context

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Algebra seminar, Charles Univ., Prague (via Zoom)

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- $\mathrm{Ext}^n_A(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$, and $n \geq 1$.

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Let $\mathcal{F} \subset A\text{-}\mathrm{Mod}$ be a resolving class. A left A-module M is said to have $\mathcal{F}$-resolution dimension $\leqslant k$ if there exists an exact sequence of left A-modules

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Let $\mathcal{S}$ and $\mathcal{T}$ be two classes of left A-modules. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $A\text{-}\mathrm{Mod}$ is said to be generated by $\mathcal{S}$ if $\mathcal{C} = \mathcal{S}^{\perp_1}$. $(\mathcal{F}, \mathcal{C})$ is said to be cogenerated by $\mathcal{T}$ if $\mathcal{F} = {}^{\perp_1}\mathcal{T}$.

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Finite $\mathcal{F}$-resolution dimension case

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In fact, we will show by explicit construction

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In fact, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences.

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In fact, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then, following the proof of the Salce lemma

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In fact, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then, following the proof of the Salce lemma, we will produce special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A,\mathcal{C}))$.

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Proof.

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Proof.

Choose a special precover sequence for M

Lemma

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Let M be a left R-module of $\mathcal{F}$-resolution dimension $\leqslant d$. Then the $\mathcal{F}$-resolution dimension of the left R-module $\mathrm{Hom}_R(A, M)$ also does not exceed d.

Proof.

Choose a special precover sequence for M, then a special precover sequence for the kernel, etc.

Lemma

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition* (††) *holds. Let M be a left R-module of $\mathcal{F}$-resolution dimension $\leqslant d$. Then the $\mathcal{F}$-resolution dimension of the left R-module* $\mathrm{Hom}_R(A, M)$ *also does not exceed d.*

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Choose a special precover sequence for M, then a special precover sequence for the kernel, etc. Proceeding in this way, we obtain an exact sequence $0 \longrightarrow C_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$

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Lemma

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Let M be a left R-module of $\mathcal{F}$-resolution dimension $\leqslant d$. Then the $\mathcal{F}$-resolution dimension of the left R-module $\mathrm{Hom}_R(A, M)$ also does not exceed d.

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Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition.

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Let M be a left A-module.

Finite $\mathcal{F}$-resolution dimension case

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Let M be a left A-module. Then there is a natural (adjunction) left A-module morphism $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$

Finite $\mathcal{F}$-resolution dimension case

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Finite $\mathcal{F}$-resolution dimension case

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Let $0 \longrightarrow C'(M) \longrightarrow F(M) \longrightarrow M \longrightarrow 0$ be a special precover sequence for the underlying left R-module of M; so $C'(M) \in \mathcal{C}$ and $F(M) \in \mathcal{F} \subset R\text{-}\mathrm{Mod}$.

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Finite $\mathcal{F}$-resolution dimension case

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Finite $\mathcal{F}$-resolution dimension case

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Let $0 \longrightarrow C'(M) \longrightarrow F(M) \longrightarrow M \longrightarrow 0$ be a special precover sequence for the underlying left R-module of M; so $C'(M) \in \mathcal{C}$ and $F(M) \in \mathcal{F} \subset R\text{-}\mathrm{Mod}$. Since $A \in {}^{\perp_1}\mathcal{C}$, the coinduced A-module map $\mathrm{Hom}_R(A, F(M)) \longrightarrow \mathrm{Hom}_R(A, M)$ is surjective. Denote by $Q(M)$ the pullback of the pair of morphisms $M \longrightarrow \mathrm{Hom}_R(A, M)$ and $\mathrm{Hom}_R(A, F(M)) \longrightarrow \mathrm{Hom}_R(A, M)$

Finite $\mathcal{F}$-resolution dimension case

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Finite $\mathcal{F}$-resolution dimension case

We get a commutative diagram of A-module morphisms, where all the three-term sequences are short exact:

Finite $\mathcal{F}$-resolution dimension case

We get a commutative diagram of A-module morphisms, where all the three-term sequences are short exact:

$$
\begin{array}{ccccc}
M & \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \| \\
Q(M) & \longrightarrow & \mathrm{Hom}_R(A, F(M)) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}_R(A, C'(M)) & = & \mathrm{Hom}_R(A, C'(M)) & &
\end{array}
$$

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

Finite $\mathcal{F}$-resolution dimension case

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Let $\mathrm{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

Let $\mathrm{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N. The claim is that $\mathrm{rd}_{\mathcal{F}} Q(M) < \mathrm{rd}_{\mathcal{F}} M$

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

Let $\mathrm{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N. The claim is that $\mathrm{rd}_{\mathcal{F}} Q(M) < \mathrm{rd}_{\mathcal{F}} M$ whenever $0 < \mathrm{rd}_{\mathcal{F}} M < \infty$.

Finite $\mathcal{F}$-resolution dimension case

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Indeed, $\mathrm{rd}_{\mathcal{F}} \mathrm{Hom}_R(A, M) \leqslant \mathrm{rd}_{\mathcal{F}} M$ by the lemma.

Finite $\mathcal{F}$-resolution dimension case

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$$0 \longrightarrow M \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_R(A, M)/M \longrightarrow 0$$

Finite $\mathcal{F}$-resolution dimension case

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splits over R, it follows that $\mathrm{rd}_{\mathcal{F}} \mathrm{Hom}_R(A, M)/M \leqslant \mathrm{rd}_{\mathcal{F}} M$.

Finite $\mathcal{F}$-resolution dimension case

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Let $\operatorname{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N. The claim is that $\operatorname{rd}_{\mathcal{F}} Q(M) < \operatorname{rd}_{\mathcal{F}} M$ whenever $0 < \operatorname{rd}_{\mathcal{F}} M < \infty$.

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$$0 \longrightarrow Q(M) \longrightarrow \mathrm{Hom}_R(A, F(M)) \longrightarrow \mathrm{Hom}_R(A, M)/M \longrightarrow 0$$

we conclude that $\mathrm{rd}_{\mathcal{F}} Q(M) < \mathrm{rd}_{\mathcal{F}}(M)$

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

Let $\operatorname{rd}_{\mathcal{F}} N$ denote the $\mathcal{F}$-resolution dimension of a left R-module N. The claim is that $\operatorname{rd}_{\mathcal{F}} Q(M) < \operatorname{rd}_{\mathcal{F}} M$ whenever $0 < \operatorname{rd}_{\mathcal{F}} M < \infty$.

Indeed, $\operatorname{rd}_{\mathcal{F}} \operatorname{Hom}_R(A, M) \leqslant \operatorname{rd}_{\mathcal{F}} M$ by the lemma. Since the short exact sequence

$$0 \longrightarrow M \longrightarrow \operatorname{Hom}_R(A, M) \longrightarrow \operatorname{Hom}_R(A, M)/M \longrightarrow 0$$

splits over R, it follows that $\operatorname{rd}_{\mathcal{F}} \operatorname{Hom}_R(A, M)/M \leqslant \operatorname{rd}_{\mathcal{F}} M$. From the short exact sequence

$$0 \longrightarrow Q(M) \longrightarrow \operatorname{Hom}_R(A, F(M)) \longrightarrow \operatorname{Hom}_R(A, M)/M \longrightarrow 0$$

we conclude that $\operatorname{rd}_{\mathcal{F}} Q(M) < \operatorname{rd}_{\mathcal{F}}(M)$, since $\operatorname{Hom}_R(A, F(M)) \in \mathcal{F}$ by (††).

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Finite $\mathcal{F}$-resolution dimension case

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To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\mathrm{rd}_{\mathcal{F}} Q(M) \leqslant k$ by assumption

Finite $\mathcal{F}$-resolution dimension case

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To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\mathrm{rd}_{\mathcal{F}} Q(M) \leqslant k$ by assumption, we have $Q^k(M) \in \mathcal{F}$.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\operatorname{rd}_{\mathcal{F}} Q(M) \leqslant k$ by assumption, we have $Q^k(M) \in \mathcal{F}$.

From the commutative diagram above, we see that $\ker(Q(M) \to M) = \operatorname{Hom}_R(A, C'(M))$.

Finite $\mathcal{F}$-resolution dimension case

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To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

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From the commutative diagram above, we see that $\ker(Q(M) \to M) = \mathrm{Hom}_R(A, C'(M))$. Thus the kernel of the map $Q^k(M) \longrightarrow M$ is cofiltered by

Finite $\mathcal{F}$-resolution dimension case

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To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

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From the commutative diagram above, we see that $\ker(Q(M) \to M) = \mathrm{Hom}_R(A, C'(M))$. Thus the kernel of the map $Q^k(M) \longrightarrow M$ is cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^{k-1}(M)))$.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\mathrm{rd}_{\mathcal{F}}\, Q(M) \leqslant k$ by assumption, we have $Q^k(M) \in \mathcal{F}$.

From the commutative diagram above, we see that $\ker(Q(M) \to M) = \mathrm{Hom}_R(A, C'(M))$. Thus the kernel of the map $Q^k(M) \longrightarrow M$ is cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^{k-1}(M)))$. So $\ker(Q^k(M) \to M) \in \mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C}))$

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition cont'd.

To every left A-module M, we have assigned a surjective morphism of left A-modules $Q(M) \longrightarrow M$. Now we iterate this construction, producing a sequence of surjective A-module maps

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^k(M).$$

Since $\mathrm{rd}_{\mathcal{F}}\, Q(M) \leqslant k$ by assumption, we have $Q^k(M) \in \mathcal{F}$.

From the commutative diagram above, we see that $\ker(Q(M) \to M) = \mathrm{Hom}_R(A, C'(M))$. Thus the kernel of the map $Q^k(M) \longrightarrow M$ is cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^{k-1}(M)))$. So $\ker(Q^k(M) \to M) \in \mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C}))$, as desired.

Finite $\mathcal{F}$-resolution dimension case

Proof of Proposition fin'd.

Finite $\mathcal{F}$-resolution dimension case

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The construction above provides the special precover sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$.

Finite $\mathcal{F}$-resolution dimension case

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The construction above provides the special precover sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$.

The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))$

Finite $\mathcal{F}$-resolution dimension case

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The construction above provides the special precover sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathsf{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$.

The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathsf{Hom}_R(A,\mathcal{C}))$ are produced from these using (the construction from the proof of) the Salce lemma

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The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))$ are produced from these using (the construction from the proof of) the Salce lemma and the fact that any A-module can be embedded into an A-module from $\mathrm{Hom}_R(A, \mathcal{C})$

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The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A,\mathcal{C}))$ are produced from these using (the construction from the proof of) the Salce lemma and the fact that any A-module can be embedded into an A-module from $\mathrm{Hom}_R(A,\mathcal{C})$ (since any R-module can be embedded into an R-module from $\mathcal{C}$).

Finite $\mathcal{F}$-resolution dimension case

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The construction above provides the special precover sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$.

The special preenvelope sequences for the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))$ are produced from these using (the construction from the proof of) the Salce lemma and the fact that any A-module can be embedded into an A-module from $\mathrm{Hom}_R(A, \mathcal{C})$ (since any R-module can be embedded into an R-module from $\mathcal{C}$). □

Finite $\mathcal{F}$-resolution dimension case

Theorem

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Let $(\mathcal{F}, \mathcal{C})$ *be a hereditary complete cotorsion pair in* $R\text{-}\mathrm{Mod}$*.*

Finite $\mathcal{F}$-resolution dimension case

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Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds.

Finite $\mathcal{F}$-resolution dimension case

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_R A \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k

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Finite $\mathcal{F}$-resolution dimension case

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Proof.

Follows from Proposition and the direct summand lemma.

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Corollary

In the assumptions of the theorem

Finite $\mathcal{F}$-resolution dimension case

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k, where $k \geqslant 0$ is a finite integer. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

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In the assumptions of the theorem, one has

$\mathcal{F}_A^{\perp_1} =$

Finite $\mathcal{F}$-resolution dimension case

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k, where $k \geqslant 0$ is a finite integer. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

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Follows from Proposition and the direct summand lemma. □

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In the assumptions of the theorem, one has
$\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$.

Finite $\mathcal{F}$-resolution dimension case

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Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k, where $k \geqslant 0$ is a finite integer. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

Proof.

Follows from Proposition and the direct summand lemma. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus =$

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In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$.

Finite $\mathcal{F}$-resolution dimension case

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_R A \in \mathcal{F}$ and the condition (††) holds. Assume further that the $\mathcal{F}$-resolution dimension of any left R-module does not exceed k, where $k \geqslant 0$ is a finite integer. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

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Follows from Proposition and the direct summand lemma. □

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In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus} = \mathrm{Cof}_{k+1}(\mathrm{Hom}_R(A, \mathcal{C}))^{\oplus}$. □

The case when $\mathcal{F}$ is closed under products

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Proposition

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Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences.

The case when $\mathcal{F}$ is closed under products

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The case when $\mathcal{F}$ is closed under products

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Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$

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Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$

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Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\operatorname{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.

The case when $\mathcal{F}$ is closed under products

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the* Ext^1*-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.*

As in the previous proof, we will show by explicit construction that

The case when $\mathcal{F}$ is closed under products

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.

As in the previous proof, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences.

The case when $\mathcal{F}$ is closed under products

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.

As in the previous proof, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then the construction of the Salce lemma will provide the special preenvelope sequences

The case when $\mathcal{F}$ is closed under products

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under the kernels of surjective morphisms and countable products in $R\text{-}\mathrm{Mod}$. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\operatorname{Hom}_R(A, \mathcal{C}))$ admits approximation sequences as well.

As in the previous proof, we will show by explicit construction that the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\operatorname{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits special precover sequences. Then the construction of the Salce lemma will provide the special preenvelope sequences for the same pair of classes.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition.

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Let M be a left A-module.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition.

Let M be a left A-module.We use the construction of the surjective left A-module morphism $Q(M) \longrightarrow M$ from the previous proof

The case when $\mathcal{F}$ is closed under products

Proof of Proposition.

Let M be a left A-module.We use the construction of the surjective left A-module morphism $Q(M) \longrightarrow M$ from the previous proof, but now we iterate it over the ordinal ω

The case when $\mathcal{F}$ is closed under products

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$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

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and consider the projective limit $\varprojlim_n Q^n(M)$.

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The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$

The case when $\mathcal{F}$ is closed under products

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and consider the projective limit $\varprojlim_n Q^n(M)$.

The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$ is ω-cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^n(M)))$, …

The case when $\mathcal{F}$ is closed under products

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The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$ is ω-cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^n(M)))$, … So this kernel belongs to $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$.

The case when $\mathcal{F}$ is closed under products

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$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

and consider the projective limit $\varprojlim_n Q^n(M)$.

The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$ is ω-cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, …, $\mathrm{Hom}_R(A, C'(Q^n(M)))$, … So this kernel belongs to $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$.

Let us show that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition.

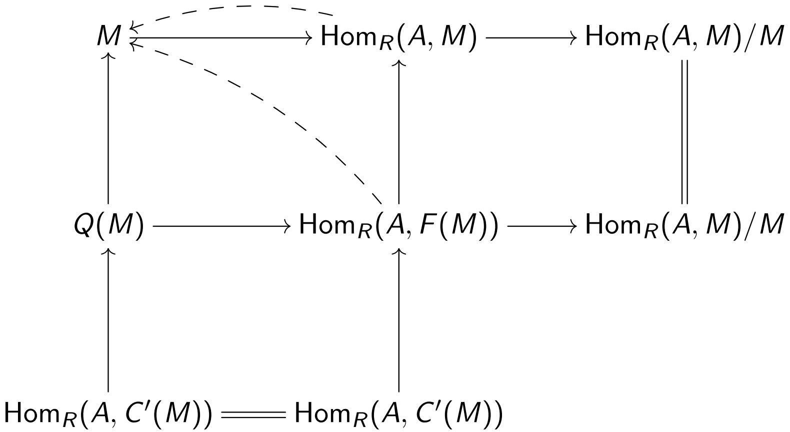
Let M be a left A-module.We use the construction of the surjective left A-module morphism $Q(M) \longrightarrow M$ from the previous proof, but now we iterate it over the ordinal ω

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

and consider the projective limit $\varprojlim_n Q^n(M)$.

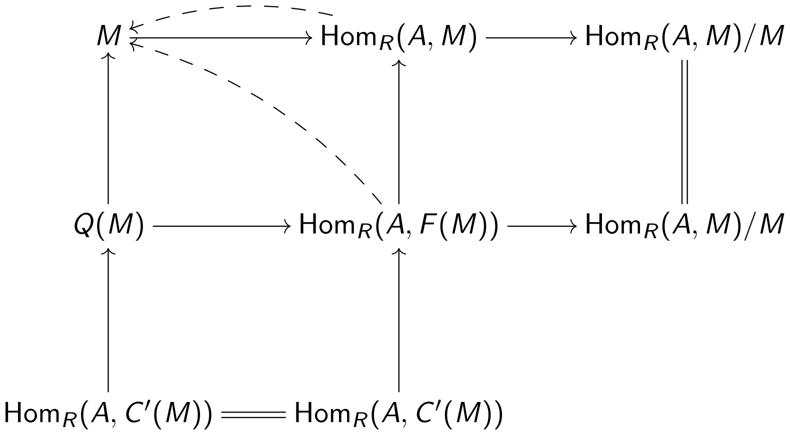
The kernel of the surjective morphism $\varprojlim_n Q^n(M) \longrightarrow M$ is ω-cofiltered by $\mathrm{Hom}_R(A, C'(M))$, $\mathrm{Hom}_R(A, C'(Q(M)))$, ..., $\mathrm{Hom}_R(A, C'(Q^n(M)))$, ... So this kernel belongs to $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))$.

Let us show that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$. Look on the diagram from the previous proof again.


M
$\mathrm{Hom}_R(A, M)$
$\mathrm{Hom}_R(A, M)/M$
$Q(M)$
$\mathrm{Hom}_R(A, F(M))$
$\mathrm{Hom}_R(A, M)/M$
$\mathrm{Hom}_R(A, C'(M))$
$\mathrm{Hom}_R(A, C'(M))$

$$
\begin{array}{ccccc}
M & \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \Vert \\
Q(M) & \longrightarrow & \mathrm{Hom}_R(A, F(M)) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}_R(A, C'(M)) & = & \mathrm{Hom}_R(A, C'(M)) & &
\end{array}
$$

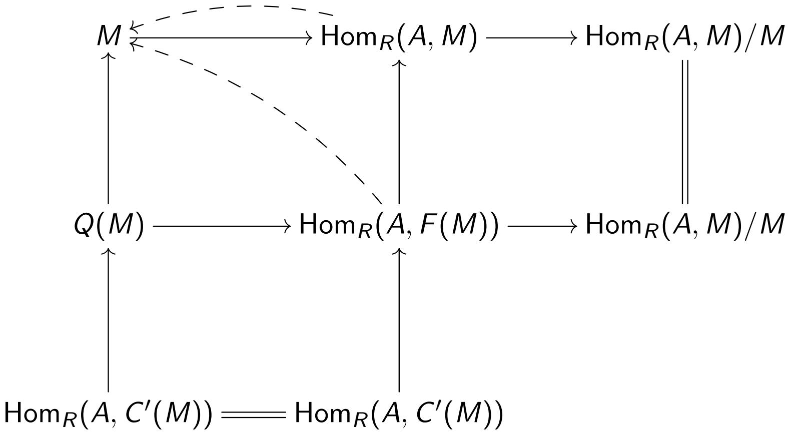
The injective morphism of A-modules $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$



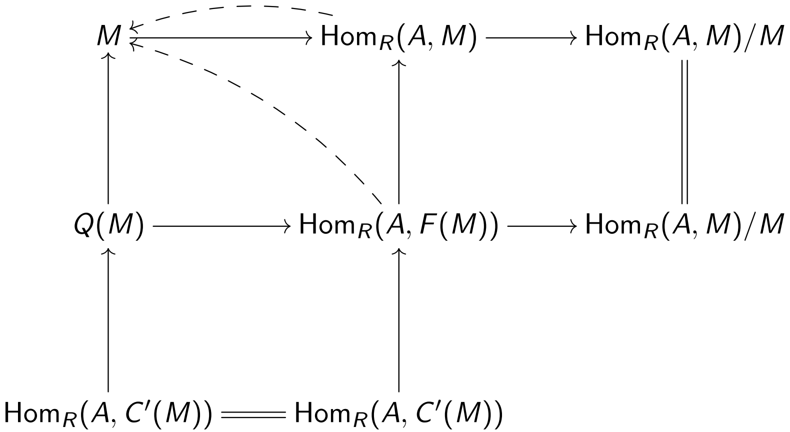
The injective morphism of A-modules $\nu_M\colon M \longrightarrow \mathrm{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M\colon \mathrm{Hom}_R(A, M) \longrightarrow M$.

$$\begin{array}{ccccc} M & \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\ \uparrow & & \uparrow & & \| \\ Q(M) & \longrightarrow & \mathrm{Hom}_R(A, F(M)) & \longrightarrow & \mathrm{Hom}_R(A, M)/M \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}_R(A, C'(M)) & = & \mathrm{Hom}_R(A, C'(M)) & & \end{array}$$

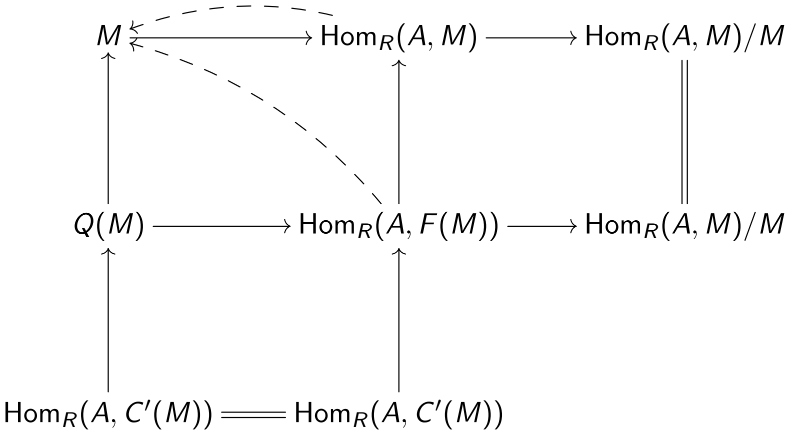
The injective morphism of A-modules $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M \colon \mathrm{Hom}_R(A, M) \longrightarrow M$. Therefore, the morphism of left A-modules $Q(M) \longrightarrow M$



The injective morphism of A-modules $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M \colon \mathrm{Hom}_R(A, M) \longrightarrow M$. Therefore, the morphism of left A-modules $Q(M) \longrightarrow M$, viewed as a morphism of left R-modules



The injective morphism of A-modules $\nu_M\colon M \longrightarrow \operatorname{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M\colon \operatorname{Hom}_R(A, M) \longrightarrow M$. Therefore, the morphism of left A-modules $Q(M) \longrightarrow M$, viewed as a morphism of left R-modules, factors through $\operatorname{Hom}_R(A, F(M))$.



The injective morphism of A-modules $\nu_M \colon M \longrightarrow \mathrm{Hom}_R(A, M)$ admits an R-linear retraction $\phi_M \colon \mathrm{Hom}_R(A, M) \longrightarrow M$. Therefore, the morphism of left A-modules $Q(M) \longrightarrow M$, viewed as a morphism of left R-modules, factors through $\mathrm{Hom}_R(A, F(M))$.

The dashed arrows show R-module maps between A-modules.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms
$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms $\mathrm{Hom}_R(A, F(M)) \longleftarrow \mathrm{Hom}_R(A, F(Q(M))) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_R(A, F(Q^n(M))) \longleftarrow \cdots$.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms $\operatorname{Hom}_R(A, F(M)) \longleftarrow \operatorname{Hom}_R(A, F(Q(M))) \longleftarrow \cdots \longleftarrow \operatorname{Hom}_R(A, F(Q^n(M))) \longleftarrow \cdots$.

Hence the derived projective limit $\varprojlim_n^1 \operatorname{Hom}_R(A, F(Q^n(M)))$ vanishes

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms $\mathrm{Hom}_R(A, F(M)) \longleftarrow \mathrm{Hom}_R(A, F(Q(M))) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_R(A, F(Q^n(M))) \longleftarrow \cdots$.

Hence the derived projective limit $\varprojlim^1_n \mathrm{Hom}_R(A, F(Q^n(M)))$ vanishes, $\varprojlim^1_n \mathrm{Hom}_R(A, F(Q^n(M))) = \varprojlim^1_n Q^n(M) = 0$

The case when $\mathcal{F}$ is closed under products

Proof of Proposition cont'd.

So the projective system of left A-module morphisms $M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$ is mutually cofinal with the projective system of left R-module morphisms $\operatorname{Hom}_R(A, F(M)) \longleftarrow \operatorname{Hom}_R(A, F(Q(M))) \longleftarrow \cdots \longleftarrow \operatorname{Hom}_R(A, F(Q^n(M))) \longleftarrow \cdots$.

Hence the derived projective limit $\varprojlim^1_n \operatorname{Hom}_R(A, F(Q^n(M)))$ vanishes, $\varprojlim^1_n \operatorname{Hom}_R(A, F(Q^n(M))) = \varprojlim^1_n Q^n(M) = 0$, since the maps $Q^n(M) \longrightarrow Q^{n-1}(M)$ are surjective.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \\ \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \\ \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

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Since $\mathrm{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††)

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

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Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

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Since $\mathrm{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

As the underlying R-module of the A-module $\varprojlim_n Q^n(M)$ is isomorphic to $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M)))$

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

As the underlying R-module of the A-module $\varprojlim_n Q^n(M)$ is isomorphic to $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M)))$, we can conclude that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$, as desired.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \\ \longrightarrow \prod_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M)))) \\ \longrightarrow \prod_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\mathrm{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

As the underlying R-module of the A-module $\varprojlim_n Q^n(M)$ is isomorphic to $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M)))$, we can conclude that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$, as desired. This finishes the construction of the special precover sequences.

The case when $\mathcal{F}$ is closed under products

Proof of Proposition fin'd.

Thererefore, we have a short exact sequence of left R-modules

$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod\nolimits_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

Since $\operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$ by (††), and the class $\mathcal{F}$ is closed under countable products and the kernels of surjective morphisms by assumption, it follows that $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \in \mathcal{F}$.

As the underlying R-module of the A-module $\varprojlim_n Q^n(M)$ is isomorphic to $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M)))$, we can conclude that $\varprojlim_n Q^n(M) \in \mathcal{F}_A$, as desired. This finishes the construction of the special precover sequences. The rest is the Salce lemma. □

The case when $\mathcal{F}$ is closed under products

Theorem

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$.

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ *be a hereditary complete cotorsion pair in* $R\text{-}\mathrm{Mod}$*. Assume that* ${}_RA \in \mathcal{F}$ *and the condition* $(\dagger\dagger)$ *holds.*

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$.

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$

The case when $\mathcal{F}$ is closed under products

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Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathsf{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$.

The case when $\mathcal{F}$ is closed under products

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Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

The case when $\mathcal{F}$ is closed under products

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Corollary

In the assumptions of the theorem

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$.

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$.

The case when $\mathcal{F}$ is closed under products

Theorem

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is closed under countable products in $R\text{-}\mathrm{Mod}$. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

Corollary

In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus = \mathrm{Cof}_\omega(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. □

Combined result

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Proposition

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences.

Combined result

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Let $\mathcal{F}$ *and* $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ *be an* Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that* ${}_RA \in \mathcal{F}$ *and the condition* $(\dagger\dagger)$ *holds.*

Combined result

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Let $\mathcal{F}$ *and* $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ *be an* Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that* ${}_RA \in \mathcal{F}$ *and the condition* $(\dagger\dagger)$ *holds. Assume further that the class* $\mathcal{F}$ *is resolving in* $R\text{-}\mathrm{Mod}$

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k.

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$

Combined result

Proposition

Let $\mathcal{F}$ *and* $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ *be an* Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that* ${}_RA \in \mathcal{F}$ *and the condition* $(\dagger\dagger)$ *holds. Assume further that the class* $\mathcal{F}$ *is resolving in* $R\text{-}\mathrm{Mod}$ *and the* $\mathcal{F}$*-resolution dimension of any countable product of modules from* $\mathcal{F}$ *does not exceed a fixed finite integer* k*. Then the* Ext^1*-orthogonal pair of classes* $\mathcal{F}_A$ *and* $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$ *admits approximation sequences.*

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences.

Proof.

Let M be a left A-module.

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_R A \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences.

Proof.

Let M be a left A-module. In order to construct a special precover sequence for M, we start from the projective system of left A-modules

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the Ext^1-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences.

Proof.

Let M be a left A-module. In order to construct a special precover sequence for M, we start from the projective system of left A-modules

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

Combined result

Proposition

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{Mod}$ be an Ext^1*-orthogonal pair of classes admitting approximation sequences. Assume that ${}_RA \in \mathcal{F}$ and the condition (††) holds. Assume further that the class $\mathcal{F}$ is resolving in $R\text{-}\mathrm{Mod}$ and the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the* Ext^1*-orthogonal pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\operatorname{Hom}_R(A,\mathcal{C})) \subset A\text{-}\mathrm{Mod}$ admits approximation sequences.*

Proof.

Let M be a left A-module. In order to construct a special precover sequence for M, we start from the projective system of left A-modules

$$M \longleftarrow Q(M) \longleftarrow Q(Q(M)) \longleftarrow \cdots \longleftarrow Q^n(M) \longleftarrow \cdots$$

indexed by the ordinal ω.

Combined result

Proof of Proposition cont'd.

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Put $N = \varprojlim_n Q^n(M)$.

Combined result

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Put $N = \varprojlim_n Q^n(M)$. As we have seen, the underlying R-module of N is isomorphic to $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M)))$.

Combined result

Proof of Proposition cont'd.

Put $N = \varprojlim_n Q^n(M)$. As we have seen, the underlying R-module of N is isomorphic to $\varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M)))$. Furthermore, we have a short exact sequence of left R-modules

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$$0 \longrightarrow \varprojlim_n \operatorname{Hom}_R(A, F(Q^n(M))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow \prod_{n=0}^{\infty} \operatorname{Hom}_R(A, F(Q^n(M)))) \longrightarrow 0.$$

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The left R-modules $\operatorname{Hom}_R(A, F(Q^n(M)))$ belong to $\mathcal{F}$ by (††)

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The left R-modules $\mathrm{Hom}_R(A, F(Q^n(M)))$ belong to $\mathcal{F}$ by (††), hence the left R-module $\prod_{n=0}^{\infty} \mathrm{Hom}_R(A, F(Q^n(M))))$ has $\mathcal{F}$-resolution dimension $\leqslant k$ by assumption.

Combined result

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As the class of all left R-modules of $\mathcal{F}$-resolution dimension $\leqslant k$ is resolving by lemma

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As the class of all left R-modules of $\mathcal{F}$-resolution dimension $\leqslant k$ is resolving by lemma, it follows that the $\mathcal{F}$-resolution dimension of $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \simeq N$

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As the class of all left R-modules of $\mathcal{F}$-resolution dimension $\leqslant k$ is resolving by lemma, it follows that the $\mathcal{F}$-resolution dimension of $\varprojlim_n \mathrm{Hom}_R(A, F(Q^n(M))) \simeq N$ does not exceed k.

Combined result

Proof of Proposition fin'd.

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Now we make a further sequence of k iterations

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Following the above argument, applying Q lowers the $\mathcal{F}$-resolution dimension, hence $Q^k(N) \in \mathcal{F}_A$.

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$Q^k(N) \longrightarrow N$

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Finally, we have surjective A-module maps $Q^k(N) \longrightarrow N = \varprojlim_n Q^n(M) \longrightarrow M$.

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Finally, we have surjective A-module maps $Q^k(N) \longrightarrow N = \varprojlim_n Q^n(M) \longrightarrow M$. The kernel of the map $Q^k(N) \longrightarrow N$ belongs to $\mathrm{Cof}_k(\mathrm{Hom}_R(A, \mathcal{C}))$

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In the assumptions of the theorem, one has
$\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A,\mathcal{C}))^{\oplus}$.

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Let $(\mathcal{F}, \mathcal{C})$ be a hereditary complete cotorsion pair in $R\text{-}\mathrm{Mod}$. Assume that ${}_RA \in \mathcal{F}$ and the condition $(\dagger\dagger)$ holds. Assume further that the $\mathcal{F}$-resolution dimension of any countable product of modules from $\mathcal{F}$ does not exceed a fixed finite integer k. Then the pair of classes $\mathcal{F}_A$ and $\mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$ is a hereditary complete cotorsion pair in $A\text{-}\mathrm{Mod}$. □

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In the assumptions of the theorem, one has $\mathcal{F}_A^{\perp_1} = \mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$. In particular, it follows that $\mathrm{Cof}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus = \mathrm{Cof}_{\omega+k}(\mathrm{Hom}_R(A, \mathcal{C}))^\oplus$.

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If the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A \otimes_R \mathcal{S}$.

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If the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A \otimes_R \mathcal{S}$. In particular, if $(\mathcal{F}, C)$ is generated by a set $\mathcal{S}$

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If the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A \otimes_R \mathcal{S}$. In particular, if $(\mathcal{F}, C)$ is generated by a set $\mathcal{S}$, then $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the set $\mathcal{S}^A$.

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Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$. Denote by $\mathcal{C}^A$ the class of all left A-modules whose underlying left R-modules belong to $\mathcal{C}$. Then there exists a cotorsion pair $(\mathcal{F}^A,\mathcal{C}^A)$ in $A\text{-}\mathrm{Mod}$ if and only if the left R-module $A^+ = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ belongs to $\mathcal{C}$. If this is the case (which we assume in the sequel) then the cotorsion pair $(\mathcal{F}^A,\mathcal{C}^A)$ is generated by the class $A\otimes_R\mathcal{F}$ of all left A-modules of the form $A\otimes_R F$ with $F\in\mathcal{F}$.

If the cotorsion pair $(\mathcal{F},\mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A,\mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A\otimes_R\mathcal{S}$. In particular, if $(\mathcal{F},C)$ is generated by a set $\mathcal{S}$, then $(\mathcal{F}^A,\mathcal{C}^A)$ is generated by the set $\mathcal{S}^A$. Hence $(\mathcal{F}^A,\mathcal{C}^A)$ is complete

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The dual results to the above three theorems are also provable using the dual constructions, but they are less surprising given the Eklof–Trlivaj theorem.

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $R\text{-}\mathrm{Mod}$. Denote by $\mathcal{C}^A$ the class of all left A-modules whose underlying left R-modules belong to $\mathcal{C}$. Then there exists a cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ in $A\text{-}\mathrm{Mod}$ if and only if the left R-module $A^+ = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ belongs to $\mathcal{C}$. If this is the case (which we assume in the sequel) then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $A \otimes_R \mathcal{F}$ of all left A-modules of the form $A \otimes_R F$ with $F \in \mathcal{F}$.

If the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is generated by a class of left R-modules $\mathcal{S}$, then the cotorsion pair $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the class $\mathcal{S}^A = A \otimes_R \mathcal{S}$. In particular, if $(\mathcal{F}, C)$ is generated by a set $\mathcal{S}$, then $(\mathcal{F}^A, \mathcal{C}^A)$ is generated by the set $\mathcal{S}^A$. Hence $(\mathcal{F}^A, \mathcal{C}^A)$ is complete and $\mathcal{F}^A = \mathrm{Fil}(\mathcal{S}^A \cup \{{}_AA\})^{\oplus}$.

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- P. C. Eklof, J. Trlifaj. How to make Ext vanish. *Bulletin of the London Math. Society* **33**, #1, p. 41–51, 2001.
- R. Göbel, J. Trlifaj. Approximations and endomorphism algebras of modules. Second Revised and Extended Edition. De Gruyter Expositions in Mathematics 41, De Gruyter, Berlin–Boston, 2012.
- L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Monografie Matematyczne IMPAN, vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. `arXiv:0708.3398 [math.CT]`
- L. Positselski. An explicit construction of complete cotorsion pairs in the relative context. Electronic preprint `arXiv:2006.01778 [math.RA]`, 29 pp.
- L. Salce. Cotorsion theories for abelian groups. *Symposia Math.* **XXIII**, Academic Press, London–New York, 1979, p. 11–32.