Derived adically complete modules and complexes

Leonid Positselski – IM AV ČR

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The coreflector $\Gamma_I \colon R\text{-}\mathrm{Mod} \longrightarrow R\text{-}\mathrm{Mod}_{I\text{-tors}}$ assigns to an R-module M its maximal I-torsion submodule $\Gamma_I(M) \subset M$.

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$R\text{-}\mathrm{Mod}_{I\text{-secom}}$ is not abelian. $R\text{-}\mathrm{Mod}_{I\text{-ctra}}^{\mathrm{qs}}$ and $R\text{-}\mathrm{Mod}_{I\text{-ctra}}$ are abelian categories (closed under kernels and cokernels in $R\text{-}\mathrm{Mod}$).

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What can one do? Replace Λ_I by a better-behaved derived functor.

Classical counterexample (A.-M. Simon, A. Yekutieli, ...)

Take $R = \mathbb{Z}$ and $I = (p)$, where p is a prime number. Let $C \subset \prod_{n=0}^{\infty} \mathbb{Z}_p$ denote the group of all sequences of p-adic integers $u_0, u_1, u_2, \dots$ converging to zero in the topology of $\mathbb{Z}_p$.

Let $D \subset C$ denote the group of all sequences of p-adic integers of the form $v_0, pv_1, p^2v_2, \dots$, where $v_n \in \mathbb{Z}_p$. Let $E \subset D$ be the subgroup of all sequences $u_n = p^n v_n$ such that $v_n \to 0$ in $\mathbb{Z}_p$ as $n \to \infty$. So $D \simeq \prod_{n=0}^{\infty} \mathbb{Z}_p$ and $E \simeq C$.

All the three groups C, D, E are (p)-adically separated and complete. So are the quotient groups C/D and D/E. But the quotient group C/E is not (p)-adically separated (though still (p)-adically complete). In fact, one has $\bigcap_{n \geqslant 1} p^n(C/E) = D/E$, so $\Lambda_{(p)}(C/E) = C/D$.

Applying $\Lambda_{(p)}$ to the short exact sequence $0 \longrightarrow E \longrightarrow C \longrightarrow C/E \longrightarrow 0$, one obtains the sequence $0 \longrightarrow E \longrightarrow C \longrightarrow C/D \longrightarrow 0$, which is not exact in the middle. The inclusion $E \longrightarrow C$, viewed as a morphism in $\mathbb{Z}\text{-}\mathrm{Mod}_{(p)\text{-secom}}$, violates the abelian category axiom.

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Derived functor $\mathbb{L}\Lambda_I$ can be also defined as $\mathbb{L}(\mathbb{L}_0\Lambda_I)$. So one can use $\mathbb{L}_0\Lambda_I$ as an improved version of Λ_I. But there is also another such improved version, denoted by Δ_I.

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Then one has $\mathbb{R}\varprojlim_{n\geqslant 1} \mathrm{Tot}(C^\bullet \xrightarrow{s^n} C^\bullet) = \mathbb{R}\mathrm{Hom}_R(K_\infty^\vee(R;s), C^\bullet)$, where $K_\infty^\vee(R;s) = (\underset{0}{R} \longrightarrow \underset{1}{R[s^{-1}]}) = \varinjlim_{n\geqslant 1}(R \xrightarrow{s^n} R)$ is the "infinite dual Koszul complex".

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So any I-contramodule can be made quotseparated by restricting the ring (replacing it by a subring).

Overview of derived complete complexes

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The weak proregularity property only depends on (the radical of) the ideal I, and not on a particular set of its generators $\mathbf{s}$.

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It follows from the second item that the functor $\mathbb{R}\Gamma_I : \mathrm{D}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{D}(R\text{-}\mathrm{Mod})$ is not idempotent. In fact, the complex $\mathbb{R}\Gamma_I(\mathbb{R}\Gamma_I(J)) = \mathbb{R}\Gamma_I(\Gamma_I(J))$ has a nonzero cohomology module in some nonzero (positive) cohomological degree.

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