Derived adically complete modules and complexes

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Algebra seminar, Charles Univ., Prague (via Zoom)

March 23, 2020

The setting

Throughout this talk, R is a commutative ring, and $I \subset R$ is a finitely generated ideal.

Torsion modules

There is one category of I-torsion R-modules. An R-module M is I-torsion if for every $s \in I$, $m \in M$ there exists $n \geqslant 1$ such that $s^n m = 0$ in M.

$R\text{-}\mathrm{Mod}_{I\text{-tors}}$ is a coreflective subcategory (in fact, a hereditary pretorsion class) in $R\text{-}\mathrm{Mod}$ and a Grothendieck abelian category.

The coreflector $\Gamma_I \colon R\text{-}\mathrm{Mod} \longrightarrow R\text{-}\mathrm{Mod}_{I\text{-tors}}$ assigns to an R-module M its maximal I-torsion submodule $\Gamma_I(M) \subset M$.

Dual analogue of torsion modules?

There are three categories of $\approx I$-adically complete R-modules.

$R\text{-}\mathrm{Mod}^{\mathrm{sep}}_{I\text{-}\mathrm{ctra}} \subset R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}} \subset R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{ctra}} \subset R\text{-}\mathrm{Mod}$

Where $R\text{-}\mathrm{Mod}^{\mathrm{sep}}_{I\text{-}\mathrm{ctra}} = R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{secom}}$ is the category of I-adically separated & complete R-modules ($=$ separated I-contramodules).

$R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{ctra}}$ is the category of I-contramodule R-modules.

$R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}}$ is the category of quotseparated I-contramodule R-modules.

All full subcategories in each other. All reflective in $R\text{-}\mathrm{Mod}$ (and in each other). All locally $\aleph_1$-presentable categories.

$R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{secom}}$ is not abelian. $R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}}$ and $R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{ctra}}$ are abelian categories (closed under kernels and cokernels in $R\text{-}\mathrm{Mod}$).

I-adic completion functor

$\Lambda_I \colon R\text{-}\mathrm{Mod} \longrightarrow R\text{-}\mathrm{Mod}$, $\Lambda_I(C) = \varprojlim_{n\geqslant 1} C/I^nC$.

Completion morphism $\ell_{I,C} \colon C \longrightarrow \Lambda_I(C)$. An R-module C is called I-adically separated if $\ell_{I,C}$ is injective, and C is called I-adically complete if $\ell_{I,C}$ is surjective.

Λ_I is the reflector onto $R\text{-}\mathrm{Mod}_{I\text{-secom}} \subset R\text{-}\mathrm{Mod}$.

Λ_I is neither left nor right exact (in fact, not exact in the middle; but it takes epimorphisms to epimorphisms).

Explanation for nonexactness: Λ_I is the composition of right exact functor $(C \longmapsto C/I^nC)_{n=1}^{\infty}$ and left exact $\varprojlim$.

$R\text{-}\mathrm{Mod}_{I\text{-secom}}$ not abelian; not closed under cokers of monos, nor under extensions in $R\text{-}\mathrm{Mod}$.

What can one do? Replace Λ_I by a better-behaved derived functor.

Classical counterexample (A.-M. Simon, A. Yekutieli, …)

Take $R = \mathbb{Z}$ and $I = (p)$, where p is a prime number. Let $C \subset \prod_{n=0}^{\infty} \mathbb{Z}_p$ denote the group of all sequences of p-adic integers $u_0, u_1, u_2, \dots$ converging to zero in the topology of $\mathbb{Z}_p$.

Let $D \subset C$ denote the group of all sequences of p-adic integers of the form $v_0, pv_1, p^2v_2, \dots$, where $v_n \in \mathbb{Z}_p$. Let $E \subset D$ be the subgroup of all sequences $u_n = p^n v_n$ such that $v_n \to 0$ in $\mathbb{Z}_p$ as $n \to \infty$. So $D \simeq \prod_{n=0}^{\infty} \mathbb{Z}_p$ and $E \simeq C$.

All the three groups C, D, E are (p)-adically separated and complete. So are the quotient groups C/D and D/E. But the quotient group C/E is not (p)-adically separated (though still (p)-adically complete). In fact, one has $\bigcap_{n \geqslant 1} p^n(C/E) = D/E$, so $\Lambda_{(p)}(C/E) = C/D$.

Applying $\Lambda_{(p)}$ to the short exact sequence $0 \longrightarrow E \longrightarrow C \longrightarrow C/E \longrightarrow 0$, one obtains the sequence $0 \longrightarrow E \longrightarrow C \longrightarrow C/D \longrightarrow 0$, which is not exact in the middle. The inclusion $E \longrightarrow C$, viewed as a morphism in $\mathbb{Z}\text{-}\mathrm{Mod}_{(p)\text{-secom}}$, violates the abelian category axiom.

Idealistic approach (terminology of [Yek20])

$$\mathbb{L}\Lambda_I\colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$$

Constructed by applying Λ_I termwise to homotopy projective complexes of R-modules (homotopy flat is enough).

In particular, $C \in R\text{-}\mathrm{Mod} \rightsquigarrow \mathbb{L}_n\Lambda_I(C) = H_n\mathbb{L}\Lambda_I(C)$.

$\mathbb{L}_0\Lambda_I \neq \Lambda_I$, because Λ_I is not right exact.

$\mathbb{L}_0\Lambda_I$ is the reflector onto $R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-ctra}} =$

$$\{\mathrm{coker}_{R\text{-}\mathrm{Mod}}(f\colon C \to D) \mid C, D \in R\text{-}\mathrm{Mod}_{I\text{-secom}}\}.$$

Functor $\mathbb{L}_0\Lambda_I$ is right exact. Category $R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-ctra}}$ is abelian (closed under kernels and cokernels in $R\text{-}\mathrm{Mod}$).

Derived functor $\mathbb{L}\Lambda_I$ can be also defined as $\mathbb{L}(\mathbb{L}_0\Lambda_I)$. So one can use $\mathbb{L}_0\Lambda_I$ as an improved version of Λ_I. But there is also another such improved version, denoted by Δ_I.

Sequential approach (terminology of [Yek20])

Assume first that $I = (s) \subset R$ is a principal ideal.

Use $C^\bullet \longmapsto \mathrm{Tot}(\underset{-1}{C^\bullet} \xrightarrow{s^n} \underset{0}{C^\bullet})$ as a derived functor of $C \longmapsto C/s^nC$ and $\mathbb{R}\varprojlim_{n\geqslant 1} \mathrm{Tot}(C^\bullet \xrightarrow{s^n} C^\bullet)$ as a derived functor of s-adic completion. Here the projective system is

$$\begin{array}{ccc} C^\bullet & \xrightarrow{s^n} & C^\bullet \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 1} \\ C^\bullet & \xrightarrow{s^{n-1}} & C^\bullet \end{array}$$

Then one has $\mathbb{R}\varprojlim_{n\geqslant 1} \mathrm{Tot}(C^\bullet \xrightarrow{s^n} C^\bullet) = \mathbb{R}\mathrm{Hom}_R(K_\infty^\vee(R;s), C^\bullet)$, where $K_\infty^\vee(R;s) = (\underset{0}{R} \longrightarrow \underset{1}{R}[s^{-1}]) = \varinjlim_{n\geqslant 1}(R \xrightarrow{s^n} R)$ is the "infinite dual Koszul complex".

Generally, suppose $I = (s_1, \dots, s_m) \subset R$. For brevity, let $\mathbf{s}$ denote the sequence $s_1, \dots, s_m$. Put

$$K_\infty^\vee(R;\mathbf{s}) = (R \to R[s_1^{-1}]) \otimes_R \cdots \otimes_R (R \to R[s_m^{-1}]).$$

So $K_\infty^\vee(R;\mathbf{s})$ is a bounded complex of flat R-modules sitting in the cohomological degrees $0, \dots, m$. Up to quasi-isomorphism, the complex $K_\infty^\vee(R;\mathbf{s})$ is determined by (the radical of) the ideal I, and does not depend on the generators $s_1, \dots, s_m$.

Use $C^\bullet \longmapsto \mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}), C^\bullet)$ as the derived functor of I-adic completion. In particular, $C \in R\text{-}\mathrm{Mod} \rightsquigarrow$

$$\Delta_I(C) = H_0\mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}), C).$$

$\Delta_I \colon R\text{-}\mathrm{Mod} \longrightarrow R\text{-}\mathrm{Mod}$ is the reflector onto $R\text{-}\mathrm{Mod}_{I\text{-ctra}} =$

$$\{C \in R\text{-}\mathrm{Mod} \mid \operatorname{Ext}_R^{0,1}(R[s^{-1}], C) = 0 \quad \forall s \in I\}$$

(it suffices to check this condition for $s = s_1, \dots, s_m$).

The functor Δ_I is right exact by construction. The category of I-contramodule R-modules $R\text{-}\mathrm{Mod}_{I\text{-ctra}}$ is abelian (closed under kernels, cokernels, and extensions in $R\text{-}\mathrm{Mod}$).

Sequential vs. idealistic: comparison

Sequential approach "ignores" the ring R and only cares about $s_1, \ldots, s_m$ (still does not depend on the choice of a sequence $\mathbf{s}$ for a fixed ideal $I \subset R$).

Idealistic approach is sensitive to R (because what are projective or flat R-modules, or homotopy projective/flat complexes of R-modules, depends very much on R).

If $R' \subset R$ is a subring containing $s_1, \ldots, s_m$, and $I' \subset R'$ is the ideal generated by $s_1, \ldots, s_m$, then sequential functors for $I \subset R$ and for $I' \subset R'$ agree. Idealistic ones don't.

Separated and quotseparated contramodules

All I-adically separated and complete R-modules are I-contramodules. All I-contramodules are I-adically complete, but they don't need to be I-adically separated.

Hence "I-adically separated and complete R-modules" $=$ "(I-adically) separated I-contramodule R-modules".

A better definition of $R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-ctra}}$: an I-contramodule is quotseparated if it is a quotient of a separated I-contramodule.

Lemma

Any I-contramodule R-module is an extension of two quotseparated I-contramodule R-modules.

Sketch of proof.

For any R-module C, one has $0 \to K \longrightarrow \Delta_I(C) \longrightarrow \Lambda_I(C) \to 0$, where K is a quotseparated I-contramodule. In particular, if C is an I-contramodule, then $\Delta_I(C) = C$, so C is an extension of one quotseparated and one separated I-contramodule. □

Separated and quotseparated contramodules

The properties of an R-module to be (1) an I-contramodule, or (2) a separated I-contramodule – are not sensitive to the ring R.

Let $R' \subset R$ be a subring containing $s_1, \dots, s_m$ and $I' \subset R'$ be the ideal generated by $s_1, \dots, s_m$.

Then an R-module is an I-contramodule if and only if it is an I'-contramodule R'-module. An R-module is a separated I-contramodule if and only if it is a separated I'-contramodule R'-module.

The property of an R-module to be (3) a quotseparated I-contramodule – is sensitive to the ring R. Choose R' as above to be Noetherian (e.g., generated by $s_1, \dots, s_m$ over $\mathbb{Z}$). Then all I'-contramodule R'-modules are quotseparated.

So any I-contramodule can be made quotseparated by restricting the ring (replacing it by a subring).

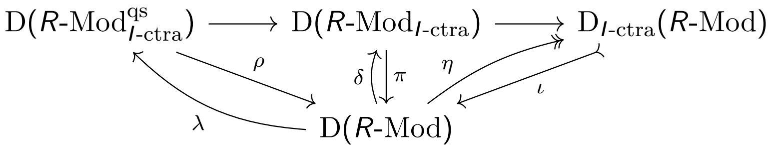
Overview of derived complete complexes

There are three reasonable triangulated categories of $\approx$ derived I-adically complete complexes of R-modules.

$\mathrm{D}(R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}})$ and $\mathrm{D}(R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{ctra}})$ are the derived categories of the respective abelian categories.

$\mathrm{D}_{I\text{-}\mathrm{ctra}}(R\text{-}\mathrm{Mod})$ is the full subcategory in $\mathrm{D}(R\text{-}\mathrm{Mod})$ consisting of all complexes with I-contramodule cohomology modules.

There is a diagram of triangulated functors



Here straight arrows form a commutative diagram. The curvilinear arrows show left adjoint functors.

Derived completion functors

The sequential derived completion functor

$$\mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}),-)\colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$$

is the composition of two adjoint functors $\mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}),-) = \iota\circ\eta$. It is the reflector onto the full subcategory $\mathrm{D}_{I\text{-}\mathrm{ctra}}(R\text{-}\mathrm{Mod}) \subset \mathrm{D}(R\text{-}\mathrm{Mod})$. Hence the sequential derived completion functor is idempotent.

The idealistic derived completion functor

$$\mathbb{L}\Lambda_I\colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$$

is the composition of two adjoint functors $\mathbb{L}\Lambda_I = \rho\circ\lambda$. The point is that applying Λ_I to every term of a homotopy flat complex of R-modules produces a complex in $R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{secom}} \subset R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}}$, which is an object of $\mathrm{D}(R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}})$.

Overview of derived complete complexed cont'd

The terminology of [Yek20]: $C^\bullet \in \mathrm{D}(R\text{-}\mathrm{Mod})$ is called derived I-adically complete in the idealistic sense if the natural morphism $C^\bullet \longrightarrow \mathbb{L}\Lambda_I(C^\bullet)$ is an isomorphism.

$C^\bullet \in \mathrm{D}(R\text{-}\mathrm{Mod})$ is called derived I-adically complete in the sequential sense if the natural morphism $C^\bullet \longrightarrow \mathbb{R}\,\mathrm{Hom}_R(K_\infty^\vee(R;\mathbf{s}), C^\bullet)$ is an isomorphism.

The full subcategory of derived complete complexes in the sequential sense is $\mathrm{D}_{I\text{-}\mathrm{ctra}}(R\text{-}\mathrm{Mod}) \subset \mathrm{D}(R\text{-}\mathrm{Mod})$.

The full subcategory of derived complete complexes in the idealistic sense is always contained in $\mathrm{D}_{I\text{-}\mathrm{ctra}}(R\text{-}\mathrm{Mod})$. This full subcategory seems to be too small in general.

"Idealistic is not realistic" or "not reasonable". Or "should be properly understood". $\mathrm{D}(R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}})$ is a good replacement of the category of derived I-adically complete complexes in the idealistic sense, but it is not a full subcategory in $\mathrm{D}(R\text{-}\mathrm{Mod})$.

Key technical condition: Weak proregularity

For a single element $s \in R$, the Koszul complex $K(R;s)$ is the two-term complex $\underset{-1}{R} \overset{s}{\longrightarrow} \underset{0}{R}$. For a sequence of elements $s_1, \ldots, s_m \in R$, the Koszul complex is

$$K(R;\mathbf{s}) = K(R;s_1) \otimes_R \cdots \otimes_R K(R;s_m).$$

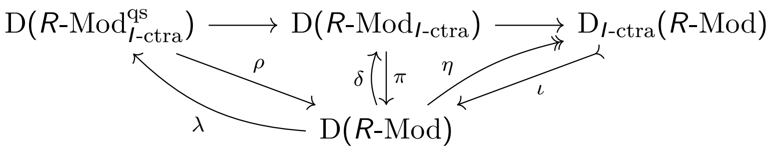
Given $n \geqslant 1$, denote by $\mathbf{s}^n$ the sequence $s_1^n, \ldots, s_m^n$. When n varies, the Koszul complexes $K(R;\mathbf{s}^n)$ form a projective system.

A finitely generated ideal $I \subset R$ is said to be weakly proregular if, for every $i < 0$, the projective system of homology modules $(H^i(K(R;\mathbf{s}^n)))_{n\geqslant 1}$ is pro-zero (= "trivial Mittag-Leffler").

Here a projective system of abelian groups $(E_n)_{n\geqslant 1}$ is said to be pro-zero if for every $j \geqslant 1$ there exists $k > j$ such that the transition map $E_k \longrightarrow E_j$ vanishes.

The weak proregularity property only depends on (the radical of) the ideal I, and not on a particular set of its generators $\mathbf{s}$.

When the ideal I is weakly proregular, the sequential and idealistic approaches agree. In the above diagram



both the horizontal arrows are triangulated equivalences. So one has

$$\rho = \pi = \iota, \qquad \lambda = \delta = \eta$$

and also

$$R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-ctra}} = R\text{-}\mathrm{Mod}_{I\text{-ctra}}, \qquad \mathbb{L}_0\Lambda_I = \Delta_I$$

as well as

$$\mathbb{L}\Lambda_I = \mathbb{R}\,\mathrm{Hom}_R(K^\vee_\infty(R;\mathbf{s}), -).$$

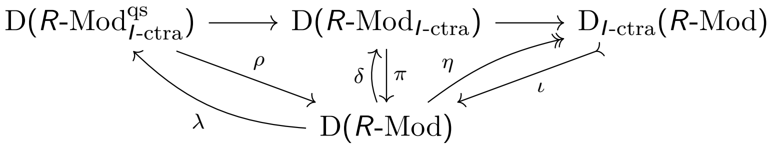
When the ideal I is not weakly proregular, one has that

- the functor ρ is not fully faithful;
- the complex $\mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}), R^{(\omega)})$ is derived I-adically complete in the sequential, but not in the idealistic sense;
- even the R-module $\Lambda_I(R^{(\omega)})$, viewed as a complex of R-modules, is derived I-adically complete in the sequential, but not in the idealistic sense.

Here $R^{(\omega)} = \bigoplus_{i=0}^{\infty} R$ is the free R-module with a countable set of generators. The R-module $\Lambda_I(R^{(\omega)})$ is sometimes called the module of decaying functions $\omega \longrightarrow \Lambda_I(R)$.

It follows (from the last item in the list) that the functor $\mathbb{L}\Lambda_I\colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$ is not idempotent. In fact, the complex $\mathbb{L}\Lambda_I(\mathbb{L}\Lambda_I(R^{(\omega)})) = \mathbb{L}\Lambda_I(\Lambda_I(R^{(\omega)}))$ has a nonzero cohomology module in some nonzero (negative) cohomological degree.

Let us return once more to the diagram



The weak proregularity condition for $I = (s_1, \dots, s_m) \subset R$ splits into $2m$ pieces. The 1st piece is responsible for

$$R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-ctra}} = R\text{-}\mathrm{Mod}_{I\text{-ctra}},$$
$$\mathbb{L}_0\Lambda_I = \Delta_I, \qquad \lambda = \delta, \qquad \rho = \pi.$$

The remaining $2m - 1$ pieces are responsible for

$$\mathrm{D}(R\text{-}\mathrm{Mod}_{I\text{-ctra}}) \simeq \mathrm{D}_{I\text{-ctra}}(R\text{-}\mathrm{Mod}),$$
$$\delta = \eta, \qquad \pi = \iota.$$

Details on weak proregularity

For any R-modules M and C, there are natural morphisms in $\mathrm{D}(R\text{-}\mathrm{Mod})$

$$\Gamma_I(M) \longrightarrow K_\infty^\vee(R;\mathbf{s}) \otimes_R M; \qquad (1)$$

$$\mathbb{R}\operatorname{Hom}_R(K_\infty^\vee(R;\mathbf{s}), C) \longrightarrow \Lambda_I(C). \qquad (2)$$

The morphism (1) is an isomorphism on H^0. Taking H^0 of the morphism (2), one obtains the natural surjective morphism $b_{I,C}\colon \Delta_I(C) \longrightarrow \Lambda_I(C)$ (mentioned above in the proof of Lemma).

Theorem

The following conditions are equivalent:

- *the ideal $I \subset R$ is weakly proregular;*
- *for every injective R-module J and $i > 0$ one has $H^i(K_\infty^\vee(R;\mathbf{s}) \otimes_R J) = 0$ (equivalently, the morphism (1) is a quasi-isomorphism for $M = J$);*
- *the morphism (2) is a quasi-isomorphism for $C = R^{(\omega)}$.*

The infinite dual Koszul complex is the direct limit of finite dual Koszul complexes, $K^\vee_\infty(R;\mathbf{s}) = \varinjlim_n \operatorname{Hom}_R(K(R;\mathbf{s}^n), R)$. Hence for any R-module C and every $-m \leqslant q \leqslant 0$ there is a short exact seq.

$$0 \longrightarrow \varprojlim\nolimits^1_{n\geqslant 1} H^{q-1}(K(R;\mathbf{s}^n)\otimes_R C) \longrightarrow H^q\,\mathbb{R}\operatorname{Hom}_R(K^\vee_\infty(R;\mathbf{s}),\ C) \longrightarrow \varprojlim\nolimits_{n\geqslant 1} H^q(K(R;\mathbf{s}^n)\otimes_R C) \longrightarrow 0.$$

For $q=0$, the middle term is $\Delta_I(C)$ and the rightmost term is $\Lambda_I(C)$. The proof of the above Theorem is partly based on

Proposition

The ideal $I \subset R$ is weakly proregular iff $\forall\, k=-1,\dots,-m$ two conditions hold:

1. $\varprojlim^1_{n\geqslant 1}(H^k(K(R;\mathbf{s}^n)\otimes_R R^{(\omega)})) = 0;$
2. $\varprojlim_{n\geqslant 1} H^k(K(R;\mathbf{s}^n)) = 0.$

These are the above-mentioned "$2m$ pieces of weak proregularity". The "1st piece" is (i) for $k=-1$.

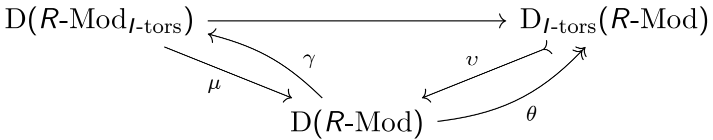
Derived torsion complexes

There are two reasonable triangulated categories of $\approx$ derived I-torsion complexes.

$\mathrm{D}(R\text{-}\mathrm{Mod}_{I\text{-tors}})$ is the derived category of the abelian category of I-torsion R-modules.

$\mathrm{D}_{I\text{-tors}}(R\text{-}\mathrm{Mod})$ is the full subcategory in $\mathrm{D}(R\text{-}\mathrm{Mod})$ consisting of all complexes with I-torsion cohomology modules.

There is a diagram of triangulated functors



Here straight arrows form a commutative triangular diagram. The curvilinear arrows show right adjoint functors. The arrow with a tail shows a fully faithful functor. The arrow with two heads shows a Verdier quotient functor.

Derived torsion functors

The sequential derived torsion functor is

$$K^\vee_\infty(R;\mathbf{s})\otimes_R - \colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod}).$$

It is the composition of two adjoint functors $K^\vee_\infty(R;\mathbf{s})\otimes_R - = \upsilon\circ\theta$. This is the reflector onto the full subcategory $\mathrm{D}_{I\text{-}\mathrm{tors}}(R\text{-}\mathrm{Mod})\subset \mathrm{D}(R\text{-}\mathrm{Mod})$. Hence the sequential derived torsion functor is idempotent.

The idealistic derived torsion functor

$$\mathbb{R}\Gamma_I\colon \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$$

is constructed by applying Γ_I termwise to homotopy injective complexes of R-modules. This functor is the composition of two adjoint functors $\mathbb{R}\Gamma_I = \mu\circ\gamma$.

Derived torsion complexes cont'd

The terminology of [Yek20]: $M^\bullet \in \mathrm{D}(R\text{-}\mathrm{Mod})$ is called derived I-torsion in the idealistic sense if the natural morphism $\mathbb{R}\Gamma_I(M^\bullet) \longrightarrow M^\bullet$ is an isomorphism.

$M^\bullet \in \mathrm{D}(R\text{-}\mathrm{Mod})$ is called derived I-torsion in the sequential sense if the natural morphism $K_\infty^\vee(R;\mathbf{s}) \otimes_R M^\bullet \longrightarrow M^\bullet$ is an isomorphism.

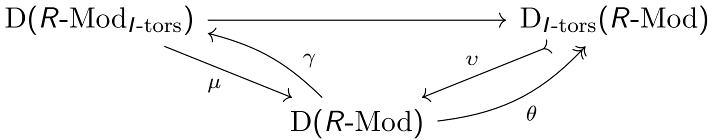
The full subcategory of derived torsion complexes in the sequential sense is $\mathrm{D}_{I\text{-tors}}(R\text{-}\mathrm{Mod}) \subset \mathrm{D}(R\text{-}\mathrm{Mod})$.

The full subcategory of derived torsion complexes in the idealistic sense is always contained in $\mathrm{D}_{I\text{-tors}}(R\text{-}\mathrm{Mod})$. This full subcategory seems to be too small in general.

"Idealistic is not realistic", or "should be properly understood". $\mathrm{D}(R\text{-}\mathrm{Mod}_{I\text{-tors}})$ is a good replacement of the category of derived I-torsion complexes in the idealistic sense, but it is not a full subcategory in $\mathrm{D}(R\text{-}\mathrm{Mod})$.

Derived torsion complexes cont'd

When the ideal $I \subset R$ is weakly proregular, the sequential and idealistic approaches agree. In the above diagram



the horizontal arrow is a triangulated equivalence. So one has

$$\mu = \upsilon \quad \text{and} \quad \gamma = \theta$$

as well as

$$\mathbb{R}\Gamma_I = K^\vee_\infty(R; \mathbf{s}) \otimes_R -.$$

When the ideal I is not weakly proregular, one has that

- the functor μ is not fully faithful;
- there exists an injective R-module J for which the complex $K_\infty^\vee(R;\mathbf{s})\otimes_R J$ is not derived I-torsion in the idealistic sense (though $K_\infty^\vee(R;\mathbf{s})\otimes_R M^\bullet$ is derived I-torsion in the sequential sense for all $M^\bullet\in\mathrm{D}(R\text{-}\mathrm{Mod})$);
- there even exists an injective R-module J for which the R-module $\Gamma_I(J)$, viewed as a complex of R-modules, is not derived I-torsion in the idealistic sense (though any complex of I-torsion R-modules is derived I-torsion in the sequential sense).

It follows from the second item that the functor $\mathbb{R}\Gamma_I\colon\mathrm{D}(R\text{-}\mathrm{Mod})\longrightarrow\mathrm{D}(R\text{-}\mathrm{Mod})$ is not idempotent. In fact, the complex $\mathbb{R}\Gamma_I(\mathbb{R}\Gamma_I(J))=\mathbb{R}\Gamma_I(\Gamma_I(J))$ has a nonzero cohomology module in some nonzero (positive) cohomological degree.

Final Questions and Conclusion

All ideals in Noetherian commutative rings are weakly proregular. The following questions, therefore, apply to non-Noetherian commutative rings R.

Given an arbitrary finitely generated ideal $I \subset R$,

- can one come up with an example of a complex of R-modules that is derived I-adically complete in the idealistic sense? Does there exist such a nonzero complex, generally speaking?
- dually, can one come up with an example of a complex of R-modules that is derived I-torsion in the idealistic sense? Does there exist such a nonzero complex?

The categories $\mathrm{D}(R\text{-}\mathrm{Mod}^{\mathrm{qs}}_{I\text{-}\mathrm{ctra}})$ and $\mathrm{D}(R\text{-}\mathrm{Mod}_{I\text{-}\mathrm{tors}})$ aren't full subcategories in $\mathrm{D}(R\text{-}\mathrm{Mod})$, when I is not weakly proregular. The advantage of considering them as the proper versions of the idealistic derived complete/torsion categories lies in the fact that they contain the objects one wants them to contain.

- J. P. C. Greenlees, J. P. May. Derived functors of I-adic completion and local homology. *Journ. of Algebra* **149**, #2, p. 438–453, 1992.
- P. Schenzel. Proregular sequences, local cohomology, and completion. *Math. Scand.* **92**, #2, p. 161–180, 2003.
- A.-M. Simon. Approximations of complete modules by complete big Cohen–Macaulay modules over a Cohen–Macaulay local ring. *Algebras and Representation Theory* **12**, #2–5, p. 385–400, 2009.
- A. Yekutieli. On flatness and completion for infinitely generated modules over noetherian rings. *Communicat. in Algebra* **39**, #11, p. 4221–4245, 2011. `arXiv:0902.4378 [math.AC]`
- M. Kashiwara, P. Schapira. Deformation quantization modules. *Astérisque* **345**, Soc. Math. de France, 2012, vi+147 pp. `arXiv:1003.3304 [math.AG]`

A. Beilinson. *p*-adic periods and derived de Rham cohomology. *Journ. of the Amer. Math. Soc.* **25**, #3, p. 715–738, 2012.

B. Bhatt, P. Scholze. The pro-étale topology for schemes. Electronic preprint arXiv:1309.1198 [math.AG].

M. Porta, L. Shaul, A. Yekutieli. On the homology of completion and torsion. *Algebras and Represent. Theory* **17**, #1, p. 31–67, 2014. arXiv:1010.4386 [math.AC]. Erratum in *Algebras and Represent. Theory* **18**, #5, p. 1401–1405, 2015. arXiv:1506.07765 [math.AC]

L. Positselski. Dedualizing complexes and MGM duality. *Journ. of Pure and Appl. Algebra* **220**, #12, p. 3866–3909, 2016. arXiv:1503.05523 [math.CT]

L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. *Moscow Math. Journ.* **17**, #3, p. 385–455, 2017. arXiv:1605.03934 [math.CT]

L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint arXiv:1705.04960 [math.CT].

A. Yekutieli. Flatness and completion revisited. *Algebras and Represent. Theory* **21**, #4, p. 717–736, 2018. arXiv:1606.01832 [math.AC]

M. Hrbek, J. Šťovíček. Tilting classes over commutative rings. *Forum Math.* **32**, #1, p. 235–267, 2020. arXiv:1701.05534 [math.AC]

A. Yekutieli. Weak proregularity, derived completion, adic flatness, and prisms. Electronic preprint arXiv:2002.04901 [math.AC].

L. Positselski. Remarks on derived complete modules and complexes. Electronic preprint arXiv:2002.12331 [math.AC].